Nonlinear methods for linear equations

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objective

Some results on linear elliptic (or parabolic) equations with Stampacchia's methods

Using these methods for the study of numerical schemes (properties of the approximate solutions, convergence of schemes. . .)

Positivity

Ω bounded open set of $\mathbb{R}^d$ with a Lipschitz continuous boundary.

$A : \Omega \to M_d(\mathbb{R})$ sym. pos. def., uniformly ($A\xi.\xi \geq \alpha\xi.\xi$ with some $\alpha > 0$), with coefficients in $L^\infty(\Omega)$.

$f \in L^2(\Omega)$.

$$-\operatorname{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

$f \leq 0$ a.e. $\Rightarrow u \leq 0$ a.e.

Positivity, proof

$$u \in H_0^1(\Omega), \int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega f v dx \text{ for all } v \in H_0^1(\Omega).$$

$f \leq 0$.

Taking u^+ as test function (possible since $u^+ \in H_0^1(\Omega)$) :

$$\alpha \| |\nabla u^+| \|_2 \leq \int_\Omega A\nabla u^+ \cdot \nabla u^+ = \int_\Omega A\nabla u \cdot \nabla u^+ = \int_\Omega f u^+ \leq 0.$$

Then, $\nabla u^+ = 0$ a.e. and $u^+ = 0$ a.e.., $u \leq 0$ a.e...

Property used : $\nabla u^+ = 1_{u>0}\nabla u = 1_{u\geq 0}\nabla u$ a.e..

Nonlinear tool (Stampacchia)

$\varphi : \mathbb{R} \to \mathbb{R}$, Lipschitz continuous function such that $\varphi(0) = 0$.

$u \in H_0^1(\Omega)$. Then, $\varphi(u) \in H_0^1(\Omega)$ and

$$\nabla\varphi(u) = \varphi'(u)\nabla u \text{ a.e.}.$$

Example : $\varphi(s) = s^+$, $\nabla u^+ = 1_{u>0}\nabla u = 1_{u\geq 0}\nabla u$ a.e.

Indeed, it is possible to use only regular function φ (C^1 functions).

Bounded solutions (Stampacchia)

$$-\operatorname{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

$f \in H^{-1}(\Omega)$. Existence and uniqueness of u solution to :

$u \in H_0^1(\Omega)$, $\displaystyle\int_\Omega A\nabla u \cdot \nabla v dx = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \; \forall v \in H_0^1(\Omega)$.

Question : $u \in L^\infty(\Omega)$ $(d \geq 2)$?

Answer :

$\rightsquigarrow$ Yes if it exists $p > \frac{d}{2}$ such that $f \in L^p(\Omega)$ (and $\langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega f v dx$).

$\rightsquigarrow$ Yes if it exists $p > d$ such that $f \in W^{-1,p}(\Omega)$.

NB: $L^{p/2}(\Omega) \subset W^{-1,p}(\Omega)$ for $p > d$.

Bounded solutions, proof (1)

Let $p > d$ s.t. $f \in W^{-1,p}(\Omega)$. Then, it exists $F \in (L^p(\Omega))^d$ s.t. $f = \mathrm{div} F$. One has:
$u \in H_0^1(\Omega)$, $\int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega F \cdot \nabla v dx$ for all $v \in H_0^1(\Omega)$.

Let $k \in \mathbb{R}_+^\star$. Take $v = \psi(u) = (u-k)^+ - (u+k)^-$ (ψ is nondecreasing). One has $\nabla\psi(u) = 1_{A_k}\nabla u$ a.e.with $A_k = \{|u| \geq k\}$ and:

$$\int_{A_k} A\nabla u \cdot \nabla u dx = \int_{A_k} F \cdot \nabla u dx.$$

Then, with Cauchy-Schwarz and Hölder inequalites ($p/2$ and its conjugate):

$$\alpha \| |\nabla u| \|_{L^2(A_k)} \leq C_1 \|f\|_{W^{-1,p}} \mathrm{mes}(A_k)^{\frac{1}{2}-\frac{1}{p}}.$$

Bounded solutions, proof (2)

Using Sobolev imbedding ($W_0^{1,1}(\Omega) \subset L^{d/(d-1)}(\Omega)$) and Cauchy-Schwarz again:

$$\mathrm{mes}(A_h) \leq \frac{C_2 \|f\|_{W^{-1,p}}^{\gamma}}{h-k} \mathrm{mes}(A_k)^{\beta}, \text{ for } 0 \leq k < h,$$

with $\gamma = d/(d-1)$ and $\beta = \frac{p-1}{p}\frac{d}{d-1} > 1$ (since $p > d$).

Since $\beta > 1$, this gives (with a little tricky computation) $\mathrm{mes}(A_h) = 0$ si $h \geq C_3 \|f\|_{W^{-1,p}}$. Then:

$$\|u\|_{\infty} \leq C_3 \|f\|_{W^{-1,p}}.$$

A further developpement of this proof leads to $u \in C(\overline{\Omega})$ and finally to the Hölder continuity of u.

