Finite volume approximation of elliptic problems and convergence of an approximate gradient

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Abstract : Using a classical finite volume piecewise constant approximation of the solution of an elliptic problem in a domain Ω, we build here an approximate gradient of the solution. It is then shown that this approximate gradient converges in $H_{div}(\Omega)$. An error estimate is given when the solution belongs to $H^2(\Omega)$.

1 Introduction

The finite volume method is a discretization method which is well suited for the numerical simulation of various types (elliptic, parabolic or hyperbolic, for instance) of conservation laws; it has been extensively used in several engineering fields, such as fluid mechanics, heat and mass transfer or petroleum engineering. Some of the important features of the finite volume method are that it may be used on arbitrary geometries, using structured or unstructured meshes, and that it leads to robust schemes. An additional feature is the local conservativity of the numerical fluxes, that is the numerical flux is conserved from one discretization cell to its neighbour. This last feature makes the finite volume method quite attractive when modelling problems for which the flux is of importance, such as in fluid mechanics, semi-conductor device simulation, heat and mass transfer...

Finite volume methods for convection-diffusion equations were introduced as early as the early sixties by Tichonov and Samarskii, see [37], [32] and [33]. The convergence theory of such schemes in several space dimensions has only recently been undertaken. In the case of vertex-centered finite volume schemes, studies were carried out by [34] in the case of Cartesian meshes, [18], [3], [5], [6] and [38] in the case of unstructured meshes; see also [28], [36], [24], [29] and [35] in the case of quadrilateral meshes in two space dimensions. Cell-centered finite volume schemes are addressed in [25], [15], [41] and [23] in the case of Cartesian meshes and in [40], [19], [20], [22], [26] in the case of triangular (in two space dimensions) or Voronoï meshes; let us also mention [8] and [9] where more general meshes are treated, with, however, a somewhat technical geometrical condition. In the pure diffusion case, the cell centered finite volume method has also been analyzed with finite element tools: [1], [4], [2] or Petrov-Galerkin tools [10]. The convergence analysis has also been performed in some cases of nonlinear convection-diffusion problems; see [14] with a combined finite element-finite volume method, [13] and [12] with a pure finite volume scheme.

Since the approximate solution constructed with a classical cell-centered finite volume scheme is piecewise constant, an approximation of the gradient of the solution may be seen to be more complex than with a finite element method. Indeed, the convergence of a reconstructed gradient has been shown in [9], for certain quadrangular meshes using a nine point scheme in two space dimensions. It has also been shown on certain meshes by rewriting the finite volume scheme as a finite element scheme [1], [39] or a Petrov-Galerkin scheme [10]. Here we show that one may construct an approximate gradient on all "admissible" meshes by using some mesh functions which are very close to those used in mixed finite element theory (see e.g. [31]); these include triangular meshes (with a four-points scheme), rectangular meshes in two space dimensions, (with a five-points scheme) and Voronoï meshes in all space dimensions. For the sake of clarity, we consider here the Laplace equation:

$$-\Delta u(x) = f(x), \quad \text{for a.e. } x \in \Omega, \tag{1}$$

with a Dirichlet boundary condition:

$$u(x) = 0, \quad \text{for a.e. } x \in \partial\Omega, \tag{2}$$

where

Assumption 1

1. *Ω is an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3,*

2. *$f \in L^2(\Omega)$.*

Here, and in the sequel, "polygonal" is used for both $d = 2$ and $d = 3$ (meaning polyhedral in the latter case). Note also that "a.e. on $\partial\Omega$" is a.e. for the $d-1$-dimensional Lebesgue measure on $\partial\Omega$.
Under Assumption 1, by the Lax-Milgram theorem, there exists a unique variational solution $u \in H_0^1(\Omega)$ of Problem (1), (2), which satisfies

$$\int_\Omega \nabla u(x) \cdot \nabla \psi(x) dx = \int_\Omega f(x)\psi(x)dx, \ \forall \psi \in H_0^1(\Omega). \tag{3}$$

1.1 Definition of the finite volume scheme

Let us first introduce admissible meshes in order to define the finite volume scheme; an example of such a mesh is depicted in Figure 1.

Definition 1 (Admissible meshes) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$, or 3. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are open polygonal convex subsets of Ω (with positive measure), a family of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, denoted by $\mathcal{E}$ (these are the edges (2D) or sides (3D) of the control volumes), with strictly positive $(d-1)$-dimensional measure, and a family of points of Ω denoted by $\mathcal{P}$ satisfying the following properties (in fact, we shall denote, somewhat incorrectly, by $\mathcal{T}$ the family of control volumes):*

(i) The closure of the union of all the control volumes is $\overline{\Omega}$;

(ii) For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \overline{K} \setminus K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. Let $\mathcal{E} = \cup_{K\in\mathcal{T}}\mathcal{E}_K$.

(iii) For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$, either the $(d-1)$-dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is 0 or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, which will then be denoted by $K|L$.

(iv) The family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ is such that $x_K \in K$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that the straight line $\mathcal{D}_{K,L}$ going through x_K and x_L is orthogonal to $K|L$.

(v) For any $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega$, let K be the control volume such that $\sigma \in \mathcal{E}_K$. If $x_K \notin \sigma$, let $\mathcal{D}_{K,\sigma}$ be the straight line going through x_K and orthogonal to σ, then the condition $\mathcal{D}_{K,\sigma} \cap \sigma \neq \emptyset$ is assumed; let $y_\sigma = \mathcal{D}_{K,\sigma} \cap \sigma$.

In the sequel, the following notations are used. The mesh size is defined by: $\text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\}$. *For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $\text{m}(K)$ is the d-dimensional Lebesgue measure of K (i.e. area if $d = 2$, volume if $d = 3$) and $\text{m}(\sigma)$ the $(d-1)$-dimensional measure of σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), that is $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we denote by $d_{K,\sigma}$ the distance from x_K to σ and set $\tau_{K,\sigma} = \frac{\text{m}(\sigma)}{d_{K,\sigma}}$.*

Remark 1
(i) The definition of y_σ for $\sigma \in \mathcal{E}_{\text{ext}}$ requires that $y_\sigma \in \sigma$. However, In many cases, this condition may be relaxed. The condition $x_K \in K$ may also be relaxed as described for instance in Example 1 below.
(ii) The condition $x_K \in K$ may in fact be relaxed so long as two center points x_K and x_L do not get "inverted", see the example of triangular meshes. We shall keep to the condition $x_K \in K$ here for the sake of simplicity. Note that in particular, if for a given mesh the two center points x_K and x_L of two neighbouring control volumes K, L happen to be equal and lying on the common edge, then these two control volumes just have to be collapsed into a new control volume M with $x_M = x_K = x_L$, and the edge $K|L$ removed from the set of edges. The new mesh thus obtained is admissible.

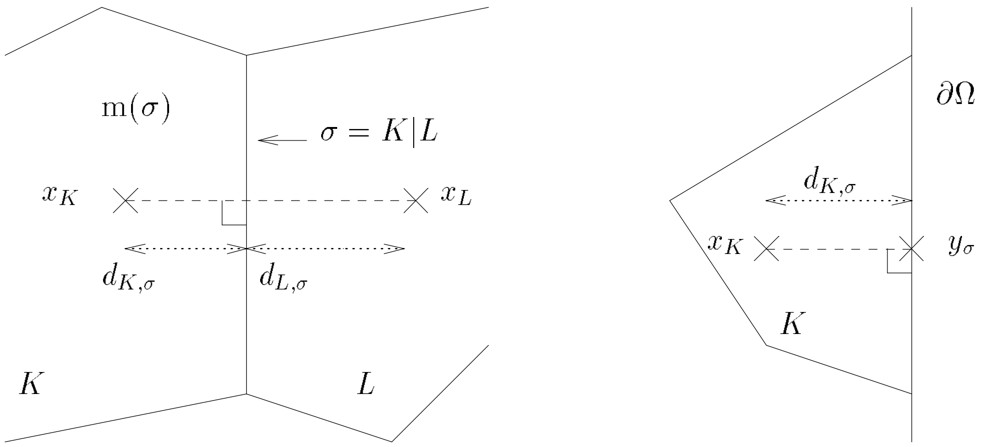

Figure 1: Admissible meshes

Example 1 (Triangular meshes) Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$. Let $\mathcal{T}$ be a family of open triangular disjoint subsets of Ω such that two triangles having a common edge have also two common vertices. Assume that all angles of the triangles are less than $\pi/2$. This last condition is sufficient for the orthogonal bisectors to intersect inside each triangle, thus naturally defining the points $x_K \in K$. One obtains an admissible mesh. In the case of an elliptic operator, the finite volume scheme defined on such a grid using differential quotients for the approximation of the normal flux yields a 4-point scheme [19]. This scheme does not lead to a finite difference scheme consistent with the continuous diffusion operator (using a Taylor expansion). The consistency is only verified for the approximation of the fluxes, but this, together with the conservativity of the scheme yields the convergence of the scheme, as it is proved below (and in other recent papers, see e.g. [11], [13]).
Note that the condition that all angles of the triangles are less than $\pi/2$ (which yields $x_K \in K$) may be relaxed (at least for the triangles the closure of which are in Ω) to the so called "strict Delaunay condition" which is that the closure of the circumscribed circle to each triangle of the mesh does not contain any other triangle of the mesh. For such a mesh, the point x_K (which is the intersection of the orthogonal bisectors of the edges of K) is not always in K, but the scheme defined below still behaves well since it yields a consistent approximation of the diffusion fluxes and the discrete maximum principle holds (see e.g. [19]).

