Resonance and nonlinearities

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1 Introduction

Let $p \in \mathbb{N}^\star$ and A be a real $p \times p$ matrix. One considers the following Cauchy problem, where the unknown is the function $\mathbf{W}$ from $\mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}^p$:

$$\begin{array}{l} \partial_t \mathbf{W}(x,t) + A\partial_x \mathbf{W}(x,t) = 0, (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ \mathbf{W}(x,0) = \mathbf{W}_0(x),\ x \in \mathbb{R}. \end{array} \tag{1}$$

A weak solution of Problem (1), for $\mathbf{W}_0 \in L^1_{loc}(\mathbb{R}, \mathbb{R}^p)$, is a function $u \in L^1_{loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^p)$ such that, for all $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^p)$,

$$\int_0^\infty \int_{\mathbb{R}} (\mathbf{W} \cdot \partial_t \boldsymbol{\varphi} + A\mathbf{W} \partial_x \boldsymbol{\varphi})(x,t) dx dt + \int_{\mathbb{R}} \mathbf{W}_0(x) \boldsymbol{\varphi}(x,0) dx = 0. \tag{2}$$

If A is diagonalizable in $\mathbb{R}$, we will say that the first equation of Problem (1) is a linear genuinely hyperbolic system. In this case, if $q \in [1, \infty]$ and $\mathbf{W}_0 \in L^q(\mathbb{R}, \mathbb{R}^p)$ (the Lebesgue space on $\mathbb{R}$ with value in $\mathbb{R}^p$)), Problem (1) has a unique weak solution $\mathbf{W}$, and $\mathbf{W}(\cdot, t) \in L^q(\mathbb{R}, \mathbb{R}^p)$ for a.e. $t \in \mathbb{R}^+$.

If the matrix A has only real eigenvalues but is not diagonalizable, the first equation of Problem (1) is said to be a linear hyperbolic resonant system. In this case, Problem (1) is ill posed in the sense that if $\mathbf{W}_0 \in L^q(\mathbb{R}, \mathbb{R}^p)$ (for some $q \in [1, \infty]$), it has, in general, no weak solution $\mathbf{W}$ (and, in particular, no weak solution with $\mathbf{W}(\cdot, t) \in L^q(\mathbb{R}, \mathbb{R}^p)$ for a.e. $t \in \mathbb{R}^+$). (However, Problem (1) is well posed in C^∞, it has a unique solution in $C^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^p)$ if the initial datum $\mathbf{W}_0$ belongs to $C^\infty(\mathbb{R}, \mathbb{R}^p)$.) This ill posedness is due to the fact that there is a lack of regularity between $\mathbf{W}(\cdot, t)$ (for $t > 0$) and $\mathbf{W}_0$. For instance, the Riemann problem, that is Problem (1) with $\mathbf{W}_0(x) = \mathbf{w}_l$ for $x < 0$ and $\mathbf{W}_0(x) = \mathbf{w}_r$ for $x > 0$ (and $\mathbf{w}_l$, $\mathbf{w}_r \in \mathbb{R}^p$), does not have a weak solution (in the sense given before) except for very particular choices of $\mathbf{W}_0$, but it has a solution in a greater space. In the case $p = 2$, it has a (unique) solution in a space allowing $\mathbf{W}(\cdot, t)$ to be, for $t > 0$, a measure on the bounded sets of $\mathbb{R}$, see Section 2 below (in the case $p \geq 3$, the solution $\mathbf{W}(\cdot, t)$ may even be less regular).

One considers now that the matrix A in (1) is depending on $\mathbf{w}$, leading to the following nonlinear system:

$$\begin{aligned} &\partial_t \mathbf{W}(x,t) + A(\mathbf{W}(x,t))\partial_x \mathbf{W}(x,t) = 0, (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ &\mathbf{W}(x,0) = \mathbf{W}_0(x),\ x \in \mathbb{R}. \end{aligned} \tag{3}$$

The unknown $\mathbf{W}$ is supposed to take values in an admissible set $D \subset \mathbb{R}^p$.

If the matrix $A(w)$ is diagonalizable in $\mathbb{R}$ for all $w \in D$, we say that the first equation of Problem (3) is a nonlinear genuinely hyperbolic system. Problem (3) is expected to be have a unique solution, with $\mathbf{W}(\cdot, t)$ belonging to a Lebesgue space, in a convenient sense (including, for instance, an entropy condition). This result could be suggested by the fact that the linear problem (1) with $A = A(\mathbf{w})$ is well posed in Lebesgue spaces for any $\mathbf{w} \in D$.

