Stationary compressible Stokes equations (+Navier-Stokes equations and nonsteady case)

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Stationary Compressible Stokes Equations

Ω is a connected bounded open set of $\mathbb{R}^N$, $N = 2$ or 3, with a Lipschitz continuous boundary,

$$\mathrm{div}(\varphi(\rho)u) = 0 \text{ in } \Omega,\ \rho \geq 0 \text{ in } \Omega,\ \int_\Omega \rho(x)dx = M,$$

$$-\Delta u + \nabla p = f(\cdot, \rho) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$p = \eta(\rho) \text{ in } \Omega$$

$M > 0$,
f is at most linear, $|f(x,\rho)| \leq B(h(x) + |\rho|)$, $h \in L^2(\Omega)$
φ is Lipschitz continuous, increasing, $\varphi(0) = 0$
η is continuous superlinear, $\liminf_{s\to+\infty} \eta(s)/s = +\infty$, increasing ($\eta(0) = 0$)

Functional spaces : $u \in H_0^1(\Omega)^N$ $p, \rho \in L^2(\Omega)$

Main example

$\varphi(\rho) = \rho$
$f(x,\rho) = \bar{f}(x) + g(x)\rho$, $\bar{f} \in L^2(\Omega)$, $g \in L^\infty(\Omega)$
$\eta(\rho) = \rho^\gamma$, $\gamma > 1$.

$$\mathrm{div}(\rho u) = 0 \ \text{ in } \Omega, \ \rho \geq 0 \ \text{ in } \Omega, \ \int_\Omega \rho(x)dx = M,$$

$$-\Delta u + \nabla p = \bar{f} + g\rho \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$p = \rho^\gamma \text{ in } \Omega \qquad \text{(Equation Of State)}$$

$M > 0$

The main difficulty is the nonlinearity of the EOS (for passing to the limit with weak convergences)

Existence of a solution

Existence of a (weak) solution can be proved passing to the limit on two different approximate solution

1. Approximate solution obtained using a convenient numerical scheme
2. Approximate solution obtained using a viscous regularization of the mass equation

The second way is essentially in previous works of P. L. Lions, E. Feirsel, A. Novotny (1990. . .)

No uniqueness result.

(Incompressible NS Equations : J. Leray, 1934)

Existence of a solution

Existence of a (weak) solution can be proved passing to the limit on two different approximate solution

1. Approximate solution obtained using a convenient numerical scheme
2. Approximate solution obtained using a viscous regularization of the mass equation. Convergence of the approximate solution proven with a simple proof which can be adapted in order to do the first method (in particular with schemes used in an industrial framework)

Steps of the proof

1. Definition of the approximate problem and existence of an approximate solution
2. Estimates on the approximate solution
3. passing to the limit

The approximate problem

$T_n(s) = \min\{\max\{s, -n\}, n\}$

Then, the regularized problem reads, for $n, l, m \in \mathbb{N}^*$,

$$u \in H_0^1(\Omega)^N,\ \rho \in H^1(\Omega),\ p \in L^2(\Omega),$$

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \, \mathrm{div}(v) \, dx = \int_\Omega f_l(x, \rho) \cdot v \, dx,\ \forall v \in H_0^1(\Omega)^N$$

$$\int_\Omega \varphi(\rho) u \cdot \nabla \psi \, dx - \frac{1}{n} \int_\Omega \nabla \rho(x) \cdot \nabla \psi(x) \, dx = 0,\ \forall \psi \in H^1(\Omega)$$

$$\rho > 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho \, dx = M,\ p = \eta_m(\rho) \text{ a.e. in } \Omega$$

where $f_l(x, s) = T_l(f(x, s))$ and $\eta_m(s) = T_m(\eta(s))$

Existence of an approximate solution

Schauder fixed point for the application T from $L^2(\Omega)$ to $L^2(\Omega)$, $T(\rho) = \rho$.

