Convergence of approximate solutions for Stationary compressible Stokes and Navier-Stokes equations

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joint work with

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Fisrt step for proving the convergence of approximate solutions for the evolution compressible Navier-Stokes equations (which gives, in particular, the existence of solutions for compressible Navier Stokes equations, $d = 3$, $p = \rho^\gamma$, $\gamma > \frac{3}{2}$).

Existence of (weak) solutions is already known since the works of P. L. Lions, E. Feirsel, A. Novotny...
No uniqueness result.

Aim : to prove the existence of solutions, passing to the limit on approximate solutions given by efficient numerical schemes (in particular, with schemes used in industrial codes).

Stationary compressible Stokes equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma \geq 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho u) = 0 \text{ in } \Omega, \; \rho \geq 0 \text{ in } \Omega, \; \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Functional spaces : $u \in H_0^1(\Omega)$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

(different spaces for p and ρ in the case of Navier-Stokes if $d = 3$ and $\gamma < 3$)

Weak solution of the stationary compressible Stokes problem

Functional spaces : $u \in H_0^1(\Omega)^d$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

- Momentum equation:

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \mathrm{div}(v) \, dx = \int_\Omega f \cdot v \, dx \text{ for all } v \in H_0^1(\Omega)^d$$

- Mass equation:

$$\int_\Omega \rho u \cdot \nabla \varphi \, dx = 0 \text{ for all } \varphi \in C_c^\infty(\Omega)$$

$$\rho \geq 0 \text{ a.e.}, \quad \int_\Omega \rho dx = M$$

- EOS: $p = \rho^\gamma$

Main result

- Two possible discretizations for the momentum equation :
  $\rightsquigarrow$ MAC scheme (most commonly used scheme for incompressible Navier Stokes equations)
  $\rightsquigarrow$ Crouzeix-Raviart Finite Element
- Discretization of the mass equation (and total mass constraint) by classical upwind Finite Volume
- Existence of solution for the discrete problem
- Proof of the convergence (up to subsequence) of the solution of the discrete problem towards a weak solution of the continuous problem (no uniqueness result for this problem) as the mesh size goes to 0

Simpler result: "continuity" with respect to the data

$$-\Delta u_n + \nabla p_n = f_n \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho_n u_n) = 0 \;\text{ in } \Omega,\; \rho \geq 0 \;\text{ in } \Omega,\; \int_\Omega \rho_n(x)dx = M_n,$$

$$p_n = \rho_n^\gamma \text{ in } \Omega$$

$\gamma > 1$, $f_n \to f$ weakly in $(L^2(\Omega))^d$ and $M_n \to M$. Then, up to a subsequence,

- $u_n \to u$ in $L^2(\Omega)^d$ and weakly in $H_0^1(\Omega)^d$,
- $p_n \to p$ in $L^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $L^2(\Omega)$,
- $\rho_n \to \rho$ in $L^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$,

where (u, p, ρ) is a weak solution of the compressible Stokes equations (with f and M as data)

Preliminary lemma

$\rho \in L^{2\gamma}(\Omega)$, $\rho \geq 0$ a.e. in Ω, $u \in (H_0^1(\Omega))^d$, $\mathrm{div}(\rho u) = 0$, then:

$$\int_\Omega \rho \mathrm{div}(u) dx = 0$$

$$\int_\Omega \rho^\gamma \mathrm{div}(u) dx = 0$$

The second part is used in order to obtain some estimates on the approximate solutions

The first part is crucial for passing to the limit on the EOS (if $\gamma > 1$)

Proof of the preliminary result, ρ regular

For simplicity : $\rho \in C^1(\bar{\Omega})$, $\rho \geq \alpha$ a.e. in Ω.
$1 < \beta \leq \gamma$. Take $\varphi = \rho^{\beta-1}$ as test function in $\mathrm{div}(\rho u) = 0$:

$$\int_\Omega \rho u \cdot \nabla \rho^{\beta-1} dx = (\beta - 1) \int_\Omega \rho^{\beta-1} u \cdot \nabla \rho dx = 0.$$

Then

$$0 = \int_\Omega u \cdot \nabla \rho^\beta dx,$$

and finally

$$\int_\Omega \rho^\beta \mathrm{div}(u) dx = 0.$$

Two cases :
$\beta = \gamma$
$\beta = 1 + \frac{1}{n}$ and $n \to \infty$ (or $\varphi = \ln(\rho)$)

Proof of the preliminary result, non regular ρ

One uses a "classical" lemma

$\gamma > 1$, $\rho \in L^{2\gamma}(\mathbb{R}^d)$, and $u \in H^1(\mathbb{R}^d)^d$.
Let $(r_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers and, for $n \in \mathbb{N}^\star$,
$\rho_n = \rho \star r_n$ and $(\rho u)_n = (\rho u) \star r_n$.