Assume now that there exists $R \subset D$, $R \neq \emptyset$, such that the matrix $A(\mathbf{w})$ is diagonalizable in $\mathbb{R}$ for all $\mathbf{w} \in D \setminus R$ and has only real eigenvalues but is not diagonalizable if $\mathbf{w} \in R$. Then, the first equation of Problem (3) is said to be a nonlinear resonant hyperbolic system. The linear problem (1) has, in general, no weak solution (in the sense given before) if $A = A(\mathbf{w})$, for any $\mathbf{w} \in R$ (since it corresponds to a linear resonant hyperbolic system). In this case, two questions seem of interest:

1. Is it possible to have an existence and uniqueness result (in Lebesgue spaces) for this nonlinear resonant hyperbolic system ?
2. What is the behaviour of numerical schemes using a linearization of the system (and then, possibly, using some linear resonant systems) ?

There are many recent works on nonlinear resonant hyperbolic systems, in particular for proving an existence and uniqueness result for the Riemann problem. See, for instance, [GL04], [GS06] (for an exemple in phase transition), [CLS04] (for the case of shallow water with topography). There are also papers devoted to the study of numerical schemes for nonlinear resonant hyperbolic systems. See [AGG04] for a quite general study and, for the case of shallow water with topography, [CLS04], [KL02], [ABB04], [GHS03]. In this latter case, it seems possible to use linearized Riemann problems for the design of numerical schemes, even if the linearized system is resonant for the computation of some fluxes (see [GHS03]).
In this paper, we focus on a simple example, coming from the modelization of a two phase flow in an heterogeneous porous medium. It leads to a scalar equation with a flux function discontinuously depending on the spatial variable. Then, it can be seen as a nonlinear hyperbolic system and this system is resonant for some values of the unknown. For this problem, it is possible to prove, for a large class of initial data, an existence and uniqueness result of an entropy weak solution (including cases where the initial datum belongs to R,

the set corresponding to resonant systems, for all $x \in \mathbb{R}$), along with the convergence of numerical schemes, following [SV03], [BV06], [Bac04], [Bac05] (for an existence and uniqueness result) and [Bac05], [Bac06] (for the convergence of numerical schemes). Many other papers are devoted to this case of a scalar conservation law with discontinuous coefficients, see, for instance, [Tow00], [KRT03]. Some contributions on this problem are in the present proceedings.

One considers in this paper that the space variable x belongs to $\mathbb{R}$, but some extensions to $x \in \mathbb{R}^d$, $d = 2$ or 3, are possible.

2 Linear resonant systems

Let $p = 2$ and A be a real 2×2 matrix which has only real eigenvalues but is not diagonalizable. Then, using a change of unknown, the Riemann problem for the linear problem (1) can be put under the following form, with some $\lambda \in \mathbb{R}$ (which is the unique eigenvalue of A):

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0,$$

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ v_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ v_r \end{bmatrix}, \text{ if } x > 0,$$

with u_l, u_r, v_l, $v_r \in \mathbb{R}$. The second equation of the system and the second initial condition are decoupled from the first ones. Then, the unique solution for v (uniqueness holds even in the larger possible space of distributions) is $v(\cdot, t) = v(\cdot - \lambda t, 0)$ for all $t > 0$. It is now possible to give the solution for u (which is also unique in in the larger possible space of distributions), it is, for all $t > 0$:

$$u(\cdot, t) = u_l 1_{\{x \in \mathbb{R},\, x < \lambda t\}} + u_r 1_{\{x \in \mathbb{R},\, x > \lambda t\}} + t(v_l - v_r)\delta_{\lambda t},$$

where 1_B is the characteristic function of B, for $B \subset \mathbb{R}$, and δ_a is the Dirac mass at point a, for $a \in \mathbb{R}$. In this example, the problem has no weak solution with $u(\cdot, t)$ in a Lebesgue space but it has a unique solution in a space allowing $u(\cdot, t)$ to be a measure on the bounded sets of $\mathbb{R}$.

If $p > 2$, the (unique) solution of the Riemann problem for a linear hyperbolic resonant system may be even less regular. Indeed, the regularity of the solution depends on the difference between the algebraic and the geometric multiplicity of the eigenvalues.

To conclude this section, one also presents the Riemann problem for the simplest example of nonlinear resonnant hyperbolic system:

$$\begin{aligned} &u_t + (au)_x = 0, \\ &a_t = 0, \end{aligned} \tag{4}$$

$$\begin{bmatrix} u(x,0) \\ a(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ a_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ a_r \end{bmatrix}, \text{ if } x > 0, \tag{5}$$

Problem (4)-(5) has no weak solution (in the natural sense, similar to (2), and even in a weaker sense allowing the solution to takes values, for $t > 0$, in a distribution space) if $a_l > 0$, $a_r < 0$ and $a_l u_l \neq a_r u_r$ and has infinitely many weak solution with $u(\cdot, t) \in L^\infty(\mathbb{R})$ for a.e. t, if $a_l < 0$ and $a_r > 0$. See [BJ98] for the study of such problems.