$$u \in H_0^1(\Omega)^N,\ \rho \in H^1(\Omega),\ p \in L^2(\Omega),$$

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \, \mathrm{div}(v) \, dx = \int_\Omega f_l(x, \rho) \cdot v \, dx,\ \forall v \in H_0^1(\Omega)^N$$

$$\int_\Omega \varphi(\rho) u \cdot \nabla \psi \, dx - \frac{1}{n} \int_\Omega \nabla \rho \cdot \nabla \psi \, dx = 0,\ \forall \psi \in H^1(\Omega)$$

$$\rho > 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho \, dx = M,\ p = \eta_m(\rho^+) \text{ a.e. in } \Omega$$

where $f_l(x, s) = T_l(f(x, s))$ and $\eta_m(s) = T_m(\eta(s))$

Intermediate problem

$u \in L^p(\Omega)^N$, $p > N$ (true if $u \in H_0^1(\Omega)^N$). $M > 0$. φ Lipschitz continuous and $\varphi(0) = 0$.

There exist a unique ρ solution of

$$\rho \in H^1(\Omega),$$
$$\int_\Omega \nabla\rho \cdot \nabla v \, dx - \int_\Omega \varphi(\rho) u \cdot \nabla v \, dx = 0, \; \forall v \in H^1(\Omega)$$

with $\int_\Omega \rho(x) dx = M$

Furthermore:

1. $\rho > 0$ a.e. on Ω
2. For any $A > 0$, there exists C such that

$$\| \, |u| \, \|_{L^p(\Omega)} \leq A \Rightarrow \|\rho\|_{H^1(\Omega)} \leq C(A, p, M, \varphi, \Omega)$$

Proof using the Leray-Schauder topological degree

Intermediate problem, main point

$u \in L^p(\Omega)^N$, $p > N$. $M > 0$. φ Lipschitz continuous and $\varphi(0) = 0$.

$$\rho \in H^1(\Omega),$$
$$\int_\Omega \nabla\rho \cdot \nabla v \, dx - \int_\Omega \varphi(\rho) u \cdot \nabla v \, dx = 0, \ \forall v \in H^1(\Omega)$$

with $\int_\Omega \rho(x) dx = M$

Proof of *a priori* positivity of ρ taking $v = T_\varepsilon(\rho^+)$ and $\varepsilon \to 0$ and uniqueness taking $v = T_\varepsilon((\rho_1 - \rho_2)^+)$, as in an old paper of Boccardo-G-Murat

Intermediate problem, Leray-Schauder topological degree

$u \in L^p(\Omega)^N$, $p > N$. $M > 0$. φ Lipschitz continuous and $\varphi(0) = 0$.
F from $[0,1] \times L^2(\Omega)$ to $L^2(\Omega)$
$F(t, \rho) = \rho$

$$\rho \in H^1(\Omega),$$
$$\int_\Omega \nabla\rho \cdot \nabla v \, dx - \int_\Omega t\varphi(\rho) u \cdot \nabla v \, dx = 0, \ \forall v \in H^1(\Omega)$$

with $\int_\Omega \rho \, dx = tM$
$L^2(\Omega)$-estimate on ρ if $F(t, \rho) = \rho$

Passing to the limit

$n, l, m \in \mathbb{N}^*$,

$$u \in H_0^1(\Omega)^N,\ \rho \in H^1(\Omega),\ p \in L^2(\Omega),$$
$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \,\mathrm{div}(v)\, dx = \int_\Omega f_l(x,\rho)\cdot v\, dx,\ \forall v \in H_0^1(\Omega)^N$$
$$\int_\Omega \varphi(\rho) u \cdot \nabla\psi \, dx - \frac{1}{n}\int_\Omega \nabla\rho(x)\cdot\nabla\psi(x)\, dx = 0,\ \forall \psi \in H^1(\Omega)$$
$$\rho > 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho\, dx = M,\ p = \eta_m(\rho) \text{ a.e. in } \Omega$$

