Discretization of nonlinear diffusion equations

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joint work with R. Eymard and R. Herbin

Non linear diffusion equation

Ω is an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \geq 1$.

$$-\operatorname{div}(\mathbf{a}(x, \nabla u)) = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

$\partial\Omega = \overline{\Omega} \setminus \Omega$.

Hypotheses on **a** and *f*

$p > 1$, $\alpha, \beta > 0$, $b \in L^{p'}(\Omega)$,

- $\mathbf{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Caratheodory function,
- $\mathbf{a}(x, \xi) \cdot \xi \geq \alpha |\xi|^p$, for all $\xi \in \mathbb{R}^d$ and a.e. $x \in \Omega$,
- $|\mathbf{a}(x, \xi)| \leq b(x) + \beta |\xi|^{p-1}$, for all $\xi \in \mathbb{R}^d$ and a.e. $x \in \Omega$.
- $(\mathbf{a}(x, \xi) - \mathbf{a}(x, \chi)) \cdot (\xi - \chi) > 0$, for all $\xi, \chi \in \mathbb{R}^d$, $\xi \neq \chi$, and a.e. $x \in \Omega$.

Main example: $\mathbf{a}(x, \xi) = |\xi|^{p-2}\xi$.

$f \in L^{p'}(\Omega)$.

Weak formulation

Existence and uniqueness of a solution:

$$u \in W_0^{1,p}(\Omega),$$

$$\int_\Omega \mathbf{a}(x, \nabla u(x)) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx, \ \forall v \in W_0^{1,p}(\Omega).$$

Objective

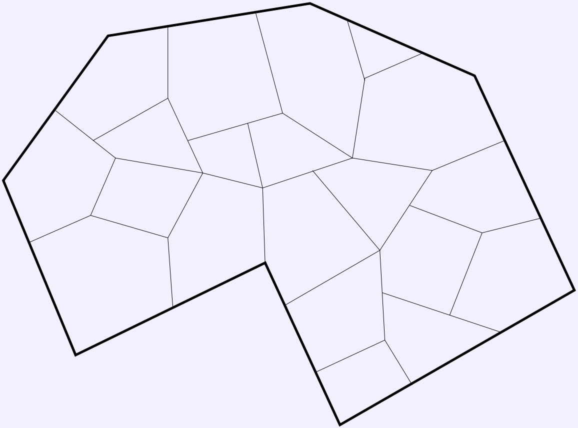
Discretization of this nonlinear diffusion problem with two constraints:

- "General" meshes (not adapted to the Finite Element method)
- One discrete unknown per cell

A good scheme will, at least, leads to:

- Existence and uniqueness of the approximate solution
- Convergence result of the approximate solution to the exact solution when the mesh size goes to 0.

Example of mesh



Main steps for proving a convergence result

- Estimates on the approximate solutions
- Compactness results on the family of approximate solutions
- Regularity of the limit of approximate solutions ($u \in W_0^{1,p}(\Omega)$)
- Prove that the limit of approximate solutions satisfies $-\operatorname{div}(\mathbf{a}(x, \nabla u)) = f$

To prove these results, one uses some "discrete functional analysis" tools, mimicking classical functional analysis (Sobolev, Kolmogorov, Minty trick, Leray-Lions trick. . .)

First approximate problem (1)

$\mathcal{T}$ is a mesh of Ω.
$H_{\mathcal{T}}$ is the set of functions $u \ : \ \Omega \to \mathbb{R}$ such that u is constant on K, for all $K \in \mathcal{T}$; this constant is denoted by u_K.

First idea:

For $u \in H_{\mathcal{T}}$, define $\nabla_{\mathcal{T}} u$ (convenient approximation of the gradient on u)

Approximate problem :

$$\int_\Omega \mathbf{a}(x, \nabla_{\mathcal{T}} u) \cdot \nabla_{\mathcal{T}} v \, dx = \int fv \, dx, \ \forall v \in H_{\mathcal{T}}.$$

Definition of the approximate solution on edges

For $K \in \mathcal{T}$, one chooses $x_K \in \mathcal{T}$ (K is "x_K star shaped").
For σ, interior edge (interface) of $\mathcal{T}$, x_σ is the center of σ.
One chooses a decomposition of x_σ : $x_\sigma = \sum_{M \in \mathcal{T}} a_{M,\sigma} x_M$.

