Convergence of a finite volume scheme for nonlinear degenerate parabolic equations.

Robert Eymard[1], Thierry Gallouët[2], Raphaèle Herbin[3], Anthony Michel[4].

Abstract

One approximates the entropy weak solution u of a nonlinear parabolic degenerate equation $u_t + \mathrm{div}(\mathbf{q}f(u)) - \Delta\varphi(u) = 0$ by a piecewise constant function $u_{\mathcal{D}}$ using a discretization $\mathcal{D}$ in space and time and a finite volume scheme. The convergence of $u_{\mathcal{D}}$ to u is shown as the size of the space and time steps tend to zero. In a first step, estimates on $u_{\mathcal{D}}$ are used to prove the convergence, up to a subsequence, of $u_{\mathcal{D}}$ to a measure valued entropy solution (called here an entropy process solution). A result of uniqueness of the entropy process solution is proved, yielding the strong convergence of $u_{\mathcal{D}}$ to u. Some numerical results on a model equation are shown.

Key words Nonlinear degenerate hyperbolic-parabolic equations, finite volume methods.

AMS Subject Classification 65M12

1 The nonlinear parabolic degenerate problem.

Let Ω be a bounded open subset of $\mathbb{R}^d$, ($d = 1, 2$ or 3) with boundary $\partial\Omega$ and let $T \in \mathbb{R}_+^*$. One considers the following problem.

$$u_t(x,t) + \mathrm{div}\Big(\mathbf{q}\ f(u)\Big)(x,t) - \Delta\varphi(u)(x,t) = 0,\ \text{ for } (x,t) \in \Omega \times (0,T). \tag{1}$$

The initial condition is formulated as follows:

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega. \tag{2}$$

The boundary condition is the following nonhomogeneous Dirichlet condition:

$$u(x,t) = \bar{u}(x,t), \quad \text{for } (x,t) \in \partial\Omega \times (0,T). \tag{3}$$

This problem arises in different physical contexts. One of them is the problem of two phase flows in a porous medium, such as the air-water flow of hydrological aquifers. In this case, Problem (1)-(3) represents the conservation of the incompressible water phase, described by the water saturation u, submitted to convective flows (first order space terms $\mathbf{q}(x,t)\ f(u)$) and capillary effects ($\Delta\varphi(u)$). The expression $\mathbf{q}(x,t)\ f(u)$ for the convective term in (1) appears to be a particular case of the more general expression $F(u,x,t)$, but since it involves the same tools as the general framework, the results of this paper could be extended to some other problems.

One supposes that the following hypotheses, globally referred to in the following as hypotheses (H), are fulfilled.

Hypotheses (H)

(H1) Ω is polygonal (if $d = 1$, Ω is an interval, and if $d = 3$, Ω is a polyhedron),

[1]Ecole Nationale des Ponts et Chaussées, Marne-la-Vallée; eymard@enpc.fr
[2]Université de Provence, Marseille; gallouet@cmi.univ-mrs.fr
[3]Université de Provence, Marseille; herbin@cmi.univ-mrs.fr
[4]Université de Provence, Marseille; michel@cmi.univ-mrs.fr

(H2) $u_0 \in L^\infty(\Omega)$ *and* $\bar{u} \in L^\infty(\partial\Omega \times (0,T))$, $\bar{u}$ *being the trace of a function of* $H^1(\Omega \times (0,T)) \cap L^\infty(\Omega \times (0,T))$ *(also denoted* $\bar{u}$*); one sets* $U_I = \min(\text{infess } u_0, \text{infess } \bar{u})$ *and* $U_S = \max(\text{supess } u_0, \text{supess } \bar{u})$,

(H3) φ *is a nondecreasing Lipschitz-continuous function, with Lipschitz constant* Φ*, and one defines a function* ζ *such that* $\zeta' = \sqrt{\varphi'}$,

(H4) $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $f' \geq 0$*; one sets* $F = \max_{s \in [U_I, U_S]} f'(s)$,

(H5) $\mathbf{q}$ *is the restriction to* $\Omega \times (0,T)$ *of a function of* $\mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d)$,

(H6) $\text{div}(\mathbf{q}(x,t)) = 0$ *for all* $(x,t) \in \mathbb{R}^d \times (0,T)$*, where* $\text{div}(\mathbf{q}(x,t)) = \sum_{i=1}^{d} \frac{\partial q_i}{\partial x_i}(x,t)$*, (*$q_i$ *is the i-eth component of* q*) and*

$$\mathbf{q}(x,t).\mathbf{n}(x) = 0, \quad \text{for a.e. } (x,t) \in \partial\Omega \times (0,T), \tag{4}$$

(for $x \in \partial\Omega$, $\mathbf{n}(x)$ *denotes the outward unit normal to* Ω *at point* x*).*

Remark 1.1 *The function* f *is assumed to be non decreasing in* $(H3)$ *for the sake of simplicity. In fact, the convergence analysis which we present here would also hold without this monotonicity assumption using for instance a flux splitting scheme for the treatment of the convective term* $\mathbf{q}f(u)$.

Under hypotheses (H), Problem (1)-(3) does not have, in the general case, strong regular solutions. Because of the presence of a non-linear convection term, the expected solution is an entropy weak solution in the sense of Definition 1.1 given below.

Definition 1.1 (Entropy weak solution) *Under hypotheses (H), a function u is said to be an entropy weak solution to Problem (1)-(3) if it satisfies:*

$$u \in L^\infty(\Omega \times (0,T)), \tag{5}$$

$$\varphi(u) - \varphi(\bar{u}) \in L^2(0,T; H_0^1(\Omega)), \tag{6}$$

and u satisfies the following Kruzkov entropy inequalities: $\forall\ \psi \in \mathcal{D}^+(\Omega \times [0,T))$, $\forall \kappa \in \mathbb{R}$,

$$\int_{\Omega \times (0,T)} \left[\begin{array}{l} |u(x,t) - \kappa|\ \psi_t(x,t) + \\ (f(u(x,t)\top\kappa) - f(u(x,t)\bot\kappa))\ \mathbf{q}(x,t) \cdot \nabla\psi(x,t) \\ -\nabla|\varphi(u)(x,t) - \varphi(\kappa)| \cdot \nabla\psi(x,t) \end{array} \right] dxdt + \int_\Omega |u_0(x) - \kappa|\psi(x,0)dx \quad \geq 0, \tag{7}$$

where one denotes by $a\top b$ *the maximum value between two real values* a *and* b*, and by* $a\bot b$ *their minimum value and where* $\mathcal{D}^+(\Omega \times [0,T)) = \{\psi \in C_c^\infty(\Omega \times \mathbb{R}, \mathbb{R}_+), \psi(\cdot, T) = 0\}$.

This notion has been introduced by several authors ([5], [20]), who proved the existence of such a solution in bounded domains. In [20], the proof of existence uses strong BV estimates in order to derive estimates in time and space for the solution of the regularized problem obtained by adding a small diffusion term. In [5], the existence of a weak solution is proved using semigroup theory (see [2]), and the uniqueness of the entropy weak solution is proved using techniques which have been introduced by S.N. Krushkov and extended by J. Carrillo.
In the present study, thanks to condition (4), boundary conditions are entirely taken into account by (6) and do not appear in the entropy inequality (7). For studies of the continuous problem, one can refer to [20], which uses the

classical Bardos-Leroux-Nédélec formulation [1], or [5] in the case of a homogeneous Dirichlet boundary condition on $\partial\Omega$ without condition (4).

Let us mention some related work in the case of infinite domains ($\Omega = \mathbb{R}^d$): In [3], the authors prove the existence in the case $\Omega = \mathbb{R}^d$, regularizing the problem with the "general kinetic BGK" framework to yield estimates on translates of the approximate solutions. Continuity of the solution with respect to the data for a more general equation was studied by Cockburn and Gripenberg [8], and convergence of the discretization with an implicit finite volume scheme was recently studied by Ohlberger [21].

We shall deal here with the case of a bounded domain. The aim of the present work is then to prove the convergence of approximate solutions obtained using a finite volume method with general unstructured meshes towards the entropy weak solution of Problem (1)-(3) as the mesh size and time step tend to 0. We state this result in Theorem 2.1 in Section 2, after presenting the finite volume scheme. Then in Section 3, the existence and uniqueness of the solution to the nonlinear set of equations resulting from the finite volume scheme is proven, along with some properties of the discrete solutions. In Section 4 we show some compactness properties of the family of approximate solutions. We show in Section 5 that there exists some subsequence of sequences of approximate solutions which tends to a so-called "entropy process solution", and in Section 6 we prove the uniqueness of this entropy process solution, which allows us to conclude to the convergence of the scheme in Section 7. We finally give an example of numerical implementation in Section 8.

2 Finite volume approximation and main convergence result

Let us first define space and time discretizations of $\Omega \times (0,T)$.

Definition 2.1 (Admissible mesh of Ω) *An admissible mesh of Ω is given by a set $\mathcal{T}$ of open bounded polygonal convex subsets of Ω called control volumes, a family $\mathcal{E}$ of subsets of $\bar\Omega$ contained in hyperplanes of $\mathbb{R}^d$ with strictly positive measure, and a family of points $(x_K)_{K\in\mathcal{T}}$ (the "centers" of control volumes) satisfying the following properties:*

(i) The closure of the union of all control volumes is $\bar\Omega$.

(ii) For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar K \backslash K = \cup_{\sigma\in\mathcal{E}_K}\bar\sigma$. Furthermore, $\mathcal{E} = \cup_{K\in\mathcal{T}}\mathcal{E}_K$.

(iii) For any $(K,L) \in \mathcal{T}^2$ with $K \neq L$, either the "length" (i.e. the $(d-1)$ Lebesgue measure) of $\bar K \cap \bar L$ is 0 or $\bar K \cap \bar L = \bar\sigma$ for some $\sigma \in \mathcal{E}$. In the latter case, we shall write $\sigma = K|L$ and $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}, \exists (K,L) \in \mathcal{T}^2, \sigma = K|L\}$. For any $K \in \mathcal{T}$, we shall denote by $\mathcal{N}_K$ the set of boundary control volumes of K, i.e. $\mathcal{N}_K = \{L \in \mathcal{T}, K|L \in \mathcal{E}_K\}$.

