

Nonlinear methods for linear equations

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objective

Some results on linear elliptic (or parabolic) equations with Stampacchia's methods

Using these methods for the study of numerical schemes (properties of the approximate solutions, convergence of schemes. . .)

Positivity

Ω bounded open set of \mathbb{R}^d with a Lipschitz continuous boundary.

$A : \Omega \rightarrow M_d(\mathbb{R})$ sym. pos. def., uniformly ($A\xi \cdot \xi \geq \alpha\xi \cdot \xi$ with some $\alpha > 0$), with coefficients in $L^\infty(\Omega)$.

$f \in L^2(\Omega)$.

$$-\operatorname{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

$f \leq 0$ a.e. $\Rightarrow u \leq 0$ a.e.

Positivity, proof

$$u \in H_0^1(\Omega), \int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \text{ for all } v \in H_0^1(\Omega).$$

$$f \leq 0.$$

Taking u^+ as test function (possible since $u^+ \in H_0^1(\Omega)$) :

$$\alpha \|\nabla u^+\|_2 \leq \int_{\Omega} A \nabla u^+ \cdot \nabla u^+ = \int_{\Omega} A \nabla u \cdot \nabla u^+ = \int_{\Omega} f u^+ \leq 0.$$

Then, $\nabla u^+ = 0$ a.e. and $u^+ = 0$ a.e., $u \leq 0$ a.e..

Property used : $\nabla u^+ = 1_{u>0} \nabla u = 1_{u \geq 0} \nabla u$ a.e..

Nonlinear tool (Stampacchia)

$\varphi : \mathbb{R} \rightarrow \mathbb{R}$, Lipschitz continuous function such that $\varphi(0) = 0$.

$u \in H_0^1(\Omega)$. Then, $\varphi(u) \in H_0^1(\Omega)$ and

$$\nabla \varphi(u) = \varphi'(u) \nabla u \text{ a.e..}$$

Example : $\varphi(s) = s^+$, $\nabla u^+ = \mathbf{1}_{u>0} \nabla u = \mathbf{1}_{u \geq 0} \nabla u$ a.e.

Indeed, it is possible to use only regular function φ (C^1 functions).

Maximum principle

$$-\operatorname{div}(A\nabla u) = 0, \text{ in } \Omega,$$

$$u = g, \text{ on } \partial\Omega.$$

$$g \in C(\partial\Omega, \mathbb{R}) \cap H^{\frac{1}{2}}(\partial\Omega).$$

Then $a = \inf g \leq u \leq \max g = b$.

Proof : Take $(u - b)^+$ and $(u - a)^-$ as test functions in the weak formulation.

Bounded solutions (Stampacchia)

$$-\operatorname{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

$f \in H^{-1}(\Omega)$. Existence and uniqueness of u solution to :

$$u \in H_0^1(\Omega), \int_{\Omega} A\nabla u \cdot \nabla v dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Question : $u \in L^\infty(\Omega)$ ($d \geq 2$) ?

Answer :

\rightsquigarrow Yes if it exists $p > \frac{d}{2}$ such that $f \in L^p(\Omega)$ (and $\langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\Omega} f v dx$).

\rightsquigarrow Yes if it exists $p > d$ such that $f \in W^{-1,p}(\Omega)$.

NB: $L^{p/2}(\Omega) \subset W^{-1,p}(\Omega)$ for $p > d$.

Bounded solutions, proof (1)

Let $p > d$ s.t. $f \in W^{-1,p}(\Omega)$. Then, it exists $F \in (L^p(\Omega))^d$ s.t. $f = \operatorname{div} F$. One has:

$$u \in H_0^1(\Omega), \int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} F \cdot \nabla v dx \text{ for all } v \in H_0^1(\Omega).$$

Let $k \in \mathbb{R}_+^*$. Take $v = \psi(u) = (u - k)^+ - (u + k)^-$ (ψ is nondecreasing). One has $\nabla \psi(u) = 1_{A_k} \nabla u$ a.e. with $A_k = \{|u| \geq k\}$ and:

$$\int_{A_k} A \nabla u \cdot \nabla u dx = \int_{A_k} F \cdot \nabla u dx.$$

Then, with Cauchy-Schwarz and Hölder inequalities ($p/2$ and its conjugate):

$$\alpha \|\nabla u\|_{L^2(A_k)} \leq C_1 \|f\|_{W^{-1,p}} \operatorname{mes}(A_k)^{\frac{1}{2} - \frac{1}{p}}.$$

Bounded solutions, proof (2)

Using Sobolev imbedding ($W_0^{1,1}(\Omega) \subset L^{d/(d-1)}(\Omega)$) and Cauchy-Schwarz again:

$$\text{mes}(A_h) \leq \frac{C_2 \|f\|_{W^{-1,p}}^\gamma}{h-k} \text{mes}(A_k)^\beta, \text{ for } 0 \leq k < h,$$

with $\gamma = d/(d-1)$ and $\beta = \frac{p-1}{p} \frac{d}{d-1} > 1$ (since $p > d$).