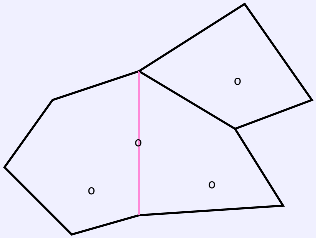
Then, if $u \in H_{\mathcal{T}}$, one sets:

$$\Pi_\sigma u = \sum_{M \in \mathcal{T}} a_{M,\sigma} u_M.$$

If σ is an edge on the boundary, one sets $\Pi_\sigma u = 0$.

Decomposition of x_σ

$$x_\sigma = \sum_{M\in\mathcal{T}} a_{M,\sigma} x_M$$



Definition of the approximate gradient

For $K \in \mathcal{T}$,

$$\frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma n_{K,\sigma} (x_\sigma - x_K)^t = Id.$$

For $u \in H_\mathcal{T}$, the value of the approximate gradient, $\nabla_\mathcal{T} u$ is, on K :

$$\nabla_K u = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (\Pi_\sigma u - u_K) n_{K,\sigma}.$$

First approximate problem (2)

$\mathcal{T}$ is a mesh of Ω.
$H_{\mathcal{T}}$ is the set of functions $u : \Omega \to \mathbb{R}$ such that u is constant on K, for all $K \in \mathcal{T}$; this constant is denoted by u_K.

One defines $\Pi_\sigma u$ for $\sigma \in \mathcal{E}$ and $\nabla_{\mathcal{T}} u$ on Ω.

First approximate problem :

$$\int_\Omega \mathbf{a}(x, \nabla_{\mathcal{T}} u) \cdot \nabla_{\mathcal{T}} v \, dx = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}.$$

But, no existence, no uniqueness, no coercivity on $H_{\mathcal{T}}$...

$\nabla_{\mathcal{T}} u = 0 \not\Rightarrow u = 0$

Stabilization using consistency estimate, $R_{\mathcal{T}}u$

$u \in H_{\mathcal{T}}$, $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, the value of $R_{\mathcal{T}}u$ is on the cone $D_{K,\sigma}$:

$$R_{K,\sigma}u = \frac{1}{d_{K,\sigma}}(\Pi_\sigma u - u_K - \nabla_K u \cdot (x_\sigma - x_K)).$$

If u is a regular function $u_K = u(x_K)$, $R_{K,\sigma}u \to 0$ as the mesh size goes to 0.

Approximate problem

$H_{\mathcal{T}}$ is the set of functions $u : \Omega \to \mathbb{R}$ such that u is constant on K, for all $K \in \mathcal{T}$; this constant is denoted by u_K.

$$u \in H_{\mathcal{T}},$$

$$\int_\Omega \mathbf{a}(x, \nabla_{\mathcal{T}} u) \cdot \nabla_{\mathcal{T}} v \, dx + b(u, v) = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}.$$

$$b(u, v) = \int_\Omega |R_{\mathcal{T}} u|^{p-2} R_{\mathcal{T}} u \, R_{\mathcal{T}} v \, dx$$

main properties:

- Coercivity of the discrete operator, existence of a solution and estimate
- Monotony of b, uniqueness of the solution
- $b(u, u) \geq 0$
- Convenient bound on u, and v such that $v_K = \psi(x_K)$ with ψ regular ($v = P_{\mathcal{T}} \psi$) gives $b(u, v) \to 0$ as the $\mathrm{size}(\mathcal{T}) \to 0$

Discrete $W_0^{1,p}$ norms on $H_{\mathcal{T}}$

$u \in H_{\mathcal{T}}$.

$$\|u\|_{p,\mathcal{T},\Pi}^p = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} m_\sigma d_{K,\sigma} \frac{|\Pi_\sigma u - u_K|^p}{d_{K,\sigma}^p}.$$

For $p = 1$, $\|u\|_{1,\mathcal{T},\Pi} \geq \|u\|_{1,\mathcal{T}}$

$$\|u\|_{1,\mathcal{T}} = \sum_{\sigma\in\mathcal{E}} m_\sigma D_\sigma u,$$

where $D_\sigma u = |u_K - u_L|$ if σ is between K and L and
$D_\sigma u = |u_K - 0|$ if σ is an edge of K on the boundary of Ω.