(iv) The family of points $(x_K)_{K\in\mathcal{T}}$ is such that $x_K \in K$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) is orthogonal to σ.
For a control volume $K \in \mathcal{T}$, we will denote by $m(K)$ its measure and $\mathcal{E}_{ext,K}$ the subset of the edges of K included in the boundary $\partial\Omega$. If $L \in \mathcal{N}_K$, $m(K|L)$ will denote the measure of the edge between K and L, $\tau_{K|L}$ the "transmissibility" through $K|L$, defined by $\tau_{K|L} = \dfrac{m(K|L)}{d(x_k, x_L)}$. Similarly, if $\sigma \in \mathcal{E}_{ext,K}$, we will denote by $m(\sigma)$ its measure and τ_σ the "transmissibility" through σ, defined by $\tau_\sigma = \dfrac{m(\sigma)}{d(x_K, \sigma)}$. One denotes $\mathcal{E}_{ext} = \cup_{K\in\mathcal{T}}\mathcal{E}_{ext,K}$ and for $\sigma \in \mathcal{E}_{ext}$, one denotes by K_σ the control volume K such that $\sigma \in \mathcal{E}_{ext,K}$. The size of the mesh $\mathcal{T}$ is defined by

$$\mathrm{size}(\mathcal{T}) = \max_{K\in\mathcal{T}} \mathrm{diam}(K), \tag{8}$$

and a geometrical factor, linked with the regularity of the mesh, is defined by

$$\mathrm{reg}(\mathcal{T}) = \max_{K\in\mathcal{T}}(\mathrm{card}\mathcal{E}_K,\ \max_{\sigma\in\mathcal{E}_K} \frac{\mathrm{diam}(K)}{d(x_K,\sigma)}). \tag{9}$$

Remark 2.1 *Assumption (iv) in the previous definition is due to the presence of the second order term. Examples of meshes satisfying these assumptions are triangular meshes satisfying the acute angle condition (in fact this condition may be weakened to the Delaunay condition), rectangular meshes or Voronoï meshes, see [14] or [13] for more details.*

Definition 2.2 (Time discretization of $(0,T)$) *A time discretization of $(0,T)$ is given by an integer value N and by an increasing sequence of real values $(t^n)_{n\in[\![0,N+1]\!]}$ with $t^0 = 0$ and $t^{N+1} = T$. The time steps are then defined by $\delta t^n = t^{n+1} - t^n$, for $n \in [\![0,N]\!]$.*

Definition 2.3 (Space-time discretization of $\Omega\times(0,T)$) *A finite volume discretization $\mathcal{D}$ of $\Omega\times(0,T)$ is the family $\mathcal{D} = (\mathcal{T},\mathcal{E},(x_K)_{K\in\mathcal{T}},N,(t^n)_{n\in[\![0,N]\!]})$, where $\mathcal{T}$, $\mathcal{E}$, $(x_K)_{K\in\mathcal{T}}$ is an admissible mesh of Ω in the sense of Definition 2.1 and N, $(t^n)_{n\in[\![0,N+1]\!]}$ is a time discretization of $(0,T)$ in the sense of Definition 2.2. For a given mesh $\mathcal{D}$, one defines:*

$$\mathrm{size}(\mathcal{D}) = \max(\mathrm{size}(\mathcal{T}),(\delta t^n)_{n\in[\![0,N]\!]}), \qquad \textit{and}\ \mathrm{reg}(\mathcal{D}) = \mathrm{reg}(\mathcal{T}).$$

We may now define the finite volume discretization of Problem (1)-(3) . Let $\mathcal{D}$ be a finite volume discretization of $\Omega\times(0,T)$ in the sense of Definition 2.3. The initial condition is discretized by:

$$U_K^0 = \frac{1}{m(K)}\int_K u_0(x)dx, \quad \forall K\in\mathcal{T}. \tag{10}$$

In order to introduce the finite volume scheme, we need to define:

$$\bar{U}_\sigma^{n+1} = \frac{1}{\delta t^n\ m(\sigma)}\int_{t^n}^{t^{n+1}}\int_\sigma \bar{u}(x,t)d\gamma(x)dt, \qquad \forall\sigma\in\mathcal{E}_{ext}, \forall n\in[\![0,N]\!], \tag{11}$$

$$q_{K,L}^{n+1} = \frac{1}{\delta t^n}\int_{t^n}^{t^{n+1}}\int_{K|L} \mathbf{q}(x,t)\cdot\mathbf{n}_{K,L}d\gamma(x)dt, \qquad \forall K\in\mathcal{T}, \forall L\in\mathcal{N}_K, \forall n\in[\![0,N]\!], \tag{12}$$

where $\mathbf{n}_{K,L}$ is the normal unit vector to $K|L$ oriented from K to L.

An **implicit finite volume scheme** for the discretization of Problem (1)-(3) is given by the following set of nonlinear equations, the discrete unknowns of which are $U = (U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$:

$$\begin{aligned}\frac{U_K^{n+1}-U_K^n}{\delta t^n}\,m(K) \quad &+ \sum_{L\in\mathcal{N}_K}\Big[(q_{K,L}^{n+1})^+ f(U_K^{n+1}) - (q_{K,L}^{n+1})^- f(U_L^{n+1})\Big] \\ &- \sum_{L\in\mathcal{N}_K}\tau_{K|L}(\varphi(U_L^{n+1}) - \varphi(U_K^{n+1})) \\ &- \sum_{\sigma\in\mathcal{E}_{ext,K}}\tau_\sigma(\varphi(\bar{U}_\sigma^{n+1}) - \varphi(U_K^{n+1})) \qquad = 0,\end{aligned} \tag{13}$$

$\forall K\in\mathcal{T}, \forall n\in[\![0,N]\!]$, where $(q_{K,L}^{n+1})^+$ and $(q_{K,L}^{n+1})^-$ denote the positive and negative parts of $q_{K,L}^{n+1}$ (*i.e.* $(q_{K,L}^{n+1})^+ = \max(q_{K,L}^{n+1}, 0)$ and $(q_{K,L}^{n+1})^- = -\min(q_{K,L}^{n+1}, 0)$).

Remark 2.2 *The upwind discretization of the flux $\mathbf{q}f(u)$ in (13) uses the monotonicity of f and should be replaced in the general case by, for instance, a flux splitting scheme.*

Remark 2.3 *Thanks to Hypothesis (H6), one gets for all $K \in \mathcal{T}$ and $n \in [\![0, N]\!]$,* $\sum_{L\in\mathcal{N}_K} q_{K,L}^{n+1} = \sum_{L\in\mathcal{N}_K} [(q_{K,L}^{n+1})^+ - (q_{K,L}^{n+1})^-] = 0$. *This leads to*

$$\sum_{L\in\mathcal{N}_K} (q_{K,L}^{n+1})^+ f(U_K^{n+1}) - (q_{K,L}^{n+1})^- f(U_L^{n+1}) = -\sum_{L\in\mathcal{N}_K} (q_{K,L}^{n+1})^- (f(U_L^{n+1}) - f(U_K^{n+1})). \tag{14}$$

This property will be used in the following.

In Section (3) we shall prove the existence (Lemma 3.1) and the uniqueness (Lemma 3.4) of the solution $U = (U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ to (11)-(13) . We may then define the approximate solution to Problem (1)-(3) associated to an admissible discretization $\mathcal{D}$ of $\Omega \times (0, T)$ by:

Definition 2.4 *Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0, T)$ in the sense of Definition 2.3. The approximate solution of Problem (1)-(3) associated to the discretization $\mathcal{D}$ is defined almost everywhere in $\Omega \times (0, T)$ by:*

$$u_{\mathcal{D}}(x, t) = U_K^{n+1},\ \forall x \in K,\ \forall t \in (t^n, t^{n+1}),\ \forall K \in \mathcal{T},\ \forall n \in [\![0, N]\!], \tag{15}$$

where $(U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ is the unique solution to (11)-(13) .

Theorem 2.1 (Convergence of the approximate solution towards the entropy weak solution)
Let $\xi \in \mathbb{R}$, consider a family of admissible discretizations of $\Omega \times (0, T)$ in the sense of Definition 2.3 such that, for all $\mathcal{D}$ in the family, one has $\xi \geq \mathrm{reg}(\mathcal{D})$. For a given admissible discretization $\mathcal{D}$ of this family, let $u_{\mathcal{D}}$ denote the associated approximate solution as defined in Definition 2.4. Then:

$$u_{\mathcal{D}} \longrightarrow u \in L^p(\Omega \times (0, T)) \text{ as } \mathrm{size}(\mathcal{D}) \longrightarrow 0,\ \forall p \in [1, +\infty),$$

where u is the unique entropy weak solution to Problem (1)-(3) .

The proof of this convergence theorem will be concluded in Section 7 after we lay out the properties of the discrete solution (sections 3 and 4), its convergence towards an “entropy process solution” (Section 5) and a uniqueness result on this entropy process solution (Section 6).

Remark 2.4 *All the results of this paper also hold for explicit schemes, under a convenient CFL condition on the time step and mesh size.*

3 Existence, uniqueness and discrete properties

We state here the properties and estimates which are satisfied by the scheme which we introduced in the previous section and prove existence and uniqueness of the solution to this scheme. All the discrete properties which we address here correspond to natural estimates which are satisfied, at least formally, by regular continuous solutions. Let us first start by an L^∞ estimate:

Lemma 3.1 (L^∞ estimate) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 2.3 and let $(U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ be a solution of scheme (11)-(13) . Then*

$$U_I \leq U_K^{n+1} \leq U_S, \quad \forall K \in \mathcal{T},\ \forall n \in [\![0,N]\!].$$

Proof.
Let $U_M = \max_{L\in\mathcal{T},m\in[\![0,N]\!]} U_L^{m+1}$ and let $n \in [\![0,N]\!]$ and $K \in \mathcal{T}$ such that $U_K^{n+1} = U_M$. Equations (13) and (14) yield

$$\begin{aligned} U_M = U_K^{n+1} = U_K^n \quad &+ \quad \frac{\delta t^n}{m(K)} \sum_{L\in\mathcal{N}_K} (q_{K,L}^{n+1})^-(f(U_L^{n+1}) - f(U_K^{n+1})) \\ &+ \quad \frac{\delta t^n}{m(K)} \sum_{L\in\mathcal{N}_K} \tau_{K|L}(\varphi(U_L^{n+1}) - \varphi(U_K^{n+1})) \\ &+ \quad \frac{\delta t^n}{m(K)} \sum_{\sigma\in\mathcal{E}_{ext,K}} \tau_\sigma(\varphi(\bar{U}_\sigma^{n+1}) - \varphi(U_K^{n+1})). \end{aligned} \tag{16}$$

If one assumes that
$U_M \geq \max_{\sigma\in\mathcal{E}_{ext},m\in[\![0,N]\!]} \bar{U}_\sigma^{m+1}$, using the monotonicity of φ and f, one gets $U_M \leq U_K^n$, and therefore $U_M \leq U_K^0$.
This shows that

$$U_M \leq \max(\max_{\sigma\in\mathcal{E}_{ext},m\in[\![0,N]\!]} \bar{U}_\sigma^{m+1}, \max_{L\in\mathcal{T}} U_L^0),$$

yielding $U_M \leq U_S$. By the same method, one shows that $\min_{L\in\mathcal{T},m\in[\![0,N]\!]} U_L^{m+1} \geq U_I$. $\square$
A corollary of Lemma 3.1 is the existence of a solution $(U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ to (11)-(13) . (Uniqueness is proven in Lemma (3.4) below).

