# Hyperbolic equations and systems with discontinuous coefficients or source terms

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Equadiff 11, 2005

Notations:

$$(\cdot)_t = \frac{\partial(\cdot)}{\partial t},$$

$$(\cdot)_x = \frac{\partial(\cdot)}{\partial x},$$

$t \in \mathbb{R}_+$.

$x \in \mathbb{R}$, but extensions to $x \in \mathbb{R}^d$, $d = 2$ or $3$ are possible.

# Hyperbolic equation with a discontinuous coefficient

Two phase flow in an heterogeneous porous medium:
unknown: $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$:

$$u_t(x,t) + (kg(u))_x = 0,$$

$k(x) = k_l$, for $x < 0$,
$k(x) = k_r$, for $x > 0$,
$k_l, k_r > 0$, $k_l \neq k_r$,
$g : [0,1] \to \mathbb{R}$, Lipschitz continuous and such that
$g(0) = g(1) = 0$. Example: $g(u) = u(1-u)$

This hyperbolic equation with a discontinuous coefficient can be viewed has a conservative $2 \times 2$ hyperbolic system, adding $k$ has an unknown and the equation $k_t = 0$.

# Hyperbolic equation with a discontinuous coefficient

$$\begin{aligned} &u_t(x,t) + (kg(u))_x = 0, \\ &k_t = 0. \end{aligned}$$

$$W = \begin{bmatrix} u \\ k \end{bmatrix} \text{ and } F(W) = \begin{bmatrix} kg(u) \\ 0 \end{bmatrix},$$

$$W_t + (F(W))_x = 0,$$

or equivalently (for regular solutions), with $A(W) = DF(W)$:

$$W_t + A(W)W_x = 0.$$

# Hyperbolic system with a source term

Saint Venant Equations with topography (nonflat bottom)
unknowns: $h, u \ : \ \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ (with $h > 0$):

$$
\begin{aligned}
&h_t + (hu)_x = 0, \\
&(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x = -ghz_x,
\end{aligned}
$$

$g$ is a given constant and $z$ is a given function of $x$.
This $2 \times 2$ conservative hyperbolic system with a source term can be viewed has a nonconservative $3 \times 3$ hyperbolic system, adding $z$ has an unknown and the equation $z_t = 0$.

# Hyperbolic system with a source term

$$
\begin{aligned}
&h_t + (hu)_x = 0,\\
&(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + ghz_x = 0,\\
&z_t = 0.
\end{aligned}
$$

$$
W = \begin{bmatrix} u \\ hu \\ z \end{bmatrix},\ F(W) = \begin{bmatrix} hu \\ \frac{(hu)^2}{h} + \frac{1}{2}gh^2 \\ 0 \end{bmatrix} \text{ and}
$$

$$
B(W) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & gh \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
W_t + (F(W))_x + B(W)W_x = 0,
$$

or equivalently (for regular solutions):

$W_t + A(W)W_x = 0,$ (with $A(W) = DF(W) + B(W)$).

# Common feature : Resonance

Unknown: $W : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^p$, $p > 1$
Equation: $W_t + A(W)W_x = 0$

For all admissible $W \in \mathbb{R}^p$, the $p \times p$ matrix $A(W)$ has only real eigenvalues, but $A(W)$ is, for some $W$, not diagonizable.

$A(W)$ strictly hyperbolic: p real distincts eigenvalues and then diagonizable (in $\mathbb{R}$).
$A(W)$ genuine-hyperbolic: real eigenvalues and diagonalizable.
$A(W)$ resonnant-hyperbolic: real eigenvalues and not diagonalizable.

First example:

$p = 2$, $W = \begin{bmatrix} u \\ k \end{bmatrix}$, $A(W) = \begin{bmatrix} kg'(u) & g(u) \\ 0 & 0 \end{bmatrix}$ which is not diagonalizable if $g'(u) = 0$ and $g(u) \neq 0$ (if $g(u) = u(1-u)$, this is the case for $u = \frac{1}{2}$).

Second example:

$$p = 3,\ W = \begin{bmatrix} u \\ hu \\ z \end{bmatrix},\ A(W) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & gh \\ 0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues of $A(W)$ are $u \pm c$ and $0$, with $c = \sqrt{gh}$.