Since $\beta > 1$, this gives (with a little tricky computation) $\text{mes}(A_h) = 0$ si $h \geq C_3 \|f\|_{W^{-1,p}}$. Then:

$$\|u\|_\infty \leq C_3 \|f\|_{W^{-1,p}}.$$

A further developpement of this proof leads to $u \in C(\bar{\Omega})$ and finally to the Hölder continuity of u .

Existence of a solution for f “measure”

$$-\operatorname{div}(A\nabla u) = f, \text{ in } \Omega,$$

$$u = 0, \text{ on } \partial\Omega.$$

f is a measure on Ω ($f \in (C(\overline{\Omega}))'$).

First method: duality method (Stampacchia, 1965)

Second method: passing to the limit on approximate solutions

Main difficulty: obtain estimates on u only depending of the L^1 -norm of f (with $f \in L^2$).

Existence of a solution for f “measure”, proof (1)

$$u \in H_0^1(\Omega), \int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \text{ for all } v \in H_0^1(\Omega).$$

For $\theta > 1$, one defines φ :

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt; \quad s \in \mathbb{R}.$$

Taking $v = \varphi(u) \in H_0^1(\Omega)$ leads to:

$$\int_{\Omega} \frac{|\nabla u|^2}{(1+|u|)^\theta} dx \leq C_\theta \|f\|_1,$$

with $C_\theta = \int_0^\infty \frac{1}{(1+|t|)^\theta} dt < \infty$.

Existence de solution with f “mesure”, proof (1)

$$u \in H_0^1(\Omega), \int_{\Omega} A \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \text{ for all } v \in H_0^1(\Omega).$$

For $\theta > 1$, one defines φ :

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt; \quad s \in \mathbb{R}.$$

taking $v = \varphi(u) \in H_0^1(\Omega)$ leads to:

$$\int_{\Omega} |\nabla \phi(u)|^2 dx = \int_{\Omega} \frac{|\nabla u|^2}{(1+|u|)^\theta} dx \leq C_\theta \|f\|_1,$$

with $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt.$

Existence of a solution for f “measure”, proof (2)

Using Hölder Inequality, Sobolev imbedding and θ close 1, one obtains, for $q < \frac{d}{d-1}$:

$$\int_{\Omega} |\nabla u|^q dx \leq C_q \|f\|_{L^1}.$$

Passing to the limit on a sequence of approximate solutions (corresponding to regular second members converging towards f), one obtains existence of a solution (in the distribution sense) if f is a measure.

This solution belongs to $W_0^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$.

Convection-diffusion without coercivity

$$\begin{aligned} -\operatorname{div} A \nabla u + \operatorname{div}(w u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial \Omega, \end{aligned}$$

with $w \in C(\bar{\Omega})^d$ and $f \in L^2(\Omega)$ (or f measure on Ω).

Existence and uniqueness of a solution.

Main step: *a priori* estimates on $\operatorname{meas}(\{|u| \geq k\})$ (this measure goes to 0 as $k \rightarrow \infty$).

(then, one obtains an $H_0^1(\Omega)$ -estimate and existence follows with a topological degree argument. Uniqueness is a consequence of an existence result for the dual problem.)

Convection-diffusion without coercivity, proof (1)

$$u \in H_0^1(\Omega),$$
$$\int_{\Omega} A \nabla u \cdot \nabla v \, dx - \int_{\Omega} uw \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \text{ for all } v \in H_0^1(\Omega).$$

Idea of Lucio Boccardo.

Taking $v = \varphi(u)$ with $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2} (\theta = 2)$:

$$\begin{aligned} \alpha \int_{\Omega} \frac{|\nabla u|^2}{(1+|u|)^2} \, dx &\leq \|f\|_1 + \int_{\Omega} \frac{|w||u||\nabla u|}{(1+|u|)^2} \, dx \\ &\leq \|f\|_1 + \|w\|_{\infty} \int_{\Omega} \frac{|\nabla u|}{1+|u|} \, dx, \end{aligned}$$

with $\|w\|_{\infty} = \sup_{x \in \Omega} |w(x)| < \infty$.

Convection-diffusion without coercivity, proof (2)

and, with Young Inequality:

$$\int_{\Omega} |\nabla \ln(1 + |u|)|^2 dx = \int_{\Omega} \frac{|\nabla u|^2}{(1 + |u|)^2} dx \leq C(\alpha, \|f\|_1, \|w\|_{\infty}).$$

$$|\nabla \ln(1 + |u|)| = |\nabla \phi(u)|.$$
$$\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt = \int_0^s \frac{1}{1+|t|} dt.$$

Since $\ln(1 + |u|) \in H_0^1(\Omega)$, one deduces an estimate on $\ln(1 + |u|)$ in $L^2(\Omega)$ and then an estimate on $\text{meas}(\{|u| \geq k\})$.

