Convergence and error estimates for the compressible Navier-Stokes equations

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Compressible (Isentropic) Navier-Stokes Equations (CNS)

Ω is a bounded open connected set of $\mathbb{R}^3$, with a Lipschitz continuous boundary, $T > 0$, $\gamma > 3/2$, $f \in L^2(]0,T[, L^2(\Omega)^3)$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad \textit{mass}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \Delta u + \nabla p = f \qquad \textit{momentum}$$

$$p = \rho^\gamma \qquad \textit{EOS}$$

Dirichlet boundary condition : $u = 0$

Initial condition on ρ and u (or on ρu)

Stationary compressible Navier Stokes equations (SCNS)

Ω is a bounded open set of $\mathbb{R}^3$, with a Lipschitz continuous boundary, $\gamma > 3/2$, $f \in L^2(\Omega)^3$ and $M > 0$

$$\begin{aligned} &\mathrm{div}(\rho u) = 0 && \textit{mass} \\ &\mathrm{div}(\rho u \otimes u) - \Delta u + \nabla p = f && \textit{momentum} \\ &p = \rho^\gamma && \textit{EOS} \end{aligned}$$

Dirichlet boundary condition : $u = 0$

$\rho \geq 0$, $\int_\Omega \rho(x) dx = M$

Functional spaces : $u \in H_0^1(\Omega)^3$, $p \in L^{\overline{q}}(\Omega)$, $\rho \in L^{\gamma\overline{q}}(\Omega)$

If $\gamma \geq 3$ then $\overline{q} = 2$

If $\frac{3}{2} < \gamma < 3$: $\overline{q} = \frac{3(\gamma-1)}{\gamma}$

$\gamma = \frac{3}{2}$, $p \in L^1$, $\rho \in L^{3/2}$, $\rho u \otimes u \in L^1$ since $u \in (L^6)^3$

Questions

In both cases (CNS and SCNS), existence of a weak solution is known (but no uniqueness)

We use a space discretization with the MAC scheme and an implicit discretization in time (for the evolution case)

Questions :

1. Is it possible to prove convergence (up to the subsequence) of the approximate solution to the exact (weak) solution as the mesh size goes to 0 (and also the time step in the evolution case) ?
2. In case of uniqueness of the exact solution, is it possible to obtain error estimates ?

Results and open problems

1. For SCNS, we prove convergence (up to the subsequence) of the approximate solution to the exact (weak) solution as the mesh size goes to 0 for $\gamma > 3$. Open problem for $3/2 < \gamma \leq 3$
   This convergence proof also gives existence of a weak solution
2. For CNS, the convergence is probably true, but we do not have a complete proof
3. For CNS, if the exact solution is regular, we obtain an error estimate (and this gives uniqueness) for $\gamma > 3/2$
4. For SCNS (and a regular solution), we are not able to obtain error estimate

Error estimate for CNS

For CNS the proof of error estimate (comparison of an exact "strong" solution and an approximate solution) is very close to the weak-strong uniqueness principle (comparison of a "strong" solution and a weak solution)

Prodi, Serrin for Incompressible NS ($\sim$ 1960)
Germain for Isentropic CNS (2011), Feireisl-Novotny (other EOS)

The proof uses the so-called "relative entropy" (introduced Dafermos for Euler Equations) or "modulated energy"

relative energy

Weak-strong uniqueness principle, simple case

Stokes Equations, $\gamma = 2$, $f = 0$

Ω is a bounded open connected set of $\mathbb{R}^3$, with a Lipschitz continuous boundary, $T > 0$, $\gamma = 2$, $f \in L^2(]0, T[, L^2(\Omega)^3)$

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}(\rho u) = 0, && \textit{mass} \\
&\partial_t u - \Delta u + \nabla p = 0, && \textit{momentum} \\
&p = \rho^2. && \textit{EOS}
\end{aligned}
$$

Dirichlet boundary condition : $u = 0$

Initial condition on ρ and u (ρ_0 and u_0)