Then, $[(\rho u)_n - \rho_n u] \to 0$ weakly in $W^{1,(2\gamma)/(\gamma+1)}(\mathbb{R}^d)^d$ (which gives, in particular, that $\mathrm{div}((\rho u)_n - \rho_n u) \to 0$ weakly in $L^{(2\gamma)/(\gamma+1)}(\mathbb{R}^d)$).

$$r \in C_c^\infty(\mathbb{R}^d, \mathbb{R}),\ \int_{\mathbb{R}^d} r dx = 1,\ r \geq 0 \text{ in } \mathbb{R}^d \\ \text{and, for } n \in \mathbb{N}^\star, x \in \mathbb{R}^d,\ r_n(x) = n^d r(nx). \tag{1}$$

Estimates on u

Taking u_n as test function in $-\Delta u_n + \nabla p_n = f_n$:

$$\int_\Omega \nabla u_n : \nabla u_n \, dx - \int_\Omega p_n \mathrm{div}(u_n) \, dx = \int_\Omega f_n \cdot u_n \, dx.$$

But $p_n = \rho_n^\gamma$ a.e. and $\mathrm{div}(\rho_n u_n) = 0$, then $\int_\Omega p_n \mathrm{div}(u_n) \, dx = 0$.
This gives an estimate on u_n:

$$\|u_n\|_{(H_0^1(\Omega))^d} \leq C_1.$$

Estimate on p, divergence Lemma

Let $q \in L^2(\Omega)$ s.t. $\int_\Omega q dx = 0$.
Then, there exists $v \in (H_0^1(\Omega))^d$ s.t.

$$\mathrm{div}(v) = q \text{ a.e. in } \Omega,$$

$$\|v\|_{(H_0^1(\Omega))^d} \leq C_2 \|q\|_{L^2(\Omega)},$$

where C_2 only depends on Ω.

Estimate on p

$m_n = \frac{1}{|\Omega|}\int_\Omega p_n dx$, $v_n \in H_0^1(\Omega)^d$, $\mathrm{div}(v_n) = p_n - m_n$.
Taking v_n as test function in $-\Delta u_n + \nabla p_n = f_n$:

$$\int_\Omega \nabla u_n : \nabla v_n \, dx - \int_\Omega p_n \mathrm{div}(v_n) \, dx = \int_\Omega f_n \cdot v_n \, dx.$$

Using $\int_\Omega \mathrm{div}(v_n) dx = 0$:

$$\int_\Omega (p_n - m_n)^2 dx = \int_\Omega (f_n \cdot v_n - \nabla u_n : \nabla v_n) dx.$$

Since $\|v_n\|_{(H_0^1(\Omega))^d} \leq C_2 \|p_n - m_n\|_{L^2(\Omega)}$ and $\|u_n\|_{(H_0^1(\Omega))^d} \leq C_1$, the preceding inequality leads to:

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3.$$

where C_3 only depends on the L^2-bound of $(f_n)_{n\in\mathbb{N}}$ and on Ω.

Estimate on p and ρ

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3.$$

$$\int_\Omega p_n^{\frac{1}{\gamma}} dx = \int_\Omega \rho_n dx \leq \sup\{M_p, p \in \mathbb{N}\}.$$

Then:

$$\|p_n\|_{L^2(\Omega)} \leq C_4;$$

where C_4 only depends on the L^2-bound of $(f_n)_{n\in\mathbb{N}}$, the bound of $(M_n)_{n\in\mathbb{N}}$, γ and Ω.

