

Boundary conditions for hyperbolic equations or systems

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Summary. Different types of boundary conditions in industrial numerical simulators involving the discretization of hyperbolic systems are presented. For some of them, one may determine the problem to which the limit of approximate solutions (as the discretization parameters tend to 0) is the unique solution. In turn, this convergence result may suggest other ways to take into account the boundary conditions.

1 Introduction

In the industrial context, efficient numerical simulators are often developed after a long “trial and error” procedure. The efficiency of the simulators may be evaluated, for instance, by the fact that the solution satisfies some natural constraints and that it is in agreement with experimental data. In some cases, estimates on the approximate solutions allow to obtain the convergence of some sequences of approximate solutions as the discretization size tends to 0. However, it is not easy to give the answer to the following question: “Which problem is the limit of the approximate solutions the unique solution to?”.

This paper will focus on the problem of boundary conditions needed in the discretization of non linear hyperbolic equations or systems of equations; this problem is not yet clearly understood in many cases. Two different cases will be presented: a two phase flow in a pipeline and a two phase flow in a porous medium.

2 A two phase flow in a pipeline

2.1 Description of the system

A “simple” model for a two phase flow in a pipeline (see [8], for instance) leads to a 3×3 system of conservations laws. The unknown w is a function from $(0, 1) \times \mathbf{R}_+$ in \mathbf{R}^3 , solution of the following system:

$$w_t + (F(w))_x = 0, \quad x \in (0, 1), \quad t \in \mathbf{R}_+, \quad (1)$$

where $(\cdot)_t$ and $(\cdot)_x$ denote the derivatives with respect to t and x variables. The first two equations of (1) give the mass conservation of the 2 phases (gas and liquid) and the third one is the momentum equation for the mixture. The expression of the given function $F : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is quite complicated. It takes into account thermodynamical laws and a hydrodynamical law. System (1) is hyperbolic: for any $w \in \mathbf{R}^3$, the Jacobian matrix $DF(w)$ is diagonalizable in \mathbf{R} . The three eigenvalues can be ordered: $\lambda_1(w) < \lambda_2(w) < \lambda_3(w)$. In real situations, the first eigenvalue, $\lambda_1(w)$ is negative and the third,

$\lambda_3(w)$, is positive (they correspond to some “pressure waves” which are related to a “sound velocity”). The second eigenvalue, $\lambda_2(w)$, corresponds to some mean velocity between the two phases and can change sign. One can also note that the field related to this second eigenvalue is quite complicated because it is not, in general, a genuinely non linear field or a linearly degenerate field. In petroleum engineering, the wave associated to this second eigenvalue is a “void fraction wave”; engineers require a good representation of this wave in the numerical simulations.

Remark 1. In real situations, the function F in System (1) also depends on x , in order to take into account, for instance, the variation in the slope of the pipeline. Moreover, some source terms have to be added to the system, in order to take into account, for instance, some friction terms.

In order to complete System (1), an initial condition is prescribed:

$$w(x, 0) = w_0(x), \quad x \in (0, 1), \quad (2)$$

and it is also necessary to give some boundary conditions. This appears to be not so easy. Indeed, classically, a general principle is that the number of boundary conditions needs to be equal to the number of positive eigenvalues of the Jacobian matrix at $x = 0$ and to the number of negative eigenvalues of the Jacobian matrix at $x = 1$ (and these boundary conditions have to satisfy some compatibility conditions). However, this principle is not so easy to understand when an eigenvalue changes sign during the simulation (or in the case of a null eigenvalue). A very interesting case is the so called “severe slugging” case in a pipeline. For this case, there are always two positive eigenvalues at $x = 0$ and two natural boundary conditions are prescribed at $x = 0$, namely the fluxes of gas and liquid; these boundary conditions can be taken constant in time. At $x = 1$, there is one natural boundary condition, namely the pressure (which is the same for the two phases, in this model), to be prescribed. It can also be constant in time. The true physical solution, which is measured by experiments (and the aim is to modelize these experiments), is periodical in time and it appears that, at $x = 1$, the first eigenvalue is always positive and the third one is always negative but the second eigenvalue changes sign during the simulation. In the sequel, one presents different ways to take into account the boundary conditions and one gives a convergence result in a simplified case.

2.2 Discretization of the problem

In order to discretize Problem (1), (2) and some boundary conditions, which will be introduced later, let $h = \frac{1}{N}$ (with $N \in \mathbf{N}^*$) be the mesh size and $k > 0$ be the time step (assumed to be constant, for the sake of simplicity). The discrete unknown are the values $w_i^n \in \mathbf{R}^3$ for $i \in \{1, \dots, N\}$ and $n \in \mathbf{N}$. The discretization of the initial condition leads to

$$w_i^0 = \frac{1}{h} \int_{(i-1)h}^{ih} w_0(x) dx, \quad i \in \{1, \dots, N\}. \quad (3)$$

For the computation of w_i^n for $n > 0$, one uses an explicit, 3-points scheme:

$$\frac{h}{k} (w_i^{n+1} - w_i^n) + F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n = 0, \quad i \in \{1, \dots, N\}, \quad n \in \mathbf{N}. \quad (4)$$

For $i \in 1, \dots, N-1$, one takes $F_{i+\frac{1}{2}}^n = g(w_i^n, w_{i+1}^n)$, where g is the numerical flux. It has to satisfy, in particular, the classical consistency condition, namely $g(a, a) = F(a)$, and needs to be chosen in order to obtain some stability properties for the numerical scheme under a so called CFL condition on the time step (see Sect. 3 for the study of a scalar model). In the case of two phase flow in a pipeline, the classical numerical fluxes such as the Godunov flux (see [9]) or the Roe flux (see [11]) may not be implemented, because of computational difficulties. A convenient choice is obtained with a simplified Roe flux, namely $g(a, b) = \frac{g(a)+g(b)}{2} + \frac{1}{2}|A(a, b)|(a-b)$, where $A(a, b)$ is some approximation of the Jacobian matrix, depending on a and b , but not satisfying the so called Roe condition, see [8].

Remark 2. In fact, for the simulation of a two phase flow in a pipeline, the magnitude of the so-called fast eigenvalues, λ_1 and λ_3 , is much greater than that of λ_2 ; the choice in [8] is to use an implicit scheme with respect to the fast eigenvalues, whereas the eigenvalue λ_2 , which corresponds to the void fraction wave, is handled with an explicit second order discretization, since the void fraction wave needs to be simulated precisely (see [8] for details).