Problem (4)-(5) correspond to a nonlinear hyperbolic resonant system since the system is equivalent (for regular solution) to:

$$\begin{bmatrix} u \\ a \end{bmatrix}_t + \begin{bmatrix} a & u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ a \end{bmatrix}_x = 0.$$

Then, resonance occurs, for this system, when $a = 0$ and $u \neq 0$ and, as it is said before, an existence and uniqueness result of a weak solution for the Riemann problem does not hold for this nonlinear system provided that 0 is between a_r and a_l (except for some particular data).

3 Hyperbolic equation with a discontinuous coefficient

The example of a nonlinear resonant hyperbolic system studied in this paper is given by a two phase flow in an heterogeneous porous medium, considering only gravity effect (without capillarity and with a total flux equal to zero). The unknown is the saturation, which is a function $u : \mathbb{R} \times \mathbb{R}_+ \to [0, 1] \subset \mathbb{R}$. The equation is (forgetting the variable (x, t)):

$$\partial_t u + \partial_x (k g(u)) = 0, \text{ in } \mathbb{R} \times \mathbb{R}_+, \tag{6}$$

where $k(x) = k_l$, for $x < 0$, and $k(x) = k_r$, for $x > 0$, $k_l, k_r > 0$, $k_l \neq k_r$, the function $g : [0, 1] \to \mathbb{R}$ is Lipschitz continuous, nonnegative and such that $g(0) = g(1) = 0$. A typical example, studied in [SV03], is $g(u) = u(1-u)$.

This hyperbolic equation with a discontinuous coefficient can be viewed has a conservative 2×2 system, adding k has an unknown and the equation $k_t = 0$:

$$\begin{aligned} &u_t + (k g(u))_x = 0, \\ &k_t = 0. \end{aligned}$$

Then, with $\mathbf{W} = \begin{bmatrix} u \\ k \end{bmatrix}$ and $F(\mathbf{W}) = \begin{bmatrix} kg(u) \\ 0 \end{bmatrix}$, this system is:

$$\mathbf{W}_t + (F(\mathbf{W}))_x = 0,$$

or equivalently (for regular solutions), with $A(\mathbf{w}) = DF(\mathbf{w})$ for $\mathbf{w} \in \mathbb{R}^2$:

$$\mathbf{W}_t + A(\mathbf{W})\mathbf{W}_x = 0.$$

This leads to problem (3) with $p = 2$, $\mathbf{W} = \begin{bmatrix} u \\ k \end{bmatrix}$, $A(\mathbf{W}) = \begin{bmatrix} kg'(u) & g(u) \\ 0 & 0 \end{bmatrix}$.

The admissibility domain is $D = \{(u,k)^t, u \in [0,1], k > 0\}$. Assuming that $g \in C^1$ (which is not a necessary hypothesis for this problem), let $R = \{(u,k)^t \in D,\ g'(u) = 0,\ g(u) \neq 0\}$. The matrix $A(\mathbf{w})$ is diagonalizable in $\mathbb{R}$ for $\mathbf{w} = (u,k)^t \in D \setminus R$ and has only 0 as eigenvalue but is not diagonalizable if $\mathbf{w} \in R$. In the case $g(u) = u(1-u)$, $R = \{1/2\} \times \mathbb{R}_+^\star$. But the domain R corresponding to resonance may be larger. In the case corresponding to Figure 1, $R = \{(u,k)^t \in D,\ g'(u) = 0,\ g(u) \neq 0\}$ contains $(1/4, 3/4) \times \mathbb{R}_+^\star$.

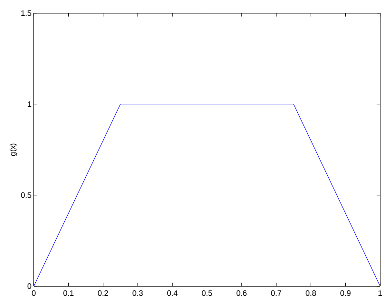

Fig. 1. Resonance occurs for all (k, u) with $u \in (\frac{1}{4}, \frac{3}{4})$

Despite this resonance phenomenon, it is possible to prove existence and uniqueness of an "entropy weak solution" of (6) with an initial condition u_0, provided that $u_0 \in L^\infty(\mathbb{R})$ takes its values in $[0,1]$. Actually, it is proven in [BV06] (previous partial results were, for instance, in [SV03], [KRT03] and [Bac04]) that there exists a unique solution of the following weak entropic formulation of (6) with the initial condition u_0:

$$\begin{aligned} & u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}),\ 0 \le u \le 1 \text{ a.e.}, \\ & \int_{\mathbb{R}_+} \int_{\mathbb{R}} [|u(x,t) - \kappa| \partial_t \varphi(x,t) + k(x) \phi(u(x,t),\kappa) \partial_x \varphi(x,t)]\, dxdt \\ & \quad + \int_{\mathbb{R}} |u_0(x) - \kappa| \varphi(x,0) dx + |k_r - k_l| \int_{\mathbb{R}_+} g(\kappa) \varphi(0,t) dt \ge 0, \\ & \qquad \forall \kappa \in [0,1],\ \forall \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+), \end{aligned} \tag{7}$$

where $\phi(s,\kappa) = \text{sign}(s-k)(g(s)-g(\kappa))$ for $s \in [0,1]$. This definition of entropy weak solution was previously given in [Tow00]. Of course, as usual, an entropy weak solution of (6) with the initial condition u_0 (that is a solution of (7)) is a weak solution, that is satisfies:

$$
\begin{aligned}
&u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}),\ 0 \le u \le 1 \text{ a.e.},\\
&\int_{\mathbb{R}_+}\int_{\mathbb{R}} (u(x,t)\partial_t\varphi(x,t) + k(x)g(u(x,t)))\partial_x\varphi(x,t))dxdt\\
&\quad + \int_{\mathbb{R}} (u_0(x)\varphi(x,0)dx \ge 0,\ \ \forall \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}_+).
\end{aligned}
\tag{8}
$$

A crucial property (see Section 4 for a sketch of proof) is that the constant functions 0 and 1 are solutions of (6). Without this hypothesis, it is sometimes possible to obtain an existence (and, possibly, with uniqueness) result for (6) with the initial condition u_0, but the solution does not takes (in general) its values in $[0,1]$ and is not a weak solution of (6) (actually, the jump condition at point $x = 0$ is not the jump condition implicitely given in (8)). Some contributions in this direction are in the present proceedings.

The fact that the flux function has, with respect to u, the same form for $x < 0$ and $x > 0$ (namely $k_l g(u)$ and $k_r g(u)$) is not necessary. A similar result of existence and uniqueness was proven recently in [Bac05] when $k(x)g(u)$ is replaced by $g(x,u)$ with $g(x,u) = g_l(u)$ pour $x < 0$ and $g(x,u) = g_r(u)$, where g_l and g_r are some Lipschitz continuous functions from $[[0,1]$ to $\mathbb{R}$. But, for this generalization, a main hypothesis is still that $g(\cdot,0)$ and $g(\cdot,1)$ are some constants functions, that is to say $g_l(0) = g_l(0)$ and $g_l(1) = g_l(1)$ (and theses values are not necessarily equal to zero).

In the following section, one gives a sketch of proof of this existence and uniqueness result.

4 An existence and uniqueness result

In Section 3, we saw the following theorem (which is proven in [BV06] and generalized in [Bac05] to a larger class of flux function):

Theorem 1. *Let $k_l, k_r > 0$, k defined (from $\mathbb{R}$ to $\mathbb{R}$) by $k(x) = k_l$, for $x < 0$, and $k(x) = k_r$, for $x > 0$, and $g\ :\ [0,1] \to \mathbb{R}$ be a Lipschitz continuous function, nonnegative and such that $g(0) = g(1) = 0$. Let $u_0 \in L^\infty(\mathbb{R})$ be such that $u_0(x) \in [0,1]$ for a.e. $x \in \mathbb{R}$. Then, there exists a unique solution to (7).*

One gives now a sketch of proof of this theorem.

There are similar methods for proving the existence part of Theorem 1. For instance:

1. Replace, in (6), k by a regular function k_ϵ (then, existence and uniqueness of the solution u_ϵ is classical, following Krushkov theory) converging pointwise to k (as $\epsilon \to 0$) and pass to limit as $\epsilon \to 0$.
2. Add a viscous term $-\epsilon\partial_x^2 u$ to (6) (with $\epsilon > 0$, then existence and uniqueness of the solution u_ϵ is also classical) and pass to limit as $\epsilon \to 0$.

3. Pass to the limit (as the discretization parameters go to 0) on the approximate solution given by "monotone" numerical schemes (such as the Godunov sheme). We will call also u_ϵ the approximate solution.

The first method was used in [SV03], [Bac04] and [BV06]. The second and third methods are very closed since the monotonicity of a numerical scheme leads to some viscosity term in the approximate equation. The third method is used in [Bac05] and [Bac06].