$m \to +\infty$
$l \to +\infty$
$n \to +\infty$

$m \to +\infty$

n, l are fixed

$$u \in H_0^1(\Omega)^N,\ \rho \in H^1(\Omega),\ p \in L^2(\Omega),$$
$$\int_\Omega \nabla u : \nabla v\, dx - \int_\Omega p\, \mathrm{div}(v)\, dx = \int_\Omega f_l(x,\rho)\cdot v\, dx,\ \forall v \in H_0^1(\Omega)^N$$
$$\int_\Omega \varphi(\rho) u \cdot \nabla\psi\, dx - \frac{1}{n}\int_\Omega \nabla\rho(x)\cdot\nabla\psi(x)\, dx = 0,\ \forall \psi \in H^1(\Omega)$$
$$\rho > 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho\, dx = M,\ p = \eta_m(\rho) \text{ a.e. in } \Omega$$

$H_0^1(\Omega)$-estimate on u thanks to $\int_\Omega \eta_m(\rho)\mathrm{div}(u) dx \le 0$
$H^1(\Omega)$-estimate on ρ
$L^2(\Omega)$-estimate on p taking $\mathrm{div}(v) = p - m$ with $v \in H_0^1(\Omega)^N$
and using $\int_\Omega \rho\, dx = M$.

$l \to +\infty$

n is fixed

$$u \in H_0^1(\Omega)^N,\ \rho \in H^1(\Omega),\ p \in L^2(\Omega),$$
$$\int_\Omega \nabla u : \nabla v\, dx - \int_\Omega p\, \mathrm{div}(v)\, dx = \int_\Omega f_l(x,\rho) \cdot v\, dx,\ \forall v \in H_0^1(\Omega)^N$$
$$\int_\Omega \varphi(\rho) u \cdot \nabla\psi\, dx - \frac{1}{n} \int_\Omega \nabla\rho(x) \cdot \nabla\psi(x)\, dx = 0,\ \forall \psi \in H^1(\Omega)$$
$$\rho > 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho\, dx = M,\ p = \eta(\rho) \text{ a.e. in } \Omega$$

$H_0^1(\Omega)$-estimate on u and $L^2(\Omega)$-estimate on p thanks to $\int_\Omega \eta(\rho)\mathrm{div}(u) dx \le 0$ and taking $\mathrm{div}(v) = p - m$ with $v \in H_0^1(\Omega)^N$ and using $\int_\Omega \rho\, dx = M$ and the superlinearity of η

$H^1(\Omega)$-estimate on ρ

$n \to +\infty$

$$
\begin{aligned}
& u_n \in H_0^1(\Omega)^N,\ \rho_n \in H^1(\Omega),\ p_n \in L^2(\Omega), \\
& \int_\Omega \nabla u_n : \nabla v \, dx - \int_\Omega p_n \, \mathrm{div}(v) \, dx = \int_\Omega f(x, \rho_n) \cdot v \, dx,\ \forall v \in H_0^1(\Omega)^N \\
& \int_\Omega \varphi(\rho_n) u_n \cdot \nabla \psi \, dx - \frac{1}{n} \int_\Omega \nabla \rho_n(x) \cdot \nabla \psi(x) \, dx = 0,\ \forall \psi \in H^1(\Omega) \\
& \rho_n > 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho_n \, dx = M,\ p_n = \eta(\rho_n) \text{ a.e. in } \Omega
\end{aligned}
$$

$H_0^1(\Omega)$-estimate on u_n and $L^2(\Omega)$-estimate on p_n thanks to $\int_\Omega \eta(\rho_n) \mathrm{div}(u_n) dx \leq 0$ and taking $\mathrm{div}(v_n) = p_n - m_n$ with $v_n \in H_0^1(\Omega)^N$ and using $\int_\Omega \rho_n \, dx = M$ and the superlinearity of η

$L^2(\Omega)$-estimate on ρ

$n \to +\infty$

$$u_n \to u \text{ in } L^2(\Omega)^N \text{ and weakly in } H_0^1(\Omega)^N$$
$$\rho_n \to \rho \text{ weakly in } L^2(\Omega)$$
$$p_n \to p \text{ weakly in } L^2(\Omega)$$