Let us now define the fluxes $F_{\frac{1}{2}}^n$ and $F_{N+\frac{1}{2}}^n$ at the boundary.

2.3 Boundary conditions for the discretized problem

In order to compute $F_{\frac{1}{2}}^n$ (and similarly $F_{N+\frac{1}{2}}^n$) a good way is to know, or to determine, some artificial value $w_0^n \in \mathbf{R}^3$ (and $w_{N+1}^n \in \mathbf{R}^3$) and to take $F_{\frac{1}{2}}^n = g_0(w_0^n, w_1^n)$ (and $F_{N+\frac{1}{2}}^n = g_1(w_N^n, w_{N+1}^n)$). The numerical fluxes g_0 and g_1 can be chosen equal to g , but this is not at all necessary (see the convergence result of Sect. 3); in fact, there are numerous situations where one should take g_0 and g_1 different from g . Indeed, the scheme is often very sensitive to the computation of the boundary fluxes and it is often worthwhile to use a more precise, but also more expensive numerical flux (such as the Godunov flux, for instance) for the computation of the boundary fluxes than for the computation of the interior fluxes. The difficulty is now to determine these artificial values, w_0^n and w_{N+1}^n .

Remark 3. In some cases, the choice of w_0^n and w_{N+1}^n is quite easy. A well known example is given by the wall-boundary condition for the Euler equations (with a perfect gas state law or a more general state law). For the sake of simplicity, let us mention the one-dimensional case; the generalization to a multi-dimensional case is quite easy. The Euler equations may be written the form (1), corresponding to conservation of mass, momentum and energy, with $w = (\rho, \rho u, E)^t$, where ρ is the density of the fluid, u its velocity, and E its energy. The wall-boundary condition at $x = 0$ is $u = 0$, and the only component to compute for the boundary condition is the second component of $F_{\frac{1}{2}}^n$ which is equal here to the pressure at $x = 0$ (since $u = 0$ at the wall), say $p_{\frac{1}{2}}^n$. The value w_1^n may be computed from the values ρ_1^n , u_1^n and p_1^n . A natural choice for w_0^n is to take $\rho_0^n = \rho_1^n$, $u_0^n = -u_1^n$ and $p_0^n = p_1^n$. The flux $F_{\frac{1}{2}}^n$ (that is the value $p_{\frac{1}{2}}^n$) is then obtained with $F_{\frac{1}{2}}^n = g_0(w_0^n, w_1^n)$ and a convenient choice of the numerical flux g_0 . We suggest to choose g_0 as the Godunov flux (or as a linearized Godunov flux, see [3] for instance). Numerical tests which were performed in [3] show that this choice is very satisfactory, even in the difficult case of a strong depressurization at the boundary. These tests also show that the pressure obtained

with the Roe flux is not so satisfactory and neither is the choice $p_{\frac{1}{2}}^n = p_1^n$ which may seem natural (in particular, in 2D simulations, using a dual mesh obtained with a finite element primal mesh).

In most cases, however, the choice of w_0^n and w_{N+1}^n is not so easy. A possible method, which is described in [4], is now layed out, for a fixed n and g_0 given:

1. Compute $DF(w_1^n)$, its eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ and a basis of \mathbf{R}^3 , $\{\varphi_1, \varphi_2, \varphi_3\}$, such that $DF(w_1^n)\varphi_i = \lambda_i\varphi_i$, $i = 1, 2, 3$.
2. Write w_1^n on the basis $\{\varphi_1, \varphi_2, \varphi_3\}$, namely $w_1^n = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \alpha_3\varphi_3$,
3. Let p be the number of positive eigenvalues, compute $w_0^n = \beta_1\varphi_1 + \beta_2\varphi_2 + \beta_3\varphi_3$ and $F_{\frac{1}{2}}^n = g_0(w_0^n, w_1^n)$, where the three unknowns β_1 , β_2 , and β_3 are determined by the p equations stating the boundary conditions (note that these equations involve the components of $F_{\frac{1}{2}}^n$) and by the $3 - p$ equalities $\beta_i = \alpha_i$ for $\lambda_i < 0$.

This method leads, at each time step, to a non linear system of 3 equations with 3 unknowns (except if $\lambda_i = 0$ for some i), namely β_1 , β_2 and β_3 ; note that some compatibility conditions are needed in order that this non linear system has a solution. Several variants of this method are possible. For instance, a boundary condition may be imposed on w_0^n rather than $F_{\frac{1}{2}}^n$. A similar method is, of course, possible at point $x = 1$ (changing the role of positive and negative eigenvalues).

This method is not always satisfactory. In the case of severe slugging for the simulation of two phase flow in a pipeline, the method seems to perform well at $x = 0$, where the eigenvalues λ_1 and λ_2 are always positive and the two boundary conditions (gas and liquid fluxes) are convenient. However, at $x = 1$, the second eigenvalue sometimes becomes negative and one needs a second boundary condition (the first one is a condition on the pressure). A natural condition seems to be $Q_l = 0$, where Q_l is the second component of the flux F , that is the liquid flux, but this condition does not lead to good results. Other possible choices of this additional boundary condition at $x = 1$ were tested and did not give good results. A possible interpretation of this problem is the fact that the sign of λ_2 is computed with w_N^n . Roughly speaking, it is “too late” when $\lambda_2(w_N^n)$ becomes negative (see Sect. 3 for the study of a simple scalar case). Indeed, good results (in agreement with experiments) are obtained with the unilateral condition $Q_l \geq 0$ (whatever the sign of $\lambda_2(w_N^n)$). It consists in using the preceeding method (for the boundary condition at $x = 1$) and in replacing, in the numerical scheme (4), the second component of $F_{N+\frac{1}{2}}^n$ by its positive part. Then, if $\lambda_2(w_N^n) < 0$, two boundary conditions are given at $x = 1$ (pressure and $Q_l = 0$) and if $\lambda_2(w_N^n) \geq 0$, one boundary condition is given at $x = 1$ (pressure) but, in (4), the second component of $F_{N+\frac{1}{2}}^n$ is replaced by its positive part.

In the following section, we will try to understand the sense of this boundary condition in a simplified scalar case.

3 The scalar case

A general convergence result is presented here in the case of a scalar equation. Then, this result will be applied to understand the sense of the boundary condition, described at $x = 1$ in the previous section, in a simplified scalar case.