Coercivity, estimate on the approximate solution

For $u \in H_{\mathcal{T}}$,

$$C\|u\|^p_{1,p,\Pi} \leq \int_\Omega \mathbf{a}(x, \nabla_{\mathcal{T}} u) \cdot \nabla_{\mathcal{T}} u \, dx + b(u, u).$$

Then, if u is solution of the approximate problem, taking $v = u$,

$$C\|u\|^p_{1,p,\Pi} \leq \int fu \, dx \leq \|f\|_{L^{p'}} \|u\|_{L^p}.$$

Estimate on u follows if $\|u\|_{L^p} \leq C\|u\|_{1,p,\Pi} \ldots$

Discrete Sobolev embedding (1)

$u \in H_{\mathcal{T}}$

$p = 1$, $1^\star = \frac{d}{d-1}$, then $\|u\|_{L^{1^\star}} \le \|u\|_{1,\mathcal{T}} = \sum_{\sigma\in\mathcal{E}} m_\sigma D_\sigma u$.
Proof by induction on d.

- $d = 1$, $\|u\|_{L^\infty} \le \|u\|_{1,\mathcal{T}} = \sum_{\sigma\in\mathcal{E}} D_\sigma u$.
- $d = 2$, $x = (x_1, x_2) \in \mathbb{R}^2$,

$$u(x_1,x_2)^2 \le \Big(\sum_{\sigma\in\mathcal{E}} \chi_{1,\sigma}(x_2) D_\sigma u\Big)\Big(\sum_{\sigma\in\mathcal{E}} \chi_{2,\sigma}(x_1) D_\sigma u\Big)$$

$\chi_{1,\sigma}(x_2) = 1$ if $\{(y,x_2), y \in \mathbb{R}\} \cap \sigma \neq \emptyset$, and 0 otherwise.
Integrating leads to $\|u\|_{L^2} \le \sum_{\sigma\in\mathcal{E}} m_\sigma D_\sigma u = \|u\|_{1,\mathcal{T}}$.

Discrete Sobolev embedding (2)

$d = 3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\int |u(x)|^{\frac{3}{2}} dx_1\, dx_2 \le \left(\int |u(x)| dx_1\, dx_2\right)^{\frac{1}{2}} \left(\int |u(x)|^2 dx_1\, dx_2\right)^{\frac{1}{2}}$$

$$|u(x)| \le \sum_{\sigma \in \mathcal{E}} \chi_\sigma(x_1, x_2) D_\sigma u,$$

with $\chi_\sigma(x_1, x_2) = 1$ if $\{(x_1, x_2, y), y \in \mathbb{R}\} \cap \sigma \ne \emptyset$ (and 0 elsewhere).

$$\left(\int |u(x)|^2 dx_1\, dx_2\right)^{\frac{1}{2}} \le \sum_{\sigma \in \mathcal{E}} l_\sigma(x_3) D_\sigma u,$$

where $l_\sigma(x_3) = \lambda_2(\{(x_1, x_2), (x_1, x_2, x_3) \in \sigma\})$. Then

$$\int |u(x)|^{\frac{3}{2}} dx_1\, dx_2 \le \left(\sum_{\sigma \in \mathcal{E}} m_\sigma D_\sigma u\right)^{\frac{1}{2}} \sum_{\sigma \in \mathcal{E}} l_\sigma(x_3) D_\sigma u.$$

$$\int |u(x)|^{\frac{3}{2}} dx \le \left(\sum_{\sigma \in \mathcal{E}} m_\sigma D_\sigma u\right)^{\frac{3}{2}}.$$

Discrete Sobolev embedding (3)

$u \in H_{\mathcal{T}}$

- $1 < p < d$. $p^\star = \frac{dp}{d-p}$, then $\|u\|_{L^{p^\star}} \le C\|u\|_{p,\mathcal{T},\Pi}$, where C depends on some $\eta > 0$ such that $\eta \le \frac{d_{K,\sigma}}{d_{L,\sigma}} \le \frac{1}{\eta}$ for all σ.