Weak-strong uniqueness, simple case, Energy Equalities

Stokes Equations, $\gamma = 2$, $f = 0$
$\bar{u}$, $\bar{p}$, $\bar{\rho}$: strong solution
u, p, ρ : "suitable" weak solution
Energy Equalities (formally taking u as test function in the momentum equation for u)

$$\frac{1}{2}\int_{\Omega}|u|^2(T)+\int_0^T\int_{\Omega}(|\nabla u|^2-p\mathrm{div}\,u)=\frac{1}{2}\int_{\Omega}|u_0|^2$$

$$\frac{1}{2}\int_{\Omega}|\bar{u}|^2(T)+\int_0^T\int_{\Omega}(|\nabla \bar{u}|^2-\bar{p}\mathrm{div}\,\bar{u})=\frac{1}{2}\int_{\Omega}|u_0|^2$$

Weak-strong uniqueness, simple case, Relative Energy

Stokes Equations, $\gamma = 2$, $f = 0$. Relative Energy using Energy Equalities gives

$$E_T(\rho, u|\bar{\rho}, \bar{u}) = \frac{1}{2}\int_\Omega |u(T) - \bar{u}(T)|^2 + \int_\Omega (\rho(T) - \bar{\rho}(T))^2 =$$

$$- \int_\Omega u(T)\cdot \bar{u}(T) - \int_0^T \int_\Omega (|\nabla u|^2 - p\,\mathrm{div}\,u + |\nabla \bar{u}|^2 - \bar{p}\,\mathrm{div}\,\bar{u})$$

$$+ \int_\Omega |u_0|^2 + \int_\Omega (\rho(T) - \bar{\rho}(T))^2$$

$$\int_\Omega (\rho(T) - \bar{\rho}(T))^2 = \int_\Omega \rho^2(T) + \int_\Omega \bar{\rho}^2(T) - 2\int_\Omega \rho(T)\bar{\rho}(T)$$

Weak-strong uniqueness, simple case, $\rho^2(T), \bar{\rho}^2(T)$

Stokes Equations, $\gamma = 2$, $f = 0$

Using Mass Equations (formally ρ as test function in the equation for ρ) gives

$$\frac{1}{2}\int_\Omega \rho^2(T) - \frac{1}{2}\int_\Omega \rho_0^2 - \int_0^T \int_\Omega \rho u \cdot \nabla \rho = 0$$

But, since $\rho^2 = p$,

$$\int_0^T \int_\Omega \rho u \cdot \nabla \rho = \frac{1}{2}\int_0^T \int_\Omega u \cdot \nabla(\rho^2) = -\frac{1}{2}\int_0^T \int_\Omega p \,\mathrm{div}\, u$$

$$\int_\Omega \rho^2(T) = -\int_0^T \int_\Omega p \,\mathrm{div}\, u + \int_\Omega \rho_0^2$$

$$\int_\Omega \bar{\rho}^2(T) = -\int_0^T \int_\Omega \bar{p} \,\mathrm{div}\, \bar{u} + \int_\Omega \rho_0^2$$

Weak-strong uniqueness, simple case, $\rho(T)\bar{\rho}(T)$

Stokes Equations, $\gamma = 2$, $f = 0$
Using Mass Equations (taking ρ as test function in the equation for $\bar{\rho}$ and $\bar{\rho}$ as test function in the equation for ρ) gives

$$\int_0^T \int_\Omega (\partial_t \bar{\rho})\rho + \int_0^T \int_\Omega \mathrm{div}(\bar{\rho}\bar{u})\rho = 0$$

$$\int_0^T \int_\Omega (\partial_t \rho)\bar{\rho} - \int_0^T \int_\Omega \rho u \cdot \nabla \bar{\rho} = 0$$

Adding the two equations leads to

$$\int_\Omega \bar{\rho}(T)\rho(T) = \int_\Omega \rho_0^2 + \int_0^T \int_\Omega \rho u \cdot \nabla \bar{\rho} - \int_0^T \int_\Omega \mathrm{div}(\bar{\rho}\bar{u})\rho$$

Weak-strong uniqueness, simple case, $u(T)\cdot\bar{u}(T)$

Stokes Equations, $\gamma = 2$, $f = 0$
Using Momentum Equations (taking $\bar{u}$ as test function in the equation for u and u as test function in the equation for $\bar{u}$) gives

$$\int_0^T \int_\Omega (\partial_t u)\bar{u} + \int_0^T \int_\Omega (\nabla u : \nabla \bar{u} - p\,\mathrm{div}(\bar{u})) = 0$$

$$\int_0^T \int_\Omega (\partial_t \bar{u})u + \int_0^T \int_\Omega (\nabla u : \nabla \bar{u} - \bar{p}\,\mathrm{div}(u)) = 0$$