3.1 A general convergence result

The unknown is now a function $u : (0, 1) \times \mathbf{R}_+ \rightarrow \mathbf{R}$. The flux is a function $f \in C^1(\mathbf{R}, \mathbf{R})$ (or $f : \mathbf{R} \rightarrow \mathbf{R}$ Lipschitz continuous) and the initial datum is $u_0 \in L^\infty((0, 1))$. Let $A, B \in \mathbf{R}$ be such that $A \leq u_0 \leq B$ a.e.. The problem to solve is:

$$u_t + (f(u))_x = 0, \quad x \in (0, 1), \quad t \in \mathbf{R}_+, \quad (5)$$

with the initial condition :

$$u(x, 0) = u_0(x), \quad x \in (0, 1), \quad (6)$$

and some boundary conditions which will be prescribed later.

As in the previous section, let $h = \frac{1}{N}$ (with $N \in \mathbf{N}^*$) be the mesh size and $k > 0$ be the time step (assumed to be constant, for the sake of simplicity). The discrete unknowns are now the values $u_i^n \in \mathbf{R}$ for $i \in \{1, \dots, N\}$ and $n \in \mathbf{N}$. In order to define the approximate solution a.e. in $(0, 1) \times \mathbf{R}$, one sets $u_{h,k}(x, t) = u_i^n$ for $x \in ((i-1)h, ih)$, $t \in (nk, (n+1)k)$, $i \in \{1, \dots, N\}$, $n \in \mathbf{N}$.

The discretization of the initial condition leads to

$$u_i^0 = \frac{1}{h} \int_{(i-1)h}^{ih} u_0(x) dx, \quad i \in \{1, \dots, N\}. \quad (7)$$

For the computation of u_i^n for $n > 0$, one uses, as before, an explicit, 3-points scheme:

$$\frac{h}{k} (u_i^{n+1} - u_i^n) + f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n = 0, \quad i \in \{1, \dots, N\}, \quad n \in \mathbf{N}. \quad (8)$$

For $i \in 1, \dots, N-1$, one takes

$$f_{i+\frac{1}{2}}^n = g(u_i^n, u_{i+1}^n), \quad (9)$$

where g is the numerical flux. Sufficient conditions on $g : [A, B]^2 \rightarrow \mathbf{R}$, in order to have a convergent scheme if $x \in \mathbf{R}$ instead of $(0, 1)$, are:

- C1: g is non decreasing with respect to its first argument and nonincreasing with respect to its second argument,
- C2: $g(s, s) = f(s)$, for all $s \in [A, B]$,
- C3: g is Lipschitz continuous.

Let L be a Lipschitz constant for g (on $[A, B]^2$) and $\zeta > 0$. If $(0, 1)$ is replaced by \mathbf{R} , it is well known (see e.g. [4]) that, if $k \leq (1 - \zeta) \frac{h}{L}$, the approximate solution $u_{h,k}$, that is the solution defined by (7)-(9) (with $i \in \mathbf{Z}$), takes its values in $[A, B]$ and converges towards the unique entropy weak solution of (5)-(6) in $L_{loc}^p(\mathbf{R} \times \mathbf{R}_+)$ as $h \rightarrow 0$.

In the case $x \in (0, 1)$ instead of $x \in \mathbf{R}$, one assumes the same conditions on g , namely (C1)-(C3). In order to complete the scheme, one has to define $f_{\frac{1}{2}}^n$ and $f_{N+\frac{1}{2}}^n$.

Let $\bar{u}, \bar{\bar{u}} \in L^\infty(\mathbf{R}_+)$ be such that $A \leq \bar{u}, \bar{\bar{u}} \leq B$, a.e. on \mathbf{R}_+ , let $g_0, g_1 : [A, B]^2 \rightarrow \mathbf{R}$, satisfying (C1)-(C3), and define:

$$\begin{aligned} f_{\frac{1}{2}}^n &= g_0(\bar{u}^n, u_1^n); \quad \bar{u}^n = \frac{1}{k} \int_{nk}^{(n+1)k} \bar{u}(t) dt \\ f_{N+\frac{1}{2}}^n &= g_1(u_N^n, \bar{\bar{u}}^n); \quad \bar{\bar{u}}^n = \frac{1}{k} \int_{nk}^{(n+1)k} \bar{\bar{u}}(t) dt, \end{aligned} \quad (10)$$

Then, a convergence theorem can be proven as in the case $x \in \mathbf{R}$, see [13]:

Theorem 1. *Let $f \in C^1(\mathbf{R}, \mathbf{R})$ (or $f : \mathbf{R} \rightarrow \mathbf{R}$ Lipschitz continuous). Let $u_0 \in L^\infty((0, 1))$, $\bar{u}, \bar{\bar{u}} \in L^\infty(\mathbf{R}_+)$ and $A, B \in \mathbf{R}$ be such that $A \leq u_0 \leq B$ a.e. on $(0, 1)$, $A \leq \bar{u}, \bar{\bar{u}} \leq B$ a.e. on \mathbf{R}_+ . Let $g_0, g_1 : [A, B]^2 \rightarrow \mathbf{R}$, satisfying (C1)-(C3). Let L be a common Lipschitz constant for g, g_0 and g_1 (on $[A, B]^2$) and let $\zeta > 0$. Then, if $k \leq (1 - \zeta)\frac{h}{L}$, the equations (7)-(10) define an approximate solution $u_{h,k}$ which takes its values in $[A, B]$ and converges towards the unique solution of (11) in $L^p_{loc}([0, 1] \times \mathbf{R}_+)$ for any $1 \leq p < \infty$, as $h \rightarrow 0$:*

$$\begin{aligned} u \in L^\infty((0, 1) \times (0, \infty)), \\ \int_0^\infty \int_0^1 [(u - \kappa)^\pm \varphi_t + \text{sign}_\pm(u - \kappa)(f(u) - f(\kappa))\varphi_x] dx dt \\ + M \int_0^\infty (\bar{u}(t) - \kappa)^\pm \varphi(0, t) dt + M \int_0^\infty (\bar{\bar{u}}(t) - \kappa)^\pm \varphi(1, t) dt \\ + \int_0^1 (u_0 - \kappa)^\pm \varphi(x, 0) dx \geq 0, \\ \forall \kappa \in [A, B], \forall \varphi \in C_c^1([0, 1] \times [0, \infty), \mathbf{R}_+). \end{aligned} \quad (11)$$

In (11), M is any bound for $|f'|$ on $[A, B]$ (and the solution of (11) does not depend on the choice of M). The definition of sign_\pm is: $\text{sign}_+(s) = 1$ if $s > 0$, $\text{sign}_+(s) = 0$ if $s < 0$, $\text{sign}_-(s) = 0$ if $s > 0$, $\text{sign}_-(s) = -1$ if $s < 0$.