Example 2 (Voronoï meshes)
Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. An admissible finite volume mesh can be built by using the so called "Voronoï" technique. Let $\mathcal{P}$ be a family of points of $\overline{\Omega}$. For example, this family may be chosen as $\mathcal{P} = \{(k_1 h, \ldots, k_d h),\ k_1, \ldots k_d \in \mathbb{Z}\} \cap \Omega$, for a given $h > 0$. The control volumes of the Voronoï mesh are defined with respect to each point x of $\mathcal{P}$ by

$$K_x = \{y \in \Omega, |x - y| < |z - y|,\ \forall z \in \mathcal{P},\ z \neq x\}.$$

Recall that $|x - y|$ denotes the euclidean distance between x and y.
An advantage of the Voronoï method is that it leads to meshes on non polygonal domains Ω.

Under Assumption 1, let $\mathcal{T}$ be an admissible mesh in the sense of Definition 1 and let $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ be defined by:

$$f_K = \frac{1}{\mathrm{m}(K)} \int_K f(x)dx,\ \forall K \in \mathcal{T}, \tag{4}$$

$$-\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}(u_\sigma-u_K)=\mathrm{m}(K)f_K,\ \forall K\in\mathcal{T},\tag{5}$$

$$\begin{cases}\tau_{K,\sigma}(u_\sigma-u_K)+\tau_{L,\sigma}(u_\sigma-u_L)=0,\ \forall\sigma\in\mathcal{E}_{\rm int},\sigma=K|L,\\ u_\sigma=0,\ \forall\sigma\in\mathcal{E}_{\rm ext}.\end{cases}\tag{6}$$

The existence and uniqueness of the solution $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ to (4)-(6) are proved in [11] or [13]; they result from a discrete Poincaré inequality and a discrete H^1_0 estimate, which we now recall for the sake of completeness (see [11] or [13] for the complete proofs). Let us first introduce two equivalent forms of the discrete H^1_0 norm which we shall use here.

Definition 2 (Discrete H^1_0 norm) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d=2$ or 3, and $\mathcal{T}$ an admissible finite volume mesh in the sense of Definition 1. Let $X(\mathcal{T})$ denote the set of piecewise constant functions on each control volume of the mesh, and let u_K denote the value of $u\in X(\mathcal{T})$ on the cell K.*
For $u\in X(\mathcal{T})$, the discrete H^1_0 norm of s is defined by:

$$\|u\|_{1,\mathcal{T}}=\Big(\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}}\frac{\mathrm{m}(\sigma)}{d_{K,\sigma}+d_{L,\sigma}}(u_L-u_K)^2+\sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma\subset\partial K\cap\partial\Omega}}\frac{\mathrm{m}(\sigma)}{d_{K,\sigma}}u_K^2\Big)^{\frac{1}{2}}.\tag{7}$$

where u_K denotes the value taken by u on the control volume K and the sets $\mathcal{E}$, $\mathcal{E}_{\rm int}$, $\mathcal{E}_{\rm ext}$ and $\mathcal{E}_K$ are defined in Definition 1.

Remark 2 *In the above definition, if $(u_\sigma)_{\sigma\in\mathcal{E}}$ are defined by (6), then one also has*

$$\frac{\mathrm{m}(\sigma)}{d_{K,\sigma}+d_{L,\sigma}}(u_K-u_L)=\tau_{K,\sigma}(u_K-u_\sigma)=-\tau_{L,\sigma}(u_L-u_\sigma)\ \textit{for any}\ \sigma=K|L\in\mathcal{E}_{\rm int}$$

so that one also has:

$$\|u\|_{1,\mathcal{T}}=\Big(\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}(u_\sigma-u_K)^2\Big)^{\frac{1}{2}}.\tag{8}$$

It is shown in [11] or [13] that for any mesh $\mathcal{T}$ if $u\in X(\mathcal{T})$ then the following discrete Poincaré inequality holds:

$$\|u\|_{L^2(\Omega)}\le \mathrm{diam}(\Omega)\|u\|_{1,\mathcal{T}}\tag{9}$$

In the same references, we also showed that if $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ satisfies (4)-(6) or more precisely an equivalent system where (5)-(6) is replaced by

$$\sum_{\sigma\in\mathcal{E}_K}F_{K,\sigma}=\mathrm{m}(K)f_K,\ \forall K\in\mathcal{T},\tag{10}$$

$$\begin{cases}F_{K,\sigma}=\dfrac{\mathrm{m}(\sigma)}{d_{K,\sigma}+d_{L,\sigma}}(u_K-u_L),\ \forall\sigma\in\mathcal{E}_{\rm int},\sigma=K|L,\\ F_{K,\sigma}=\dfrac{\mathrm{m}(\sigma)}{d_{K,\sigma}}u_K,\ \forall\sigma\in\mathcal{E}_{\rm ext},\sigma\in\mathcal{E}_K,\end{cases}\tag{11}$$

then the discrete $H^1_0(\Omega)$ estimate holds:

$$\|u_\mathcal{T}\|^2_{1,\mathcal{T}}\le \mathrm{diam}(\Omega)^2\|f\|^2_{L^2(\Omega)},\tag{12}$$

where $u_\mathcal{T}$ is defined by $u_\mathcal{T}(x)=u_K$ for all $x\in K$ where $(u_K)_{K\in\mathcal{T}}$ is the assumed solution of (11); hence by Remark 2, if $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ denotes the solution to the finite volume scheme (4)-(6), one has:

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}(u_\sigma-u_K)^2\le \mathrm{diam}(\Omega)^2\|f\|^2_{L^2(\Omega)}.\tag{13}$$

Note that in the above form of the scheme, $F_{K,\sigma}$ represents the numerical approximation of the flux through an edge σ outward to K, and that for $\sigma = K|L \in \mathcal{E}_{\rm int}$, one has: $F_{K,\sigma} = \frac{{\rm m}(\sigma)}{d_{K,\sigma}+d_{L,\sigma}}(u_K - u_L) = \tau_{K,\sigma}(u_K - u_\sigma)$.
These two estimates yield the existence of a unique solution. For a given admissible mesh $\mathcal{T}$, we may therefore define the finite volume approximation of the exact solution u by

$$u_{\mathcal{T}} = \sum_{K\in\mathcal{T}} u_K 1_K, \tag{14}$$

where 1_K is the characteristic function of K, that is $1_K(x) = 1$ if $x \in K$, 0 otherwise.
The convergence of the finite volume solution was already proven in [13] or [11]. The aim here is to construct an approximate gradient and prove its convergence.

1.2 Definition of an approximate gradient

In order to define an approximate gradient of u, we introduce, for $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ the variational solution $\phi_{K,\sigma} \in H^1(K)$ of the following Neumann problem.

$$\begin{array}{l} \Delta\phi_{K,\sigma}(x) = \frac{{\rm m}(\sigma)}{{\rm m}(K)}, \quad \text{for a.e. } x \in K, \\ \int_K \phi_{K,\sigma}(x)dx = 0, \\ \nabla\phi_{K,\sigma}(y)\cdot \mathbf{n}_{K,\sigma} = 1, \text{ for a.e. } y \in \sigma, \\ \nabla\phi_{K,\sigma}(y)\cdot \mathbf{n}_{K,\tilde\sigma} = 0, \text{ for a.e. } y \in \tilde\sigma, \tilde\sigma \in \mathcal{E}_K, \tilde\sigma \neq \sigma, \end{array} \tag{15}$$

where $\mathbf{n}_{K,\sigma}$ denotes the unit normal vector to σ outward to K. The function $\phi_{K,\sigma} \in H^1(K)$ is therefore the unique solution such that $\int_K \phi_{K,\sigma}(x)dx = 0$ of the variational formulation

$$\int_K \nabla\phi_{K,\sigma}(x)\nabla\psi(x)dx = -\frac{{\rm m}(\sigma)}{{\rm m}(K)}\int_K \psi(x)dx + \int_\sigma \psi(y)d\gamma(y), \quad \forall\psi \in H^1(K), \tag{16}$$

where $d\gamma$ denotes the integration with respect to the $d-1$ Lebesgue measure on the boundary ∂K.
Let us mention that in the case of triangular or rectangular control volumes, the functions $\phi_{K,\sigma}$ are the usual basis functions for the flux in the classical mixed finite element approximation, see e.g. [31], the gradient of which may be written explicitly (note that the expression of the basis functions themselves is of no real interest):

- Let K be a **triangular** control volume, a be one of its vertices and σ be the side opposite this vertex, then
$$\nabla\phi_{K,\sigma}(x) = \frac{x-a}{d(a,\sigma)} \text{ for any } x \in K$$
where $d(a,\sigma)$ denotes the distance between a and σ.