  Proof using $p = 1$ with $|u|^\alpha$, $\alpha 1^\star = p^\star$.
- $p \ge d$, $q < \infty$, then $\|u\|_{L^q} \le C\|u\|_{p,\mathcal{T},\Pi}$.

Estimate on the approximate solution

$$C\|u\|_{1,p,\Pi}^{p} \leq \int fu\,dx \leq \|f\|_{L^{p'}}\|u\|_{L^p}.$$

Using sobolev embedding $\|u\|_{L^p} \leq C\|u\|_{1,p,\Pi}$, then

$$\|u\|_{L^p} \leq C,$$

$$\|u\|_{1,p,\Pi} \leq C,$$

$$\|\nabla_{\mathcal{T}} u\|_{L^p} \leq C.$$

Compactness of the family of approximate solutions

If $u \in H_{\mathcal{T}}$, $\xi \in \mathbb{R}^d$.

$$\int_{\mathbb{R}^d} |u(x+\xi) - u(x)| \, dx \leq |\xi| \|u\|_{1,\mathcal{T}}.$$

(In general, no similar result for $p > 1$, except particular meshes).

Then, since the family of approximate solutions is bounded for the norm $\| \cdot \|_{1,p,\Pi}$, it is bounded for the norm $\| \cdot \|_{1,\mathcal{T}}$ and then, by the Kolmogorov compacness theorem, the family of approximate solution is relatively compact in L^1.

Since, the family of approximate solution is bounded in L^q for some $q > p$, it is also relatively compact in L^p.

Regularity of the limit (1)

$u_{\mathcal{T}} \to u$ in $L^p(\Omega)$ or $L^p(\mathbb{R}^d)$ as the mesh size goes to 0. (Indeed, this is true for subsequences of sequences of approximate solutions).

Since $\nabla_{\mathcal{T}} u_{\mathcal{T}}$ is bounded in $(L^p)^d$, one also has (up to subsequences),

$$\nabla_{\mathcal{T}} u_{\mathcal{T}} \to g, \text{ weakly in } (L^p(\Omega))^d \text{ or } (L^p(\mathbb{R}^d))^d.$$

But, for $\psi \in (C^\infty(\mathbb{R}^d))^d$, let $P_{\mathcal{T}}\psi \in H_{\mathcal{T}}$ define by $P_{\mathcal{T}}\psi = \psi(x_K)$ on K.

$$\int_{\mathbb{R}^d} \nabla_{\mathcal{T}} u_{\mathcal{T}} \cdot P_{\mathcal{T}}\psi \, dx = -\int_{\mathbb{R}^d} u_{\mathcal{T}} \mathrm{div}_{\mathcal{T}} P_{\mathcal{T}}\psi \, dx$$

Regularity of the limit (2)

$u_{\mathcal{T}} \to u$ in $L^p(\mathbb{R}^d)$, $\|u\|^p_{1,p,\Pi} \leq C$.

$\nabla_{\mathcal{T}} u_{\mathcal{T}} \to g$, weakly in $(L^p(\mathbb{R}^d))^d$

$$\int_{\mathbb{R}^d} \nabla_{\mathcal{T}} u_{\mathcal{T}} \cdot P_{\mathcal{T}} \psi \, dx \to \int_{\mathbb{R}^d} g \cdot \psi \, dx.$$

The consistency of the definition of $\nabla_{\mathcal{T}}$ gives

$$\int_{\mathbb{R}^d} u_{\mathcal{T}} \mathrm{div}_{\mathcal{T}} P_{\mathcal{T}} \psi \, dx \to \int_{\mathbb{R}^d} u \mathrm{div} \psi \, dx.$$

Then, $\int_{\mathbb{R}^d} g \cdot \psi \, dx = -\int_{\mathbb{R}^d} u \mathrm{div} \psi \, dx$, and $g = \nabla u$.

This proves that $u \in W_0^{1,p}(\Omega)$.