Remark 4.

1. It is interesting to remark that this convergence result is also true if the function g depends on i and n , provided that L is a common Lipschitz constant for all these functions.
2. The definition (11) of solution of (5)-(6) with the “weak” boundary conditions \bar{u} and $\bar{\bar{u}}$ at $x = 0$ and $x = 1$ is essentially due to F. Otto, see [10].
3. It is interesting also to remark that if one replaces, in (11), the two entropies $(u - \kappa)^\pm$ by the sole entropy $|u - \kappa|$, one has an existence result (since $|u - \kappa| = (u - \kappa)^+ + (u - \kappa)^-$) but no uniqueness result, see [13] for a counter-example to uniqueness.
4. This convergence result can be generalized to the multidimensional case, see Sect. 5 and [13].

If u , solution of (11), is regular enough (say $u \in C^1([0, 1] \times \mathbf{R}_+)$, for instance), u satisfies $u(0, t) = \bar{u}(t)$ and $u(1, t) = \bar{\bar{u}}(t)$ in the weak sense given in [1]. This condition is very simple if f is monotone:

- If $f' > 0$, then $u(0, \cdot) = \bar{u}$ and u does not depend on $\bar{\bar{u}}$.
- If $f' < 0$, then $u(1, \cdot) = \bar{\bar{u}}$ and u does not depend on \bar{u} .

3.2 A very simple example

One considers here Equation (5), with initial condition (6) and weak boundary condition \bar{u} and $\bar{\bar{u}}$ at $x = 0$ and $x = 1$, that is in the sense of (11), in the particular case $f' > 0$. In this case, the main example of numerical flux is $g = g_0 = g_1$, $g(a, b) = f(a)$, which leads to the well known upstream scheme. With this choice of g_0 and g_1 , using the notations of Sect. 3.1, the boundary conditions are taken into account in the form:

$$f_{\frac{1}{2}}^n = f(\bar{u}^n), f_{N+\frac{1}{2}}^n = f(u_N^n), \quad (12)$$

with $\bar{u}^n = \frac{1}{k} \int_{nk}^{(n+1)k} \bar{u}(t) dt$. One may apply the general convergence theorem. The approximate solutions converge (as $h \rightarrow 0$) towards the solution of (11). In this case, the approximate solutions, as well as the solution of (11), do not depend on \bar{u} .

In the case $f' < 0$ the main example is $g = g_0 = g_1$, $g(a, b) = f(b)$, which also leads to the upstream scheme. The boundary conditions are taken into account in the following way:

$$f_{\frac{1}{2}}^n = f(u_1^n), f_{N+\frac{1}{2}}^n = f(\bar{u}^n), \quad (13)$$

with $\bar{u}^n = \frac{1}{k} \int_{nk}^{(n+1)k} \bar{u}(t) dt$.

These simple cases suggest the following scheme for any f , which is the scalar version of the scheme described in Sect. 2.2 (note that $f'(u)$ is the Jacobian matrix at point $u \in \mathbf{R}$):

– Boundary condition at $x = 0$:

$$\begin{cases} f_{\frac{1}{2}}^n = f(\bar{u}^n), & \text{if } f'(u_1^n) > 0, \\ f_{\frac{1}{2}}^n = f(u_1^n), & \text{if } f'(u_1^n) < 0. \end{cases} \quad (14)$$

– Boundary condition at $x = 1$:

$$\begin{cases} f_{N+\frac{1}{2}}^n = f(\bar{u}^n), & \text{if } f'(u_N^n) < 0, \\ f_{N+\frac{1}{2}}^n = f(u_N^n), & \text{if } f'(u_N^n) > 0. \end{cases} \quad (15)$$

This solution is not always satisfactory as can be shown on the following simple example with the Burgers equation:

Let $f(s) = s^2$, $u_0 = 1$ a.e. on $(0, 1)$, $\bar{u} = 1$ a.e. on \mathbf{R}_+ and $\bar{u} = -2$ a.e. on \mathbf{R}_+ . The exact solution which has to be approached by the numerical scheme is the unique solution of (11) with these values of f , u_0 , \bar{u} and \bar{u} . Computing the approximate solution with (7)-(9), the function g satisfying (C2), and with (14)-(15), leads to an approximate solution which is constant and equal to 1 for any h and k . Then, it does not converge (as h and k go to 0) towards the exact solution which is not constant and equal to 1 since, for the exact solution, a shock wave with a negative speed starts from the point $x = 1$ at time $t = 0$. Indeed, one can also remark that this approximate solution is the exact solution of (11) with the same values of f , u_0 , \bar{u} and with any \bar{u} satisfying $\bar{u} \geq -1$ a.e. on \mathbf{R}_+ . In order to obtain a convergent approximation of the exact solution corresponding to $\bar{u} = -2$, a good choice is, instead of (15), $f_{N+\frac{1}{2}}^n = g_1(u_N^n, -2)$ with g_1 satisfying (C1)-(C3).

3.3 A simplified model for two phase flows in pipelines

It is now possible to understand the treatment of the boundary described in Sect. 2 on a simplified model. This simplified model for two phase flows in pipelines is given in [12]. For this model, the densities are constant so that there are no longer pressure waves but only the void fraction wave, corresponding to the second eigenvalue of the original system (1). It is also easy to see that for this model, the total flux (that is the sum of the fluxes of

the two phases) is constant in space. One also assumes that this total flux is constant in time (and positive). System (1) is then reduced to a scalar equation, Equation (5), where the unknown, $u : (0, 1) \times \mathbf{R} \rightarrow \mathbf{R}$, is the gas fraction which takes its values between 0 and 1.

The function f can be taken as $f(s) = as - bs^2$, where $a, b \in \mathbf{R}$ are given and such that $0 < b < a < 2b$. In (5), the quantity $f(u)$ is the flux of gas. Then, $f(1) - f(u)$ is the flux of liquid. The function f is increasing between 0 and $u_M = a/(2b)$ and decreasing between u_M and 1. An important value is $u_m \in [0, u_M]$ such that $f(u_m) = f(1)$.