For these three methods, an L^∞ estimate on u_ϵ is quite easy. Actually, it holds $0 \le u_\epsilon \le 1$ a.e.. Then, it is possible to assume, at least for a subsequence of a sequence of approximate solutions, that $u_\epsilon \to u$ for the weak-$\star$ topology of $L^\infty(\mathbb{R} \times \mathbb{R}_+)$ and $0 \le u \le 1$ a.e..

The main difficulty (even if u_0 is regular) in order to pass to the limit (as ϵ goes to 0) and to obtain the existence part of Theorem 1 is to prove the a.e. convergence of u_ϵ towards u, at least also for a subsequence of a sequence of approximate solutions. This a.e. convergence is useful for proving that $h(u_\epsilon)$ converges towards $h(u)$ for all bounded continuous function h from $\mathbb{R}$ to $\mathbb{R}$.

The main difficulty for the uniqueness part of Theorem 1 is to prove the existence of traces for u on the line $\{(0,t), t > 0\}$.

In order to get rid of this two difficulties, some authors ([Tow00], [KRT03], [SV03], [Bac04]...) use the following hypothesis of genuine nonlinearity for the flux function (where "meas" stands for the Lebesgue measure on $\mathbb{R}$):

$$g \in C^2 \text{ and } \text{meas}(\{x \in [0,1];\, g''(s) = 0\}) = 0. \tag{9}$$

With this hypothesis, the existence part of Theorem 1 follows by proving the a.e. convergence of u_ϵ towards u, using some tools as "Temple function" or "compensated compacness". The uniqueness part of Theorem 1 is obtain using the existence of traces for an entropy weak solution on the line $\{(0,t), t > 0\}$.

Without Assumption (9) on g, the proof of Theorem 1 is much more tricky (see [BV06], [Bac05]). Passing to the limit as ϵ goes to 0 leads to the existence of a solution in a very weak sense, namely it gives a "kinetic process solution" (see the definition below). Then, an uniqueness result proves the fact that this kinetic process solution is indeed an entropy weak solution (and that this entropy weak solution is unique). This gives the existence part and the uniqueness part of Theorem 1. A by product of the proof is that u_ϵ converges towards u (as ϵ goes to 0) in $L^p_{loc}(\mathbb{R} \times \mathbb{R}_+)$, for all $1 \le p < \infty$ (and then a.e., at least for subsequences of sequences of approximate solutions). One gives now some details on this proof.

Since a sequence of approximate solutions is bounded in $L^\infty(\mathbb{R} \times \mathbb{R}_+)$, one can assume that, up to a subsequence, it converges towards a Young measure (see, for instance, [DiP85]) or equivalently towards some $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+ \times (0,1))$ in the "nonllnear weak-$\star$ sense", using the following result (see [EGH00], [EGH95]):

Theorem 2. *Let $N \geq 1$, Ω be an open set of $\mathbb{R}^N$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\Omega)$. Then, there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, and there exists $u \in L^\infty(\Omega \times (0,1))$ such that, for all $\psi \in L^1(\Omega)$ and all $\theta \in C(\mathbb{R},\mathbb{R})$:*

$$\int_\Omega \theta(u_n(y))\psi(y)dy \to \int_0^1 \int_\Omega \theta(u(y,\alpha))\psi(y)dyd\alpha, \text{ as } n \to \infty.$$

Then, it is quite easy to prove that this function u is an entropy process solution, that is a solution of:

$$\begin{aligned}
&u \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1)),\ 0 \leq u \leq 1 \text{ a.e.},\\
&\int_0^1 \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left[|u(x,t,\alpha) - \kappa|\partial_t\varphi(x,t) + k(x)\phi(u(x,t,\alpha),\kappa)\partial_x\varphi(x,t)\right] dxdtd\alpha\\
&\qquad + \int_{\mathbb{R}} |u_0(x) - \kappa|\varphi(x,0)dx + |k_r - k_l| \int_{\mathbb{R}_+} g(\kappa)\varphi(0,t)dt \geq 0,\\
&\qquad\qquad \forall\kappa \in [0,1],\ \forall\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+).
\end{aligned}$$

If k is a regular function (then, with the hypotheseses of Theorem 1, k is a constant function, but similar results are also true if k is not a constant function), it is possible to prove an uniqueness result for this entropy process solution and the fact that u does not depends on α, by using the doubling variable technique of Krushkov (see [EGH00], for instance). This gives that u is an entropy weak solution and concludes the proof of Theorem 1 for k regular (i.e. constant). Unfortunately, the doubling variable technique does not seem easily generalizable to the case of a discontinuous function k.