- Let K be a **rectangular** control volume the sides of which may be assumed to be parallel to the axis, and let σ and σ' be two parallel edges of this rectangle, then
$$\nabla\phi_{K,\sigma}(x) = \frac{d(x,\sigma')}{d(\sigma,\sigma')}\mathbf{n}_{K,\sigma} \text{ for any } x \in K,$$
where $d(x,\sigma')$ (resp. $d(\sigma,\sigma')$) denotes the distance between x and σ' (resp. σ and σ') and $\mathbf{n}_{K,\sigma}$ denotes the unit normal vector to σ outward K.

Note that in the general case, $\nabla\phi_{K,\sigma}$ can only be approximated.

Next we define the function $U_K \in H^1(K)$ by

$$U_K = \sum_{\sigma\in\mathcal{E}_K} \frac{u_\sigma - u_K}{d_{K,\sigma}}\phi_{K,\sigma}, \tag{17}$$

and we define the approximate gradient $G_{\mathcal{T}} \in H_{div}(\Omega) = \{v \in (L^2(\Omega))^d; \mathrm{div} v \in L^2(\Omega)\}$ by

$$G_{\mathcal{T}}(x) = \nabla U_K(x), \forall x \in K, \forall K \in \mathcal{T}. \tag{18}$$

It is clear from the definition of $G_{\mathcal{T}}$ that

$$G_{\mathcal{T}}.\mathbf{n}_{K,\sigma}(x) = \frac{u_\sigma - u_K}{d_{K,\sigma}}, \text{ for any } x \in \sigma, \text{ for any } \sigma \in \mathcal{E}_K, \text{ for any } K \in \mathcal{T}.$$

(Recall that $\mathbf{n}_{K,\sigma}$ denotes the unit normal vector to σ outward K.) The result $G_{\mathcal{T}} \in H_{div}(\Omega)$ is then a consequence of (6) which expresses the principle of conservativity of the finite volume scheme. Indeed, $G_{\mathcal{T}}|_K \in H^1(K)$ for any $K \in \mathcal{T}$ and the normal flux of $G_{\mathcal{T}}$ through and edge $\sigma = K|L$ is continuous thanks to (6). Furthermore, the restriction $(\mathrm{div} G_{\mathcal{T}})|_K$ of $\mathrm{div} G_{\mathcal{T}}$ to any control volume K of the mesh is constant:

$$(\mathrm{div} G_{\mathcal{T}})|_K = \mathrm{div}\Big(\sum_{\sigma \in \mathcal{E}_K} \frac{u_\sigma - u_K}{d_{K,\sigma}} \nabla \phi_{K,\sigma}\Big) = \sum_{\sigma \in \mathcal{E}_K} \frac{u_\sigma - u_K}{d_{K,\sigma}} \frac{\mathrm{m}(\sigma)}{\mathrm{m}(K)} = f_K.$$

2 Convergence of the approximate gradient

2.1 The convergence result

Theorem 1 *Under Assumption 1, let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1). Let $(u_K)_{K \in \mathcal{T}}$, $(u_\sigma)_{\sigma \in \mathcal{E}}$ be the unique solution of the system given by equations (4)-(6) and $u_{\mathcal{T}}$ be the approximate finite volume solution to Problem (1), (2) as defined by (14), (4)-(6). Then $u_{\mathcal{T}} \to u$ and $\|u_{\mathcal{T}}\|_{1,\mathcal{T}} \to \|\nabla u\|_{(L^2(\Omega))^2}$ as $\mathrm{size}(\mathcal{T})$ tends to 0, where u is the unique variational solution $u \in H_0^1(\Omega)$ of Problem (1), (2).*

Furthermore, for a given admissible mesh $\mathcal{T}$, let $\zeta_{\mathcal{T}} \in \mathbb{R}_+^*$ and $M_{\mathcal{T}} \in \mathbb{N}^*$ be such that

$$\frac{\mathrm{diam}(K)}{d_{K,\sigma}} \geq \zeta_{\mathcal{T}}, \text{ and } \mathrm{card}(\mathcal{E}_K) \leq M_{\mathcal{T}}, \forall \sigma \in \mathcal{E}_K, \forall K \in \mathcal{T}.$$

Let $\zeta > 0$ and $M \in \mathbb{N}^*$ be given and let us consider admissible meshes such that $\zeta_{\mathcal{T}} \geq \zeta$ and $M_{\mathcal{T}} \leq M$. Let $G_{\mathcal{T}}$ be the approximate gradient which is defined by equations (16)-(18). Then $G_{\mathcal{T}}$ belongs to the Hilbert space $H_{\mathrm{div}}(\Omega) = \{v \in (L^2(\Omega))^d; \mathrm{div} v \in L^2(\Omega)\}$, equipped with the norm defined by $\|v\|^2_{H_{\mathrm{div}}(\Omega)} = \|v\|^2_{(L^2(\Omega))^d} + \|\mathrm{div} v\|^2_{(L^2(\Omega))}$, and $G_{\mathcal{T}}$ converges in $H_{\mathrm{div}}(\Omega)$ to the gradient of the unique variational solution $u \in H_0^1(\Omega)$ of Problem (1), (2) as $\mathrm{size}(\mathcal{T}) \to 0$.

Remark 3 *In the case of a triangular mesh, the condition that $\zeta_{\mathcal{T}} \geq \zeta$ is equivalent to an angle condition which is often encountered in the study of finite element schemes and which states that there must exist $\beta > 0$ such that any angle of any triangle of the mesh lies between 0 and $\frac{\pi}{2} - \beta$.*

Proof
Step 1 We first prove that $u_{\mathcal{T}} \to u$ and $\|u_{\mathcal{T}}\|_{1,\mathcal{T}} \to \|\nabla u\|_{L^2(\Omega)}$ as $\mathrm{size}(\mathcal{T})$ tends to 0. This result is already contained in [13] or [11], but we give here a slightly different proof which prepares the proof of convergence of the approximate gradient.

We use here the fact that the space $C_c^\infty(\Omega)$ of functions of class C^∞ with compact support in Ω is dense in the Hilbert space $H_0^1(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$. Let $K \in \mathcal{T}$, from the Stokes formula we have that

$$\int_K \Delta\varphi(x)dx = \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla\varphi(y).\mathbf{n}_{K,\sigma} d\gamma(y),$$

where $\mathbf{n}_{K,\sigma}$ denotes the unit normal vector to σ outward K. Hence if one approximates the norma derivative $\nabla\varphi(y).\mathbf{n}_{K,\sigma}$ by the consistent choice $\frac{1}{d_{K,\sigma}}(\varphi_\sigma - \varphi(x_K))$ where the values φ_σ are defined for all $\sigma \in \mathcal{E}$ by

$$
\begin{aligned}
&\tau_{K,\sigma}(\varphi_\sigma - \varphi(x_K)) + \tau_{L,\sigma}(\varphi_\sigma - \varphi(x_L)) = 0, \ \forall \sigma \in \mathcal{E}_{\rm int}, \sigma = K|L, \\
&\varphi_\sigma = 0, \ \forall \sigma \in \mathcal{E}_{\rm ext},
\end{aligned} \tag{19}
$$

one obtains

$$
\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(\varphi_\sigma - \varphi(x_K)) = \int_K \Delta\varphi(x)dx - \sum_{\sigma\in\mathcal{E}_K} {\rm m}(\sigma) R_{K,\sigma}, \tag{20}
$$

where for any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ the value $R_{K,\sigma}$ is the consistency error on the approximation of the normal derivative or φ through σ outward to K which is defined by

$$
R_{K,\sigma} = \frac{\varphi_\sigma - \varphi(x_K)}{d_{K,\sigma}} - \frac{1}{{\rm m}(\sigma)} \int_\sigma \nabla\varphi(y)\cdot \mathbf{n}_{K,\sigma} d\gamma(y).
$$

Note that thanks to (19), if $\sigma = K|L \in \mathcal{E}_{\rm int}$ then $R_{K,\sigma} + R_{L,\sigma} = 0$ and

$$
R_{K,\sigma} = \frac{\varphi(x_L) - \varphi(x_K)}{d_{K|L}} - \frac{1}{{\rm m}(\sigma)} \int_\sigma \nabla\varphi(y)\cdot \mathbf{n}_{K,\sigma} d\gamma(y).
$$