$A(W)$ is not diagonalizable if $u - c = 0$ or $u + c = 0$ (and $h > 0$).

# Linear resonant problem, ill posed

The Cauchy problem for a linear resonant problem is ill-posed in $L^\infty$(or in $L^1$, $L^2$ ..., but well posed in $C^\infty$).

Riemann problem for a typical example:

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0,$$

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ v_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ v_r \end{bmatrix}, \text{ if } x > 0,$$

The solution is , for all $t > 0$, $v(\cdot,t) = v(\cdot,0)$ and

$$u(\cdot,t) = u_l 1_{\mathbb{R}_-} + u_r 1_{\mathbb{R}_+} + t(v_l - v_r)\delta_0.$$

# Nonlinear resonant problem, sometimes ill posed

Academic simple example:

$$\begin{aligned} u_t + (au)_x &= 0, \\ a_t &= 0, \end{aligned}$$

$$\left[\begin{array}{c} u(x,0) \\ a(x,0) \end{array}\right] = \left[\begin{array}{c} u_l \\ a_l \end{array}\right], \text{ if } x < 0, \text{ and } \left[\begin{array}{c} u_r \\ a_r \end{array}\right], \text{ if } x > 0,$$

has no weak solution in $L^\infty$ if $a_l > 0$, $a_r < 0$ and $a_l u_l \neq a_r u_r$ (and has infinetely many solution if $a_l < 0$ and $a_r > 0$).

# Nonlinear resonant problem, sometimes ill posed

$$u_t + (au)_x = 0,$$
$$a_t = 0,$$

is equivalent (for regular solution) to $W_t + A(W)W_x = 0$, with $W = \begin{bmatrix} u \\ a \end{bmatrix}$ and $A(W) = \begin{bmatrix} a & u \\ 0 & 0 \end{bmatrix}$. Resonance occurs when $a = 0$ and $u \neq 0$.

The Riemann problem for the nonlinear system is ill posed in $L^\infty$ provided that $0$ is between $a_r$ and $a_l$ (except if $u_r = u_l = 0$).

This nonlinear case is “worse” than the linear case.

First example

$$u_t + (kg(u))_x = 0,$$
$$k_t = 0,$$

$g : [0,1] \to \mathbb{R}$, Lipschitz continuous and such that $g(0) = g(1) = 0$. The Cauchy problem is well posed in the following sense:

$k(x) = k_l$ if $x < 0$, $k(x) = k_r$ if $x > 0$, $u(\cdot, 0) = u_0 \in L^\infty$, $0 \leq u_0 \leq 1$. Then, the Cauchy problem has a unique entropy weak solution.

Karlsen-Risebro-Towers

Seguin-Vovelle, Bachmann-Vovelle

Bachmann: $k(x)g(u) \rightsquigarrow g(x,u)$, $g(x,0) = g(x,1) = 0$.

# Nonlinear resonant problem, sometimes well posed

## Second example

Saint Venant Equations with topography. The Riemann problem:

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + ghz_x = 0, \\ &z_t = 0 \end{aligned}$$

$$\begin{bmatrix} h \\ hu \\ z \end{bmatrix}(x,0) = \begin{bmatrix} h_l \\ (hu)_l \\ z_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} h_r \\ (hu)_r \\ z_r \end{bmatrix}, \text{ if } x > 0,$$

has one (sometimes three. . . ) solution (composed of constant states and waves), satisfying a classical entropy condition, assuming continuity of (the Riemann invariants) $hu$ and $\psi$ at the contact discontinuity (at $x = 0$) with $\psi = \frac{1}{2}u^2 + g(h+z)$.
Chinnayya-LeRoux-Seguin, Goatin-LeFloch

# Nonlinear resonant problem, sometimes well posed

## Third example

Isentropic Euler Equations with an EOS taking into account a simple model of “phase transition”, that is:

$$
\begin{aligned}
&p = a_1\rho, \text{ if } 0 < \rho < \rho_1,\\
&p = a_1\rho_1, \text{ if } \rho_1 \le \rho \le \rho_2,\\
&p = a_2\rho, \text{ if } \rho_2 < \rho,
\end{aligned}
$$

with $\rho_1, \rho_2, a_1, a_2$ given constants, $0 < \rho_1 < \rho_2$, $0 < a_1 < a_2$.