Adding the two equations leads to

$$\int_\Omega \bar{u}(T)\cdot u(T) = \int_\Omega |u_0|^2 + \int_0^T \int_\Omega (-2\nabla u : \nabla \bar{u} + p\,\mathrm{div}(\bar{u}) + \bar{p}\,\mathrm{div}(u))$$

Weak-strong uniqueness, simple case, Relative Energy(2)

Stokes Equations, $\gamma = 2$, $f = 0$. Relative Energy using Energy Equalities gives

$$E_T(\rho, u|\bar{\rho}, \bar{u}) = \frac{1}{2}\int_\Omega |u(T) - \bar{u}(T)|^2 + \int_\Omega (\rho(T) - \bar{\rho}(T))^2 =$$

$$- \int_\Omega u(T) \cdot \bar{u}(T) - \int_0^T \int_\Omega (|\nabla u|^2 - p\,\mathrm{div}\, u + |\nabla \bar{u}|^2 - \bar{p}\,\mathrm{div}\, \bar{u})$$

$$+ \int_\Omega |u_0|^2 + \int_\Omega (\rho(T) - \bar{\rho}(T))^2$$

$$\int_\Omega (\rho(T) - \bar{\rho}(T))^2 = \int_\Omega \rho^2(T) + \int_\Omega \bar{\rho}^2(T) - 2\int_\Omega \rho(T)\bar{\rho}(T)$$

Weak-strong uniqueness, simple case, Relative Energy(3)

Replacing the red quantities using the previous slides and using the EOS lead to

$$E_T(\rho, u|\bar\rho, \bar u) = \frac{1}{2}\int_\Omega |u(T) - \bar u(T)|^2 + \int_\Omega (\rho(T) - \bar\rho(T))^2 =$$

$$\int_0^T \int_\Omega (-|\nabla u - \nabla \bar u|^2 - (\rho - \bar\rho)^2 \mathrm{div}(\bar u) + 2(\bar\rho - \rho)(\bar u - u)\cdot\nabla\bar\rho)$$

$\mathrm{div}\, \bar u \in L^\infty(]0, T[\times\Omega)$, $\nabla\bar\rho \in L^\infty(]0, T[\times\Omega)$

$$\varphi(t) = E_t(\rho, u|\bar\rho, \bar u) = \frac{1}{2}\int_\Omega |u(t) - \bar u(t)|^2 + \int_\Omega (\rho(t) - \bar\rho(t))^2$$

Then previous equality (for any $0 \le t \le T$) gives

$$\varphi(t) \le C\int_0^t \varphi(s)ds$$

This gives (Gronwall Inequality) $\varphi(t) \le \varphi(0)e^{-Ct}$ and then $\varphi = 0$

Error estimate for CNS

We mimic the previous proof of uniqueness at the discrete level to obtain error estimate, (ρ, u) is now the solution of a numerical scheme

$$E_t(\rho, u|\bar{\rho}, \bar{u}) \leq C(h^\alpha + k^{1/2}) \text{ for } 0 \leq t \leq T$$

h is the mesh size, k is the time step

$\alpha = \min(\frac{2\gamma - 3}{\gamma}, \frac{1}{2})$

$\gamma > 3/2$

For $\gamma = 2$ and CNS, $\alpha = 1/2$, E_t is the L^2-norm of $(\rho - \bar{\rho})$ + the L^2-norm of $(u - \bar{u})$ weighted by ρ (and we have $\rho > 0$)

Error estimate for SCNS

No error estimate for SCNS

No weak-strong uniqueness principle

No Gronwall inequality

Question : What can play the role of Gronwall Inequality for stationary problems ?

Uniqueness for stationary problems, a simple example

$\varphi \in C(\mathbb{R}, \mathbb{R})$, Lipschitz continuous, $w \in L^\infty(\Omega)$,
$f \in L^2(]0, T[, L^2(\Omega))$, $u_0 \in L^2(\Omega)$
(No hypothesis on $div(w)$)

$$\begin{aligned} \partial_t u + \mathrm{div}(w\varphi(u)) - \Delta u = f \\ u(\cdot, t) = 0 \text{ on } \partial\Omega \\ u(\cdot, 0) = u_0 \end{aligned}$$