$p_n = \rho_n^\gamma$ a.e. in Ω, then:

$$\|\rho_n\|_{L^{2\gamma}(\Omega)} \leq C_5 = C_4^{\frac{1}{\gamma}}.$$

Weak-convergence on u_n, p_n, ρ_n

Thanks to the estimates on u_n, p_n, ρ_n, it is possible to assume (up to a subsequence) that, as $n \to \infty$:

$$u_n \to u \text{ in } L^2(\Omega)^d \text{ and weakly in } H_0^1(\Omega)^d,$$

$$p_n \to p \text{ weakly in } L^2(\Omega),$$

$$\rho_n \to \rho \text{ weakly in } L^{2\gamma}(\Omega).$$

Passage to the limit on the equations, except EOS

Linear equation :

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

Strong times weak convergence

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega,$$

L^1-weak convergence of ρ_n gives positivity of ρ and convergence of mass:

$$\rho \geq 0 \text{ in } \Omega, \int_\Omega \rho(x) dx = M.$$

Passage to the limit in EOS

- No problem if $\gamma = 1$, $p = \rho$
- If $\gamma > 1$, question:

$$p = \rho^\gamma \text{ in } \Omega \text{ ?}$$

p_n and ρ_n converge only weakly...

Idea : prove $\int_\Omega p_n \rho_n \to \int_\Omega p\rho$ and deduce a.e. convergence (of p_n and ρ_n) and $p = \rho^\gamma$.

$\nabla : \nabla = \mathrm{div}\mathrm{div} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v}).$$

Then, for all $\bar{v}$ in $H_0^1(\Omega)^d$

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(u_n) \cdot \mathrm{curl}(\bar{v}) - \int_\Omega p_n \mathrm{div}(\bar{v}) = \int_\Omega f_n \cdot \bar{v}.$$

Choice of $\bar{v}$? $\bar{v} = \bar{v}_n$ with $\mathrm{curl}(\bar{v}_n) = 0$, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\bar{v}_n$ bounded in H_0^1 (unfortunately, 0 is impossible).

Then, up to a subsequence,

$\bar{v}_n \to v$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof using $\bar{v}_n$ (1)

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}_n) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(\bar{v}_n) - \int_\Omega p_n \mathrm{div}(\bar{v}_n) = \int_\Omega f_n \cdot \bar{v}_n.$$

But, $\mathrm{div}(\bar{v}_n) = \rho_n$ and $\mathrm{curl}(\bar{v}_n) = 0$. Then:

$$\int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n = \int_\Omega f_n \cdot \bar{v}_n.$$

Weak convergence of f_n in $L^2(\Omega)^d$ to f and convergence of $\bar{v}_n$ in $L^2(\Omega)^d$ to v :

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n = \int_\Omega f \cdot v.$$

Proof using $\bar{v}_n$ (2)

But, since $-\Delta u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v) + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v) - \int_\Omega p\,\mathrm{div}(v) = \int_\Omega f\cdot v.$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):
$\displaystyle\int_\Omega (\mathrm{div}(u) - p)\rho = \int_\Omega f\cdot v$. Then:

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))\rho_n = \int_\Omega (p - \mathrm{div}(u))\rho.$$

Finally, the preliminary lemma gives, thanks to the mass equations, $\int_\Omega \rho_n \mathrm{div}(u_n) = 0$ and $\int_\Omega \rho\,\mathrm{div}(u) = 0$. Then,

$$\lim_{n\to\infty} \int_\Omega p_n\rho_n = \int_\Omega p\rho.$$

Unfortunately, it is impossible to have $\bar{v}_n \in H^1_0$

Curl-free test function

Let $w_n \in H^1_0(\Omega)$, $-\Delta w_n = \rho_n$,
One has $w_n \in H^2_{loc}(\Omega)$ since, for $\varphi \in C^\infty_c(\Omega)$, one has $\Delta(w_n\varphi) \in L^2(\Omega)$ and

$$\sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_j(w_n\varphi)\,\partial_i\partial_j(w_n\varphi) = \sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_i(w_n\varphi)\,\partial_j\partial_j(w_n\varphi)$$
$$= \int_\Omega (\Delta(w_n\varphi))^2 = C_\varphi < \infty$$

Then, taking $v_n = \nabla w_n$

- $v_n \in (H^1_{loc}(\Omega))^d$,
- $\mathrm{div}(v_n) = \rho_n$ a.e. in Ω,
- $\mathrm{curl}(v_n) = 0$ a.e. in Ω,
- $H^1_{loc}(\Omega)$-estimate on v_n with respect to $\|\rho_n\|_{L^2(\Omega)}$.