To overcome this difficulty, one introduces once again a new variable, denoted ξ, and one remarks that u is also a "kinetic process solution". It means that, for all $\xi \in \mathbb{R}$, there exist two positive Radon measures on $\mathbb{R} \times \mathbb{R}_+$, denoted $m_{\xi,\pm}$, continuously depending on ξ in the sense that $\xi \mapsto \int \varphi dm_{\xi,\pm}$ is continuous for all $\varphi \in C_c(\mathbb{R} \times \mathbb{R}_+)$, such that, for all $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$:

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}_+} \int_{\mathbb{R}} h_\pm(x,t,\alpha,\xi)(\partial_t\varphi(x,t,\xi) + k(x)a(\xi)\partial_x\varphi(x,t,\xi))dxdtd\xi d\alpha\\
&\quad + \int_{\mathbb{R}} \int_{\mathbb{R}} h_\pm^{(0)}(x,\xi)\varphi(x,0,\xi)dxd\xi + \int_{\mathbb{R}} \int_{\mathbb{R}_+} (k_r - k_l)^\pm a(\xi)\varphi(0,t,\xi)dtd\xi\\
&\qquad = \textstyle\int_{\mathbb{R}} \int_{\mathbb{R}\times\mathbb{R}_+} \partial_\xi\varphi(x,t,\xi)dm_{\xi,\pm}(x,t)d\xi,
\end{aligned}$$

where $a(\xi) = g'(\xi)$, $h_\pm(x,t,\alpha,\xi) = \text{sign}_\pm(u(t,x,\alpha) - \xi)$ and $h_\pm^{(0)}(x,\xi) = \text{sign}_\pm(u_0(x) - \xi)$ (with $\text{sign}_+(s) = 1$ if $s > 0$ and 0 if $s < 0$, $\text{sign}_-(s) = -1$ if $s < 0$ and 0 if $s > 0$). The functions $h_\pm$ and $h_\pm^{(0)}$ are the equilibrium functions associated to u and u_0. This definition of a kinetic process solution is a natural generalization of the definition of a kinetic solution for a nonlinear hyperbolic equation, see, for instance, [Per98] (the generalization appears in the variable α and in the discontinuity of k). One can now adapt the proof of uniqueness of the kinetic solution given in [Per98]. It gives here that u does not depends on α and is the unique kinetic solution of our problem and then the unique entropy weak solution (i.e. the unique solution of (7)). This concludes the proof of Theorem 1. Note also that the preceding proof gives that u_ϵ converges towards u (as ϵ goes to 0) in $L^p_{loc}(\mathbb{R}\times\mathbb{R}_+)$, for all $1 \le p < \infty$ (and then a.e., at least for subsequences of sequences of approximate solutions).

5 Numerical schemes and numerical results

The presentation of the numerical schemes is restricted here to the case of a system under a conservative form, which is the case of the simple system presented in Section 3. An additional work has to be done for a system with a nonconservative term (this is the case for Shallow Water with topography, see [GHS03] for instance). Then, the system reads, with the same notations are before:

$$\begin{aligned} &\partial_t \mathbf{W} + \partial_x F(\mathbf{W}) = 0, \\ &\mathbf{W}(\cdot,0) = \mathbf{W}_0, \end{aligned} \tag{10}$$

where F is a Lipschitz continuous function from $D\subset\mathbb{R}^p$ to $\mathbb{R}^p$. Recall that the unknown $\mathbf{W}$ is a function from $\mathbb{R}\times\mathbb{R}_+$ to $D\subset\mathbb{R}^p$, where D is the so-called admissible domain.

The time and space steps are denoted by δt and δx. For simplicity, they are assumed to be constant. Let $t_n = n\delta t$ and $x_{i+1/2} = ih$ for $n\in\mathbb{N}$ and $i\in\mathbb{Z}$. The approximate solution is defined by the family $\{\mathbf{w}_i^n,\ i\in\mathbb{Z},\ n\in\mathbb{N}\}\subset\mathbb{R}$, where $\mathbf{w}_i^n$ is the value of the approximate solution for $t\in(t_n,t_{n+1})$ and in the control volume $M_i = (x_{i-1/2}, x_{i+1/2})$.

The initial condition is used to compute $\{\mathbf{w}_i^0,\ i\in\mathbb{Z}\}$:

$$\mathbf{w}_i^0 = \frac{1}{\delta x}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \mathbf{W}_0(x)dx, \text{ for } i\in\mathbb{Z}. \tag{11}$$

One describes now two possibilities for the computation of $\{\mathbf{w}_i^{n+1},\ i\in\mathbb{Z}\}$ using $\{\mathbf{w}_i^n,\ i\in\mathbb{Z}\}$. The first one uses the resolution of the Riemann problem associated to (10), it is the Godunov scheme. The second one uses a linearized Riemann problem.