One takes $u_0 = 0$ a.e. on $[0, 1]$ as an initial condition. At $x = 0$, the gas flux is given (as in the complete model, see Sect. 2), one takes $f(u(0, \cdot)) = \bar{f}$ with $\bar{f}(t) = c$ for $t \leq T$ and $\bar{f}(t) = 0$ for $t \geq T$, where c and T are given with $c > f(1)$ and T large enough so that f' changes sign at $x = 1$ during the simulation. Indeed, in this simplified model, it is also necessary to take T not too large in order to avoid a problem at $x = 0$ (for T too large, f' will also changes sign at $x = 0$). The boundary condition at $x = 1$ will be described on the discrete problem below.

The discretization of the problem is performed as before with (7)-(9), with g satisfying (C1)-(C3).

For the discretization of the boundary condition at $x = 0$, the method described in Sect. 2 leads here to

$$f_{\frac{1}{2}}^n = \bar{f}(nk), \quad (16)$$

which is indeed in accordance with the fact that $f'(u_1^n) > 0$ for all n , at least if T is not too large.

For the discretization of the boundary condition at $x = 1$, the first method described in Sect. 2 and given in (15), using the sign of $f'(u_N^n)$ leads to

$$\begin{cases} f_{N+\frac{1}{2}}^n = f(u_N^n), & \text{if } u_N^n < u_M, \\ f_{N+\frac{1}{2}}^n = f(1), & \text{if } u_N^n > u_M, \end{cases} \quad (17)$$

and does not lead to the desired results. Note also that $f_{N+\frac{1}{2}}^n$, given by (17), is a discontinuous function of u_N^n .

The second method, described in Sect. 2, uses the fact that the liquid flux cannot be negative at $x = 1$. Since the liquid flux at $x = 1$ is $f(1) - f_{N+\frac{1}{2}}^n$ and since $f(u_m) = f(1)$, this method leads to

$$\begin{cases} f_{N+\frac{1}{2}}^n = f(u_N^n), & \text{if } u_N^n \leq u_m, \\ f_{N+\frac{1}{2}}^n = f(u_m), & \text{if } u_N^n > u_m, \end{cases} \quad (18)$$

Note that $f_{N+\frac{1}{2}}^n$, given by (18), is a continuous function of u_N^n . We shall apply the convergence theorem, Theorem 1 given in Sect. 3.1, for the boundary conditions (16) and (18), and understand the boundary conditions satisfied by the limit of the approximate solutions. In order to do so, we need to find g_0 and g_1 , satisfying (C1)-(C3), and $\bar{u}, \bar{\bar{u}} \in L^\infty(\mathbf{R}_+)$ such that $f_{\frac{1}{2}}^n$ and $f_{N+\frac{1}{2}}^n$, respectively defined by (16) and (18), satisfy (10). Indeed, it is shown in [7] that both boundary fluxes $f_{\frac{1}{2}}^n$ and $f_{N+\frac{1}{2}}^n$ may be expressed with the Godunov flux in the following way:

- Boundary flux at $x = 1$. One takes $\bar{\bar{u}} = 1$ a.e. on \mathbf{R}_+ and g_0 equal to the Godunov flux, that is $g_0 = g_G$ with

$$g_G(\alpha, \beta) = \begin{cases} \min\{f(s), s \in [\alpha, \beta]\} & \text{if } \alpha \leq \beta, \\ \max\{f(s), s \in [\beta, \alpha]\} & \text{if } \alpha > \beta. \end{cases}$$

The formula (18) reads

$$f_{N+\frac{1}{2}}^n = g_G(u_N^n, 1) = \begin{cases} f(u_N^n) & \text{if } u_N^n \leq u_m, \\ f(1) & \text{if } u_N^n > u_m. \end{cases} \quad (19)$$

– Boundary flux at $x = 0$. One assumes (for simplicity) that $\frac{T}{k} \in \mathbf{N}$. let $\alpha, \beta \in (0, 1)$ such that $\alpha < \beta$ and $f(\alpha) = f(\beta) = c$. One takes

$$\bar{u}(t) = \begin{cases} \alpha & \text{if } t < T, \\ 0 & \text{if } t > T, \end{cases} \quad (20)$$

so that, recalling that $\bar{u}^n = \frac{1}{k} \int_{nk}^{(n+1)k} \bar{u}(t) dt$,

$$f(\bar{u}^n) = \begin{cases} c & \text{if } nk < T, \\ 0 & \text{if } nk \geq T, \end{cases}$$

Then, if $u_1^n \leq \beta$, the formula (16) reads

$$f_{\frac{1}{2}}^n = g_G(\bar{u}^n, u_1^n), \quad (21)$$

since, in this case, $g_G(\bar{u}^n, u_1^n) = f(\bar{u}^n)$. The fact that $u_1^n \leq \beta$ is true for all n if T is not too large. If T is too large, the convergence result can be applied with (21) instead of (16).

It is now possible to apply Theorem 1. Let L be a common Lipschitz constant for g and g_G (on $[0, 1]^2$) and let $\zeta > 0$. If $k \leq (1 - \zeta) \frac{h}{T}$, the approximate solution $u_{h,k}$, that is the solution defined by (7)-(9), with the boundary fluxes (19)-(21) (and $u_0 = 0$, $\bar{u} = 1$ and \bar{u} given by (20)), takes its values in $[0, 1]$ and converges towards the unique solution of (22) in $L_{loc}^p([0, 1] \times \mathbf{R}_+)$ for any $1 \leq p < \infty$, as $h \rightarrow 0$:

$$\begin{aligned} u &\in L^\infty((0, 1) \times (0, \infty)), \\ &\int_0^\infty \int_0^1 [(u - \kappa)^\pm \varphi_t + \text{sign}_\pm(u - \kappa)(f(u) - f(\kappa))\varphi_x] dx dt \\ &+ M \int_0^\infty (\bar{u}(t) - \kappa)^\pm \varphi(0, t) dt + M \int_0^\infty (1 - \kappa)^\pm \varphi(1, t) dt \\ &+ \int_0^1 (0 - \kappa)^\pm \varphi(x, 0) dx \geq 0, \\ &\forall \kappa \in [0, 1], \forall \varphi \in C_c^1([0, 1] \times [0, \infty), \mathbf{R}_+). \end{aligned} \quad (22)$$

If u , solution of (22), is regular enough on $[0, 1] \times (0, T)$, then, it is possible to prove that u satisfies the boundary conditions, for $0 < t < T$, in the following sense (see [13] and [7]):