Corollary 3.1 (Existence of the solution to the scheme) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 2.3. Then there exists a solution $(U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ to the scheme (11)-(13) .*

The proof of this corollary is an adaptation of the technique which was used in [11] for the existence of the solution to an implicit finite volume scheme for the discretization of a pure hyperbolic equation.
The two following lemmas express the monotonicity of the scheme. Both are used to derive continuous entropy inequalities.

Lemma 3.2 (Regular convex discrete entropy inequalities) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 2.3 and let $U = (U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ be a solution to (11)-(13) .*

Then, for all $\eta \in \mathcal{C}^2(\mathbb{R},\mathbb{R})$, with $\eta'' \geq 0$, for all μ and ν in $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ with $\mu' = \eta'(\varphi)$ and $\nu' = \eta'(\varphi)f'$, for all $K \in \mathcal{T}$, and $n \in [\![0,N]\!]$, there exist $(U_{K,L}^{n+1})_{L\in\mathcal{N}_K}$ with $U_{K,L}^{n+1} \in (\min(U_K^{n+1}, U_L^{n+1}), \max(U_K^{n+1}, U_L^{n+1}))$ for all $L \in \mathcal{N}_K$ and $(U_{K,\sigma}^{n+1})_{\sigma\in\mathcal{E}_{ext,K}}$ with $U_{K,\sigma}^{n+1} \in (\min(U_K^{n+1}, \bar{U}_\sigma^{n+1}), \max(U_K^{n+1}, \bar{U}_\sigma^{n+1}))$ for all $\sigma \in \mathcal{E}_{ext,K}$ satisfying

$$
\begin{aligned}
\frac{\mu(U_K^{n+1})-\mu(U_K^n)}{\delta t^n}\,m(K) \quad &+ \sum_{L\in\mathcal{N}_K}(q_{K,L}^{n+1})^+\nu(U_K^{n+1})-(q_{L,K}^{n+1})^-\nu(U_L^{n+1})\\
&- \sum_{L\in\mathcal{N}_K}\tau_{K|L}(\eta(\varphi(U_L^{n+1}))-\eta(\varphi(U_K^{n+1})))\\
&- \sum_{\sigma\in\mathcal{E}_{ext,K}}\tau_\sigma(\eta(\varphi(\bar U_\sigma^{n+1}))-\eta(\varphi(U_K^{n+1})))\\
&+ \frac{1}{2}\sum_{L\in\mathcal{N}_K}\tau_{K|L}\eta''(\varphi(U_{K,L}^{n+1}))(\varphi(U_L^{n+1})-\varphi(U_K^{n+1}))^2\\
&+ \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{ext,K}}\tau_\sigma\eta''(\varphi(U_{K,\sigma}^{n+1}))(\varphi(\bar U_\sigma^{n+1})-\varphi(U_K^{n+1}))^2 \le 0
\end{aligned}
\tag{17}
$$

Proof.
In order to prove (17), one multiplies Equation (13) by $\eta'(\varphi(U_K^{n+1}))$.
The convexity of μ yields

$$
m(K)\,\frac{U_K^{n+1}-U_K^n}{\delta t^n}\,\eta'(\varphi(U_K^{n+1}))\ge m(K)\,\frac{\mu(U_K^{n+1})-\mu(U_K^{n+1})}{\delta t^n}. \tag{18}
$$

Using the convexity of ν and Remark 2.3, one gets

$$
\begin{aligned}
-\sum_{L\in\mathcal{N}_K}(q_{K,L}^{n+1})^-(f(U_L^{n+1})-f(U_K^{n+1}))\eta'(\varphi(U_K^{n+1})) \quad &\ge \quad -\sum_{L\in\mathcal{N}_K}(q_{K,L}^{n+1})^-(\nu(U_L^{n+1})-\nu(U_K^{n+1}))\\
&\ge \quad \sum_{L\in\mathcal{N}_K}(q_{K,L}^{n+1})^+\nu(U_K^{n+1})-(q_{K,L}^{n+1})^-\nu(U_L^{n+1})
\end{aligned}
$$

The Taylor-Lagrange formula gives, for all $L\in\mathcal{N}_K$ and all $\sigma\in\mathcal{E}_{ext,K}$, the existence of
$U_{K,L}^{n+1}\in(\min(U_K^{n+1},U_L^{n+1}),\max(U_K^{n+1},U_L^{n+1}))$ and $U_{K,\sigma}^{n+1}\in(\min(U_K^{n+1},\bar U_\sigma^{n+1}),\max(U_K^{n+1},\bar U_\sigma^{n+1}))$
such that

$$
-(\varphi(U_L^{n+1})-\varphi(U_K^{n+1}))\eta'(\varphi(U_K^{n+1}))=-(\eta(\varphi(U_L^{n+1}))-\eta(\varphi(U_K^{n+1})))+\frac{1}{2}\eta''(\varphi(U_{K,L}^{n+1}))(\varphi(U_L^{n+1})-\varphi(U_K^{n+1}))^2,
$$

$$
-(\varphi(\bar U_\sigma^{n+1})-\varphi(U_K^{n+1}))\eta'(\varphi(U_K^{n+1}))=-(\eta(\varphi(\bar U_\sigma^{n+1}))-\eta(\varphi(U_K^{n+1})))+\frac{1}{2}\eta''(\varphi(U_{K,\sigma}^{n+1}))(\varphi(\bar U_\sigma^{n+1})-\varphi(U_K^{n+1}))^2.
$$

Then collecting the previous inequalities gives Inequality (17) . □

Lemma 3.3 (Kruzkov's discrete entropy inequalities) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega\times(0,T)$ in the sense of Definition 2.3 and let $U=(U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ be a solution of the scheme (11)-(13) .*

Then, for all $\kappa\in\mathbb{R}$, $K\in\mathcal{T}$ and $n\in[\![0,N]\!]$,

$$
\begin{aligned}
\frac{|U_K^{n+1}-\kappa|-|U_K^n-\kappa|}{\delta t^n}\,m(K) \quad &+ \sum_{L\in\mathcal{N}_K}\left[\begin{array}{c}(q_{K,L}^{n+1})^+|f(U_K^{n+1})-f(\kappa)|\\ -(q_{K,L}^{n+1})^-|f(U_L^{n+1})-f(\kappa)|\end{array}\right]\\
&- \sum_{L\in\mathcal{N}_K}\tau_{K|L}(|\varphi(U_L^{n+1})-\varphi(\kappa)|-|\varphi(U_K^{n+1})-\varphi(\kappa)|)\\
&- \sum_{\sigma\in\mathcal{E}_{ext,K}}\tau_\sigma(|\varphi(\bar U_\sigma^{n+1})-\varphi(\kappa)|-|\varphi(U_K^{n+1})-\varphi(\kappa)|)\le 0
\end{aligned}
\tag{19}
$$

Proof. In order to prove Kruzkov's entropy inequalities, one follows [11]. Equation (13) is rewritten as

$$B(U_K^{n+1}, U_K^n, (U_L^{n+1})_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1})_{\sigma\in\mathcal{E}_{ext,K}}) = 0, \tag{20}$$

where B is nonincreasing with respect to each of its arguments except U_K^{n+1}. Consequently,

$$B(U_K^{n+1}, U_K^n \top \kappa, (U_L^{n+1}\top\kappa)_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1}\top\kappa)_{\sigma\in\mathcal{E}_{ext,K}}) \leq 0. \tag{21}$$

Since $B(\kappa, \kappa, (\kappa)_{L\in\mathcal{N}_K}, (\kappa)_{\sigma\in\mathcal{E}_{ext,K}}) = 0$, one gets

$$B(\kappa, U_K^n \top \kappa, (U_L^{n+1}\top\kappa)_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1}\top\kappa)_{\sigma\in\mathcal{E}_{ext,K}}) \leq 0. \tag{22}$$

Using the fact that $U_K^{n+1}\top\kappa = U_K^{n+1}$ or κ, (21) and (22) give

$$B(U_K^{n+1}\top\kappa, U_K^n \top \kappa, (U_L^{n+1}\top\kappa)_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1}\top\kappa)_{\sigma\in\mathcal{E}_{ext,K}}) \leq 0. \tag{23}$$

In the same way one obtains

$$B(U_K^{n+1}\bot\kappa, U_K^n \bot \kappa, (U_L^{n+1}\bot\kappa)_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1}\bot\kappa)_{\sigma\in\mathcal{E}_{ext,K}}) \geq 0. \tag{24}$$

Substracting (24) from (23) and remarking that for any nondecreasing function g and all real values a, b, $g(a\top b) - g(a\bot b) = |g(a) - g(b)|$ yields Inequality (19). $\square$

Let us now prove the uniqueness of the solution to (11)-(13) and define the approximate solution.

Lemma 3.4 (Uniqueness of the approximate solution) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 2.3. Then there exists a unique solution $(U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ to (11)-(13) .*

Proof.