Then, up to a subsequence, as $n \to \infty$, $v_n \to v$ in $L^2_{loc}(\Omega)$ and weakly in $H^1_{loc}(\Omega)$, $\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_n\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_n\varphi$:

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(v_n\varphi) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(v_n\varphi) - \int_\Omega p_n \mathrm{div}(v_n\varphi) = \int_\Omega f_n\cdot(v_n\varphi).$$

Using a proof similar to that given if $\varphi = 1$ (with additionnal terms involving φ), we obtain :

$$\lim_{n\to\infty} \int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi,$$

Proving $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_n\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_n\varphi$:

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(v_n\varphi) + \int_\Omega \mathrm{curl}(u_n)\cdot\mathrm{curl}(v_n\varphi) - \int_\Omega p_n\mathrm{div}(v_n\varphi) \\ = \int_\Omega f_n \cdot (v_n\varphi).$$

But, $\mathrm{div}(v_n\varphi) = \rho_n\varphi + v_n\cdot\nabla\varphi$ and $\mathrm{curl}(v_n\varphi) = L(\varphi)v_n$, where $L(\varphi)$ is a matrix involving the first order derivatives of φ. Then:

$$\int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n\varphi = \int_\Omega f_n\cdot(v_n\varphi) \\ - \int_\Omega \mathrm{div}(u_n)v_n\cdot\nabla\varphi - \int \mathrm{curl}(u_n)\cdot L(\varphi)v_n + \int_\Omega p_n v_n\cdot\nabla\varphi.$$

Weak convergence of u_n in $H_0^1(\Omega)^d$, weak convergence of p_n and f_n in $L^2(\Omega)$ and convergence of v_n in $L^2_{loc}(\Omega)^d$:

$$\lim_{n\to\infty}\int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n\varphi = \int_\Omega f\cdot(v\varphi) \\ - \int_\Omega \mathrm{div}(u)v\cdot\nabla\varphi - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi.$$

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

But, since $-\Delta u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v\varphi) + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v\varphi) - \int_\Omega p\,\mathrm{div}(v\varphi) = \int_\Omega f\cdot(v\varphi).$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):

$$\int_\Omega (\mathrm{div}(u) - p)\rho\varphi = \int_\Omega f\cdot(v\varphi) - \int_\Omega \mathrm{div}(u) v\cdot\nabla\varphi - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi.$$

Then:

$$\lim_{n\to\infty}\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi.$$

Proof of $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \rightarrow \int_\Omega (p - \mathrm{div}(u))\rho$

Lemma : $F_n \rightarrow F$ in $D'(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$. Then $F_n \rightarrow F$ weakly in L^1.

With $F_n = (p_n - \mathrm{div}(u_n))\rho_n$, $F = (p - \mathrm{div}(u))\rho$ and since $\gamma > 1$, the lemma gives

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \rightarrow \int_\Omega (p - \mathrm{div}(u))\rho.$$

Proving $\int_\Omega p_n \rho_n \longrightarrow \int_\Omega p\rho$

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

But thanks to the mass equations, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(u_n)\rho_n = 0, \int_\Omega \mathrm{div}(u)\rho = 0;$$

Then:

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n = \int_\Omega p\rho.$$

a.e. convergence of ρ_n and p_n

Let $G_n = (\rho_n^\gamma - \rho^\gamma)(\rho_n - \rho) \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω.
Futhermore $G_n = (p_n - \rho^\gamma)(\rho_n - \rho) = p_n\rho_n - p_n\rho - \rho^\gamma\rho_n + \rho^\gamma\rho$ and:

$$\int_\Omega G_n = \int_\Omega p_n\rho_n - \int_\Omega p_n\rho - \int_\Omega \rho^\gamma\rho_n + \int_\Omega \rho^\gamma\rho.$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ρ_n and $\lim_{n\to\infty} \int_\Omega p_n\rho_n = \int_\Omega p\rho$:

$$\lim_{n\to\infty} \int_\Omega G_n = 0,$$

Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \rho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_n \to \rho$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$,

$p_n = \rho_n^\gamma \to \rho^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$,

and $p = \rho^\gamma$.