The regularity of φ is crucial here to control the consistency error $R_{K,\sigma}$. We first note that since φ has a compact support, if $\sigma \in \mathcal{E}_{\rm ext}$, one has $R_{K,\sigma} = 0$ for size($\mathcal{T}$) small enough; furthermore, since $\varphi \in C^\infty(\Omega)$, a straightforward Taylor expansion yields that there exists $C_\varphi \geq 0$ depending only on φ such that

$$
|R_{K,\sigma}| \leq C_\varphi {\rm size}(\mathcal{T}). \tag{21}
$$

We set, for all $K \in \mathcal{T}$, $e_K = u_K - \varphi(x_K)$ and for all $\sigma \in \mathcal{E}$, $e_\sigma = u_\sigma - \varphi_\sigma$. We may therefore write:

$$
\begin{aligned}
&\tau_{K,\sigma}(e_\sigma - e_K) + \tau_{L,\sigma}(e_\sigma - e_L) = 0, \ \forall \sigma \in \mathcal{E}_{\rm int}, \sigma = K|L, \\
&e_\sigma = 0, \ \forall \sigma \in \mathcal{E}_{\rm ext}.
\end{aligned}
$$

We then get, adding (5) and (20),

$$
-\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K) = {\rm m}(K) f_K + \int_K \Delta\varphi(x)dx - \sum_{\sigma\in\mathcal{E}_K} {\rm m}(\sigma) R_{K,\sigma}.
$$

Multiplying this last equation by e_K and summing on $K \in \mathcal{T}$ gives

$$
\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K)^2 = \sum_{K\in\mathcal{T}} \left({\rm m}(K) f_K + \int_K \Delta\varphi(x)dx\right) e_K + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} {\rm m}(\sigma) R_{K,\sigma}(e_K - e_\sigma).
$$

By Young's inequality, one has: $R_{K,\sigma}(e_K - e_\sigma) \leq \frac{1}{2} d_{K,\sigma} R_{K,\sigma}^2 + \frac{1}{2d_{K,\sigma}}(e_K - e_\sigma)^2$, so that

$$
\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K)^2 \leq \sum_{K\in\mathcal{T}} \left({\rm m}(K) f_K + \int_K \Delta\varphi(x)dx\right) e_K + \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} {\rm m}(\sigma) d_{K,\sigma} R_{K,\sigma}^2. \tag{22}
$$

Using a compactness result which was proven in [13] or [11], the family $\{u_\mathcal{T}$, where $\mathcal{T}$ is an admissible mesh$\}$ is relatively compact in L^2 and any convergent subsequence converges to a function $w \in H_0^1(\Omega)$ as size($\mathcal{T}$) tends to 0. Furthermore, the piecewise constant function $\varphi_\mathcal{T}$ defined by $\varphi_\mathcal{T}(x) = \varphi_K$ if $x \in K$ converges uniformly to φ as size($\mathcal{T}$) tends to 0.

Hence passing to the limit as size($\mathcal{T}$) $\to 0$, we get that $\sum_{K\in\mathcal{T}} \Big({\rm m}(K) f_K + \int_K \Delta\varphi(x)dx\Big) e_K$ converges to $L = \int_\Omega (f(x) + \Delta\varphi(x))(w(x) - \varphi(x))dx$. From (3), we get that $L = \int_\Omega (\nabla w(x) - \nabla\varphi(x))(\nabla u(x) - \nabla\varphi(x))dx$.

Noting that $\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d_{K,\sigma} = d\,\mathrm{m}(\Omega)$ and thanks to (21), the quantity $\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d_{K,\sigma} R_{K,\sigma}^2$ converges to 0 as size$(\mathcal{T}) \to 0$. Hence a passage to the limit as size$(\mathcal{T})$ tends to 0 in (22) and the discrete Poincaré inequality (9) yield:

$$\|w-\varphi\|_{L^2(\Omega)}^2 \leq 2(\mathrm{diam}(\Omega))^2 \|\nabla(u-\varphi)\|_{L^2(\Omega)} \|\nabla(w-\varphi)\|_{L^2(\Omega)}.$$

Letting φ tend to u in the H^1 norm yields that $w = u$. Hence the approximate finite volume solution $u_{\mathcal{T}}$ converges to u as size$(\mathcal{T})$ tends to 0.

From this convergence result and from (22) one obtains the existence of $F_1(\Omega, \varphi, \mathcal{T}) > 0$ which tends to 0 as size$(\mathcal{T}) \to 0$ such that

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K)^2 \leq 2\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + F_1(\Omega, \varphi, \mathcal{T}).$$

Using the regularity of φ, we have, for $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, and $y \in \sigma$,

$$\left(\frac{u_\sigma - u_K}{d_{K,\sigma}} - \nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma}\right)^2 \leq \frac{2}{{d_{K,\sigma}}^2}(e_\sigma - e_K)^2 + C_\varphi \mathrm{size}(\mathcal{T})^2.$$

Using the two previous inequalities, we get the existence of $F_2(\Omega, \varphi, \mathcal{T}) > 0$ which tends to 0 as size$(\mathcal{T}) \to 0$ such that

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} d_{K,\sigma}\int_\sigma \left(\frac{u_\sigma - u_K}{d_{K,\sigma}} - \nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma}\right)^2 d\gamma(y) \leq 4\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + F_2(\Omega, \varphi, \mathcal{T}). \quad (23)$$

Step 2

Let us now turn to the study of the convergence of the approximate gradient, assuming $\zeta_{\mathcal{T}} > \zeta$ and $M_{\mathcal{T}} \leq M$. Let $\varphi \in C_c^\infty(\Omega)$ and for all $K \in \mathcal{T}$ let

$$A_K = \int_K (G_{\mathcal{T}}(x) - \nabla\varphi(x))^2 dx.$$

Let us first rewrite A_K as

$$A_K = \int_K (\nabla U_K(x) - \nabla\varphi(x))^2 dx = \int_K (\nabla w_K(x))^2 dx$$

where w_K is a function from K to $\mathbb{R}$ defined by:

$$w_K(x) = \varphi(x) - \frac{1}{\mathrm{m}(K)}\int_K \varphi(y)dy - U_K(x) \text{ for all } x \in K.$$

(Recall that $U_K = \sum_{\sigma\in\mathcal{E}_K} \frac{u_\sigma - u_K}{d_{K,\sigma}}\phi_{K,\sigma}$.)

Integrating by parts, we get

$$A_K = \sum_{\sigma\in\mathcal{E}_K}\int_\sigma w_K(y)(\nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})d\gamma(y)$$
$$-\int_K \Delta\varphi(x) w_K(x)dx + \sum_{\sigma\in\mathcal{E}_K}\int_K \frac{\mathrm{m}(\sigma)}{\mathrm{m}(K)}\frac{u_\sigma - u_K}{d_{K,\sigma}}\int_K w_K(x)dx.$$

From the definition of w_K and $\phi_{K,\sigma}$ one has $\int_K w_K(x)dx = -\int_K U_K(x)dx = 0$ and therefore one obtains

$$A_K = \sum_{\sigma\in\mathcal{E}_K} \int_\sigma w_K(y)(\nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})d\gamma(y) - \int_K \Delta\varphi(x) w_K(x)dx.$$

Using Young's inequality, we get, for some non negative α and β which will be chosen later,

$$A_K \leq \frac{\alpha}{2}B_K + \frac{1}{2\alpha}C_K + \frac{\beta}{2}D_K + \frac{1}{2\beta}E_K, \tag{24}$$

where

$$B_K = \sum_{\sigma\in\mathcal{E}_K} \int_\sigma w_K^2(y)d\gamma(y),$$

$$C_K = \sum_{\sigma\in\mathcal{E}_K} \int_\sigma (\nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})^2 d\gamma(y),$$

$$D_K = \int_K w_K^2(x)dx,$$

and

$$E_K = \int_K \Big(\Delta\varphi(x)\Big)^2 dx.$$

Since w_K belongs to $H^1(K)$ and is such that $\int_K w_K(x)dx = 0$, one may use the local trace inequality (28) of Lemma 1 (which is stated and proven below) to state that there exists $C_{\zeta,d} \geq 0$ depending only on ζ and d such that

$$\int_\sigma w_K^2(y)d\gamma(y) \leq C_{\zeta,d}\mathrm{diam}(K) \int_K (\nabla w_K(x))^2 dx.$$