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&(\rho u)_t + (\rho u^2 + p)_x = 0.
\end{aligned}
$$

For $\rho_1 \le \rho \le \rho_2$ and any $u$, the system is resonant (with $u$ as eigenvalue, and the 2 genuinely nonlinear fields lead to a linear degenerate field). But the Riemann problem is well posed. Recent result of Seguin.

# Discretization by Finite Volume Shemes

$$W_t + (F(W))_x + B(W)W_x = 0,$$

$$W(\cdot, 0) = W_0.$$

Time step: $k$, $t_n = nk$

Space step: $h$, $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h$

Approximate solution for $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $t = t_n$: $W_i^n$

$$W_i^0 = \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W_0(x)dx, \ i \in \mathbb{Z}.$$

# Riemann problem

$$W_t + (F(W))_x + B(W)W_x = 0,$$

$$W(x,0) = \begin{cases} W_l \text{ if } x < 0, \\ W_r \text{ if } x > 0. \end{cases}$$

Let $W$ be the (or a) self similar solution, that is:

$$W(x,t) = R(\frac{x}{t}, W_l, W_r)$$

and set

$$W^{\star,\pm}(W_l, W_r) = R(0^{\pm}, W_l, W_r).$$

# Godunov scheme for nonconservative system

For $i \in \mathbb{Z}$, $n \geq 0$:

$$\frac{W_i^{n+1} - W_i^n}{k} + F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+} + B(W_i^n)(W_{i+\frac{1}{2}}^{n,-} - W_{i-\frac{1}{2}}^{n,+}) = 0,$$

with $F_{i+\frac{1}{2}}^{n,\pm} = F(W_{i+\frac{1}{2}}^{n,\pm})$,
$W_{i+\frac{1}{2}}^{n,\pm} = W^{\star,\pm}(W_i^n, W_{i+1}^n)$.

CFL condition: $k \leq Ch$.

# Godunov scheme, particular cases

$$\frac{W_i^{n+1} - W_i^n}{k} + F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+} + B(W_i^n)(W_{i+\frac{1}{2}}^{n,-} - W_{i-\frac{1}{2}}^{n,+}) = 0,$$

1. conservative case: $F_{i+\frac{1}{2}}^{n,+} = F_{i+\frac{1}{2}}^{n,-}$, even if $W_{i+\frac{1}{2}}^{n,+} \neq W_{i+\frac{1}{2}}^{n,-}$ (thanks to Rankine-Hugoniot condition on the Riemann problem).
2. In the case of Saint Venant equations with topography, one has $z_{i+\frac{1}{2}}^{n,-} = z_{i-\frac{1}{2}}^{n,+} = z_i^n$ and then $B(W_i^n)(W_{i+\frac{1}{2}}^{n,-} - W_{i-\frac{1}{2}}^{n,+}) = 0$. The non conservativity of the equation (that is the source term) appears only in the fact that, generally, $F_{i+\frac{1}{2}}^{n,+} \neq F_{i+\frac{1}{2}}^{n,-}$.

# Linearized system

- Initial system: $W_t + A(W)W_x = 0$, $A(W) = DF(W) + B(W)$.
- Linearized system: $W_t + A(\overline{W})W_x = 0$, with some fixed $\overline{W} \in \mathbb{R}^p$.
- Initial system with a change of unknown: $Y = \phi(W)$. $\phi$ is not necessarily invertible, but one assumes that there exists $C, G, D$, such that $A(W) = C(Y)$, $F(W) = G(Y)$, $B(W)W_x$=$D(W)Y_x$. Then $W_t + A(W)W_x$ leads to

$$Y_t + C(W)Y_x = 0.$$

- Linearized system with a change of unknown: $Y = \phi(W)$,

$$Y_t + C(\overline{W})Y_x = 0,$$

with some fixed $\overline{W} \in \mathbb{R}^p$.