Positivity for convection-diffusion without coercivity

$$\begin{aligned} -\operatorname{div} A \nabla u + \operatorname{div}(w u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial \Omega, \end{aligned}$$

with $w \in C(\bar{\Omega})^d$ and $f \in L^2(\Omega)$ (or f measure on Ω).

Then $f \geq 0$ a.e. implies $u \geq 0$ a.e..
(Or $f \leq 0$ a.e. implies $u \leq 0$ a.e..)

Positivity for conv.-diff. without coercivity, proof (1)

$$u \in H_0^1(\Omega),$$
$$\int_{\Omega} A \nabla u \cdot \nabla v \, dx - \int_{\Omega} u w \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \text{ for all } v \in H_0^1(\Omega).$$

Assume $f \leq 0$ a.e.

Taking $v = T_{\varepsilon}(u^+) = \begin{cases} 0 & \text{if } u \leq 0 \\ u & \text{if } 0 < u < \varepsilon \\ \varepsilon & \text{if } u \geq \varepsilon \end{cases}$ leads to

$$\alpha \|\nabla T_{\varepsilon} u^+\|_2^2 - \int_{\Omega} u w \cdot \nabla T_{\varepsilon}(u^+) \leq 0.$$

Then, with Cauchy-Schwarz Inequality,

$$\alpha \|\nabla T_{\varepsilon} u^+\|_2^2 \leq \varepsilon a_{\varepsilon} \|\nabla T_{\varepsilon} u^+\|_2,$$

with $a_{\varepsilon}^2 = \int_{0 < u < \varepsilon} |w|^2 \, dx \rightarrow 0$, as $\varepsilon \rightarrow 0$.

Positivity for conv.-diff. without coercivity, proof (2)

$$\alpha \|\nabla T_\varepsilon u^+\|_2^2 \leq \varepsilon a_\varepsilon \|\nabla T_\varepsilon u^+\|_2$$

gives

$$\|\nabla T_\varepsilon u^+\|_1 \leq C_1 \|\nabla T_\varepsilon u^+\|_2 \leq C_2 \varepsilon a_\varepsilon.$$

Then, with Sobolev Inequality,

$$\varepsilon \text{meas}\{u > \varepsilon\}^{\frac{1}{1^*}} \leq \|T_\varepsilon(u^+)\|_{1^*} \leq C_3 \|\nabla T_\varepsilon u^+\|_1 \leq C_4 \varepsilon a_\varepsilon,$$

and we obtain $\text{meas}\{u > \varepsilon\} \rightarrow 0$, as $\varepsilon \rightarrow 0$, that is $u \leq 0$ a.e..

Similar ideas are in Gilbarg-Trudinger and Boccardo-G-Murat.

Stampacchia methods with FV schemes

Finite Volumes schemes with the so-called “admissible meshes for A ”.

1. positivity, maximum principle
2. Bounded solutions, Hölder continuous solutions: Thomas Rey thesis
3. “Measure” data: G-Herbin
4. Convection-diffusion without coercivity (and with measure data) : Droniou-G-Herbin

“Non admissible” meshes ?

Stampacchia methods with FE schemes

\mathcal{M} is a mesh of Ω , with triangles ($d = 2$) or tetrahedra ($d = 3$).

$$H = \{u \in C(\bar{\Omega}); u|_K \in P^1\}.$$

$$H_0 = \{u \in H; u = 0 \text{ on } \partial\Omega\}.$$

$$u_{\mathcal{M}} \in H_0,$$

$$\int_{\Omega} A \nabla u_{\mathcal{M}} \cdot \nabla v dx - \int_{\Omega} u_{\mathcal{M}} w \cdot \nabla v dx = T(v), \text{ for all } v \in H_0.$$

$$T(v) = \int_{\Omega} f v dx \text{ (examples 1, 2, 4)}$$

$$T(v) = \int_{\Omega} v df \text{ (examples 3, 4)}$$

Difficulty : $u \in H_0 \not\Rightarrow u^+, (u - b)^+, \psi(u), \varphi(u), T_{\varepsilon}(u^+) \in H_0$.

Choice of the test function

Idea: take as test function the interpolate of the test function of the “continuous” case.

If $v \in C(\bar{\Omega})$, $\Pi_{\mathcal{M}}(v) \in H$ and $\Pi_{\mathcal{M}}(v) = v$ at the vertices of the mesh.