Therefore, since the number of edges of each control volume K is uniformly bounded by M, one has:

$$B_K \leq M C_{\zeta,d}\mathrm{diam}(K) A_K,$$

Using the same properties on w_K and the Poincaré-Wirtinger inequality (29) of Lemma 1, one has:

$$D_K \leq C_{\zeta,d}\mathrm{diam}(K)^2 A_K.$$

Hence, choosing $\alpha = \dfrac{1}{2MC_{\zeta,d}\mathrm{diam}(K)}$ and $\beta = \dfrac{1}{2C_{\zeta,d}\mathrm{diam}(K)^2}$ in (24) yields that

$$\frac{1}{2}A_K \leq MC_{\zeta,d}\mathrm{diam}(K)C_K + C_{\zeta,d}\mathrm{diam}(K)^2 E_K. \tag{25}$$

Using (23) and $d_{K,\sigma} \geq \zeta\mathrm{diam}(K)$, we have

$$\sum_{K\in\mathcal{T}} \mathrm{diam}(K)C_K \leq \frac{4}{\zeta}\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + \frac{1}{\zeta}F_2(\Omega,\varphi,\mathcal{T}). \tag{26}$$

Furthermore,

$$\sum_{K\in\mathcal{T}} \mathrm{diam}(K)^2 E_K \leq (\mathrm{size}(\mathcal{T}))^2 \sum_{K\in\mathcal{T}} \mathrm{m}(K)\|\Delta\varphi\|_\infty^2 \leq (\mathrm{size}(\mathcal{T}))^2 \mathrm{m}(\Omega)\|\Delta\varphi\|_\infty^2. \tag{27}$$

From (25), (26) and (27), we get the existence of $F_3(\Omega,\varphi,\zeta,M,\mathcal{T}) \geq 0$ which tends to 0 as $\mathrm{size}(\mathcal{T}) \to 0$ such that

$$\sum_{K\in\mathcal{T}} A_K = \int_\Omega (G_\mathcal{T}(x) - \nabla\varphi(x))^2 dx \leq \frac{8MC_{\zeta,d}}{\zeta}\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + F_3(\Omega,\varphi,\zeta,M,\mathcal{T}).$$

Since

$$(G_{\mathcal{T}}(x)-\nabla u(x))^2 \le 2(G_{\mathcal{T}}(x)-\nabla\varphi(x))^2+2(\nabla\varphi(x)-\nabla u(x))^2$$

setting $C_{M,\zeta,d}=\dfrac{16MC_{\zeta,d}}{\zeta}+2$, we get

$$\int_\Omega (G_{\mathcal{T}}(x)-\nabla u(x))^2dx \le C_{M,\zeta,d}\int_\Omega(\nabla u(x)-\nabla\varphi(x))^2dx+2F_3(\Omega,f,\varphi,\zeta,M,\mathcal{T}).$$

Let $\varepsilon>0$ and let us choose φ such that

$$\int_\Omega(\nabla u(x)-\nabla\varphi(x))^2dx\le\varepsilon.$$

We can now choose size($\mathcal{T}$) small enough such that $2F_3(\Omega,\varphi,\zeta,M,\mathcal{T})\le\varepsilon$. Then

$$\int_\Omega(G_{\mathcal{T}}(x)-\nabla u(x))^2dx\le(C_{M,\zeta,d}+1)\varepsilon,$$

which concludes the proof of the convergence of $G_{\mathcal{T}}$ to ∇u in $(L^2(\Omega))^2$.

Since $\mathrm{div}G_{\mathcal{T}}(x)=f_K$, for all $x\in K$, the convergence of $\mathrm{div}G_{\mathcal{T}}$ to Δu in $L^2(\Omega)$ is proved. This concludes the proof of the convergence of $G_{\mathcal{T}}$ to ∇u in $H_{\mathrm{div}}(\Omega)$. ■

Let us conclude this section by giving the trace result and Poincaré-Wirtinger inequality which we used in the above proof. In the lemma below, the only novelty with respect to the usual trace theorems and the usual Poincaré-Wirtinger inequality is that we give the dependency of the "constants" in the inequalities on the geometrical parameters defining the control volume K: the knowledge of this dependency is crucial when passing to the limit in the above proof of convergence and therefore we may not use the usual proof by contradiction to prove these inequalities.

Lemma 1 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. Let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) such that, for some $\zeta>0$, the inequality $d_{K,\sigma}\ge\zeta\mathrm{diam}(K)$ holds for all control volume $K\in\mathcal{T}$ and for all $\sigma\in\mathcal{E}_K$. Let $K\in\mathcal{T}$ be a given control volume and let $g\in H^1(K)$, such that $\int_K g(x)dx=0$. Let us denote the trace of g on ∂K by g. Let $\sigma\in\mathcal{E}_K$. Then there exists $C(d,\zeta)\in\mathbb{R}_+$, only depending on d and ζ, such that*

$$\int_\sigma g^2(y)d\gamma(y)\le C(d,\zeta)\mathrm{diam}(K)\int_K(\nabla g(x))^2dx, \tag{28}$$

and

$$\int_K g^2(x)dx\le C(d,\zeta)\mathrm{diam}(K)^2\int_K(\nabla g(x))^2dx, \tag{29}$$

Proof of Lemma 1

We assume the hypotheses of Lemma 1. Again, by a classical argument of density, one may assume that $g\in C^1(\overline{K},\mathbb{R})$. We have, for $y\in\sigma$ and $x\in K$,

$$g(y)=g(y)-g(x)+g(x).$$

We integrate the previous equation on $x\in K$, use that $\int_K g(x)dx=0$. We get

$$\mathrm{m}(K)g(y)=\int_K(g(y)-g(x))dx.$$

We take the square of the previous equation and apply the Cauchy-Schwarz inequality. We get

$$\mathrm{m}(K)^2 g^2(y) \leq \mathrm{m}(K) \int_K (g(y) - g(x))^2 dx. \tag{30}$$

An integration of the previous equation with respect to $y \in \sigma$ and the technical result which is stated in the appendix (Lemma 3) give

$$\mathrm{m}(K) \int_\sigma g^2(y) d\gamma(y) \leq F(d,\zeta) \mathrm{diam}(K)^{d+1} \int_K (\nabla g(x))^2 \, dx. \tag{31}$$

Since $\mathrm{m}(K) \geq \left(\frac{\zeta \mathrm{diam}(K)}{\sqrt{d}}\right)^d$, this last inequality yields (28).

We now prove (29). Let $y \in K$, $x \in K$. Using the convexity of K, we deduce from (30) that

$$\mathrm{m}(K) g^2(y) \leq \int_0^1 \int_K (\nabla g(x + \theta(y-x)) \cdot (y-x))^2 dx d\theta.$$

We now integrate w.r.t. y. We get

$$\mathrm{m}(K) \int_K g^2(y) dy \leq (\mathrm{diam}(K))^2 \int_0^1 \int_K \int_K (\nabla g(x + \theta(y-x)))^2 dx dy d\theta.$$

We make the change of variable $z = y - x$ and remark that $z \in R_d$ with $R_d = [-\mathrm{diam}(K), \mathrm{diam}(K)]^d$. We get

$$\mathrm{m}(K) \int_K g^2(y) dy \leq 2^d \mathrm{diam}(K)^{d+2} \int_K (\nabla g(x))^2 dx,$$

which gives (29). ∎

2.2 Error estimate on the approximate gradient

Theorem 2 *Under Assumption 1 let $\zeta > 0$ and $M > 0$ be given values and $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) such that the inequalities $d_{K,\sigma} \geq \zeta \mathrm{diam}(K)$ and $M \geq card(\mathcal{E}_K)$ hold for all control volume $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}_K$.*
Let $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ be the unique solution of the system given by equations (4)-(6). Let $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ be the unique solution of the system given by equations (4)-(6). Let $u_\mathcal{T} : \Omega \to \mathbb{R}$ be the approximate solution defined in Ω by $u_\mathcal{T}(x) = u_K$ for a.e. $x \in K$, for all $K \in \mathcal{T}$ and let $G_\mathcal{T}$ be defined by equations (16)-(18). Assume that the unique solution u of (3) belongs to $H^2(\Omega)$. For each control volume K, let $e_K = u(x_K) - u_K$ and let $e_\mathcal{T} \in X(\mathcal{T})$ be defined by $e_\mathcal{T}(x) = e_K$ for a.e. $x \in K$, for all $K \in \mathcal{T}$.
Then, there exists c, only depending on f, ζ and Ω, such that

$$||e_\mathcal{T}||_{1,\mathcal{T}} \leq c\,\mathrm{size}(\mathcal{T}), \tag{32}$$

and

$$||e_\mathcal{T}||_{L^2(\Omega)} \leq c\,\mathrm{size}(\mathcal{T}) \tag{33}$$

Furthermore, there exists $C > 0$ which only depends on Ω, f, ζ, M such that

$$||G_\mathcal{T} - \nabla u||_{L^2(\Omega)} \leq C\mathrm{size}(\mathcal{T}). \tag{34}$$