# linearized Riemann problem

$$Y_t + C(\overline{W})Y_x = 0,$$

$$Y(x,0) = \begin{cases} Y_l = \phi(W_l) \text{ if } x < 0, \\ Y_r = \phi(W_r) \text{ if } x > 0. \end{cases}$$

For instance, $\overline{W} = \frac{W_r + W_l}{2}$ or another mean value between $W_l$ and $W_r$.
Let $Y$ be the self similar solution of this problem (when it exists...), that is:

$$Y(x,t) = LR(\frac{x}{t}, Y_l, Y_r)$$

and set $Y^{\star,\pm}(W_l, W_r) = LR(0^{\pm}, Y_l, Y_r)$.

# VFRoe-ncv scheme

For $i \in \mathbb{Z}$, $n \geq 0$:

$$\frac{W_i^{n+1} - W_i^n}{k} + F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+} + D(W_i^n)(Y_{i+\frac{1}{2}}^{n,-} - Y_{i-\frac{1}{2}}^{n,+}) = 0,$$

with:

$F_{i+\frac{1}{2}}^{n,\pm} = G(Y_{i+\frac{1}{2}}^{n,\pm})$ $(F(W) = G(\phi(W)),\ B(W)W_x = D(W)Y_x)$,

$Y_{i+\frac{1}{2}}^{n,\pm} = Y^{\star,\pm}(W_i^n, W_{i+1}^n)$

CFL condition: $k \leq Ch$.

# VFroe-ncv scheme

1. Modification for conservativity (null eigenvalue)
2. Discontinuity of the numerical flux when there is $0$ as eigenvalue. $F(W^\star(W_l, W_r))$ is a discontinuous fonction of $W_l$ and $W_r$).
3. Choice of $Y$.
   $Y = (kg(u), k)^t$ for porous media,
   $Y = (2c, u, z)^t$ or $Y = (q, \psi, z)^t$ for Saint Venant with topography.

# Remark on resonance, porous media

$$u_t(x,t) + (kg(u))_x = 0,$$
$$k_t = 0.$$

With the choice $Y = (kg(u), k)^t$ the linearized system is $Y_t + C(\overline{W})Y_x = 0$, $\overline{W} = (\overline{u}, \overline{k})^t$,

$$C(\overline{W}) = \begin{bmatrix} \overline{k}g'(\overline{u}) & 0 \\ 0 & 0 \end{bmatrix},$$

which is never resonnant…
Eigenvalues: $\lambda_1 = \overline{k}g'(\overline{u})$, $\lambda_2 = 0$
Eigenvectors: $e_1 = (1,0)^t$, $e_2 = (0,1)^t$

$$
\begin{aligned}
&h_t + (hu)_x = 0,\\
&(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + ghz_x = 0,\\
&z_t = 0
\end{aligned}
$$

With the choice $Y = (q, \psi, z)^t$, the linearized system is
$Y_t + C(\overline{W})Y_x = 0$, $\overline{W} = (\overline{h}, \overline{hu}, \overline{z})^t$, $\overline{c} = \sqrt{g\overline{h}}$,

$$
C(\overline{W}) = \begin{bmatrix} \overline{u} & \overline{h} & 0 \\ g & \overline{u} & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

which is never resonnant (for $\overline{h} > 0$)...
Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = \overline{u} + \overline{c}$, $\lambda_2 = \overline{u} - \overline{c}$
Eigenvectors: $e_1 = (0, 0, 1)^t$, $e_2 = (\overline{h}, \overline{c}, 0)^t$, $e_3 = (-\overline{h}, \overline{c}, 0)^t$

# Remark on resonance, Saint Venant with topography

With the choice $Y = (2c, u, z)^t$, the linearized system is

$Y_t + C(\overline{W})Y_x = 0$, $\overline{W} = (\overline{h}, \overline{h}\overline{u}, \overline{z})^t$, $\overline{c} = \sqrt{g\overline{h}}$,

$$C(\overline{W}) = \begin{bmatrix} \overline{u} & \overline{c} & 0 \\ \overline{c} & \overline{u} & g \\ 0 & 0 & 0 \end{bmatrix},$$