Denoting by \mathcal{V} the set of vertices of the mesh:

$\Pi_{\mathcal{M}}(v) = \sum_{K \in \mathcal{V}} v(K) \phi_K$, where ϕ_K is the basis function associated to K .

re-writing the scheme (1)

$$u_M \in H_0,$$

$$\int_{\Omega} A \nabla u_M \cdot \nabla v dx = \int_{\Omega} f v dx, \text{ for all } v \in H_0.$$

With $u_M = \sum_{K \in \mathcal{V}} u_K \phi_K$ and $v = \sum_{L \in \mathcal{V}} v_L \phi_L$, this gives

$$\sum_{K \in \mathcal{V}} \sum_{L \in \mathcal{V}} (-T_{K,L}) u_K v_L = \int f v dx,$$

with $T_{K,L} = - \int_{\Omega} A \nabla \phi_K \cdot \nabla \phi_L dx$.

or, since $\sum_{L \in \mathcal{V}} T_{K,L} = 0$,

$$\sum_{K \in \mathcal{V}} \sum_{L \in \mathcal{V}} (-T_{K,L}) (u_K) (v_L - v_K) = \int f v dx,$$

re-writing the scheme (2)

and, finally,

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(v_K - v_L) = \int f v dx.$$

Taking $v = \Pi_{\mathcal{M}}\varphi(u)$ leads to:

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int f v dx.$$

positivity (1)

$f \leq 0$ a.e.

$$\varphi(s) = s^+.$$

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int f v dx,$$

yields:

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(u_K^+ - u_L^+) = \int f v dx,$$

If $T_{K,L} \geq 0$, one has:

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)^2 \leq \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K - u_L)(u_K^+ - u_L^+),$$

positivity (2)

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)(u_K^+ - u_L^+) \leq \int f v dx \leq 0,$$

then:

$$\int_{\Omega} A \nabla \Pi_{\mathcal{M}} u^+ \cdot \nabla \Pi_{\mathcal{M}} u^+ dx = \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)^2 = 0,$$

from which one deduces $u^+ = 0$.

nondecreasing function φ

Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ Lipschitz continuous and nondecreasing.

Define ϕ by $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$.

For $a, b \in \mathbb{R}$, one has (thanks to Cauchy-Schwarz Inequality) :

$$(\phi(a) - \phi(b))^2 \leq (a - b)(\varphi(a) - \varphi(b)).$$

then, IF $T_{K,L} \geq 0$ (for all (K, L)), one has:

$$\begin{aligned} \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (\phi(u_K) - \phi(u_L))^2 &\leq \\ \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (u_K - u_L) (\varphi(u_K) - \varphi(u_L)). \end{aligned}$$

$$\int_{\Omega} A \nabla \Pi_{\mathcal{M}} \phi(u) \cdot \nabla \Pi_{\mathcal{M}} \phi(u) = \sum_{(K,L) \in (\mathcal{V})^2} T_{K,L} (\phi(u_K) - \phi(u_L))^2.$$

“measure” data

For $\theta > 1$, define φ :

$$\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt; \quad s \in \mathbb{R}.$$

Taking $v = \Pi_{\mathcal{M}}\varphi(u) \in H_0$:

$$\int_{\Omega} |\nabla \Pi_{\mathcal{M}}\phi(u)|^2 dx \leq C_\theta \|f\|_1,$$

with $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$.

convection-diffusion without coercivity

Take $v = \Pi_{\mathcal{M}}\varphi(u)$ with $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2} (\theta = 2)$.

If the mesh size is small enough (or using an “upwinding” for convection part), one obtains an $H_0^1(\Omega)$ –estimate on $\Pi_{\mathcal{M}} \ln(1 + |u|) \in H_0^1(\Omega)$, then, an estimate on $\ln(1 + |u|)$ in $L^2(\Omega)$ and finally, as in the “continuous” case, an estimate on $\text{meas}(\{|u| \geq k\})$.

Conclusion

If $T_{K,L} \geq 0$, for all K, L , the methods of Stampacchia can be used for the study of numerical schemes (EF and VF)...

They give the desired properties on the approximate solution in Examples 1 and 2 (positivity, L^∞ -bound), Estimates and Convergence of the approximate solution in Examples 3 and 4 (measure data and convection-diffusion without coercivity).

Ongoing work

Without the condition $T_{K,L} \geq 0$, it seems not easy to use the methods of Stampacchia. . .

For Finite Volumes schemes with a “non admissible” meshes, a possible solution is (perhaps) to discretize this elliptic linear problem with a nonlinear scheme taking some $T_{K,L}(u)$ depending on the approximate solution, that is a scheme under the form:

$$\sum_{(K,L) \in (\mathcal{V})^2} T_{K,L}(u)(u_K - u_L)(v_K - v_L) = T(v),$$

and with $T_{K,L}(u) \geq 0$, for all K, L .