But, we do not have an $H^1(\Omega)$ estimate on ρ_n

$$h_n = f(\cdot, \rho_n) \to h \text{ weakly in } L^2(\Omega)^N$$
$$q_n = \varphi(\rho_n) \to q \text{ weakly in } L^2(\Omega)$$

We need some additional tricks to prove $h = f(\cdot, \rho)$, $q = \varphi(\rho)$, $p = \eta(\rho)$

Passage to the limit in the momentum equation

$v \in C_c^\infty(\Omega)^N$,

$$\int_\Omega \nabla u_n : \nabla v \, dx - \int_\Omega p_n \mathrm{div}(v) \, dx = \int_\Omega h_n \cdot v \, dx.$$

$n \to \infty$

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \mathrm{div}(v) \, dx = \int_\Omega h \cdot v \, dx.$$

$h = f(\cdot, \rho)$?

Passage to the limit in the mass equation

$v \in C_c^\infty(\mathbb{R}^N)$

$$\int_\Omega q_n u_n \cdot \nabla v \, dx - \frac{1}{n} \int_\Omega \nabla \rho_n \cdot \nabla v \, dx = 0,$$

$n \to \infty$

$$\int_\Omega q u \cdot \nabla v \, dx = 0$$

$q \geq 0$ a.e.
$q = \varphi(\rho)$?

Passage to the limit in the nonlinear functions of ρ

$h = f(\cdot, \rho)$? (easy if $f(x, \rho) = \bar{f}(x) + g(x)\rho$)
$q = \varphi(\rho)$? (easy if $\varphi(\rho) = \rho$)
$p = \eta(\rho)$
Idea: prove $\int_\Omega p_n q_n \to \int_\Omega pq$ and deduce a.e. convergence (of p_n and ρ_n)

$\nabla : \nabla = \mathrm{divdiv} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^N$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v})$$

Then, for all $\bar{v}$ in $H_0^1(\Omega)^N$, the momentum equation is

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(u_n) \cdot \mathrm{curl}(\bar{v}) \\ - \int_\Omega p_n \mathrm{div}(\bar{v}) = \int_\Omega h_n \cdot \bar{v}$$

Choice of $\bar{v}$? $\bar{v} = \bar{v}_n$ with $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = q_n$ and $\bar{v}_n$ bounded in $H_0^1(\Omega)^N$ (unfortunately, 0 is impossible).

Then, up to a subsequence,

$\bar{v}_n \to v$ in $L^2(\Omega)^N$ and weakly in $H_0^1(\Omega)^N$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = q$.

Proof using $\bar{v}_n$ (1)

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}_n) + \int_\Omega \mathrm{curl}(u_n) \cdot \mathrm{curl}(\bar{v}_n) - \int_\Omega p_n \mathrm{div}(\bar{v}_n) = \int_\Omega h_n \cdot \bar{v}_n$$

But, $\mathrm{div}(\bar{v}_n) = q_n$ and $\mathrm{curl}(\bar{v}_n) = 0$. Then:

$$\int_\Omega (\mathrm{div}(u_n) - p_n) q_n = \int_\Omega h_n \cdot \bar{v}_n$$

$n \to \infty$

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}(u_n) - p_n) q_n = \int_\Omega h \cdot v$$

Proof using $\bar{v}_n$ (2)

But, since $-\Delta u + \nabla p = h$

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v) + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v) - \int_\Omega p\,\mathrm{div}(v) = \int_\Omega h\cdot v$$

which gives (using $\mathrm{div}(v) = q$ and $\mathrm{curl}(v) = 0$)
$\int_\Omega (\mathrm{div}(u) - p)q = \int_\Omega h\cdot v$ Then

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))q_n = \int_\Omega (p - \mathrm{div}(u))q$$

thanks to the mass equations, $\int_\Omega q_n \mathrm{div}(u_n) \le 0$ and $\int_\Omega q\,\mathrm{div}(u) = 0$. Then,

$$\limsup_{n\to\infty} \int_\Omega p_n q_n \le \int_\Omega pq$$

Error in the preceding proof

In the preceding proof, we used $\bar{v}_n$ such that $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\bar{v}_n$ bounded in $H_0^1(\Omega)^N$.