Proof The proof of (32) and (33) may be found in [11] or [17]. We shall give here the proof of (34). This proof strongly uses the hypothesis $d \leq 3$ and the assumption on $d_{K,\sigma}$, necessary to make use of the $H^2(\Omega)$ regularity of the solution (see [11]). Let us study the expression

$$A_K = \int_K (G_{\mathcal{T}}(x) - \nabla u(x))^2 dx.$$

We follow the same steps as in the proof of Theorem 1 with u instead of φ; indeed here u is regular enough to admit a normal derivative on the boundary. Hence we may write:

$$\frac{1}{2} A_K \leq M C_\zeta \mathrm{diam}(K) C_K + C_\zeta \mathrm{diam}(K)^2 \int_K f^2(x) dx. \tag{35}$$

where

$$C_K = \sum_{\sigma \in \mathcal{E}_K} \int_\sigma (\nabla u(y) \cdot \mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})^2 d\gamma(y) \tag{36}$$

Now summing (35) over $K \in \mathcal{T}$, using the fact that $\mathrm{diam}(K) \leq \mathrm{size}(\mathcal{T})$ and $\mathrm{diam}(K) \leq \frac{d_{K,\sigma}}{\zeta}$ one obtains:

$$\int_\Omega (G_{\mathcal{T}}(x) - \nabla u(x))^2 dx \leq \frac{2MC_\zeta}{\zeta} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} \int_\sigma (\nabla u(y) \cdot \mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})^2 d\gamma(y) + C_\zeta \mathrm{size}(\mathcal{T})^2 \|\Delta u\|^2_{L^2(\Omega)}. \tag{37}$$

In order to obtain a bound of the right hand side of (37), we write

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} \int_\sigma (\nabla u(y) \cdot \mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})^2 d\gamma(y) \leq 2(Y + Z), \tag{38}$$

with

$$Y = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} \int_\sigma (\nabla u(y) \cdot \mathbf{n}_{K,\sigma} - \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla u(y') \cdot \mathbf{n}_{K,\sigma} d\gamma(y'))^2 d\gamma(y)$$

and

$$Z = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} \mathrm{m}(\sigma) (\frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla u(y') \cdot \mathbf{n}_{K,\sigma} d\gamma(y')) - \frac{u_\sigma - u_K}{d_{K,\sigma}})^2 \tag{39}$$

By Lemma 2 there exists $c_{\zeta,d} \in \mathbb{R}_+$ only depending on ζ and d such that

$$Y \leq c_{\zeta,d} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} d_{K,\sigma} \mathrm{diam}(K) \|u\|^2_{H^2(K)} \leq M c_{\zeta,d} \mathrm{size}(\mathcal{T})^2 \|u\|^2_{H^2(\Omega)} \tag{40}$$

where M is the uniform upper bound of the number of edges of each control volume K.
Now by (32) (which was proven in [11] or [17]), one has:

$$\begin{aligned} &\sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}} \\ \sigma = K|L}} \mathrm{m}(\sigma) d_\sigma \Big(\frac{u_L - u_K}{d_\sigma} - \frac{u(x_L) - u(x_K)}{d_\sigma} \Big)^2 + \\ &\sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{ext}} \\ \sigma \subset \partial K \cap \partial \Omega}} \mathrm{m}(\sigma) d_\sigma \Big(-\frac{u_K}{d_\sigma} + \frac{u(x_K)}{d_\sigma} \Big)^2 \leq (\mathrm{csize}(\mathcal{T}))^2, \end{aligned} \tag{41}$$

where $K|L$ denotes the edge which is common to the control volumes K and L, $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L$ and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\mathrm{ext}}$.
Now if $R_{K,\sigma}$ denotes the consistency error on the approximation of $\frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla u(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x)$ which was proven in [11] or [17] to be of "order 1". for the finite volume scheme (4)- (6) if $u \in H^2(\Omega)$, one may write that there exists C_1 depending only on u, ζ and Ω such that :

$$\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} \mathrm{m}(\sigma)d_\sigma\Big(\frac{u(x_L)-u(x_K)}{d_\sigma}-\frac{1}{\mathrm{m}(\sigma)}\int_\sigma \nabla u(x)\cdot\mathbf{n}_{K,\sigma}d\gamma(x)\Big)^2+$$
$$\sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma\in\overline{K}\cap\partial\Omega}} \mathrm{m}(\sigma)d_\sigma\Big(\frac{-u(x_K)}{d_\sigma}-\frac{1}{\mathrm{m}(\sigma)}\int_\sigma \nabla u(x)\cdot\mathbf{n}_{K,\sigma}d\gamma(x)\Big)^2 = \tag{42}$$
$$\sum_{\sigma\in\mathcal{E}}\mathrm{m}(\sigma)d_\sigma R_\sigma^2\le d\,\mathrm{m}(\Omega)C_1^2(\mathrm{size}(\mathcal{T}))^2.$$

where for any $\sigma=K|L\in\mathcal{E}_{\rm int}$, $R_\sigma=|R_{K,\sigma}|(=|R_{L,\sigma}|)$ and for any $\sigma\in\mathcal{E}_{\rm ext}\cap\mathcal{E}_K$, $R_\sigma=|R_{K,\sigma}|$.
Reordering the summations in the expression of Z in (39) we have that:

$$Z=\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=K|L}} \mathrm{m}(\sigma)d_\sigma(\frac{1}{\mathrm{m}(\sigma)}\int_\sigma \nabla u(y')\cdot\mathbf{n}_{K,\sigma}d\gamma(y'))-\frac{u_L-u_K}{d_\sigma})^2+$$
$$\sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma\in\overline{K}\cap\partial\Omega}} \mathrm{m}(\sigma)d_\sigma\Big(\frac{-u_K}{d_\sigma}-\frac{1}{\mathrm{m}(\sigma)}\int_\sigma \nabla u(x)\cdot\mathbf{n}_{K,\sigma}d\gamma(x)\Big)^2 \tag{43}$$

Hence from (43), (41) and (42) one obtains that there exists C_2 depending only on u,ζ and Ω such that

$$Z\le C_2\mathrm{size}(\mathcal{T})^2 \tag{44}$$

Therefore, from (37), (38), (40) and (44), we may state that there exists $C\in\mathbb{R}_+$ depending only on M,ζ,f,Ω such that

$$\int_\Omega (G_{\mathcal{T}}(x)-\nabla u(x))^2dx\le C^2\mathrm{size}(\mathcal{T})^2.$$

This concludes the proof of Theorem (2). ■

We conclude this section by giving a trace result on the normal derivative on the boundary of a function which belongs to H^2.
Again, the only novelty with respect to the usual trace theorems on the normal derivative is that we give the dependency of the "constant" in the inequality on the geometrical parameters defining the control volume K.

Lemma 2 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. Let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) such that, for some $\zeta>0$, the inequality $d_{K,\sigma}\ge\zeta\mathrm{diam}(K)$ holds for all control volume $K\in\mathcal{T}$ and for all $\sigma\in\mathcal{E}_K$. Let $K\in\mathcal{T}$ be a given control volume and let $u\in H^2(K)$. Let $\sigma\in\mathcal{E}_K$. We set $g_{K,\sigma}=\frac{1}{\mathrm{m}(\sigma)}\int_\sigma\nabla u(y)\cdot\mathbf{n}_{K,\sigma}d\gamma(y)$. Then there exists $c(d,\zeta)\in\mathbb{R}_+$, only depending on d and ζ, such that*

$$\int_\sigma(\nabla u(y)\cdot\mathbf{n}_{K,\sigma}-g_{K,\sigma})^2d\gamma(y)\le c(d,\zeta)\mathrm{diam}(K)\|u\|^2_{H^2(K)}. \tag{45}$$

Proof of Lemma 2 We assume that $u\in C^2(\overline{K})$, which will be sufficient to conclude with a density argument. Again, without loss of generality, one assumes that $\sigma=\{0\}\times J_0$, with J_0 is a closed interval of $\mathbb{R}$ and $K\subset\mathbb{R}_+\times\mathbb{R}$. We set $g=D_1u$. Therefore $g\in C^1(\overline{K},\mathbb{R})$ and for all $y\in\sigma$, $g(y)=-\nabla u(y)\cdot\mathbf{n}_{K,\sigma}$, and $g_{K,\sigma}=-\frac{1}{\mathrm{m}(\sigma)}\int_\sigma g(y)d\gamma(y)$. Let $y\in\sigma$ and $x\in K$. From $g(y)=g(y)-g(x)+g(x)$, one obtains

$$-g_{K,\sigma}=\frac{1}{\mathrm{m}(\sigma)}\int_\sigma(g(y')-g(x))d\gamma(y')+g(x).$$