$\mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}})$ is bounded in $L^{p'}(\Omega)$. Then one can assume

$$\mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) \to h, \text{ weakly in } (L^{p'})^d.$$

Let $\psi \in C_c^\infty(\Omega)$,

$$\int_\Omega \mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) \cdot \nabla_{\mathcal{T}} P_{\mathcal{T}} \psi \, dx + b(u_{\mathcal{T}}, P_{\mathcal{T}} \psi) = \int_\Omega f \, P_{\mathcal{T}} \psi \, dx.$$

Thanks to estimate on $u_{\mathcal{T}}$ (for $\| \cdot \|_{1,p,\mathcal{T}}$) and $\nabla_{\mathcal{T}} u_{\mathcal{T}}$ in L^p), the regularity of ψ gives $b(u_{\mathcal{T}}, P_{\mathcal{T}} \psi) \to 0$. Then

$$\int_\Omega h \cdot \nabla \psi \, dx = \int_\Omega f \psi \, dx.$$

By density, this is also true if $\psi \in W_0^{1,p}(\Omega)$. It remains to prove $\int_\Omega h \cdot \nabla \psi \, dx = \int_\Omega a(\nabla u) \cdot \nabla \psi \, dx$.

$-\operatorname{div}(\mathbf{a}(x, \nabla u)) = f$, Minty trick (2)

Let $\psi \in C_c^\infty(\Omega)$,

$$\int_\Omega (\mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) - \mathbf{a}(\nabla_{\mathcal{T}} P_{\mathcal{T}}\psi)) \cdot (\nabla_{\mathcal{T}} u_{\mathcal{T}} - \nabla_{\mathcal{T}} P_{\mathcal{T}}\psi)\, dx \geq 0.$$

$$\int_\Omega \mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) \cdot \nabla_{\mathcal{T}} u_{\mathcal{T}}\, dx \leq \int_\Omega \mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) \cdot \nabla_{\mathcal{T}} u_{\mathcal{T}} + b(u_{\mathcal{T}}, u_{\mathcal{T}}) = \int_\Omega f u_{\mathcal{T}}\, dx$$

$$\limsup \int_\Omega \mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) \cdot \nabla_{\mathcal{T}} u_{\mathcal{T}}\, dx \leq \int_\Omega f u\, dx = \int_\Omega h \cdot \nabla u\, dx.$$

Then, taking the limsup of the first inequality leads to

$$\int_\Omega (h - \mathbf{a}(\nabla\psi)) \cdot (\nabla u - \nabla\psi)\, dx \geq 0.$$

This last inequality is also true, by density, if $\psi \in W_0^{1,p}(\Omega)$.
Then, Taking $\psi = u \pm t\varphi$ leads to $-\operatorname{div}(\mathbf{a}(x, \nabla u)) = f$

$-\operatorname{div}(\mathbf{a}(x,\nabla u)) = f$, Minty trick (3)

$$\int_\Omega (h - \mathbf{a}(\nabla\psi)) \cdot (\nabla u - \nabla\psi)\, dx \geq 0,$$

for all $\psi \in W_0^{1,p}$. Let $\varphi \in C_c^\infty(\Omega)$ and $t \in \mathbb{R}_+^\star$. Take $\psi = u + t\varphi$:

$$\int_\Omega (h - \mathbf{a}(\nabla u + t\nabla\varphi)) \cdot \nabla\varphi\, dx \geq 0,$$

this gives, as $t \to 0$,

$$\int_\Omega (h - \mathbf{a}(\nabla u)) \cdot \nabla\varphi\, dx \geq 0.$$

Changing φ in $-\varphi$:

$$(\int_\Omega f\varphi\, dx =) \int_\Omega h \cdot \nabla\varphi\, dx = \int_\Omega \mathbf{a}(\nabla u) \cdot \nabla\varphi\, dx.$$

Strong convergence of the gradient, Leray lions trick

$\nabla_{\mathcal{T}} u_{\mathcal{T}} \to \nabla u$ weakly in L^p.