Uniqueness easily follows from Gronwall Inequality

Uniqueness for stationary problems, a simple example

$\varphi \in C(\mathbb{R}, \mathbb{R})$, Lipschitz continuous, $w \in L^\infty(\Omega)$,
$f \in L^2(\Omega)$
(No hypothesis on $div(w)$, no coercivity)

$$\begin{aligned} \mathrm{div}(w\varphi(u)) - \Delta u = f \\ u(\cdot, t) = 0 \text{ on } \partial\Omega \end{aligned}$$

Uniqueness can be proven taking $T_\varepsilon(u - \bar{u})$ ($\varepsilon > 0$) as test function and letting $\varepsilon \to 0$

$T_\varepsilon(s) = \max(-\varepsilon, \min(s, \varepsilon))$ for $s \in \mathbb{R}$

Convergence for SCNS

For the stationary compressible Navier-Stokes equations discretized with a MAC scheme, we prove convergence of the approximate solution (up to a subsequence) to a weak solution, in the case $\gamma > 3$, following the idea of P.L. Lions for proving existence of a solution.

Steps for proving the convergence result

1. Estimates on the approximate solution (u_n, p_n, ρ_n)
2. Compactness result (convergence of the approximate solution, up to a subsequence)
3. Passage to the limit in the approximate equations

Main difficulty: Passage to the limit in the EOS ($p = \rho^\gamma$) since the EOS is a non linear function and Step 2 only leads to weak convergences of p_n and ρ_n.

Convergence of u_n, p_n, ρ_n

Thanks to the estimates on u_n, p_n, ρ_n, it is possible to assume (up to a subsequence) that, as $n \to \infty$:

$$u_n \to u \text{ in } L^2(\Omega)^3,\ u \in H_0^1(\Omega)^3$$
$$p_n \to p \text{ weakly in } L^2(\Omega)$$
$$\rho_n \to \rho \text{ weakly in } L^{2\gamma}(\Omega)$$

Bound on u_n : $\|u_n\|_{H_n} \leq C$,
$\|\cdot\|_{H_n}$ is a so-called "discrete H_0^1-norm" (but depending on n)

Passage to the limit in the mass equation

$v \in C_c^\infty(\mathbb{R}^3)$

$$\int_\Omega \rho_n u_n \cdot \nabla v + R = 0$$

$\rho_n \to \rho$ weakly in $L^{2\gamma}(\Omega)$, with $2\gamma > \frac{3}{2}$, $u_n \to u$ in $L^q(\Omega)^3$ for all $q < 6$. Then $\rho_n u_n \to \rho u$ weakly in $L^1(\Omega)^3$. This gives $\int_\Omega \rho u \cdot \nabla v = 0$.

L^1-weak convergence of ρ_n gives positivity of ρ and convergence of mass:

$$\rho \geq 0 \text{ in } \Omega, \int_\Omega \rho(x) dx = M.$$

Passage to the limit in the momentum equation

$v \in C_c^\infty(\Omega)^3$,

$$\int_\Omega \nabla_n u_n : \nabla v\, dx - \int_\Omega \rho_n u_n \otimes u_n : \nabla v\, dx - \int_\Omega p_n \mathrm{div}(v) dx + R = \int_\Omega f_n \cdot v\, dx$$

$\nabla_n u_n \to \nabla u$ weakly in $L^2(\Omega)^3$

$\rho_n \to \rho$ weakly in $L^{2\gamma}(\Omega)$, with $2\gamma > \frac{3}{2}$,
$u_n \to u$ in $L^q(\Omega)^3$ for all $q < 6$ (and $\frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$). Then
$\rho_n u_n \otimes u_n \to \rho u \otimes u$ weakly in $L^1(\Omega)^{3\times 3}$.

$p_n \to p$ weakly in $L^2(\Omega)$

$f_n \to f$ weakly in $L^2(\Omega)^3$

Then, as $n \to \infty$,

$$\int_\Omega \nabla u : \nabla v\, dx - \int_\Omega \rho u \otimes u : \nabla v\, dx - \int_\Omega p \mathrm{div}(v)\, dx = \int_\Omega f \cdot v\, dx$$

First conclusion

(ρ, u, p) is solution of the momentum equation and of the mass equation ($+$ positivity of ρ and total mass). It remains to prove $p = \rho^\gamma$.