Thus we get

$$\left\{ \quad (g(y)+g_{K,\sigma})^2 \le 2(g(y)-g(x))^2 + \frac{2}{\mathrm{m}(\sigma)} \int_\sigma (g(y')-g(x))^2 d\gamma(y'). \right.$$

Integrating the previous equation with respect to y, we get

$$\int_\sigma (g(y)+g_{K,\sigma})^2 d\gamma(y) \le 4 \int_\sigma (g(y)-g(x))^2 d\gamma(y).$$

We integrate the previous equation on $x \in K$ and apply Lemma 3 (see Appendix). We get

$$\mathrm{m}(K) \int_\sigma (g(y)+g_{K,\sigma})^2 d\gamma(y) \le 4F(d,\zeta)\mathrm{diam}(K)^{d+1} \int_K (\nabla g(x))^2 \, dx,$$

which gives (45), using $\mathrm{m}(K) \ge \mathrm{m}(Q) = \left(\dfrac{\zeta \mathrm{diam}(K)}{\sqrt{d}}\right)^d$. ■

3 Numerical tests

The computation of the approximate gradient was implemented in a numerical code which uses the finite volume method with triangles in two dimensions. The first case is the following. The domain is defined by $\Omega = (0,1) \times (0,1)$, and the right hand side is defined by $f(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$, so that the exact solution is $u(x,y) = \sin(\pi x)\sin(\pi y)$. The mesh is initially built with coarse triangles each of which is refined uniformly, see Figure 2. The advantage of this procedure is that the parameter ζ is preserved independently of the size of the mesh. In the case of Figure 2 the value ζ is about 0.1. The approximate solution is represented on the right part of Figure 2, for a refinement involving ten divisions on each coarse triangle (the darkest points correspond to $u = 0$, the whitest ones to $u = 1$).

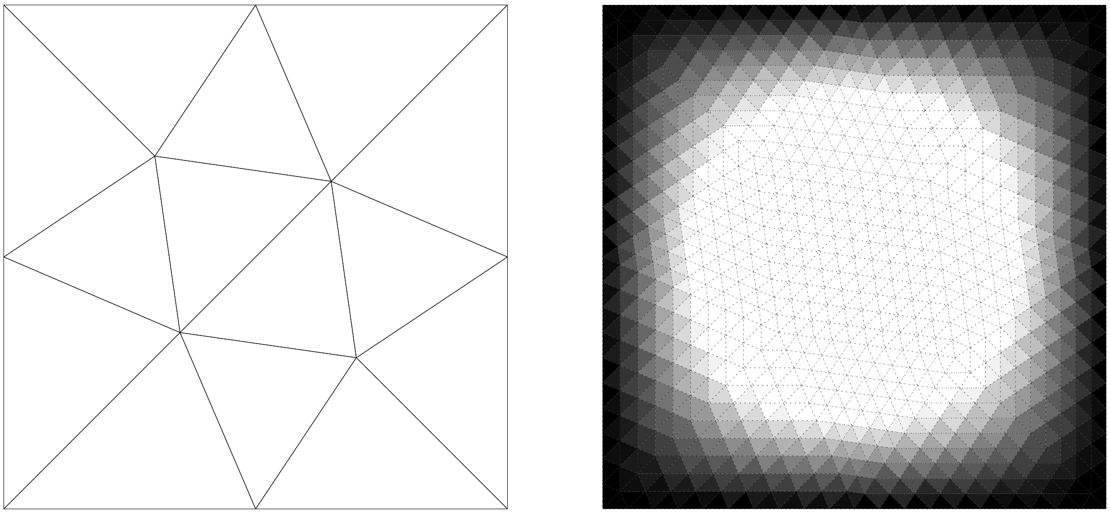

Figure 2: Meshes with 1 and 10 divisions

Figure 3 represents the norms of the error on the solution and its gradient as function of the size of the mesh. One observes the L^2-error of the approximate solution and its gradient to be of order 1, and the "discrete L^2 norm" of the error to be of order 2: the discrete L^2 norm is the L^2 norm of the function $u_\mathcal{T} - \overline{u}_\mathcal{T}$, where $u_\mathcal{T}$ is the approximate (piecewise constant) function given by the solution of the finite volume scheme and $\overline{u}_\mathcal{T}$ is the piecewise constant function defined by $\overline{u}_\mathcal{T}(x) = u(x_K)$ if $x \in K$, where x_K is the intersection of the orthogonal bisectors of K.

The order 2 obtained for the discrete norm of the error confirms the theoretical error estimate. Indeed, if the consistency error on the approximation of the flux is of order 2, then it is readily shown (see [11] or [17]) that the order of convergence of the discrete L^2 norm is 2. Now in the present case, we may note that the process of refinement leads to triangles which respect a similarity property. As a consequence, for each refined triangle of a given coarse triangle, the edge $\sigma = K|L$ between two refined triangles K and L is located on the orthogonal bisector of the line segment $x_K\ x_L$. Therefore, thanks also to the fact that the intersection between the edge $K|L$ and the line segment $x_K x_L$ is the midpoint of the edge $K|L$, a Taylor expansion shows that the consistency error of the approximation of the normal derivative to σ which is used in the finite volume scheme is of order 2 on such an edge. Now on the edges of the coarse triangles, the consistency error remains of order 1; however, there are only $O(1/h)$ edges located on the edges of the coarse triangles while there are $O(1/h^2)$ edges parting two similar refined triangles. Hence an easy adaptation of the error estimate which was given in [19] or [17] yields an error estimate of order 2.

Of course, this order 2 is never attained for the L^2 norm of the error since the approximate solution is piecewise constant.

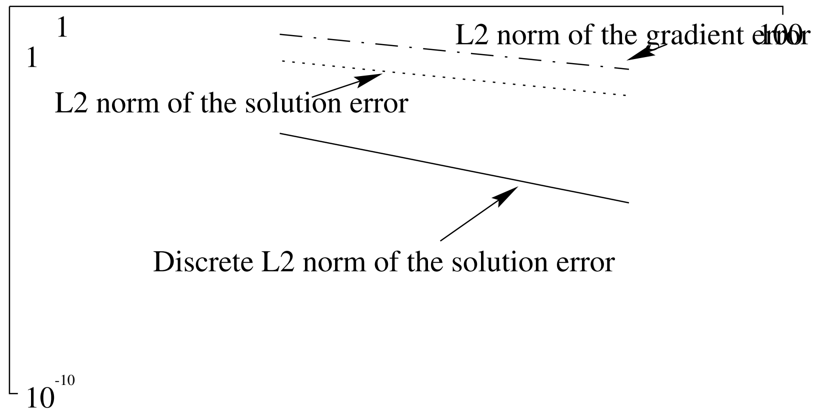

Figure 3: L^2 error as a function of the number of divisions (log scale)

The second case concerns a non convex domain for which the solution does not belong to $H^2(\Omega)$. The domain is defined by $\Omega = (0,1) \times (0,1) \setminus (\frac{1}{2}, 1) \times (\frac{1}{2}, 1)$, and a nonhomogeneous Dirichlet problem is solved, for which the solution if $u(x,y) = r^{2/3}\sin(2A/3)$, defining r and A as shown in figure 4. The right hand side is then $f(x,y) = 0$. The meshes are again refined as desired, starting with a coarse mesh presented on the left part of the figure 4 (in this case, the value ζ is about 0.04). An example of the obtained solution is presented on the right part of the figure 4 (the darkest points correspond to $u = 0$, the whitest ones to $u = 2^{-1/3}$).
(figure 4).
Figure 5 represents the norm of the error of the solution and its gradient as functions of the size of the mesh. One observes the L^2 norm of the error of the approximate solution to be of order 1 with respect to the mesh size, the L^2 norm of the error of the approximate gradient to be of order 2/3, and the discrete L^2 norm of the error (obtained by taking a constant value $u_K - u(x_K)$ in each control volume, as in the above example) to be of order 4/3. These numerical results agree with the theoretical results, since in the case of a solution which does not belong to H^2, we no longer have an order 2 convergence for the discrete L^2 norm of the error (nor an order 1 convergence for the continuous norm of the error). However it may be shown that the order of convergence depends on the regularity of the exact solution, see e.g. [11], [9] and [16]. Note that the scheme behaves well for both values of ζ which were considered here.