Unfortunately, it is impossible to have $\bar{v}_n \in H_0^1(\Omega)^d$ but only $\bar{v}_n \in H^1(\Omega)^N$.

Curl-free test function

Let $w_n \in H_0^1(\Omega)$, $-\Delta w_n = q_n$,
One has $w_n \in H^2_{loc}(\Omega)$ since, for $\psi \in C_c^\infty(\Omega)$, one has $\Delta(w_n\psi) \in L^2(\Omega)$ and

$$\sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_j(w_n\psi)\,\partial_i\partial_j(w_n\psi) = \sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_i(w_n\psi)\,\partial_j\partial_j(w_n\psi)$$
$$= \int_\Omega (\Delta(w_n\psi))^2 = C_\psi < \infty$$

Then, taking $v_n = \nabla w_n$

- $v_n \in (H^1_{loc}(\Omega))^N$,
- $\mathrm{div}(v_n) = q_n$ a.e. in Ω,
- $\mathrm{curl}(v_n) = 0$ a.e. in Ω,
- $H^1_{loc}(\Omega)$-estimate on v_n with respect to $\|q_n\|_{L^2(\Omega)}$.

Then, up to a subsequence, as $n \to \infty$, $v_n \to v$ in $L^2_{loc}(\Omega)^N$ and weakly in $H^1_{loc}(\Omega)^N$, $\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = q$.

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))q_n\psi \to \int_\Omega (p - \mathrm{div}(u))q\psi$

Let $\psi \in C_c^\infty(\Omega)$ (so that $v_n\psi \in H_0^1(\Omega)^N$)). Taking $\bar{v} = v_n\psi$:

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(v_n\psi) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(v_n\psi) - \int_\Omega p_n\mathrm{div}(v_n\psi) = \int_\Omega h_n\cdot(v_n\psi).$$

Using a proof similar to that given if $\psi = 1$ (with additionnal terms involving ψ), we obtain :

$$\lim_{n\to\infty}\int_\Omega (p_n - \mathrm{div}(u_n))q_n\psi = \int_\Omega (p - \mathrm{div}(u))q\psi$$

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))q_n \to \int_\Omega (p - \mathrm{div}(u))q$

$F_n = (p_n - \mathrm{div}(u_n))q_n$, $F = (p - \mathrm{div}(u))q$
$F_n \to F$ in $D'(\Omega)$
The sequence F_n is equiintegrable (since $p_n - \mathrm{div}(u_n)$ is bounded in L^2 and q_n^2 is equiintegrable thanks to p_n bounded in L^2 and η superlinear)
Then $F_n \to F$ weakly in $L^1(\Omega)$

Proving $\int_\Omega p_n q_n \to \int_\Omega pq$

$$\int_\Omega (p_n - \mathrm{div}(u_n)) q_n \to \int_\Omega (p - \mathrm{div}(u)) q$$

But thanks to the mass equations:

$$\int_\Omega \mathrm{div}(u_n) q_n \leq 0, \ \int_\Omega \mathrm{div}(u) q = 0;$$

Then:

$$\limsup_{n\to\infty} \int_\Omega p_n q_n \leq \int_\Omega pq$$

a.e. convergence of ρ_n and p_n,. Leray-Lions trick

Simple case: $\varphi(\rho) = \rho$ and assuming $\eta(\rho) \in L^2(\Omega)$.
Let $G_n = (\eta(\rho_n) - \eta(\rho))(\rho_n - \rho) \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω.
Futhermore
$G_n = (p_n - \eta(\rho))(\rho_n - \rho) = p_n\rho_n - p_n\rho - \eta(\rho)\rho_n + \eta(\rho)\rho$ and

$$\int_\Omega G_n = \int_\Omega p_n\rho_n - \int_\Omega p_n\rho - \int_\Omega \eta(\rho)\rho_n + \int_\Omega \eta(\rho)\rho$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ρ_n and
$\lim_{n\to\infty} \int_\Omega p_n\rho_n \leq \int_\Omega p\rho$

$$\lim_{n\to\infty} \int_\Omega G_n = 0,$$

Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \rho$ a.e. (since η is increasing function). Finally