Passage to the limit in EOS

Question: $p = \rho^\gamma$ in Ω ?

p_n and ρ_n converge only weakly... and $\gamma > 1$

Idea :
Prove $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$ (it is sufficient to prove $\liminf \int_\Omega p_n \rho_n \leq \int_\Omega p\rho$) and deduce a.e. convergence (of p_n and ρ_n) and $p = \rho^\gamma$

(For $\gamma \leq 3$, use $p_n \rho_n^\theta$)

Proof in the continuous setting

$\nabla : \nabla = \mathrm{divdiv} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^3$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v}).$$

Then, for all $\bar{v}$ in $H_0^1(\Omega)^3$, the momentum equation is

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(u_n) \cdot \mathrm{curl}(\bar{v}) - \int_\Omega (\rho_n u_n \otimes u_n) : \nabla \bar{v} dx - \int_\Omega p_n \mathrm{div}(\bar{v}) = \int_\Omega f_n \cdot \bar{v}.$$

Choice of $\bar{v}$? $\bar{v} = \bar{v}_n$ with $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\bar{v}_n$ bounded in $H_0^1(\Omega)^3$ (unfortunately, 0 is impossible).

Then, up to a subsequence,

$\bar{v}_n \to v$ in $L^2(\Omega)^3$ and weakly in $H_0^1(\Omega)^3$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof using $\bar{v}_n$ (1)

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}_n) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(\bar{v}_n) - \int_\Omega p_n \mathrm{div}(\bar{v}_n) = \int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n + \int_\Omega f_n \cdot \bar{v}_n.$$

But, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\mathrm{curl}(\bar{v}_n) = 0$. Then:

$$\int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n = \int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n + \int_\Omega f_n \cdot \bar{v}_n.$$

If we prove that $\int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n \to \int_\Omega \rho u \otimes u : \nabla v$ then:

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n = \int_\Omega \rho u \otimes u : \nabla v + \int_\Omega f \cdot v.$$

Proof using $\bar{v}_n$ (2)

But, since $-\Delta u + \mathrm{div}(\rho u \otimes u) + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v) + \int_\Omega \mathrm{curl}(u) \cdot \mathrm{curl}(v) - \int_\Omega p\mathrm{div}(v) = \int_\Omega \rho u \otimes u : \nabla v + \int_\Omega f \cdot v,$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):
$\int_\Omega (\mathrm{div}(u) - p)\rho = \int_\Omega \rho u \otimes u : \nabla v + \int_\Omega f \cdot v$. Then:

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))\rho_n = \int_\Omega (p - \mathrm{div}(u))\rho.$$

Finally, thanks to the mass equations, $\int_\Omega \rho_n \mathrm{div}(u_n) = 0$ and $\int_\Omega \rho\mathrm{div}(u) = 0$. Then,

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n = \int_\Omega p\rho.$$

Proof using $\bar{v}_n$ (3)

It remains to prove $\int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n \to \int_\Omega \rho u \otimes u : \nabla v$.

We remark that (since $\mathrm{div}(\rho_n u_n) = 0$)

$$\int_\Omega \rho_n u_n \otimes u_n : \nabla \bar{v}_n = \int_\Omega (\rho_n u_n \cdot \nabla) u_n \cdot \bar{v}_n,$$

and the sequence $((\rho_n u_n \cdot \nabla) u_n)_{n \in \mathbb{N}}$ is bounded in $L^r(\Omega)^3$ with $\frac{1}{r} = \frac{1}{2} + \frac{1}{6} + \frac{1}{2\gamma}$, and $r > \frac{6}{5}$ since $\gamma > 3$.
Then, up to a subsequence $(\rho_n u_n \cdot \nabla) u_n \to G$ weakly in $L^r(\Omega)^3$.
and (since $\bar{v}_n \to \bar{v}$ in $L^r(\Omega)^3$ for all $r < 6$),

$$\int_\Omega (\rho_n u_n \cdot \nabla) u_n \cdot \bar{v}_n \to \int_\Omega G \cdot \bar{v}$$

But, $G = (\rho u \cdot \nabla) u$, since for a fixed $w \in H_0^1(\Omega)^3$,

$$\int_\Omega (\rho_n u_n \cdot \nabla) u_n \cdot w = \int_\Omega \rho_n u_n \otimes u_n : \nabla w \to \int_\Omega \rho u \otimes u : \nabla w.$$

Error in the preceding proof

In the preceding proof, we used $\bar{v}_n$ such that $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\bar{v}_n$ bounded in $H_0^1(\Omega)^3$.