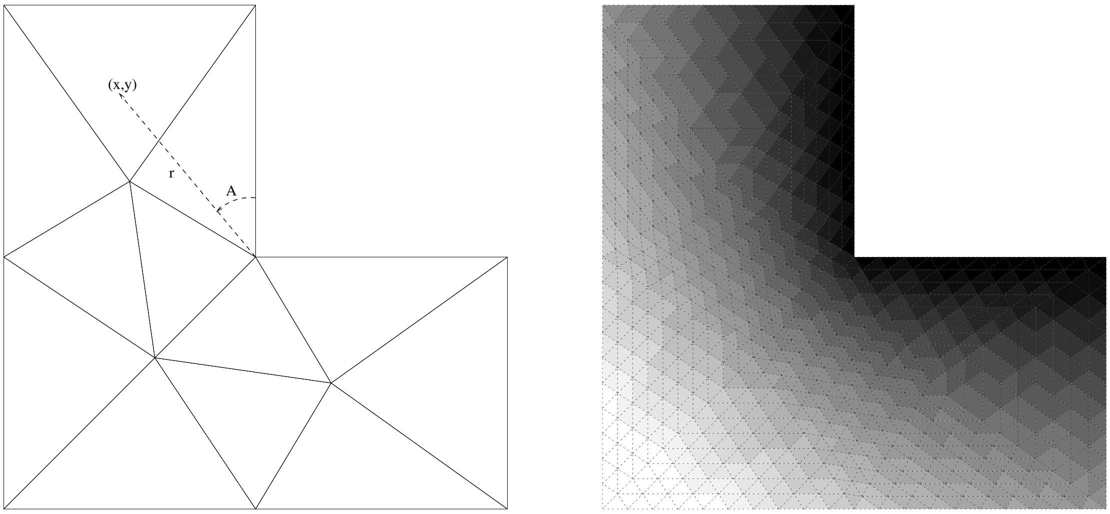

Figure 4: Coarse mesh and approximate result (with 10 divisions)

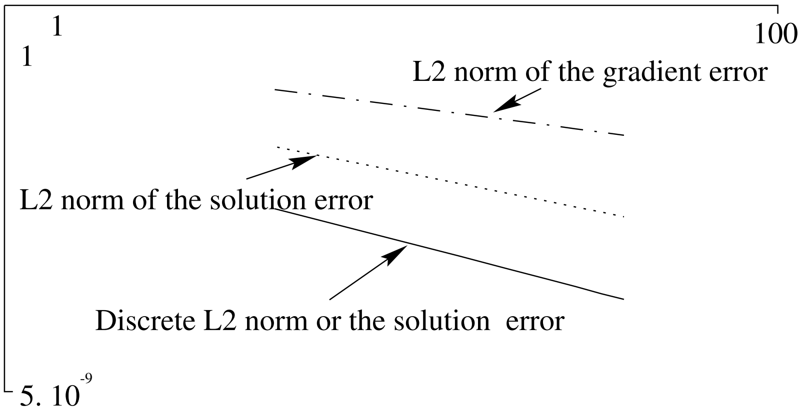

Figure 5: L^2 norm of the error as a function of the number of divisions (log scale)

4 Appendix: a technical lemma

We give here for the sake of complteness the proof of a technical result which was itself used in the proof of the "constructive" trace results and Poincaré-Wirtinger inequality of lemmas 1 and 2.

Lemma 3 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. Let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) such that, for some $\zeta > 0$, the inequality $d_{K,\sigma} \geq \zeta \mathrm{diam}(K)$ holds for all control volume $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}_K$. Let $K \in \mathcal{T}$ be a given control volume and let $g \in H^1(K)$. The trace of g on ∂K exists and is still denoted by g. Let $\sigma \in \mathcal{E}_K$. Then there exists $F(d,\zeta) \in \mathbb{R}_+$, only depending on ζ and d, such that*

$$\int_\sigma \int_K (g(y) - g(x))^2 dx d\gamma(y) \leq F(d,\zeta) \mathrm{diam}(K)^{d+1} \int_K (\nabla g(x))^2 dx, \tag{46}$$

Proof of Lemma 3

Let us assume the hypotheses of Lemma 3. By a classical argument of density, one may assume that $g \in C^1(\overline{K}, \mathbb{R})$. Without loss of generality, one assumes that $\sigma = \{0\} \times J_0$, with $J_0 \subset \mathbb{R}^{d-1}$. For $x \in [0, \mathrm{diam}(K)$, let $J_x = \{y \in \mathbb{R}^{d-1}; (x,y) \in \overline{K}\}$, then $\overline{K} = \{(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1},\ 0 \le x \le \mathrm{diam}(K),\ y \in J_x\}$. Thanks to the hypotheses of Lemma 3, the ball $B(x_K, \zeta\mathrm{diam}(K))$ is included in K. Hence the cube Q with center x_K and side $\zeta\mathrm{diam}(K)/\sqrt{d}$ is included in K. If we write $Q = [a,b] \times J$, with $J \subset \mathbb{R}^{d-1}$, we have

$$\begin{aligned} b &= a + \frac{\zeta\mathrm{diam}(K)}{\sqrt{d}} \\ a &\ge \zeta\mathrm{diam}(K)(1 - \frac{1}{2\sqrt{d}}) \\ \mathrm{m}(Q) &= \left(\frac{\zeta\mathrm{diam}(K)}{\sqrt{d}}\right)^d . \end{aligned} \tag{47}$$

We can now write, for all $z \in Q$,

$$\int_\sigma \int_K (g(y) - g(x))^2 dx d\gamma(y) \le 2 \int_\sigma \int_K (g(y) - g(z))^2 dx d\gamma(y) + 2 \int_\sigma \int_K (g(z) - g(x))^2 dx d\gamma(y). \tag{48}$$

An integration with respect to $z \in Q$ leads to

$$\mathrm{m}(Q) \int_\sigma \int_K (g(y) - g(x))^2 dx d\gamma(y) \le 2\mathrm{m}(K) A + 2\mathrm{m}(\sigma) B, \tag{49}$$

with

$$A = \int_\sigma \int_Q (g(y) - g(z))^2 dz d\gamma(y), \tag{50}$$

and

$$B = \int_Q \int_K (g(z) - g(x))^2 dx dz. \tag{51}$$

Let us first study A. By definition,

$$A = \int_{J_0} \int_J \int_a^b (g((0,y)) - g((x,z)))^2 \, dx dz dy, \tag{52}$$

and therefore,

$$A = \int_{J_0} \int_J \int_a^b \left(\int_0^1 \nabla g((\theta x, y + \theta(z-y))) \cdot (x, z-y) d\theta \right)^2 dx dz dy. \tag{53}$$

Using the Cauchy-Schwarz inequality, one gets

$$A \le \mathrm{diam}(K)^2 \int_{J_0} \int_J \int_a^b \int_0^1 (\nabla g((\theta x, y + \theta(z-y))))^2 \, d\theta dx dz dy. \tag{54}$$

which gives, remarking that $z - y \in R_{d-1}$ with $R_{d-1} = [-\mathrm{diam}(K), \mathrm{diam}(K)]^{d-1}$ and using Fubini's theorem that

$$A \le \mathrm{diam}(K)^2 \int_{R_{d-1}} \int_a^b \int_0^1 \int_{J_{\theta x}} (\nabla g((\theta x, y)))^2 \, dy d\theta dx dz. \tag{55}$$

We now change the variable θ into $t = \theta x$. This yields:

$$A \leq \operatorname{diam}(K)^2 2^{d-1} \operatorname{diam}(K)^{d-1} \int_a^b \int_0^x \int_{J_t} (\nabla g((t,y)))^2 \frac{1}{x} dy dt dx, \tag{56}$$

and therefore

$$A \leq \frac{2^{d-1}\operatorname{diam}(K)^{d+1}}{a} \int_a^b \int_K (\nabla g(y))^2 \, dy dx. \tag{57}$$

Using (47) which gives $\frac{b-a}{a} \leq \frac{2}{2\sqrt{d}-1}$, we get

$$A \leq F_1(d) \operatorname{diam}(K)^{d+1} \int_K (\nabla g(y))^2 \, dy, \tag{58}$$

with $F_1(d) = \frac{2^d}{2\sqrt{d}-1}$.
Let us now study B. We have

$$B \leq \operatorname{diam}(K)^2 \int_Q \int_K \int_0^1 (\nabla g(x + \theta(z-x)))^2 \, d\theta dx dz. \tag{59}$$

Remarking that $z - x \in R_d$ with $R_d = [-\operatorname{diam}(K), \operatorname{diam}(K)]^d$ and using Fubini's theorem, we get

$$B \leq \operatorname{diam}(K)^2 \int_{R_d} \int_K (\nabla g(x))^2 \, dx dz, \tag{60}$$

which gives

$$B \leq 2^d \operatorname{diam}(K)^{d+2} \int_K (\nabla g(x))^2 \, dx. \tag{61}$$

Thus, we get, using (49), (58) and (61), that

$$\mathrm{m}(Q) \int_\sigma \int_K (g(y) - g(x))^2 dx d\gamma(y) \leq (2F_1(d) + 2^{d+1}) \operatorname{diam}(K)^{2d+1} \int_K (\nabla g(x))^2 \, dx. \tag{62}$$

This concludes the proof of (46), using (47) for the value of $\mathrm{m}(Q)$.

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