The existence of $(U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ was established in Corollary 3.1. There only remains to prove the uniqueness of the solution. Let $(U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ and $(V_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ (setting $V_K^0 = U_K^0$) be two solutions to the scheme (11)-(13) . Following the proof of Lemma 3.3, one gets, for all $K \in \mathcal{T}$ and all $n \in [\![0, N]\!]$,

$$B(U_K^{n+1}\top V_K^{n+1}, U_K^n \top V_K^n, (U_L^{n+1}\top V_L^{n+1})_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1})_{\sigma\in\mathcal{E}_{ext,K}}) \leq 0, \tag{25}$$

and

$$B(U_K^{n+1}\bot V_K^{n+1}, U_K^n \bot V_K^n, (U_L^{n+1}\bot V_L^{n+1})_{L\in\mathcal{N}_K}, (\bar{U}_\sigma^{n+1})_{\sigma\in\mathcal{E}_{ext,K}}) \geq 0, \tag{26}$$

which by substraction give

$$\begin{aligned} \frac{|U_K^{n+1} - V_K^{n+1}| - |U_K^n - V_K^n|}{\delta t^n} m(K) \quad &+ \sum_{L\in\mathcal{N}_K} \begin{bmatrix} (q_{K,L}^{n+1})^+ |f(U_K^{n+1}) - f(V_K^{n+1})| \\ -(q_{K,L}^{n+1})^- |f(U_L^{n+1}) - f(V_L^{n+1})| \end{bmatrix} \\ &- \sum_{L\in\mathcal{N}_K} \tau_{K|L} \begin{bmatrix} |\varphi(U_L^{n+1}) - \varphi(V_L^{n+1})| - \\ |\varphi(U_K^{n+1}) - \varphi(V_K^{n+1})| \end{bmatrix} \\ &+ \sum_{\sigma\in\mathcal{E}_{ext,K}} \tau_\sigma |\varphi(U_K^{n+1}) - \varphi(V_K^{n+1})| \qquad \leq 0. \end{aligned} \tag{27}$$

For a given $n \in [\![0, N]\!]$, one sums (27) on $K \in \mathcal{T}$ and multiplies by δt^n. All the exchange terms between neighbouring control volume disappear, and because of the sign of the boundary terms, one gets

$$\sum_{K\in\mathcal{T}} |U_K^{n+1} - V_K^{n+1}|\ m(K) \le \sum_{K\in\mathcal{T}} |U_K^n - V_K^n|\ m(K). \tag{28}$$

Since $U_K^0 = V_K^0$, one concludes $\sum_{K\in\mathcal{T}} |U_K^{n+1} - V_K^{n+1}|\ m(K) = 0$, for all $n \in [\![0, N]\!]$, which concludes the proof of uniqueness. □

Let us now give two discrete estimates on the approximate solution $u_{\mathcal{D}}$ which will be crucial in the convergence analysis. The first estimate (29) is a discrete $L^2(0, T, H^1(\Omega))$ estimate on the function $\zeta(u_{\mathcal{D}})$ where $\zeta' = \sqrt{\varphi'}$. This estimate will yield some compactness on $\zeta(u_{\mathcal{D}})$.
The second estimate is the weak BV inequality (30) on $f(u_{\mathcal{D}})$. Such an inequality also holds for the continuous problem with an additional diffusion term $-\varepsilon\Delta f(u)$. This inequality does not give any compactness property (to our knowledge, no BV estimate is known in the case of unstructured meshes); however it it plays an essential role in the proof of convergence, where it is used to control the numerical diffusion introduced by the upstream weighting scheme (see Section 5 and references [7], [9], [11] and [6].

Proposition 3.1 (Discrete H^1 estimate and weak BV inequality) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega \times (0, T)$ in the sense of Definition 2.3. Let $\xi \in \mathbb{R}$ be such that $\xi \ge \mathrm{reg}(\mathcal{D})$; let $(U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ be the solution of the scheme (11)-(13) .*
Then there exists a real number $C > 0$, only depending on $\Omega, T, u_0, \bar{u}, f, \mathbf{q}, \varphi$ and ξ such that

$$\begin{aligned}(\mathcal{N}_{\mathcal{D}}(\zeta(u_{\mathcal{D}})))^2 &= \sum_{n=0}^{N} \delta t^n \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(\zeta(U_K^{n+1}) - \zeta(U_L^{n+1}))^2 \\ &+ \sum_{n=0}^{N} \delta t^n \sum_{\sigma\in\mathcal{E}_{ext}} \tau_\sigma(\zeta(\bar{U}_\sigma^{n+1}) - \zeta(U_{K_\sigma}^{n+1}))^2 \le C \end{aligned} \tag{29}$$

$$(\mathcal{B}_{\mathcal{D}}(f(u_{\mathcal{D}})))^2 = \sum_{n=0}^{N} \delta t^n \sum_{K|L\in\mathcal{E}_{int}} ((q_{K,L}^{n+1})^- + (q_{K,L}^{n+1})^+)(f(U_K^{n+1}) - f(U_L^{n+1}))^2 \le C \tag{30}$$

Proof. One first defines discrete values by averaging, in each control volume, the function $\bar{u}$, whose trace on $\partial\Omega$ defines the Dirichlet boundary condition. Note that this proof uses $\bar{u} \in H^1(\Omega\times(0, T))$ and not only $\bar{u} \in L^2(0, T; H^1(\Omega))$ and $\bar{u}_t \in L^2(0, T; H^{-1}(\Omega))$, since we use below the fact that $\bar{u}_t \in L^2(0, T; L^1(\Omega))$. Let

$$\bar{U}_K^0 = \frac{1}{m(K)} \int_K \bar{u}(x, 0)dx, \qquad \forall K \in \mathcal{T}, \tag{31}$$

$$\bar{U}_K^{n+1} = \frac{1}{\delta t^n\ m(K)} \int_{t^n}^{t^{n+1}} \int_K \bar{u}(x, t)dxdt, \qquad \forall K \in \mathcal{T}, \forall n \in [\![0, N]\!], \tag{32}$$

Setting $V = U - \bar{U}$, one multiplies (13) by $\delta t^n V_K^{n+1}$ and sums over $K \in \mathcal{T}$ and $n \in [\![0, N]\!]$. This yields $E1 + E2 + E3 = 0$ with

$$E1 = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)(U_K^{n+1}-U_K^n)V_K^{n+1}, \tag{33}$$

$$E2 = \sum_{n=0}^{N}\delta t^n \sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}((q_{K,L}^{n+1})^+ f(U_K^{n+1}) - (q_{K,L}^{n+1})^- f(U_L^{n+1}))V_K^{n+1}, \tag{34}$$

$$E3 = \sum_{n=0}^{N}\delta t^n \sum_{K\in\mathcal{T}}\big(\sum_{L\in\mathcal{N}_K}\tau_{K|L}(\varphi(U_L^{n+1})-\varphi(U_K^{n+1}))V_K^{n+1} + \sum_{\sigma\in\mathcal{E}_{ext,K}}\tau_\sigma(\varphi(\bar{U}_\sigma^{n+1})-\varphi(U_K^{n+1}))V_K^{n+1}\big). \tag{35}$$

Using $U = V + \bar{U}$ yields $E1 = E11 + E12$ with

$$E11 = \frac{1}{2}\sum_{K\in\mathcal{T}} m(K)((V_K^{N+1})^2-(V_K^0)^2) + \frac{1}{2}\sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)(V_K^{n+1}-V_K^n)^2 \tag{36}$$

$$E12 = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)(\bar{U}_K^{n+1}-\bar{U}_K^n)V_K^{n+1}. \tag{37}$$

Setting

$$A_{n,K} = \bar{U}_K^{n+1} - \frac{1}{m(K)}\int_K \bar{u}(x,t^n)dx \text{ and } B_{n,K} = \frac{1}{m(K)}\int_K \bar{u}(x,t^n) - \bar{U}_K^n,$$

one has

$$E12 = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)A_{n,K}V_K^{n+1} + \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} m(K)B_{n,K}V_K^{n+1}.$$

By a classical density argument one gets:

$$|A_{n,K}| \le \frac{1}{m(K)}\|\bar{u}_t\|_{L^1(K\times(t^n,t^{n+1}))},\ \forall n\in[\![0,N]\!],\ \forall K\in\mathcal{T}$$

and

$$|B_{n,K}| \le \frac{1}{m(K)}\|\bar{u}_t\|_{L^1(K\times(t^{n-1},t^n))},\ \forall n\in[\![1,N]\!],\ \forall K\in\mathcal{T}$$

(note that $B_{0,K} = 0$ for all $K\in\mathcal{T}$). Using these two inequalities and the L^∞ stability of the scheme (Lemma 3.1) yields:

$$|E12| \le 2\|\bar{u}_t\|_{L^1(\Omega\times(0,T))}(U_S - U_I).$$

Now remarking that

$$E11 \ge -\frac{1}{2}\sum_{K\in\mathcal{T}} m(K){V_K^0}^2 \ge -\frac{1}{2}\|u_0-\bar{u}(\cdot,0)\|^2_{L^2(\Omega)}$$

the previous inequality allows us to obtain the existence of $C1 > 0$, only depending on Ω, T, u_0 and $\bar{u}$, such that $E1 \geq C1$.

The term $E2$ can be decomposed in $E2 = E21 + E22$ with

$$\begin{aligned} E21 &= \sum_{n=0}^{N} \delta t^n \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_K} ((q_{K,L}^{n+1})^+ f(U_K^n) - (q_{K,L}^{n+1})^- f(U_L^{n+1})) U_K^{n+1}, \\ E22 &= -\sum_{n=0}^{N} \delta t^n \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_K} ((q_{K,L}^{n+1})^+ f(U_K^n) - (q_{K,L}^{n+1})^- f(U_L^{n+1})) \bar{U}_K^{n+1}, \end{aligned}$$

Using Remark 2.3, one gets

$$E21 = \sum_{n=0}^{N} \delta t^n \sum_{K \in \mathcal{T}} (q_{K,L}^{n+1})^- (f(U_K^{n+1}) - f(U_L^{n+1})) U_K^{n+1}. \tag{38}$$

Let $\tilde{g}$ be a primitive of f and $g(s) = sf(s) - \tilde{g}(s)$ for all real s. The following inequality holds for all pairs of real values (a, b) (see [13] and [6]).

$$g(b) - g(a) \leq b(f(b) - f(a)) - \frac{1}{2F}(f(b) - f(a))^2 \tag{39}$$

Using (39) for $(a, b) = (U_L^{n+1}, U_K^{n+1})$ and (38) yield

$$E21 \geq \sum_{n=0}^{N} \delta t^n \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_K} (q_{K,L}^{n+1})^- (g(U_K^{n+1}) - g(U_L^{n+1})) + \frac{1}{2F}(\mathcal{B}_{\mathcal{D}}(f(u_{\mathcal{D}})))^2.$$

Using Remark 2.3 with g instead of f gives

$$\sum_{n=0}^{N} \delta t^n \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_K} (q_{K,L}^{n+1})^- (g(U_K^{n+1}) - g(U_L^{n+1})) = 0, \tag{40}$$

and therefore

$$E21 \geq \frac{1}{2F}(\mathcal{B}_{\mathcal{D}}(f(u_{\mathcal{D}})))^2. \tag{41}$$

A discrete space integration by parts in $E22$ does not yield any boundary term since $\mathbf{q} \cdot \mathbf{n} = 0$ on $\partial\Omega$, and gives, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
E22 &= -\sum_{n=0}^{N}\delta t^n \sum_{K|L\in\mathcal{E}_{int}} ((q_{K,L}^{n+1})^+ f(U_K^{n+1}) - (q_{K,L}^{n+1})^- f(U_L^{n+1}))(\bar{U}_K^{n+1} - \bar{U}_L^{n+1}) \\
&\geq -\|\mathbf{q}\|_{L^\infty(\Omega\times(0,T))} \max_{s\in[U_I,U_S]} |f(s)| \sum_{n=0}^{N}\delta t^n \sum_{K|L\in\mathcal{E}_{int}} m(K|L)|\bar{U}_K^{n+1} - \bar{U}_L^{n+1}| \\
&\geq -\|\mathbf{q}\|_{L^\infty(\Omega\times(0,T))} \max_{s\in[U_I,U_S]} |f(s)| \mathcal{N}_{\mathcal{D}}(\bar{u}_{\mathcal{D}})[\sum_{n=0}^{N}\delta t^n \sum_{K|L\in\mathcal{E}_{int}} m(K|L) d(x_K,x_L)]^{\frac{1}{2}} \\
&\geq -\mathcal{N}_{\mathcal{D}}(\bar{u}_{\mathcal{D}})\|\mathbf{q}\|_{L^\infty(\Omega\times(0,T))} \max_{s\in[U_I,U_S]} |f(s)|(d\ m(\Omega)\ T)^{\frac{1}{2}}.
\end{aligned}$$

The following estimate for $\mathcal{N}_{\mathcal{D}}(\bar{u}_{\mathcal{D}})$ holds:

$$\mathcal{N}_{\mathcal{D}}(\bar{u}_{\mathcal{D}}) \leq F(\xi)\|\bar{u}\|_{L^2(0,T,H^1(\Omega))}, \tag{42}$$

where $F \geq 0$ only depends on ξ (Inequality (42) is proved in [14], with a different definition of the regularity factor of the mesh), leading to a lower bound of $E22$ denoted by $C22$, only depending on $\Omega, T, u_0, \bar{u}, f, \mathbf{q}$ and ξ.