Unfortunately, it is impossible to have $\bar{v}_n \in H_0^1(\Omega)^3$ but only $\bar{v}_n \in H^1(\Omega)^3$.

Curl-free test function

Let $w_n \in H_0^1(\Omega)$, $-\Delta w_n = \rho_n$,
One has $w_n \in H^2_{loc}(\Omega)$ since, for $\varphi \in C_c^\infty(\Omega)$, one has $\Delta(w_n\varphi) \in L^2(\Omega)$ and

$$\sum_{i,j=1}^{3} \int_\Omega \partial_i\partial_j(w_n\varphi)\,\partial_i\partial_j(w_n\varphi) = \sum_{i,j=1}^{3} \int_\Omega \partial_i\partial_i(w_n\varphi)\,\partial_j\partial_j(w_n\varphi)$$
$$= \int_\Omega (\Delta(w_n\varphi))^2 = C_\varphi < \infty$$

Then, taking $v_n = \nabla w_n$

- $v_n \in (H^1_{loc}(\Omega))^3$,
- $\mathrm{div}(v_n) = \rho_n$ a.e. in Ω,
- $\mathrm{curl}(v_n) = 0$ a.e. in Ω,
- $H^1_{loc}(\Omega)$-estimate on v_n with respect to $\|\rho_n\|_{L^2(\Omega)}$.

Then, up to a subsequence, as $n \to \infty$, $v_n \to v$ in $L^2_{loc}(\Omega)$ and weakly in $H^1_{loc}(\Omega)$, $\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_n\varphi \in H_0^1(\Omega)^3$)). Taking $\bar{v} = v_n\varphi$:

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(v_n\varphi) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(v_n\varphi) - \int_\Omega p_n\mathrm{div}(v_n\varphi)$$
$$= \int_\Omega \rho_n u_n \otimes u_n : \nabla(v_n\varphi) + \int_\Omega f_n \cdot (v_n\varphi).$$

Using a proof similar to that given if $\varphi = 1$ (with additionnal terms involving φ), we obtain :

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi,$$

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho$

Lemma : $F_n \to F$ in $D'(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$. Then $F_n \to F$ weakly in L^1.

With $F_n = (p_n - \mathrm{div}(u_n))\rho_n$, $F = (p - \mathrm{div}(u))\rho$ and since $p_n - \mathrm{div}(u_n)$ is bounded in $L^2(\Omega)$ and ρ_n is bounded in $L^r(\Omega)$ with some $r > 2$, the lemma gives

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

Proving $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

But thanks to the mass equations, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(u_n)\rho_n = 0,\ \int_\Omega \mathrm{div}(u)\rho = 0;$$

Then:

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n = \int_\Omega p\rho.$$

(Discrete case $\int_\Omega \mathrm{div}(u_n)\rho_n \leq Ch_n^\alpha$, $\limsup_{n\to\infty} \int_\Omega p_n\rho_n = \int_\Omega p\rho$)

a.e. convergence of ρ_n and p_n

Let $G_n = (\rho_n^\gamma - \rho^\gamma)(\rho_n - \rho) \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω.
Futhermore $G_n = (p_n - \rho^\gamma)(\rho_n - \rho) = p_n\rho_n - p_n\rho - \rho^\gamma\rho_n + \rho^\gamma\rho$ and:

$$\int_\Omega G_n = \int_\Omega p_n\rho_n - \int_\Omega p_n\rho - \int_\Omega \rho^\gamma\rho_n + \int_\Omega \rho^\gamma\rho.$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ρ_n and $\lim_{n\to\infty}\int_\Omega p_n\rho_n = \int_\Omega p\rho$:

$$\lim_{n\to\infty}\int_\Omega G_n = 0,$$

Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \rho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_n \to \rho$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$,

$p_n = \rho_n^\gamma \to \rho^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$,

and $p = \rho^\gamma$.