$p_n = \eta(\rho_n) \to \eta(\rho) \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2,$

$p_n \to p$ weakly in $L^2(\Omega)$

then $p = \eta(\rho)$, similarly $h = f(\cdot, \rho)$

a.e. convergence of ρ_n and p_n, general case

The function η is a one-to-one function from $\mathbb{R}_+$ onto $\mathbb{R}_+$. We denote by $\bar{\eta}$ the reciprocal function of η. ($\bar{\eta}$ sublinear)
Since $p \in L^2(\Omega)$, one has $\bar{\eta}(p) \in L^2(\Omega)$ and we set $\bar{\rho} = \bar{\eta}(p)$

$$G_n = (\varphi(\rho_n) - \varphi(\bar{\rho}))(\eta(\rho_n) - \eta(\bar{\rho}))$$

so that $G_n \in L^1(\Omega)$, $G_n \geq 0$ a.e.

$$0 \leq \int_\Omega G_n = \int_\Omega (q_n - \varphi(\bar{\rho}))(p_n - p)$$

Then

$$\lim_{n\to\infty} \int_\Omega G_n \, dx \leq \int_\Omega (q - \varphi(\bar{\rho}))(p - p) \, dx = 0.$$

This gives $G_n \to 0$ in $L^1(\Omega)$ and then, up to a subsequence,

$$G_n = (\varphi(\rho_n) - \varphi(\bar{\rho}))(\eta(\rho_n) - \eta(\bar{\rho})) \to 0 \text{ a.e. in } \Omega$$

We now use the fact that φ and η are increasing, $\rho_n \to \bar{\rho}$ a.e. in Ω
Then $\bar{\rho} = \rho$, $q = \varphi(\rho)$, $h = f(\cdot, \rho)$, $p = \eta(\rho)$

η non decreasing instead of increasing

Simple case $f(.,\rho) = \bar{f} + g\rho$, $\varphi(\rho) = \rho$
One proves $p = \eta(\rho)$ with the Minty trick (but no a.e. convergence)
We set $\eta(s) = 0$ for $s < 0$ and $\bar{\eta}$ is the reciprocal function of $s \mapsto \eta(s) + s$ (which is a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$)
Let $\bar{p} \in L^2(\Omega)$ and $\bar{\rho} = \bar{\eta}(\bar{p})$ so that $\bar{\rho} \in L^2(\Omega)$

$$0 \leq \int_\Omega (\rho_n - \bar{\rho})(\eta(\rho_n) - \eta(\bar{\rho})) = \int_\Omega (\rho_n - \bar{\rho})(p_n - \bar{p} + \bar{\rho})$$

$$0 \leq \int_\Omega (\rho - \bar{\rho})(p - \bar{p} + \bar{\rho})$$

which gives also

$$0 \leq \int_\Omega (\rho - \bar{\rho})(p - \bar{p} + \rho) = \int_\Omega (\rho - \bar{\eta}(\bar{p}))(p - \bar{p} + \rho)$$

η non decreasing instead of increasing

For all $\bar{p} \in L^2(\Omega)$

$$0 \leq \int_\Omega (\rho - \bar{\eta}(\bar{p}))(p - \bar{p} + \rho)\, dx$$

Let $\psi \in C_c^\infty(\Omega)$, $\epsilon > 0$. Taking $\bar{p} = p + \rho + \epsilon\psi$, letting $\epsilon \to 0$ leads to, with the Dominated Convergence Theorem,

$$0 \leq -\int_\Omega (\rho - \bar{\eta}(p + \rho)))\psi\, dx.$$

Since ψ is arbitrary in $C_c^\infty(\Omega)$, we then conclude that $\rho = \bar{\eta}(p + \rho)$ which gives $\eta(\rho) + \rho = p + \rho$ and then $p = \eta(\rho)$.