- Boundary condition at $x = 0$ (recall that \bar{u} is given by (20)): $u(0, t) = \alpha$ or $u(0, t) \geq \beta$.
In fact, if T is not too large, one has $u(0, t) = \alpha$.
- Boundary condition at $x = 1$: $u(1, t) \leq u_m$ or $u(1, t) = 1$,

Thanks to Theorem 1, it is possible to give other choices for $f_{N+\frac{1}{2}}^n$ for which the approximate solutions obtained with this new choice of $f_{N+\frac{1}{2}}^n$ converge towards the same function u , which is the unique solution of (22). Indeed, let $\psi : [0, 1] \rightarrow \mathbf{R}$ be a nondecreasing function such that $\psi \leq f$ and $\psi(1) = f(1)$ and take:

$$f_{N+\frac{1}{2}}^n = \psi(u_N^n). \quad (23)$$

One may construct a function g_1 satisfying (C1)-(C3) such that $\psi(s) = g_1(s, 1)$, for all $s \in [0, 1]$, and then use Theorem 1. Let L be a common Lipschitz constant for g and g_G and g_1 (on $[0, 1]^2$) and let $\zeta > 0$. If $k \leq (1 - \zeta)\frac{h}{L}$, the approximate solution $u_{h,k}$, that is the solution defined by (7)-(9), with the boundary fluxes (23) and (21) (and $u_0 = 0$, $\bar{u} = 1$ and \bar{u} given by (20)), takes its values in $[0, 1]$ and converges towards the unique solution of (22) in $L_{loc}^p([0, 1] \times \mathbf{R}_+)$ for any $1 \leq p < \infty$, as $h \rightarrow 0$.

Turning back to the complete system described in Sect. 2, the analysis of this simplified model for two phase flows in pipelines may also suggest another way to take into account the boundary condition at $x = 1$ (with a given numerical flux g_1):

1. Compute $DF(w_N^n)$, its eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ and a basis of \mathbf{R}^3 , $\{\varphi_1, \varphi_2, \varphi_3\}$, such that $DF(w_N^n)\varphi_i = \lambda_i\varphi_i$, $i = 1, 2, 3$,
2. write w_N^n on the basis $\{\varphi_1, \varphi_2, \varphi_3\}$, namely $w_N^n = \alpha_1\varphi_1 + \alpha_2\varphi_2 + \alpha_3\varphi_3$,
3. Since $\lambda_3 < 0$ and since one wants $Q_l \geq 0$, compute $w_{N+1}^n = \beta_1\varphi_1 + \beta_2\varphi_2 + \beta_3\varphi_3$ and $F_{N+\frac{1}{2}}^n = g_1(w_N^n, w_{N+1}^n)$ with the following 3 conditions on the components of w_{N+1}^n : usual condition on the pressure, $\beta_3 = \alpha_3$ and $R_{N+1}^n = 1$ where R_{N+1}^n is the gas fraction computed with w_{N+1}^n .

4 Two phase flow in a porous medium

A second example is given by the modelization of a two phase flow, oil and water (for instance), in a porous medium. Phases are immiscible. Compressibility and capillarity effects are neglected. The model is obtained using the conservation of mass for each phase and Darcy's law. This study is limited to the one dimensional case. In this case the pressure can be eliminated and the problem is reduced to a single equation, namely (5) with :

$$f(u) = \frac{f_1(u)(\alpha + \beta f_2(u))}{f_1(u) + f_2(u)}. \quad (24)$$

The unknown is the saturation of one phase, say water, and is denoted by u . The quantity α is the total flux, which is constant in space, thanks to the incompressibility of the phases. One assumes also that it is constant in time and positive. The quantity β is the difference between the densities of the phases. The functions f_1 and f_2 are the mobilities of the phases. The function f_1 is nondecreasing, regular and satisfies $f_1(0) = 0$. The function f_2 is nonincreasing, regular and satisfies $f_2(1) = 0$. The function $f_1 + f_2$ is bounded from below by a positive number.

Remark 5. For the equivalent two or three dimensional model, the pressure cannot be eliminated and the resulting model is a coupled system of two partial differential equations and two unknowns (pressure and saturation). The problem to which the limit of

the approximate solutions is solution is then much more complicated to determine. See [6] for a partial study of this question.

Here again, an initial condition is prescribed, namely (6), with $u_0 \in L^\infty((0,1))$, $0 \leq u_0 \leq 1$ a.e.. The boundary condition will be given later.

The numerical scheme is as in Sect. 3.1; it is given by (7) and (8) with (9). The choice of the numerical flux, g , satisfying (C1)-(C3), is usually given, for this model, using an “upwinding phase by phase”, that is (see [2], for instance) :

$$\begin{aligned} g(a,b) &= \frac{f_1(a)(\alpha + \beta f_2(a))}{f_1(a) + f_2(a)} \text{ if } -\alpha + \beta f_1(a) \leq 0 \\ g(a,b) &= \frac{f_1(a)(\alpha + \beta f_2(b))}{f_1(a) + f_2(b)} \text{ if } -\alpha + \beta f_1(a) > 0. \end{aligned} \quad (25)$$

Let us then define $f_{\frac{1}{2}}^n$ and $f_{N+\frac{1}{2}}^n$. On considers here the case of an injection of pure water at $x = 0$. Then :

$$f_{\frac{1}{2}}^n = \alpha, \quad n \geq 0. \quad (26)$$

At $x = 1$, The boundary condition is quite complicated. A simple example is (see [7] for a more complete study):

$$f_{N+\frac{1}{2}}^n = \frac{f_1(u_N^n)\alpha}{f_1(u_N^n) + f_2(u_N^n)}. \quad (27)$$

Then, the approximate solution is given with (7)-(9), g given by (25), and (26)-(27).

In order to prove that the approximate solutions converge, as h and k go to zero, and to determine the problem which the limit of the approximate solutions is the unique solution to, one proceeds as in Sect. 3.3. One has to find g_0 and g_1 satisfying (C1)-(C3) and $\bar{u}, \bar{u} \in L^\infty(\mathbf{R}_+)$ such that $f_{\frac{1}{2}}^n$ and $f_{N+\frac{1}{2}}^n$, respectively defined by (26) and (27), satisfy (10). This is again performed in [7]. The most interesting case is obtained for $\beta f_1(1) > \alpha$ and when the function f is increasing on $(0, u_M)$ and decreasing on $(u_M, 1)$, as in Sect. 3.3. In fact, the main point is the existence of a unique $u_m \in (0, 1)$ such that $f(u_m) = f(1) = \alpha$ and that f is increasing on $[0, u_m]$ and greater or equal to α on $[u_m, 1]$. Then, it is quite easy to prove that (26) gives

$$f_{\frac{1}{2}}^n = \alpha = g_G(u_m, u_1^n),$$

where g_G is the Godunov flux given in Sect. 3.3.