Passage to the limit in the EOS with the Mac scheme

Miracle with the Mac scheme:

1. There exists a discrete counterpart of

$$\int_\Omega \nabla u : \nabla v dx = \int_\Omega (\mathrm{div}(u)\mathrm{div}(v) + \mathrm{curl}(u) \cdot \mathrm{curl}(v)) dx$$

2. $w_n \in H_n$, $-\Delta_n w_n = \rho_n$,
   Estimate on a "discrete local H^2-norm" of w_n in term of the L^2-norm of ρ_n.
   If $\gamma \leq 3$, we have to work with the L^p-norm, $p > 2$, of the second dicrete derivatives of w_n

Convergence for SCNS

Open problem : convergence of approximate solutions (given by the MAC scheme) if $\frac{3}{2} < \gamma \leq 3$

Convergence for CNS

Ω : bounded open connected set of $\mathbb{R}^3$
$T > 0$, $\gamma > 3/2$, $f \in L^2(]0, T[, L^2(\Omega))$

$$\begin{aligned} \partial_{n,t}\rho + \operatorname{div}_n(\rho_n u_n) &= 0, \\ \partial_{n,t}(\rho_n u_n) + \operatorname{div}_n(\rho_n u_n \otimes u_n) - \Delta_n u_n + \nabla_n p_n &= f_n, \\ p_n &= \rho_n^\gamma. \end{aligned}$$

- Estimates on u_n, ρ_n, p_n
  u_n bounded in $L^2(]0, T[, H_n)$ and then in $L^2(]0, T[, L^q(\Omega))$
  ρ_n bounded in $L^2(]0, T[, L^\gamma(\Omega)))$
- Passing to the limit on $\rho_n u_n$ and $\rho_n u_n \otimes u_n$
- Passing to the limit on $p_n = \rho_n^\gamma$.

For nonlinear terms, weak convergences are not sufficient

SCNS, Mass equation, other method

$u_n \to u$ in $L^q(\Omega)^3$ for some $q < 6$, $\rho_n \to \rho$ weakly in $L^{q'}(\Omega)$
$(q' = q/(q-1) > 6/5)$
Then $\rho_n u_n \to \rho u$ in $L^1(\Omega)^3$

Other method :
$u_n \to u$ weakly in $H_0^1(\Omega)^3$
$\rho_n \to \rho$ in $H^{-1}(\Omega)$ (compact imbedding of $L^{q'}$ in H^{-1})

$$\int_\Omega \rho_n u_n \cdot \varphi = \langle \rho_n, u_n \cdot \varphi \rangle_{H^{-1}, H_0^1} \to \langle \rho, u \cdot \varphi \rangle_{H^{-1}, H_0^1} = \int_\Omega \rho u \cdot \varphi$$

for regular φ

For the discrete setting, we also have to replace the $H_0^1(\Omega)$-norm by the so-called discrete-H_0^1-norm which depends on n

Mass equation, in the evolution case, CNS

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

Estimates on u_n in $L^2(H_0^1(\Omega)^3)$ and ρ_n in $L^2(L^{q'}(\Omega))$ $(q' > 6/5)$.
Only weak compactness on u_n

But $\partial_t \rho_n$ is bounded in $L^2(W^{-1,1}(\Omega))$. Then
ρ_n compact in $L^2(H^{-1}(\Omega))$ (Aubin-Lions-Simon compactness results, since $L^{q'}$ compact in H^{-1})

$u_n \to u$ weakly in $L^2(H_0^1(\Omega)^3)$
$\rho_n \to \rho$ in $L^2(H^{-1}(\Omega))$

$$\int_0^T \int_\Omega \rho_n u_n \cdot \varphi = \int_0^T \langle \rho_n, u_n \cdot \varphi \rangle_{H^{-1}, H_0^1} \to = \int_0^T \int_\Omega \rho u \cdot \varphi$$

SCNS, Momentum equation, other method for $\rho u \otimes u$

$u_n \to u$ in $L^q(\Omega)^3$ for all $q < 6$
$\rho_n u_n \to \rho u$ weakly in $L^{q'}(\Omega)^3$, with $q' > \frac{6}{5}$,
Then $\rho_n u_n \otimes u_n \to \rho u \otimes u$ weakly in $L^1(\Omega)^{d\times d}$.

Other method :
$u_n \to u$ weakly in $H_0^1(\Omega)^3$
$\rho_n u_n \to \rho u$ in $H^{-1}(\Omega)^3$ (compact imbedding of $L^{q'}$ in H^{-1})
Then

$$\int_\Omega \rho_n u_n \otimes u_n : \nabla v \, dx \to \int_\Omega \rho u \otimes u : \nabla v \, dx$$

The generalization for the evolution case is possible

Convergence for CNS

It remains to pass to the limit on the EOS ($p_n = \rho_n^\gamma$)