There only remains to deal with $E3$. A discrete space integration by parts, using the fact that $V_\sigma^{n+1} = 0, \forall \sigma \in \mathcal{E}_{ext}, \forall n \in [\![0, N]\!]$, yields

$$\begin{aligned}
E3 = \sum_{n=0}^{N}\delta t^n (& \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(\varphi(U_L^{n+1}) - \varphi(U_K^{n+1}))(V_L^{n+1} - V_K^{n+1}) \\
&+ \sum_{\sigma\in\mathcal{E}_{ext}} \tau_\sigma(\varphi(\bar{U}_\sigma^{n+1}) - \varphi(U_{K_\sigma}^{n+1}))(V_\sigma^{n+1} - V_{K_\sigma}^{n+1}).
\end{aligned} \tag{43}$$

Writing again V into $U - \bar{U}$ leads to $E3 = E31 + E32$ where

$$\begin{aligned}
E31 = \sum_{n=0}^{N}\delta t^n (& \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(\varphi(U_L^{n+1}) - \varphi(U_K^{n+1}))(U_L^{n+1} - U_K^{n+1}) \\
&+ \sum_{\sigma\in\mathcal{E}_{ext}} \tau_\sigma(\varphi(\bar{U}_\sigma^{n+1}) - \varphi(U_{K_\sigma}^{n+1}))(\bar{U}_\sigma^{n+1} - U_{K_\sigma}^{n+1})
\end{aligned} \tag{44}$$

$$\begin{aligned}
E32 = -\sum_{n=0}^{N}\delta t^n (& \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(\varphi(U_L^{n+1}) - \varphi(U_K^{n+1}))(\bar{U}_L^{n+1} - \bar{U}_K^{n+1}) \\
&+ \sum_{\sigma\in\mathcal{E}_{ext}} \tau_\sigma(\varphi(\bar{U}_\sigma^{n+1}) - \varphi(U_{K_\sigma}^{n+1}))(\bar{U}_\sigma^{n+1} - \bar{U}_{K_\sigma}^{n+1})
\end{aligned} \tag{45}$$

One has for all pairs of real numbers (a, b) the inequality $(\zeta(a) - \zeta(b))^2 \leq (a - b)(\varphi(a) - \varphi(b))$. Also using $\varphi' \leq \sqrt{\Phi}\zeta'$ (recall that $\Phi = \|\varphi'\|_\infty$), one gets

$$E31 \geq (\mathcal{N}_{\mathcal{D}}(\zeta(u_{\mathcal{D}})))^2, \tag{46}$$

$$E32 \geq -\sqrt{\Phi}\mathcal{N}_{\mathcal{D}}(\zeta(u_{\mathcal{D}}))\mathcal{N}_{\mathcal{D}}(\bar{u}_{\mathcal{D}}). \tag{47}$$

Using the Young inequality and (42), one gets the existence of $C32$ only depending on $\Omega, T, u_0, \bar{u}, f, \mathbf{q}, \varphi$ and ξ such that

$$E32 \geq -\frac{1}{2}(\mathcal{N}_{\mathcal{D}}(\zeta(u_{\mathcal{D}})))^2 + C32. \tag{48}$$

Gathering the previous inequalities, one gets

$$C1 + \frac{1}{2F}(\mathcal{B}_{\mathcal{D}}(f(u_{\mathcal{D}})))^2 + C22 + \frac{1}{2}(\mathcal{N}_{\mathcal{D}}(\zeta(u_{\mathcal{D}})))^2 + C32 \leq 0, \tag{49}$$

which completes the proof. □

Remarking that from the estimate of Lemma 2 in [14], one has $\mathcal{N}_{\mathcal{D}}(\zeta(\bar{u}_{\mathcal{D}})) \leq \sqrt{\Phi} C \|\bar{u}\|_{L^2(0,T,H^1(\Omega))}$, where $C \geq 0$ only depends on ξ, one gets

Corollary 3.2 (Discrete H_0^1 estimate) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega \times (0,T)$ in the sense of Definition 2.3. Let $\xi \in \mathbb{R}$ be such that $\xi \geq \mathrm{reg}(\mathcal{D})$, let $U = (U_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ be the solution of the scheme (11)-(13) and let $\bar{U} = (\bar{U}_K^{n+1})_{K\in\mathcal{T}, n\in[\![0,N]\!]}$ be defined by (32). Then, setting $Z = \zeta(U) - \zeta(\bar{U})$, there exists $C' \in \mathbb{R}_+$, only depending on $\Omega, T, u_0, \bar{u}, \varphi, \mathbf{q}, f$ and ξ such that*

$$\sum_{n=0}^{N} \delta t^n \Big(\sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(Z_K^{n+1} - Z_L^{n+1})^2 + \sum_{\sigma\in\mathcal{E}_{ext}} \tau_\sigma (Z_{K_\sigma}^{n+1})^2 \Big) \leq C' \tag{50}$$

4 Compactness of a family of approximate solutions

From Lemma 3.1, we know that for any sequence of admissible discretizations $(\mathcal{D}_m)_{m\in\mathbb{N}}$, of $\Omega \times (0,T)$ in the sense of Definition 2.3, the associated sequence of approximate solutions $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded in $L^\infty(\Omega \times (0,T))$. Therefore one may extract a subsequence which converges for the weak star topology of $L^\infty(\Omega \times (0,T))$ as m tends to infinity. This convergence is unfortunately insufficient to pass to the limit in the nonlinearities. In order to pass to the limit, we shall use two tools:

1. the nonlinear weak star convergence which was introduced in [11] and which is equivalent to the notion of convergence towards a Young measure as developped in [10].
2. Kolmogorov's compactness theorem, which was used in [14] in the case of a semilinear elliptic equation.

Theorem 4.1 (Nonlinear weak star convergence) *Let Q be a Borelian subset of $\mathbb{R}^k$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(Q)$. Then there exists $u \in L^\infty(Q \times (0,1))$, such that up to a subsequence, u_n tends to u "in the nonlinear weak star sense" as $n \longrightarrow \infty$, i.e.:*

$$\forall g \in C(\mathbb{R},\mathbb{R}),\ g(u_n) \rightharpoonup \int_0^1 g(u(\cdot,\alpha))d\alpha \text{ for the weak star topology of } L^\infty(Q) \text{ as } n \longrightarrow \infty. \tag{51}$$

We refer to [10, 11] for details and proof of Theorem 4.1.

This compactness result allows us to exhibit a limit (in the nonlinear weak star sense) $u \in L^\infty(\Omega \times (0,T) \times (0,1))$ of a subsequence of the sequence $u_{\mathcal{D}_m}$ which we considered above. Of course, in order to show that this function u is the unique entropy weak solution to Problem (1)-(3) , we shall need to show that it does not depend on its argument α and that it satisfies the boundary condition (6) and the entropy inequalities (7) of Definition 1.1.

Let us now turn to the the Riesz-Fréchet-Kolmogorov compactness criterion (see e.g. [4]) which will allow us to pass to the limit in the nonlinear second order terms.

Theorem 4.2 (Riesz-Fréchet-Kolmogorov) *Let Q be an open bounded subset of $\mathbb{R}^k$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^k)$ such that*

$$\lim_{|\delta|\to 0}\big[\sup_{n\in\mathbb{N}}\|u_n(\cdot+\delta)-u_n(\cdot)\|_{L^2(Q)}\big]=0, \tag{52}$$

then there exists $u\in L^2(Q)$ such that, up to a subsequence,

$$u_n\to u \text{ in } L^2(Q) \text{ as } n\longrightarrow\infty. \tag{53}$$

Let us now show that we are in position to apply the Riesz-Fréchet-Kolmogorov to $(\zeta(u_{\mathcal{D}_m}))_{m\in\mathbb{N}}$. From the discrete estimates Proposition 3.1 and Corollary 3.2, one can state the following continuous estimates on $z_{\mathcal{D}}$, where $z_{\mathcal{D}}$ is defined almost everywhere in $\Omega\times(0,T)$ by

$$z_{\mathcal{D}}(x,t)=\zeta(U_K^{n+1})-\zeta(\bar{U}_K^{n+1}) \text{ for } x\in K \text{ and } t\in(t^n,t^{n+1}) \tag{54}$$

where $(U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ is the solution to (11)-(13) and $(\bar{U}_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ is defined by (32).

Corollary 4.1 (Space and time translates estimates) *Under hypotheses (H), let $\mathcal{D}$ be a discretization of $\Omega\times(0,T)$ in the sense of Definition 2.3. Let ξ be a real number such that $\xi\geq\operatorname{reg}(\mathcal{D})$; let U be the solution of scheme (11)-(13) , and let $u_{\mathcal{D}}$ be defined by (15). Let $\bar{U}$ be defined by (32), let $z_{\mathcal{D}}$ be defined by (54), and be prolonged by zero on $(0,T)\times\Omega^c$. Then there exists C_1 only depending on $\Omega, T, u_0, \bar{u}, \varphi, \mathbf{q}, f$ and ξ, and there exists C_0, only depending on Ω, such that*

$$\forall\xi\in\mathbb{R}^d,\int_0^T\int_{\mathbb{R}^d}(z_{\mathcal{D}}(x+\xi,t)-z_{\mathcal{D}}(x,t))^2dxdt\leq C_1|\xi|(|\xi|+C_0\operatorname{size}(\mathcal{T})), \tag{55}$$

and there exists C_2 only depending on $\Omega, T, u_0, \bar{u}, \varphi, \mathbf{q}, f$ and ξ such that

$$\forall s>0,\int_0^{T-s}\int_{\mathbb{R}^d}(\zeta(u_{\mathcal{D}})(x,t+s)-\zeta(u_{\mathcal{D}})(x,t))^2dxdt\leq C_2\ s. \tag{56}$$

The use of space translate estimates for the study of numerical schemes for elliptic problems was recently introduced in [14]. The technique of [14] may easily be adapted here to prove (55), using the estimates of Corollary 3.2. A time translate estimate was introduced in [16] to obtain some compactness in the study of finite volume schemes for parabolic equations. The proof of (56) follows the technique of [16] and uses estimate (29) and the discrete equation (13).