Generalizations

- (Easy) Complete Stokes problem:
  $-\mu\Delta u - \frac{\mu}{3}\nabla(\operatorname{div} u) + \nabla P = f$, with $\mu \in \mathbb{R}_+^\star$ given
- (Ongoing work) Navier-Stokes Equations with $\gamma > 1$ if $d = 2$ and $\gamma > \frac{3}{2}$ if $d = 3$ (probably sharp result with respect to γ without changing the diffusion term or the EOS)
- (Open question) Other boundary condition. Addition of an energy equation
- (Open question) Evolution equation (Stokes and Navier-Stokes)

Additional difficulty for stat. comp. NS equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \mathrm{div}(\rho u \otimes u) + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x) = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

$d = 2$: no aditional difficulty

$d = 3$: no additional difficulty if $\gamma \geq 3$. But for $\gamma < 3$, no estimate on p in $L^2(\Omega)$.

Estimates in the case of NS equations, $\frac{3}{2} < \gamma < 3$

Estimate on u : Taking u as test function in the momentum leads to an estimate on u in $(H_0^1(\Omega))^d$ since

$$\int_\Omega \rho u \otimes u : \nabla u = 0.$$

Then, we have also an estimate on u in $L^6(\Omega)^d$ (using Sobolev embedding).

Estimate on p in $L^q(\Omega)$, with $1 < q = \frac{3(\gamma-1)}{\gamma} < 2$ and $q \to 1$ when $\gamma \to \frac{3}{2}$ (using the divergence lemma in L^r instead of L^2, $r = \frac{q}{q-1}$).

Estimate on ρ in $L^q(\Omega)$, with $\frac{3}{2} < q = 3(\gamma - 1) < 6$ and $q \to \frac{3}{2}$ when $\gamma \to \frac{3}{2}$ (since $p = \rho^\gamma$).

Remark : $\rho u \otimes u \in L^1(\Omega)$, since $u \in L^6(\Omega)^d$ and $\rho \in L^{\frac{3}{2}}(\Omega)$ (and $\frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1$).

NS equations, $\gamma < 3$, how to pass to the limit in the EOS

We prove

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n^\theta = \int_\Omega p \rho^\theta,$$

with some convenient choice of $\theta > 0$ instead of $\theta = 1$.

This gives, as for $\theta = 1$, the a.e. convergence (up to a subsequence) of p_n and ρ_n.

Preliminary lemma with the numerical scheme (1)

Roughly speaking, upwinding replaces $\mathrm{div}(\rho u) = 0$ and $\int_\Omega \rho dx = M$ by

$$\mathrm{div}(\rho u) - h\mathrm{div}(|u|\nabla\rho) + h^\alpha(\rho - \rho^\star) = 0$$

with $\rho^\star = \frac{M}{|\Omega|}$
This equation as (for a given u) a solution $\rho > 0$ and we prove

$$\int_\Omega \rho_n^\gamma \mathrm{div}_n\, u_n dx \leq Ch^\alpha,$$

$$\int_\Omega \rho_n \mathrm{div}_n\, u_n dx \leq Ch^\alpha.$$

C depends on Ω, M and γ
Ch^α is due to $h^\alpha(\rho - \rho^\star)$
$\leq$ is due to upwinding

The first inequality leads to the estimate on the approx. solution.

Preliminary lemma with the numerical scheme (2)

For the passage to the limit on the EOS

$$\int_\Omega \rho_n \mathrm{div}_n\, u_n dx \le Ch^\alpha$$

$$\int_\Omega \rho \mathrm{div}\, u dx = 0$$

give $\lim_{n\to\infty} \int_\Omega p_n \rho_n dx \le \int_\Omega p\rho dx = 0$,
which is sufficient to prove the a.e. convergence (up to a subsequence) of p_n and ρ_n

Passage to the limit in the EOS with the numerical scheme

- Miracle with the Mac scheme. There exists a discrete counterpart of $\int_\Omega \nabla u : \nabla v dx = \int_\Omega (\mathrm{div}(u)\mathrm{div}(v) + \mathrm{curl}(u) \cdot \mathrm{curl}(v)) dx$
- No discrete counterpart with Crouzeix-Raviart. Two possible solutions
  – Use the continuous equality. This is possible with an additional regularization term in the mass equation (not needed from the numerical point of view, only needed to prove the convergence)
  –Discretize $\int_\Omega (\mathrm{div}(u)\mathrm{div}(v) + \mathrm{curl}(u) \cdot \mathrm{curl}(v)) dx$ instead of $\int_\Omega \nabla u : \nabla v dx$. Better for passing to the limit in the EOS but the discretized momentum equation is not coercive (with Crouzeix-Raviart Finite Element). One needs a penalization term in the discrete momentum equation (crucial from the numerical point of view)