Generalizations

- (Easy) Complete Diffusion term: $-\mu\Delta u - \frac{\mu}{3}\nabla(\operatorname{div} u)$, with $\mu \in \mathbb{R}_+^\star$ given, instead of $-\Delta u$.
- Stationary compressible Navier Stokes equation $\eta(\rho) = \rho^\gamma$, $\gamma > 3$ if $N = 3$.
- Navier-Stokes Equations with $N = 3$ and $\frac{3}{2} < \gamma \leq 3$. (probably sharp result with respect to γ without changing the diffusion term or the EOS)
- Evolution equation (Stokes and Navier-Stokes)

Stationary compressible Navier Stokes equations

Ω is a bounded open set of $\mathbb{R}^N$, $N = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^N$ and $M > 0$

$$-\Delta u + (\rho u \cdot \nabla)u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x)dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Functional spaces : $u \in H_0^1(\Omega)^N$, $p \in L^{\overline{q}}(\Omega)$, $\rho \in L^{\gamma\overline{q}}(\Omega)$.

If $d = 2$ or if $d = 3$ and $\gamma \geq 3$: $\overline{q} = 2$.

If $d = 3$ and $\frac{3}{2} < \gamma < 3$: $\overline{q} = \frac{3(\gamma-1)}{\gamma}$.

$\gamma = \frac{3}{2}$, $\frac{3\gamma-1}{\gamma} = 1$, $3(\gamma - 1) = \frac{3}{2}$

What about uniqueness?

- No uniqueness result for the stationary case
- "weak-strong" uniqueness for the evolution case (Stokes and Navier-Stokes equations, $p = \rho^\gamma$, $\gamma > 3/2$ for $N = 3$). Recent result by E. Feireisl, B. J. Jin and A. Novotny (2012).
- Adaptation of the proof of the "weak-strong" uniqueness result leads to an error estimate between a strong solution and an approximate solution given by a convenient numerical scheme (T.G, R. Herbin, D. Maltese and A. Novotny).

weak-strong uniqueness, simple case

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \ \text{ in } \Omega\times]0,T[, \ \rho \geq 0 \ \text{ in } \Omega,$$

$$\partial_t u - \Delta u + \nabla p = f \text{ in } \Omega\times]0,T[, \ \ u = 0 \text{ on } \partial\Omega,$$

$$u(x,0) = u_0, \rho(x,0) = \rho_0,$$

$$p = \rho^2 \text{ in } \Omega\times]0,T[.$$

ρ, u : weak solution

$\bar{\rho}$, $\bar{u}$: regular solution

Relative Energy :

$$E(t) = \frac{1}{2}\int_\Omega |u - \bar{u}|^2 dx + \int_\Omega |\rho - \bar{\rho}|^2 dx.$$

A similar idea was first introduced by C. Dafermos for hyperbolic systems (Relative Entropy)

weak-strong uniqueness, Gronwall Inequality

$$E(t) = \frac{1}{2}\int_\Omega |u-\bar{u}|^2 dx + \int_\Omega |\rho-\bar{\rho}|^2 dx.$$

$$E(t)-E(0) \leq -\int_0^t \int_\Omega (\rho-\bar{\rho})^2 \operatorname{div}\bar{u}\, dxdt + 2\int_0^t \int_\Omega (\rho-\bar{\rho})(u-\bar{u})\cdot\nabla\bar{\rho}\, dxdt$$

Since $\operatorname{div}(\bar{u})$ and $\nabla\bar{\rho}$ are bounded and $E(0)=0$, this gives

$$E(t) \leq C\int_0^t E(s)ds$$

Then $E(t)=0$ for all t

Proof of the energy inequality

Take u and $\bar{u}$ as test functions in the two momemtum equations
Take ρ and $\bar{\rho}$ as test functions in the two mass equations
We obtain

$$E(t)-E(0) \leq -\int_0^t \int_\Omega (\rho-\bar{\rho})^2 \mathrm{div}\,\bar{u}\,dxdt + 2\int_0^t \int_\Omega (\rho-\bar{\rho})(u-\bar{u})\cdot\nabla\bar{\rho}\,dxdt$$