From Theorem 4.2 and the estimates (55) and (56) of Corollary 4.1 we deduce the following compactness result:

Corollary 4.2 (Compactness of a family of approximate solutions) *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of $\Omega\times(0,T)$ in the sense of Definition 2.3 such that there exists $\xi\in\mathbb{R}$ with $\xi\geq \mathrm{reg}(\mathcal{D}_m)$ for all $m\in\mathbb{N}$. For all $m\in\mathbb{N}$, let $u_{\mathcal{D}_m}$ be defined by the scheme (11)-(13) and (15) with $\mathcal{D}=\mathcal{D}_m$, and let $z_{\mathcal{D}_m}$ be defined by (54) with $\mathcal{D}=\mathcal{D}_m$ and (32). Then there exists $u\in L^\infty(\Omega\times(0,T)\times(0,1))$ and $z\in L^2(\Omega\times(0,T))$ such that, up to a subsequence, $u_{\mathcal{D}_m}$ tends to u in the nonlinear weak star sense and $z_{\mathcal{D}_m}$ tends to z in $L^2(\Omega\times(0,T))$ as $m\to\infty$. Furthermore one has $z\in L^2(0,T,H_0^1(\Omega))$, $\zeta(u)=z+\zeta(\bar{u})$, and $\zeta(u)=\zeta(\bar{u})$ a.e. on $\partial\Omega$.*

Proof. The convergence of $u_{\mathcal{D}_m}$ towards $u\in L^\infty(\Omega\times(0,T)\times(0,1))$ in the nonlinear weak star sense is a consequence of Lemma 3.1 and Theorem 4.1. The convergence of $z_{\mathcal{D}_m}$ to z in $L^2(\Omega\times(0,T)$ is a consequence of Theorem 4.2 and the estimates (55) and (56) of Corollary 4.1.

Following [13] or [14], one then deduces from (56) that $D_iz\in L^2(\Omega\times(0,T))$ for $i=1,\ldots,d$ and since $z_{\mathcal{D}_m}(x,t)=0$ on $\Omega^c\times(0,T)$ for all $m\in\mathbb{N}$, one has $z\in L^2(0,T,H_0^1(\Omega))$.

Now since $u_{\mathcal{D}_m}$ converges to u in the nonlinear weak star sense and that the function $\bar{u}_{\mathcal{D}_m}$ defined a.e. by $\bar{u}_{\mathcal{D}_m}(x,t)=\bar{U}_K^{n+1}$ for (x,t) in $K\times(t^n,t^{n+1})$ converges uniformly to $\bar{u}$, one deduces that $\zeta(u_{\mathcal{D}_m})$ converges to $\zeta(u)$ in the nonlinear weak star sense and to $z+\zeta(\bar{u})$ in $L^2(\Omega\times(0,T))$ as m tends to infinity. Therefore, by Lemma 4.1 below, one obtains that $\zeta(u)=z+\zeta(\bar{u})$ and $\zeta(u)$ does not depend on α. Furthermore, since $z\in L^2(0,T,H_0^1(\Omega))$, it follows that $\zeta(u)=\zeta(\bar{u})$ a.e. on $\partial\Omega$ which ends the proof of the corollary. □

Lemma 4.1 *Let Q be a Borelian subset of $\mathbb{R}^k$ and let $(u_n)_{n\in\mathbb{N}}\subset L^\infty(Q)$ be such that u_n converges to $u\in L^\infty(Q\times(0,1))$ in the nonlinear weak star sense, and to w in $L^2(Q)$, as n tends to infinity, then $u(x,\alpha)=w(x)$, for a.e. $(x,\alpha)\in Q\times(0,1)$ and u does not depend on α.*

Proof. With the notations of the lemma, we have

$$\int_0^1\int_Q(u(x,\alpha)-w(x))^2dxd\alpha=\int_0^1\int_Q(u(x,\alpha))^2dxd\alpha-2\int_0^1\int_Q u(x,\alpha)w(x)dxd\alpha+\int_0^1\int_Q w(x)^2dxd\alpha.$$

Since u_n tends to u in the nonlinear weak star sense , one has

$$\int_0^1\int_Q(u(x,\alpha))^2dxd\alpha=\lim_{n\longrightarrow+\infty}\int_Q(u_n(x))^2dx \text{ and } \int_0^1\int_Q u(x,\alpha)w(x)dxd\alpha=\lim_{n\longrightarrow+\infty}\int_Q u_n(x)w(x)dx,$$

and since u_n tends to w in $L^2(Q)$, one deduces that $u(x,\alpha)=w(x)$, for a.e. $(x,\alpha)\in Q\times(0,1)$ and u does not depend on α. □

5 Convergence towards an entropy process solution

This section is mainly devoted to the proof of the convergence theorem 5.1, which states the convergence of the approximate solution to a measure valued solution as introduced in [10], which is also called entropy process solution [11], and defined as follows.

Definition 5.1 *Under hypotheses (H), an entropy process solution to Problem (1)-(3) is a function u such that,*

$$u\in L^\infty(\Omega\times(0,T)\times(0,1)), \tag{57}$$

$$\varphi(u)-\varphi(\bar{u})\in L^2(0,T;H_0^1(\Omega)), \tag{58}$$

(note that $\varphi(u)$ does not depend on α), and u satisfies the following inequalities :

1. *Regular convex entropy inequalities :*

$$\int_{\Omega\times(0,T)} \left[\begin{array}{l} \int_0^1 \mu(u(x,t,\alpha))d\alpha\ \psi_t(x,t)+ \\ \int_0^1 \nu(u(x,t,\alpha))d\alpha\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t) \\ -\nabla\eta(\varphi(u)(x,t))\cdot\nabla\psi(x,t) \\ -\eta''(\varphi(u)(x,t))(\nabla\varphi(u)(x,t))^2\psi(x,t) \end{array} \right] dxdt + \int_\Omega \mu(u_0(x))\psi(x,0)dx \geq 0, \tag{59}$$

$\forall\ \psi \in \mathcal{D}^+(\Omega\times[0,T)),\ \forall\eta\in\mathcal{C}^2(\mathbb{R},\mathbb{R}), \eta''\geq 0,\ \mu' = \eta'(\varphi(\cdot)),\ \nu' = \eta'(\varphi(\cdot))f'(\cdot).$

2. *Kruzkov's entropy inequalities :*

$$\int_{\Omega\times(0,T)} \left[\begin{array}{l} \int_0^1 |u(x,t,\alpha)-\kappa|d\alpha\ \psi_t(x,t)+ \\ \int_0^1 (f(u(x,t,\alpha)\top\kappa) - f(u(x,t,\alpha)\bot\kappa))d\alpha\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t) \\ -\nabla|\varphi(u)(x,t)-\varphi(\kappa)|\cdot\nabla\psi(x,t) \end{array} \right] dxdt + \int_\Omega |u_0(x)-\kappa|\psi(x,0)dx \geq 0, \tag{60}$$

$\forall\ \psi \in \mathcal{D}^+(\Omega\times[0,T)),\ \forall\kappa\in\mathbb{R}.$

In the previous definition, we use two types of entropies, since in the proof (given below) of the uniqueness theorem one should make use of terms $\eta''(\varphi(u))$. In [5], these terms are obtained from the equation satisfied by a weak solution, which itself can be obtained from the Krushkov entropy inequalities. We have prefered here to keep this slightly more complex definition since the following theorem shows that (59) and (60) are both obtained by the natural limit of the approximate solutions.

Theorem 5.1 (Convergence towards an entropy process solution) *Under hypotheses (H), let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of discretizations of $\Omega\times(0,T)$ in the sense of Definition 2.3, with $\mathrm{size}(\mathcal{D}_m)\to 0$ as $m\to\infty$, such that there exists $\xi\in\mathbb{R}$ with $\xi\geq\mathrm{reg}(\mathcal{D}_m)$ for all $m\in\mathbb{N}$. For all $m\in\mathbb{N}$, let $u_{\mathcal{D}_m}$ be defined by the scheme (11)-(13) and (15) with $\mathcal{D}=\mathcal{D}_m$.*
Then, there exists an entropy processus solution of Problem (1)-(3) in the sense of Definition 5.1 and a subsequence of $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$, again denoted by $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$, such that $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ converges to u in the nonlinear weak star sense and $(\zeta(u_{\mathcal{D}_m}))_{m\in\mathbb{N}}$ converges in $L^2(\Omega\times(0,T))$ to $\zeta(u)\in L^2(0,T;H^1(\Omega))$ as m tends to ∞.

Proof. By Lemma 4.2, there exist $u\in L^\infty(\Omega\times(0,T)\times(0,1))$ and a subsequence of $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$, again denoted $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$, such that $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ converges to u in the nonlinear weak star sense and $(\zeta(u_{\mathcal{D}_m}))_{m\in\mathbb{N}}$ converges in $L^2(\Omega\times(0,T))$ to $\zeta(u)\in L^2(0,T;H^1(\Omega))$. There remains to show that the function $u\in L^\infty(\Omega\times(0,T)\times(0,1))$ is an entropy process solution.
A number of the arguments involved in order to do so may be found in [11] or [16] and therefore will be given with few details. The main new argument introduced here concerns the term $\int_{\Omega\times(0,T)} \eta''(\varphi(u)(x,t))(\nabla\varphi(u)(x,t))^2\psi(x,t)dxdt$ in equation (59). The passage to the limit to obtain this nonlinearity motivates the use of the technical lemma 5.2 below (a related technique was used in [18] in the case of a variational inequality).