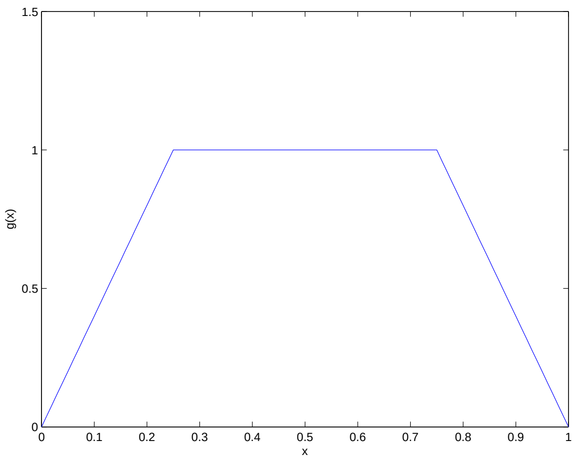
which is resonnant if $\overline{u} + \overline{c} = 0$ or $\overline{u} - \overline{c} = 0$.
Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = \overline{u} + \overline{c}$, $\lambda_2 = \overline{u} - \overline{c}$
Eigenvectors:

- If $\overline{u} \pm \overline{c} \neq 0$,
  $e_1 = (\overline{c}g, -\overline{u}g, \overline{u}^2 - \overline{c}^2)^t$, $e_2 = (1, 1, 0)^t$, $e_3 = (1, -1, 0)^t$
- If $\overline{u} = \overline{c}$, $e_1 = (1, -1, 0)^t$, $e_2 = (1, 1, 0)^t$
- If $\overline{u} = -\overline{c}$, $e_1 = (1, 1, 0)^t$, $e_3 = (1, -1, 0)^t$

# Numerical results

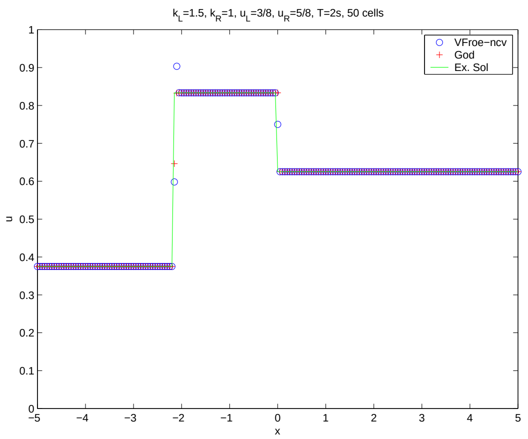
1. Perfect with Godunov. In particular, for Saint Venant with topography, one has preservation of all steady state solutions : $q = hu$ and $\psi = \frac{u^2}{2} + g(h+z)$ constant (and not only those with $u = 0$ which are called ”lake at rest”)
2. Perfect also with VFRoe-ncv, sometimes with one “incorrect point”. For Saint Venant with topography: Preservation of steady state solution with $u = 0$ (for any choice of the variable of linearization $Y$) and preservation of all steady state solutions for $Y = (q, \psi, z)^t$

# numerical results, two phase flow in porous media



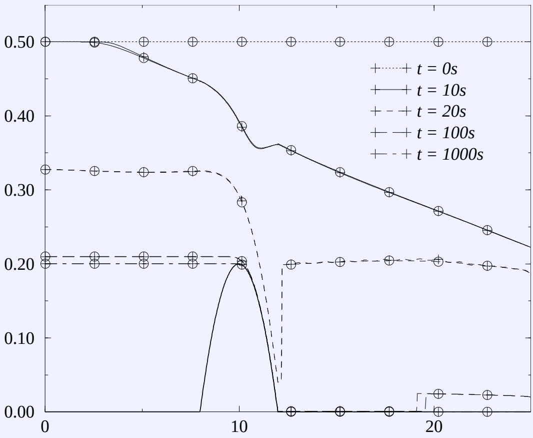

Resonance occurs for all $(k, u)$ with $u \in (\frac{1}{4}, \frac{3}{4})$.

# numerical results, two phase flow in porous media




Resonance occurs for all $(k, u)$ with $u \in (\frac{1}{4}, \frac{3}{4})$.

# numerical results, Saint Venant with topography




A case with $h = 0$, steady state, sonic point... VFRoe-ncv with $Y = (2c, u, z)^t$ (best choice for the problem of vanishing $h$).

# References

1. Porous medium: Towers, Seguin-Vovelle, Bachmann
2. Saint Venant: Kurganov-Levy, Simeoni-Perthame, Chinnayya-LeRoux-Seguin, Goutal-Karni Hérard-Gallouët-Seguin…