Existence of a solution for f "measure"

$$-\operatorname{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

f is a measure on Ω ($f \in (C(\overline{\Omega}))'$).

First method: duality method (Stampacchia, 1965)

Second method: passing to the limit on approximate solutions
Main difficulty: obtain estimates on u only depending of the L^1-norm of f (with $f \in L^2$).

Existence of a solution for f "measure", proof (1)

$u \in H_0^1(\Omega)$, $\int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega f v dx$ for all $v \in H_0^1(\Omega)$.

For $\theta > 1$, one defines φ :

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt;\ s \in \mathbb{R}.$$

Taking $v = \varphi(u) \in H_0^1(\Omega)$ leads to:

$$\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\theta} dx \le C_\theta \|f\|_1,$$

with $C_\theta = \int_0^\infty \frac{1}{(1+|t|)^\theta} dt < \infty$.

Existence de solution with f "mesure", proof (1)

$u \in H_0^1(\Omega)$, $\int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega f v dx$ for all $v \in H_0^1(\Omega)$.

For $\theta > 1$, one defines φ :

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt; \ s \in \mathbb{R}.$$

taking $v = \varphi(u) \in H_0^1(\Omega)$ leads to:

$$\int_\Omega |\nabla \phi(u)|^2 dx = \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\theta} dx \leq C_\theta \|f\|_1,$$

with $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$.

Existence of a solution for f “measure”, proof (2)

Using Hölder Inequality, Sobolev imbedding and θ close 1, one obtains, for $q < \frac{d}{d-1}$:

$$\int_\Omega |\nabla u|^q dx \leq C_q \|f\|_{L^1}.$$

Passing to the limit on a sequence of approximate solutions (corresponding to regular second members converging towards f), one obtains existence of a solution (in the disctribution sense) if f is a measure.

This solution belongs to $W_0^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$.

Convection-diffusion without coercivity

$$-\operatorname{div} A\nabla u + \operatorname{div}(wu) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$

with $w \in C(\overline{\Omega})^d$ and $f \in L^2(\Omega)$ (or f measure on Ω).

Existence and uniqueness of a solution (J. Droniou).

Main step: *a priori* estimates on meas($\{|u| \geq k\}$) (this measure goes to 0 as $k \to \infty$).

(then, one obtains an $H_0^1(\Omega)-$estimate and existence follows with a topological degree argument. Uniqueness is a consequence of an existence result for the dual problem.)

$$u \in H_0^1(\Omega),$$
$$\int_\Omega A\nabla u \cdot \nabla v dx - \int_\Omega u w \cdot \nabla v dx = \int_\Omega f v dx, \text{ for all } v \in H_0^1(\Omega).$$

Taking $v = \varphi(u)$ with $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2}$ ($\theta = 2$):

$$\alpha \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} dx \leq \|f\|_1 + \int_\Omega \frac{|w||u||\nabla u|}{(1+|u|)^2} dx$$
$$\leq \|f\|_1 + \|w\|_\infty \int_\Omega \frac{|\nabla u|}{1+|u|} dx,$$

with $\|w\|_\infty = \sup_{x\in\Omega} |w(x)| < \infty$.

Convection-diffusion without coercivity, proof (2)

and, with Young Inequality:

$$\int_\Omega |\nabla \ln(1+|u|)|^2 dx = \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} dx \le C(\alpha, \|f\|_1, \|w\|_\infty).$$

$|\nabla \ln(1+|u|)| = |\nabla \phi(u)|$.
$\phi(s) == \int_0^s \sqrt{\varphi'(t)} dt = \int_0^s \frac{1}{1+|t|} dt$.

Since $\ln(1+|u|) \in H_0^1(\Omega)$, one deduces an estimate on $\ln(1+|u|)$ in $L^2(\Omega)$ and then an estimate on meas($\{|u| \ge k\}$).

Stampacchia methods with FV schemes

Finite Volumes schemes with the so-called “admissible meshes for A”.

1. positivity
2. Bounded solutions, Hölder continuous solutions: Thomas Rey thesis
3. “Measure” data: G-Herbin
4. Convection-diffusion without coercivity (and with measure data) : Droniou-G-Herbin

“Non admissible” meshes ?

$\mathcal{M}$ is a mesh of Ω, with triangles ($d = 2$) or tetrahedra ($d = 3$).

$H = \{u \in C(\overline{\Omega}); u_{|_K} \in P^1\}$.
$H_0 = \{u \in H; u = 0 \text{ on } \partial\Omega\}$.

$$u_{\mathcal{M}} \in H_0,$$
$$\int_\Omega A\nabla u_{\mathcal{M}} \cdot \nabla v dx \; (-\int_\Omega u_{\mathcal{M}} w \cdot \nabla v dx) = T(v), \text{ for all } v \in H_0.$$

$T(v) = \int_\Omega f v dx$ (examples 1, 2, 4)
$T(v) = \int_\Omega v df$ (examples 3, 4)

Difficulty : $u \in H_0 \not\Rightarrow u^+, \psi(u), \varphi(u) \in H_0$.