The idea of the proof is to derive the continuous inequalities (59) and (60) for the limit u by multiplying the discrete entropy inequalities (17) and (19) by regular test functions and passing to the limit. Indeed, let $\psi\in\mathcal{D}^+(\Omega\times[0,T)) = \{\psi\in C_c^\infty(\Omega\times\mathbb{R},\mathbb{R}_+), \psi(\cdot,T)=0\}$. For a given m, let us denote $\mathcal{D}=\mathcal{D}_m$, and let $(U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$ be the solution of the scheme (11)-(13) associated to $\mathcal{D}$. Let $\Psi=(\Psi_K^n)_{K\in\mathcal{T},n\in[\![0,N+1]\!]}$ be defined by

$$\Psi_K^n = \psi(x_K,t^n) \qquad \forall K\in\mathcal{T},\forall n\in[\![0,N+1]\!]. \tag{61}$$

Remark 5.1 *One cannot use for Ψ_K^n the mean value of ψ on $K \times (t^n, t^{n+1})$; indeed, in order to pass to the limit on the term $A3_{\mathcal{D}}$ below (see (66) and (67)), we shall use the consistency of the approximation $\frac{\Psi_K^n - \Psi_L^n}{d(x_K, x_L)}$ to the normal derivative $\nabla\psi \cdot \mathbf{n}_{K,L}$. This consistency holds if $\Psi_K^n = \psi(x_K, t^n)$ thanks to the assumption on the family $(x_K)_{K\in\mathcal{T}}$ in Definition 2.3, but does not generally hold if Ψ_K^n is the mean value of ψ on $K \times (t^n, t^{n+1})$. Note that discrete values using the mean values were used for $\bar{u}$ when studying an upper bound of $\mathcal{N}_{\mathcal{D}}(\bar{U})$ with respect to the $L^2(0,T;H^1(\Omega))$ norm of $\bar{u}$. However we did not have to use the consistency of the flux on $\bar{u}$.*

With the notations of lemmas 3.2 and 3.3, let us multiply the discrete entropy inequalities (17) and (19) by $\delta t^n \Psi_K^n$ and sum over $K \in \mathcal{T}$ and $n \in [\![0, N]\!]$. From (17), one gets

$$A1_{\mathcal{D}} + A2_{\mathcal{D}} + A3_{\mathcal{D}} + A4_{\mathcal{D}} \le 0 \tag{62}$$

with

$$\begin{aligned}
A1_{\mathcal{D}} &= \sum_{n=0}^{N} \delta t^n \sum_{K\in\mathcal{T}} m(K) \frac{\mu(U_K^{n+1}) - \mu(U_K^n)}{\delta t^n} \Psi_K^n \\
A2_{\mathcal{D}} &= -\sum_{n=0}^{N} \delta t^n \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}_K} ((q_{K,L}^{n+1})^- (\nu(U_L^{n+1}) - \nu(U_K^{n+1}))) \Psi_K^n \\
A3_{\mathcal{D}} &= -\sum_{n=0}^{N} \delta t^n \sum_{K\in\mathcal{T}} \Big(\sum_{L\in\mathcal{N}_K} \tau_{K|L} (\eta(\varphi(U_L^{n+1})) - \eta(\varphi(U_K^{n+1}))) \Psi_K^n \\
&\qquad + \sum_{\sigma\in\mathcal{E}_{ext,K}} \tau_\sigma (\eta(\varphi(\bar{U}_\sigma^{n+1}) - \eta(\varphi(U_K^{n+1}))) \Psi_K^n \Big) \\
A4_{\mathcal{D}} &= \sum_{n=0}^{N} \delta t^n \sum_{K\in\mathcal{T}} \Big(\frac{1}{2} \sum_{L\in\mathcal{N}_K} \tau_{K|L} \eta''(\varphi(U_{K,L}^{n+1}))(\varphi(U_L^{n+1}) - \varphi(U_K^{n+1}))^2 \Psi_K^n \\
&\qquad + \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{ext,K}} \tau_\sigma \eta''(\varphi(U_{K,\sigma}^{n+1}))(\varphi(\bar{U}_\sigma^{n+1}) - \varphi(U_K^{n+1}))^2 \Psi_K^n \Big)
\end{aligned}$$

Each of these terms will be shown to converge to the corresponding continuous terms of Inequality (59) by passing to the limit on the space and time steps, i.e. letting $m \to \infty$.

Since $\psi(\cdot, T) = 0$, one has $\Psi_K^{N+1} = 0$ and therefore:

$$\begin{aligned}
A1_{\mathcal{D}} &= \sum_{n=0}^{N} \delta t^n \sum_{K\in\mathcal{T}} m(K) \mu(U_K^{n+1}) \frac{\Psi_K^n - \Psi_K^{n+1}}{\delta t^n} \\
&\quad - \sum_{K\in\mathcal{T}} m(K) \Psi_K^0 \mu(u_K^0)
\end{aligned}$$

The sequence $\mu(u_{\mathcal{D}})$ converges weakly to $\int_0^1 \mu(u(\cdot,\alpha))d\alpha$ as $m \to \infty$. Let $\chi_{\mathcal{D}}$ be the function defined almost everywhere on $\Omega \times (0,T)$ by $\chi_{\mathcal{D}}(x,t) = \frac{\Psi_K^n - \Psi_K^{n+1}}{\delta t^n}$ if $(x,t) \in K \times (t^n, t^{n+1})$; then $\chi_{\mathcal{D}}$ converges to ψ_t in $L^1(\Omega \times (0,T))$ as $m \longrightarrow +\infty$. Furthermore, let $\psi_{\mathcal{T}}^0$ (resp $u_{\mathcal{T}}^0$) be defined almost everywhere on Ω by $\psi_{\mathcal{T}}^0 = \Psi_K^0$ (resp. $u_{\mathcal{T}}^0 = U_K^0$) if $x \in K$. Then, $\mu(u_{\mathcal{T}}^0)$ converges to $\mu(u_0)$ in $L^p(\Omega)$ for any $p \in [1, +\infty)$ and $\psi_{\mathcal{T}}^0$ converges to $\psi(.,0)$ uniformly as $m \longrightarrow +\infty$. Hence passing to the limit as $m \longrightarrow +\infty$ in $A1_{\mathcal{D}}$ yields:

$$\lim_{m\to\infty} A1_{\mathcal{D}_m} = -\int_0^T\int_\Omega\int_0^1 \mu(u(x,t,\alpha))d\alpha\,\psi_t(x,t)dxdt - \int_0^T\int_\Omega \mu(u_0(x))\psi(x,0)dx. \tag{63}$$

Let us now rewrite $A2_{\mathcal{D}}$ as:

$$A2_{\mathcal{D}} \quad = \quad -\sum_{n=0}^{N}\delta t^n \sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K} \nu(U_K^{n+1})((q_{K,L}^{n+1})^+\Psi_L^n - (q_{K,L}^{n+1})^-\Psi_K^n). \tag{64}$$

We replace the term $(q_{K,L}^{n+1})^+\Psi_L^n - (q_{K,L}^{n+1})^-\Psi_K^n$ by $\frac{1}{\delta t^n}\int_{t^n}^{t^{n+1}}\int_{K|L}\psi(x,t)\,\mathbf{q}(x,t)\cdot\mathbf{n}_{K,L}d\gamma(x)dt$. When doing so, we commit an error which may be controlled (see the details in [11]) thanks to the consistency and the conservativity of the scheme and thanks to the the weak BV inequality (30). Using the weak convergence of $\nu(u_{\mathcal{T}})$ to $\int_0^1\nu(u(\cdot,\alpha))d\alpha$ as $m\to\infty$, we then obtain:

$$\begin{aligned}\lim_{m\to\infty} A2_{\mathcal{D}_m} \quad &= \quad -\int_0^T\int_\Omega\int_0^1 \nu(u(x,t,\alpha))d\alpha\nabla(\mathbf{q}(x,t)\,\psi(x,t))dxdt \\ &= \quad -\int_0^T\int_\Omega\int_0^1 \nu(u(x,t,\alpha))d\alpha\,\mathbf{q}(x,t)\cdot\nabla\psi(x,t)dxdt. \end{aligned}\tag{65}$$

Turning now to the study of $A3_{\mathcal{D}}$, one remarks that for size($\mathcal{T}$) small enough, the support of ψ does not intersect the control volumes with edges on $\partial\Omega$. Then for all control volumes $K\in\mathcal{T}$ the sum over $\sigma\in\mathcal{E}_{ext,K}$ vanishes and thus

$$A3_{\mathcal{D}} \quad = \quad -\sum_{n=0}^{N}\delta t^n\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\tau_{K|L}\eta(\varphi(U_K^n))(\Psi_L^n-\Psi_K^n) \tag{66}$$

Using the consistency of $\tau_{K|L}(\Psi_L^n-\Psi_K^n)$ with $\frac{1}{\delta t^n}\int_{t^n}^{t^{n+1}}\int_{K|L}\nabla\psi(x,t)\cdot\mathbf{n}_{K,L}d\gamma(x)dt$, Estimate (29) and the convergence of $\eta(\varphi(u_{\mathcal{D}}))$ to $\eta(\varphi(u))$ as $m\to\infty$, one gets with computations similar as in [14]:

$$\lim_{m\to\infty} A3_{\mathcal{D}_m} = -\int_0^T\int_\Omega \eta(\varphi(u))(x,t)\Delta\psi(x,t)dxdt = \int_0^T\int_\Omega \nabla\eta(\varphi(u))(x,t)\cdot\nabla\psi(x,t)dxdt. \tag{67}$$

One now deals with $A4_{\mathcal{D}}$. The second term of $A4_{\mathcal{D}}$ vanishes if size($\mathcal{T}$) is again sufficiently small. Then $A4_{\mathcal{D}}$ reduces to its first term which writes, after gathering by edges:

$$A4_{\mathcal{D}} \quad = \quad \sum_{n=0}^{N}\delta t^n\sum_{K|L\in\mathcal{E}_{int}}\tau_{K|L}\frac{\eta''(\varphi(U_{K,L}^{n+1}))\Psi_K^n+\eta''(\varphi(U_{L,K}^{n+1}))\Psi_L^n}{2}(\varphi(U_L^{n+1})-\varphi(U_K^{n+1}))^2 \tag{68}$$

Let us now introduce the sets $\mathcal{V}_\sigma$ for $\sigma\in\mathcal{E}$. Let K be a control volume and $\sigma\in\mathcal{E}_K$. One defines $\mathcal{V}_{K,\sigma}=\{tx_K+(1-t)x, x\in\sigma, t\in(0,1)\}$. For $\sigma=K|L$, $\mathcal{V}_\sigma=\mathcal{V}_{K,\sigma}\cup\mathcal{V}_{L,\sigma}$ and for $\sigma\in\mathcal{E}_{ext,K}$, $\mathcal{V}_\sigma=\mathcal{V}_{K,\sigma}$. One denotes by $H_{K|L}^{n+1}$ the discrete approximation of $\eta''(u)\psi$ on $\mathcal{V}_{K|L}$ which appears in (68), namely:

$$H_{K|L}^{n+1} = \frac{\eta''(\varphi(U_{K,L}^{n+1}))\Psi_K^n + \eta''(\varphi(U_{L,K}^{n+1}))\Psi_L^n}{2} \tag{69}$$