Godunov scheme

Let $\mathbf{w}_l$, $\mathbf{w}_r\in D$. The Riemann problem associated to $\mathbf{w}_l$ and $\mathbf{w}_r$ is (10) with $\mathbf{W}(x,0) = \mathbf{w}_l$ if $x<0$ and $\mathbf{W}(x,0) = \mathbf{w}_r$ if $x>0$. One assumes that this Riemann problem has a self similar function, which one denotes by $\mathbf{W}(x,t) =$

$\mathbf{R}(x/t, \mathbf{w}_l, \mathbf{w}_r)$ and that it is possible to compute this solution. This is the case, in particular, for the three nonlinear resonant hyperbolic systems given in Section 1 (phase transition [GS06], Shallow Water with topography [CLS04] and two phase flow in an heterogeneous porous medium [Bac05]) although this solution is sometimes not unique (in the case of shallow water, see [CLS04]). One sets $\mathbf{w}^{\star,\pm}(\mathbf{w}_l, \mathbf{w}_r) = \mathbf{R}(0^\pm, \mathbf{w}_l, \mathbf{w}_r)$. The values $\mathbf{w}^{\star,\pm}(\mathbf{w}_l, \mathbf{w}_r)$ are always well defined, even if $\{(0,t),\, t > 0\}$ is a line of discontinuity for $\mathbf{W}$. Then, the Godunov scheme is defined by:

$$\frac{\mathbf{w}_i^{n+1} - \mathbf{w}_i^n}{k} + \mathbf{F}^n_{i+\frac{1}{2}} - \mathbf{F}^n_{i-\frac{1}{2}} = 0,\ i \in \mathbb{Z},\ n \in \mathbb{N}, \tag{12}$$

with $\mathbf{F}^n_{i+\frac{1}{2}} = F(\mathbf{w}^{n,\pm}_{i+\frac{1}{2}})$ and $\mathbf{w}^{n,\pm}_{i+\frac{1}{2}} = \mathbf{w}^{\star,\pm}(\mathbf{w}_i^n, \mathbf{w}_{i+1}^n)$. The definition of $\mathbf{F}^n_{i+\frac{1}{2}}$ is correct, since $F(\mathbf{w}^{n,+}_{i+\frac{1}{2}}) = F(\mathbf{w}^{n,-}_{i+\frac{1}{2}})$ even if $\mathbf{w}^{n,+}_{i+1/2} \neq \mathbf{w}^{n,-}_{i+1/2}$, thanks to the Rankine-Hugoniot condition for the solution of the Riemann problem. This scheme is very efficient. It uses, as usual for an explicit scheme, a CFL condition which reads $\delta t \leq c\delta x$ where c is computed with the eigenvalues of $A(\mathbf{w}) = DF(\mathbf{w})$ ($DF(\mathbf{w})$ is the jacobian matrix of F at point $\mathbf{w} \in D$, assuming F continuously differentiable). It is sometimes too expansive and it is the reason of the introduction of a modified scheme, using a linearized Riemann problem.

The VFRoe-ncv scheme

Assuming, for simplicity, that F is continuously differentiable, one sets $A(\mathbf{w}) = DF(\mathbf{w})$ for $\mathbf{w} \in D$, where $DF(\mathbf{w})$ is the jacobian matrix of F at point $\mathbf{w} \in D$. Let ϕ be a regular function of $D \subset \mathbb{R}^p$ to $\mathbb{R}^p$. It is not necessary to assume that ϕ is one-to-one from D to $Ra(\phi) = \{\phi(\mathbf{w}),\ \mathbf{w} \in D\}$, but one assumes that there exists a continuous function C, from D to the set of $p \times p$ matrix with real entries, and a continuous function $\tilde{F}$, from $Ra(\phi)$ to $\mathbb{R}^p$ such that $D\phi(\mathbf{w})A(\mathbf{w}) = C(\mathbf{w})D\phi(\mathbf{w})$ and $F(\mathbf{w}) = \tilde{F}(\phi(\mathbf{w}))$ for all $\mathbf{w} \in D$.