Choice of the test function

Idea: take as test function the interpolate of the test function of the “continuous” case.

If $v \in C(\overline{\Omega})$, $\Pi_{\mathcal{M}}(v) \in H$ and $\Pi_{\mathcal{M}}(v) = v$ at the vertices of the mesh.

Denoting by $\mathcal{V}$ the set of vertices of the mesh:
$\Pi_{\mathcal{M}}(v) = \sum_{K \in \mathcal{V}} v(K)\phi_K$, where ϕ_K is the basis function associated to K.

$$u_{\mathcal{M}} \in H_0,$$
$$\int_\Omega A\nabla u_{\mathcal{M}} \cdot \nabla v dx = \int_\Omega f v dx, \text{ for all } v \in H_0.$$

With $u_{\mathcal{M}} = \sum_{K\in\mathcal{V}} u_K \phi_K$ and $v = \sum_{L\in\mathcal{V}} v_L \phi_L$, this gives

$$\sum_{K\in\mathcal{V}} \sum_{L\in\mathcal{V}} (-T_{K,L}) u_K v_L = \int f v dx,$$

with $T_{K,L} = -\int_\Omega A\nabla\phi_K \cdot \nabla\phi_L dx$.
or, since $\sum_{L\in\mathcal{V}} T_{K,L} = 0$,

$$\sum_{K\in\mathcal{V}} \sum_{L\in\mathcal{V}} (-T_{K,L})(u_K)(v_L - v_K) = \int f v dx,$$

and, finally,

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(v_K - v_L) = \int f v dx.$$

Taking $v = \Pi_{\mathcal{M}}\varphi(u)$ leads to:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int f v dx.$$

positivity (1)

$f \leq 0$ a.e.
$\varphi(s) = s^+$.

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int fvdx,$$

yields:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(u_K^+ - u_L^+) = \int fvdx,$$

If $T_{K,L} \geq 0$, one has:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)^2 \leq \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(u_K^+ - u_L^+),$$

positivity (2)

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)(u_K^+ - u_L^+) \le \int fv dx \le 0,$$

then:

$$\int_\Omega A\nabla\Pi_{\mathcal{M}}u^+ \cdot \nabla\Pi_{\mathcal{M}}u^+ dx = \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)^2 = 0,$$

from which one deduces $u^+ = 0$.

nondecreasing function φ

Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ Lipschitz continuous and nondecreasing.
Define ϕ by $\phi(s) = \int_0^s \sqrt{\varphi'(t)}dt$.
For $a, b \in \mathbb{R}$, one has (thanks to Cauchy-Schwarz Inequality) :

$$(\phi(a) - \phi(b))^2 \leq (a - b)(\varphi(a) - \varphi(b)).$$

then, IF $T_{K,L} \geq 0$ (for all (K, L)), one has:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(\phi(u_K) - \phi(u_L))^2 \leq \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)).$$

$$\int_\Omega A\nabla\Pi_{\mathcal{M}}\phi(u) \cdot \nabla\Pi_{\mathcal{M}}\phi(u) = \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(\phi(u_K) - \phi(u_L))^2.$$

For $\theta > 1$, define φ:

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt;\ s \in \mathbb{R}.$$

Taking $v = \Pi_{\mathcal{M}}\varphi(u) \in H_0$:

$$\int_\Omega |\nabla \Pi_{\mathcal{M}}\phi(u)|^2 dx \leq C_\theta \|f\|_1,$$

with $\phi(s) = \int_0^s \sqrt{\varphi'(t)}dt$.

Take $v = \Pi_{\mathcal{M}}\varphi(u)$ with $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2}$ ($\theta = 2$).
If the mesh size is small enough (or using an "upwinding" for convection part), one obtains an $H_0^1(\Omega)-$estimate on $\Pi_{\mathcal{M}} \ln(1+|u|) \in H_0^1(\Omega)$, then, an estimate on $\ln(1+|u|)$ in $L^2(\Omega)$ and finally, as in the "continuous" case, an estimate on meas($\{|u| \geq k\}$).

Conclusion

If $T_{K,L} \geq 0$, for all K, L, the methods of Stampacchia can be used for the study of numerical schemes (EF and VF)...

They gives the desired properties on the approximate solution in Examples 1 and 2 (positivity, $L^\infty-$bound), Estimates and Convergence of the approximate solution in Examples 3 and 4 (measure data and convection-diffusion without viscosity).

Further work ?

Without the condition $T_{K,L} \geq 0$, it seems not easy to use the methods of Stampacchia...

Without changing the mesh (EF of VF with "non admissible" meshes), a possible solution is perhaps to discretize this elliptic linear problem with a nonlinear scheme taking some $T_{K,L}(u)$ depending on the approximate solution, that is under the form:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u)(u_K - u_L)(v_K - v_L) = T(v),$$

and with $T_{K,L}(u) \geq 0$, for all K, L.