For the boundary condition at $x = 1$, it is possible to construct (see [7]) a function $g_1 : [0, 1]^2 \rightarrow \mathbf{R}$ satisfying (C1)-(C3) such that (27) gives :

$$f_{N+\frac{1}{2}}^n = g_1(u_N^n, 1).$$

It is now possible to use Theorem 1.

Let L be a common Lipschitz constant for g (given by (25)), g_G and g_1 (on $[0, 1]^2$) and let $\zeta > 0$. If $k \leq (1 - \zeta) \frac{h}{L}$, the approximate solution $u_{h,k}$, that is the solution defined by (7)-(9) (with g given by 25), and by the boundary fluxes (26)-(27), takes its values in $[0, 1]$ and converges towards the unique solution of (28) in $L_{loc}^p([0, 1] \times \mathbf{R}_+)$ for any $1 \leq p < \infty$, as $h \rightarrow 0$:

$$\begin{aligned}
& u \in L^\infty((0, 1) \times (0, \infty)), \\
& \int_0^\infty \int_0^1 [(u - \kappa)^\pm \varphi_t + \text{sign}_\pm(u - \kappa)(f(u) - f(\kappa))\varphi_x] dx dt \\
& + M \int_0^\infty (u_m - \kappa)^\pm \varphi(0, t) dt + M \int_0^\infty (1 - \kappa)^\pm \varphi(1, t) dt \\
& + \int_0^1 (u_0 - \kappa)^\pm \varphi(x, 0) dx \geq 0, \\
& \forall \kappa \in [0, 1], \forall \varphi \in C_c^1([0, 1] \times [0, \infty), \mathbf{R}_+),
\end{aligned} \tag{28}$$

where M is a bound for $|f'|$ on $[0, 1]$ (f is given by (24)). As in Sect. 3.3. It is possible to give the sense of the boundary condition if u is regular enough. Indeed, let u be a regular solution of (28). Then, u satisfies the boundary conditions in the sense given by [1], that is :

$$\text{sign}(u(0, t) - u_m)(f(u(0, t)) - f(\kappa)) \leq 0, \quad \forall \kappa \in [u_m, u(0, t)], \text{ for a.e. } t \in \mathbf{R}_+,$$

$$\text{sign}(u(1, t) - 1)(f(u(1, t)) - f(\kappa)) \geq 0, \quad \forall \kappa \in [1, u(1, t)], \text{ for a.e. } t \in \mathbf{R}_+,$$

with $[a, b] = \{ta + (1 - t)b, t \in [0, 1]\}$ and $\text{sign}(s) = 1$ for $s > 0$, $\text{sign}(s) = -1$ for $s < 0$, $\text{sign}(0) = 0$.

This gives $u(0, t) = u_m$ or $u(0, t) = 1$ and $u(1, t) \leq u_m$ or $u(1, t) = 1$. In particular, at $x = 0$, one has $f(u(0, t)) = \alpha$ (only water is injected) and, at $x = 1$, $f(u(1, t)) < \alpha$ if $u(1, t) < u_m$ (which states that there is some oil production).

5 The multidimensional scalar case

In this section, a generalization of Theorem 1 is presented for the multidimensional scalar case together with a rough sketch of proof. For the sake of simplicity, one considers $d = 2$ (the extension to $d = 3$ is straightforward) and a flux function under the form $v(x, t)f(u)$, with $\text{div}(v(\cdot, t)) = 0$ (see [13] for the general case of a flux function $f(x, t, u)$). This leads to the following equation:

$$u_t + \text{div}(vf(u)) = 0, \quad \text{in } \Omega \times (0, T), \tag{29}$$

where Ω is a bounded polygonal open set of \mathbf{R}^2 , $T > 0$, $f \in C^1(\mathbf{R}, \mathbf{R})$ (or $f : \mathbf{R} \rightarrow \mathbf{R}$ Lipschitz continuous) and $v \in C^1(\mathbf{R}^2 \times [0, T]) \rightarrow \mathbf{R}^2$ with $\text{div}(v(\cdot, t)) = 0$ in \mathbf{R}^2 for all $t \in [0, T]$. The unknown is $u : \Omega \times (0, T) \rightarrow \mathbf{R}$.

Let $u_0 \in L^\infty(\Omega)$ and $\bar{u} \in L^\infty(\partial\Omega \times (0, T))$. Let $A, B \in \mathbf{R}$ be such that $A \leq u_0 \leq B$ a.e. on Ω and $A \leq \bar{u} \leq B$ a.e. on $\partial\Omega \times (0, T)$. Following the work of [10], an entropy weak solution of (29) with the initial condition u_0 and the (weak) boundary condition \bar{u} is a solution of (30):

$$\begin{aligned}
& u \in L^\infty(\Omega \times (0, T)), \\
& \int_0^T \int_\Omega [(u - \kappa)^\pm \varphi_t + \text{sign}_\pm(u - \kappa)(f(u) - f(\kappa))v \cdot \text{grad}\varphi] dx dt \\
& \quad + M \int_0^T \int_{\partial\Omega} (\bar{u}(t) - \kappa)^\pm \varphi(x, t) d\gamma(x) dt \\
& \quad \quad + \int_\Omega (u_0 - \kappa)^\pm \varphi(x, 0) dx \geq 0, \\
& \quad \forall \kappa \in [A, B], \forall \varphi \in C_c^1(\bar{\Omega} \times [0, T], \mathbf{R}_+),
\end{aligned} \tag{30}$$

where $d\gamma(x)$ stands for the integration with respect to the one dimensional Lebesgue measure on the boundary of Ω and M is such that $\|v\|_\infty |f(s_1) - f(s_2)| \leq M |s_1 - s_2|$ for all $s_1, s_2 \in [A, B]$, where $\|v\|_\infty = \sup_{(x,t) \in \Omega \times [0, T]} |v(x, t)|$ (and $|\cdot|$ denotes here the Euclidean norm in \mathbf{R}^2).

Remark 6.