Let $\mathbf{W} : \mathbb{R} \times \mathbb{R}_+ \to D$ be a regular solution of $\partial_t \mathbf{W} + A(\mathbf{W})\partial_x \mathbf{W} = 0$. Then, $\mathbf{Y} = \phi(\mathbf{W})$ satisfy $\partial_t \mathbf{Y} + D\phi(\mathbf{W})A(\mathbf{W})\partial_x \mathbf{W} = 0$ and, thanks to the hypothesis on ϕ, the function $\mathbf{Y}$ satisfies:

$$\partial_t \mathbf{Y} + C(\mathbf{W})\partial_x \mathbf{Y} = 0. \tag{13}$$

It is now possible to describes the VFRoe-ncv scheme associated to ϕ. For $\mathbf{w}_l,\ \mathbf{w}_r \in \mathbb{R}^p$, one sets $\mathbf{w}_{l,r} = (\mathbf{w}_r + \mathbf{w}_l)/2$ (it is possible to take another mean value between $\mathbf{w}_l$ and $\mathbf{w}_r$) and considers the following linear Riemann problem:

$$\begin{aligned} &\partial_t \mathbf{Y} + C(\mathbf{w}_{l,r})\partial_x \mathbf{Y} = 0, \\ &\mathbf{Y}(x,0) = \begin{cases} \mathbf{y}_l = \phi(\mathbf{w}_l) \text{ if } x < 0, \\ \mathbf{y}_r = \phi(\mathbf{w}_r) \text{ if } x > 0. \end{cases} \end{aligned} \tag{14}$$

If $C(\mathbf{w}_{l,r})$ is diagonalizable in $\mathbb{R}$, Problem (14) has a unique solution. It is a self similar function: $\mathbf{Y}(x,t) = \mathbf{R}_\phi(\frac{x}{t}, \mathbf{y}_l, \mathbf{y}_r)$. Then one sets:

$$\mathbf{y}^{\star,\pm}(\mathbf{w}_l, \mathbf{w}_r) = \mathbf{R}_\phi(0^\pm, \mathbf{y}_l, \mathbf{y}_r).$$

If $C(\mathbf{w}_{l,r})$ has only real eigenvalues but is not diagonalizable in $\mathbb{R}$, the first equation of (14) is a linear resonant hyperbolic system. In this case, Problem (14) has also a unique solution but it is not, in general, a function (in the example of Section 2, if $\lambda = 0$, there is a Dirac mass at $x = 0$ for any $t > 0$). However, $\mathbf{R}_\phi(0^\pm, \mathbf{y}_l, \mathbf{y}_r)$ is always well defined (forgetting the Dirac mass in the example of Section 2) and it is also possible to set $\mathbf{y}^{\star,\pm}(\mathbf{w}_l, \mathbf{w}_r) = \mathbf{R}_\phi(0^\pm, \mathbf{y}_l, \mathbf{y}_r)$. The VFRoe-ncv scheme associated to ϕ is (11)-(12) but with $\mathbf{F}^n_{i+\frac{1}{2}} = (1/2)(\tilde{F}(\mathbf{y}^{n,+}_{i+\frac{1}{2}}) + \tilde{F}(\mathbf{y}^{n,-}_{i+\frac{1}{2}}))$ (assuming that $\mathbf{y}^{n,\pm}_{i+\frac{1}{2}} \in Ra(\phi)$), $\mathbf{y}^{n,\pm}_{i+\frac{1}{2}} = \mathbf{y}^{\star,\pm}(\mathbf{w}^n_i, \mathbf{w}^n_{i+1})$. A possible drawback of the method seems to be the fact that the numerical flux of the scheme is not a continuous function of its arguments when an eigenvalue changes sign (namely, $\mathbf{F}^n_{i+1/2}$ does not depends continuously of $\mathbf{w}^n_i$ and $\mathbf{w}^n_{i+1}$). In practice, this drawback does not seem to be so important. As for the Godunov scheme, the scheme uses a CFL condition which reads $\delta t \le c\delta x$.

In the case studied in Section 3, for $\mathbf{w} = (u,k)^t \in D = [0,1] \times \mathbb{R}^\star_+$, one has $F(\mathbf{w}) = (kg(u), 0)^t$. A simple choice of ϕ is $\phi(\mathbf{w}) = (kg(u), k)^t$ for $\mathbf{w} = (u,k)^t$. With this choice of ϕ, the matrix $C(\mathbf{w})$ is, for any $\mathbf{w} \in D$, diagonal and System (13) is not a resonant system.

One presents in Figure 2 a numerical result (given in [Bac05]) with this two schemes, the function g given in Figure 1, and $u_0(x) = 3/8$, for $x < 0$, $u_0(x) = 5/8$ for $x > 0$. This result shows the good behaviour of the two schemes (with only one "wrong point" with the VFRoe-ncv scheme).

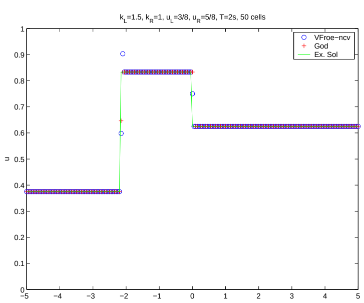

Fig. 2. Numerical results for the Godunov scheme and the VFRoe-ncv scheme

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