$\limsup \int_\Omega \mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) \cdot \nabla_{\mathcal{T}} u_{\mathcal{T}} \, dx \leq \int_\Omega f u \, dx = \int_\Omega \mathbf{a}(\nabla u) \cdot \nabla u \, dx.$

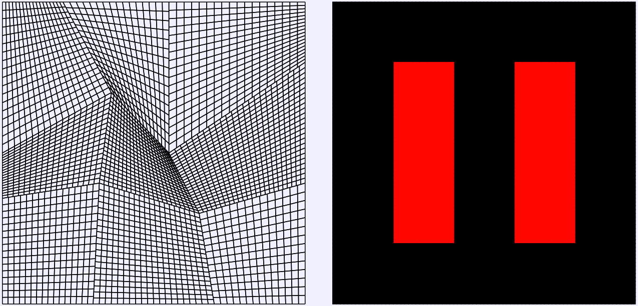
Then $\displaystyle \limsup \int_\Omega (\mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) - \mathbf{a}(\nabla u)) \cdot (\nabla_{\mathcal{T}} u_{\mathcal{T}} - \nabla u) \, dx \leq 0$

Since $(\mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) - \mathbf{a}(\nabla u)) \cdot (\nabla_{\mathcal{T}} u_{\mathcal{T}} - \nabla u) \geq 0$ a.e.,

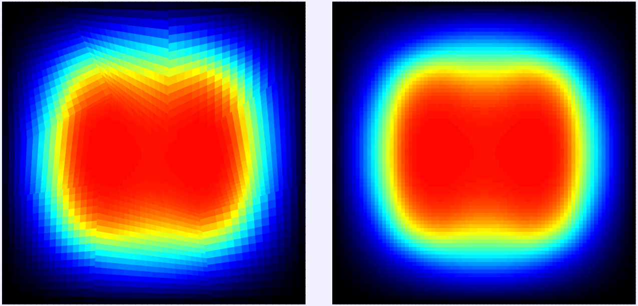
$$(\mathbf{a}(\nabla_{\mathcal{T}} u_{\mathcal{T}}) - \mathbf{a}(\nabla u)) \cdot (\nabla_{\mathcal{T}} u_{\mathcal{T}} - \nabla u) \to 0 \text{ in } L^1,$$

and therefore a.e. for a subsequence. Then, Leray-Lions trick shows that $\nabla_{\mathcal{T}} u_{\mathcal{T}} \to \nabla u$ a.e., at least for the same subsequence.
Finally, strong convergence of the gradient follows.

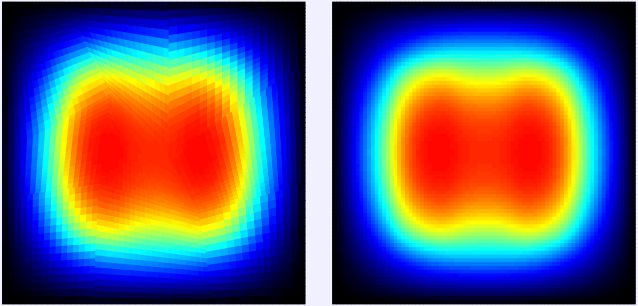
Numerical example, two grids, p-laplacian



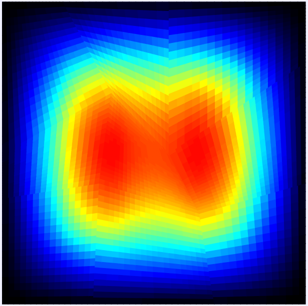
Numerical example,two grids, p=1.3

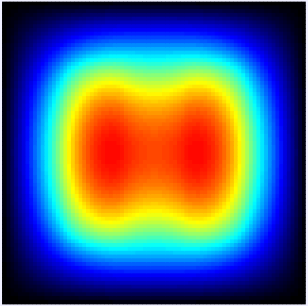


Numerical example,two grids, p=1.6

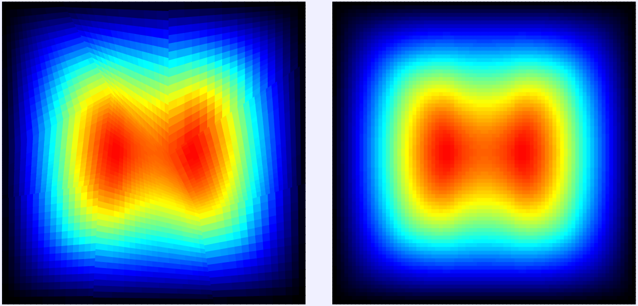


Numerical example,two grids, p=2





Numerical example,two grids, p=3



Numerical example,two grids, p=6