1. If u satisfies the family of inequalities (30), it is possible to prove that u is a solution of (29) (on a weak form), u satisfies some entropy inequalities in $\Omega \times (0, T)$, namely $|u - \kappa|_t + \text{div}(v(f(\max(u, \kappa)) - f(\min(u, \kappa)))) \leq 0$ for all $\kappa \in \mathbf{R}$, but also on the boundary of $\partial\Omega$ and on $t = 0$. u satisfies the initial condition $(u(\cdot, 0) = u_0)$ and u partially satisfies the boundary condition. For instance, if $f' > 0$ and u is regular enough, then $u(x, t) = \bar{u}(x, t)$ if $x \in \partial\Omega$, $t \in (0, T)$ and $v(x, t) \cdot n(x, t) < 0$, where n is the outward normal vector to $\partial\Omega$.
2. Let $\bar{M} \geq 1$. It is interesting to remark that u is solution of (30) if and only if u is solution of (30) where the term $\int_\Omega (u_0 - \kappa)^\pm \varphi(x, 0) dx$ is replaced by $\bar{M} \int_\Omega (u_0 - \kappa)^\pm \varphi(x, 0) dx$.

A sketch of proof of existence and uniqueness of the solution of (30) together with the convergence of numerical approximations is now given, following [13].

STEP 1: APPROXIMATE SOLUTION. With a quite general mesh of Ω (with triangles, for instance), denoted by \mathcal{T} , and a time step k , it is possible to define an approximate solution, denoted by $u_{\mathcal{T}, k}$, using some numerical fluxes (on the edges of the mesh) satisfying conditions similar to (C1)-(C3) in Sect. 3.1. Under a so called CFL condition (like $k \leq (1 - \zeta) \frac{h}{L}$ in Sect. 3.1), it is easy to prove that $A \leq u_{\mathcal{T}, k} \leq B$ a.e. on $\Omega \times (0, T)$. Unfortunately, it does not seem easy to obtain directly a strong compactness result on the family of approximate solutions (although this strong compactness result is true, as we shall see below).

STEP 2: WEAK COMPACTNESS. Using only this L^∞ bound on $u_{\mathcal{T}, k}$, one can assume (for convenient subsequences of sequences of approximate solutions) that $u_{\mathcal{T}, k} \rightarrow u$, as the mesh size goes to zero (with the CFL condition), in a “non linear weak- \star sense” (similar to the convergence towards young measures, see [4] for instance), that is $u \in L^\infty(\Omega \times (0, T) \times (0, 1))$ and

$$\begin{aligned}
& \int_0^T \int_\Omega \phi(u_{\mathcal{T}, k}(x, t)) \varphi(x, t) dx dt \rightarrow \int_0^1 \int_0^T \int_\Omega \phi(u(x, t, \alpha)) \varphi(x, t) dx dt d\alpha, \\
& \quad \text{for all } \varphi \in L^1(\Omega \times (0, T)) \text{ and all } \phi \in C(\mathbf{R}, \mathbf{R}).
\end{aligned}$$

STEP 3: PASSING TO THE LIMIT. Using the monotonicity of the numerical fluxes, the approximate solutions satisfy some discrete entropy inequalities. Passing to the limit in

these inequalities gives that u (defined in Step 2) is solution of some inequalities which are very similar to (30), namely:

$$\begin{aligned}
u &\in L^\infty(\Omega \times (0, T) \times (0, 1)), \\
&\int_0^1 \int_0^T \int_\Omega [(u - \kappa)^\pm \varphi_t + \text{sign}_\pm(u - \kappa)(f(u) - f(\kappa))v \cdot \text{grad}\varphi] dx dt d\alpha \\
&\quad + M \int_0^T \int_{\partial\Omega} (\bar{u}(t) - \kappa)^\pm \varphi(x, t) d\gamma(x) dt \\
&\quad \quad + \int_\Omega (u_0 - \kappa)^\pm \varphi(x, 0) dx \geq 0, \\
&\forall \kappa \in [A, B], \forall \varphi \in C_c^1(\bar{\Omega} \times [0, T], \mathbf{R}_+),
\end{aligned} \tag{31}$$

For this step, one chooses M not only greater than the Lipschitz constant of $\|v\|_\infty f$ on $[A, B]$, but also greater than the Lipschitz constant (on $[A, B]^2$) of the numerical fluxes associated to the edges of the meshes (the equivalent of L in Theorem 1). This choice of M is possible since the unique solution of (30) does not depend on M provided that M is greater than the Lipschitz constant of $\|v\|_\infty f$ on $[A, B]$ and since it is possible to choose numerical fluxes (namely, Godunov flux, for instance) such as the Lipschitz constant of these numerical fluxes is bounded by the Lipschitz constant of $\|v\|_\infty f$ (then, the present method leads to an existence result with M only greater than the Lipschitz constant of $\|v\|_\infty f$ on $s \in [A, B]$, passing to the limit on approximate solutions given with these numerical fluxes).

STEP 4: UNIQUENESS OF THE SOLUTION OF (31). In this step, the “doubling variables” method of Krushkov is used to prove the uniqueness of the solution of (31). Indeed, if u and w are two solutions of (31), the doubling variables method leads to:

$$\begin{aligned}
&\int_0^1 \int_0^1 \int_0^T \int_\Omega |u(x, t, \alpha) - w(x, t, \beta)| \varphi_t dx dt d\alpha d\beta \\
&\quad + \int_0^1 \int_0^1 \int_0^T \int_\Omega (f(\max(u, w)) - f(\min(u, w))) v \cdot \text{grad}\varphi dx dt d\alpha d\beta \geq 0 \\
&\quad \quad \quad \forall \varphi \in C_c^1(\bar{\Omega} \times [0, T], \mathbf{R}_+),
\end{aligned} \tag{32}$$

Taking $\varphi(x, t) = (T - t)^+$ in (32) (which is, indeed, possible) gives that u does not depend on α , w does not depend on β and $u = w$ a.e. on $\Omega \times (0, T)$. As a result, u is also the unique solution of (30).

STEP 5: CONCLUSION. Step 4 gives, in particular, the uniqueness of the solution of (30). It gives also that the non linear weak- \star limit of sequences of approximate solutions is solution of (30) and, therefore, the existence of the solution of (30). Furthermore, since the non linear weak- \star limit of sequences of approximate solution does not depend on α , it is quite easy to deduce that this limit is “strong” in $L^p(\Omega \times (0, T))$ for any $p \in [1, \infty)$ (see [4], for instance) and, thanks to the uniqueness of the limit, the convergence holds without extraction of subsequences.

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