One defines the function $h_{\mathcal{D}}^{\diamond}$ for a.e. $(x,t) \in \Omega \times (0,T)$ by

$$h_{\mathcal{D}}^{\diamond}(x,t) = H_{K|L}^{n+1}, \qquad x \in \mathcal{V}_{K|L},\ t \in (t^n, t^{n+1}) \tag{70}$$

$$h_{\mathcal{D}}^{\diamond}(x,t) = 0, \qquad x \in \mathcal{V}_{\sigma},\ t \in (t^n, t^{n+1}) \text{ if } \sigma \in \mathcal{E}_{ext}. \tag{71}$$

Let $\psi_{\mathcal{D}}$ be defined almost everywhere on $\Omega \times (0,T)$ by $\psi_{\mathcal{D}}(x,t) = \Psi_K^n$ for all $(x,t) \in K \times (t^n, t^{n+1})$, for all $K \in \mathcal{T}$ and $n \in [\![0, N]\!]$. The function $\eta''(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}}$ tends to $\eta''(\varphi(u))\psi$ in $L^p(\Omega \times (0,T))$ for all $p \in [1, +\infty)$ as $m \to \infty$. Therefore one only needs to compare $h_{\mathcal{D}}^{\diamond}$ and $\eta''(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}}$. Since size$(\mathcal{T})$ is small enough, one has

$$\|h_{\mathcal{D}}^{\diamond} - \eta''(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}}\|_{L^2(\Omega\times(0,T))}^2 = \sum_{n=0}^{N} \delta t^n \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}_K} m(\mathcal{V}_{K,K|L})(H_{K|L}^{n+1} - \eta''(\varphi(U_K^{n+1}))\Psi_K^n)^2. \tag{72}$$

Let $\varepsilon > 0$. The function η'' may be approximated by a function $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that $|g(s) - \eta''(s)| < \varepsilon$ for all $s \in [\varphi(U_I), \varphi(U_S)]$. Defining $\tilde{H}_{K|L}^{n+1}$ and $\tilde{h}_{\mathcal{D}}^{\diamond}$ using g instead of η'' in the definition of $H_{K|L}^{n+1}$ and $h_{\mathcal{D}}^{\diamond}$ respectively, one has $\|h_{\mathcal{D}}^{\diamond} - \tilde{h}_{\mathcal{D}}^{\diamond}\|_{L^2(\Omega\times(0,T))}^2 \le C_\psi \varepsilon$ and $\|g(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}} - \eta''(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}}\|_{L^2(\Omega\times(0,T))}^2 \le C_\psi\varepsilon$ where $C_\psi \ge 0$ only depends on ψ. Thanks to Young's inequality, one gets

$$\begin{aligned}(\tilde{H}_{K|L}^{n+1} - g(\varphi(U_K^{n+1}))\Psi_K^n)^2 \le{} & \left(\max_{s\in[\varphi(U_I),\varphi(U_S)]} g(s)\right)^2 (\Psi_K^n - \Psi_L^n)^2 \\ & + \frac{3}{2}\|\psi\|_{L^\infty(\Omega\times(0,T))}^2 \left(\max_{s\in[\varphi(U_I),\varphi(U_S)]} g'(s)\right)^2 (\varphi(U_K^{n+1}) - \varphi(U_L^{n+1}))^2. \end{aligned} \tag{73}$$

Using (73), the regularity of the function ψ and Estimate (29), one gets

$$\|\tilde{h}_{\mathcal{D}}^{\diamond} - g(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}}\|_{L^2(\Omega\times(0,T))}^2 \le c(g, \psi, \varphi)\text{size}(\mathcal{T}),$$

where $c(g, \psi, \varphi) \ge 0$ depends only on g, ψ and φ. Hence for size$(\mathcal{T})$ small enough, one has

$$\|\tilde{h}_{\mathcal{D}}^{\diamond} - g(\varphi(u_{\mathcal{D}}))\psi_{\mathcal{D}}\|_{L^2(\Omega\times(0,T))}^2 \le C_\psi\varepsilon,$$

which proves that one can take $m \in \mathbb{N}$ large enough such that

$$\|h_{\mathcal{D}}^{\diamond} - \eta''(u_{\mathcal{D}})\psi_{\mathcal{D}}\|_{L^2(\Omega\times(0,T))} \le 2C_\psi\varepsilon. \tag{74}$$

Hence $h_{\mathcal{D}_m}^{\diamond}$ tends to $\eta''(\varphi(u))\psi$ in $L^2(\Omega \times (0,T))$ as $m \to \infty$.

Using the straightforward generalization of Lemma 5.2 (stated below) for space-time dependent functions, one gets:

$$\liminf_{m\to\infty} A4_{\mathcal{D}_m} \geq \int_0^T\int_\Omega (\nabla\varphi(u)(x,t))^2\eta''(\varphi(u)(x,t))\psi(x,t)dxdt. \quad (75)$$

Gathering (62), (63), (65), (67) and (75), the proof that u satisfies (59) is therefore complete.

The same steps are completed in a similar way in order to show that u satisfies (60), without the difficult problem of the treatment of η''. This also completes the proof of Theorem 5.1. □

To complete the proof of Theorem 2.1 there only remains to show the uniqueness of an entropy process solution. This is the aim of Section 6.

Lemma 5.2 which was used in the above proof is a discrete equivalent of the following continuous classical lemma.

Lemma 5.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions of $H^1(\Omega)$ which converges weakly to u in $H^1(\Omega)$ and g a nonnegative function essentially bounded from Ω to $\mathbb{R}$. Then*

$$\int_\Omega (\nabla u(x))^2 g(x)dx \leq \liminf_{n\to\infty}\int_\Omega (\nabla u_n(x))^2 g(x)dx.$$

A discrete version of this lemma is now stated:

Lemma 5.2 ("Limit inf" lemma)
Under hypotheses (H), let $g \in L^\infty(\Omega)$ with $g \geq 0$, let $u \in H^1(\Omega)$ and let $M \in \mathbb{R}_+$. Consider a family of admissible meshes of Ω in the sense of Definition 2.1, such that for all $\mathcal{D} = (\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}})$ in the family:

- *there exists a family $(G_\sigma)_{\sigma\in\mathcal{E}}$ of nonnegative values such that the function $G_\mathcal{D}$ defined by $G_\mathcal{D}(x) = G_\sigma$ for all $\sigma \in \mathcal{E}$ and all $x \in \mathcal{V}_\sigma$ satisfies $G_\mathcal{D} \longrightarrow g$ in $L^2(\Omega)$ as $\text{size}(\mathcal{D}) \longrightarrow 0$,*
- *there exists a family $(u_K)_{K\in\mathcal{T}}$ of real values such that the function $u_\mathcal{D}$ defined by $u_\mathcal{D}(x) = u_K$ for all $K \in \mathcal{T}$ and all $x \in K$ satisfies $u_\mathcal{D} \longrightarrow u$ in $L^2(\Omega)$ as $\text{size}(\mathcal{D}) \longrightarrow 0$.*
- *the value $\mathcal{N}_\mathcal{D}$ defined by $\mathcal{N}_\mathcal{D}^2 = \sum_{K|L\in\mathcal{E}_{int}} \tau_{K|L}(u_K - u_L)^2$ satisfies $\mathcal{N}_\mathcal{D} \leq M$.*

Then, denoting $D_{\mathcal{D},u_\mathcal{D},G_\mathcal{D}} = \sum_{K\in\mathcal{T}} \frac{1}{2}\sum_{L\in\mathcal{N}_K} \tau_{K|L} G_{K|L}(u_K - u_L)^2$, the following inequality holds:

$$\int_\Omega (\nabla u(x))^2 g(x)dx \leq \liminf_{\text{size}(\mathcal{D})\longrightarrow 0} D_{\mathcal{D},u_\mathcal{D},G_\mathcal{D}} \quad (76)$$

Proof. The proof of this lemma is given in [18] in the particular case $g = 1$. Let $w \in C^\infty(\bar{\Omega}, \mathbb{R})$ (the function w is meant to tend to u in $H^1(\Omega)$) and let $\tilde{g} \in C_c^\infty(\Omega, \mathbb{R})$ be a nonnegative function (which is meant to tend to g in $L^2(\Omega)$).

Let $\mathcal{D}$ be one discretization of the considered family, let W be the family of values defined by $W_K = w(x_K)$ for all $K \in \mathcal{T}$, and let $\tilde{g}_\mathcal{D}^\diamond \in L^2(\Omega)$ be defined by the mean value (denoted $\tilde{G}_\sigma$) of $\tilde{g}$ on the diamond $\mathcal{V}_\sigma$ for all $\sigma \in \mathcal{E}$. One defines $Q(g)$ and $Q_\mathcal{D}(g_\mathcal{D}^\diamond)$ by

$$Q(g) = \int_\Omega g(x)\nabla u(x)\nabla w(x)dx, \tag{77}$$

$$Q_{\mathcal{D}}(g_{\mathcal{D}}^{\diamond}) = \sum_{K\in\mathcal{T}}\frac{1}{2}\sum_{L\in\mathcal{N}_K}\tau_{K|L}(u_K-u_L)(W_k-W_L)G_{K|L}, \tag{78}$$

and one similarly defines $Q(\tilde{g})$ and $Q_{\mathcal{D}}(\tilde{g}_{\mathcal{D}}^{\diamond})$.
One has

$$\begin{aligned} Q(\tilde{g}) &= -\int_\Omega u(x)\operatorname{div}(\tilde{g}\nabla w)(x)dx \\ &= -\int_\Omega u_{\mathcal{D}}(x)\operatorname{div}(\tilde{g}\nabla w)(x)dx + \int_\Omega (u_{\mathcal{D}}(x)-u(x))\operatorname{div}(\tilde{g}\nabla w)(x)dx. \end{aligned} \tag{79}$$

Using the fact that $u_{\mathcal{D}}$ is piecewise constant, one gets

$$\begin{aligned} -\int_\Omega u_{\mathcal{D}}(x)\operatorname{div}(\tilde{g}\nabla w)(x)dx &= -\sum_{K\in\mathcal{T}}u_K\sum_{L\in\mathcal{N}_K}\int_{K|L}\tilde{g}(x)\nabla w(x)\cdot\mathbf{n}_{K,L}d\gamma(x) \\ &= \sum_{K\in\mathcal{T}}\frac{1}{2}\sum_{L\in\mathcal{N}_K}(u_L-u_K)\int_{K|L}\tilde{g}(x)\nabla w(x)\cdot\mathbf{n}_{K,L}d\gamma(x). \end{aligned} \tag{80}$$

Using the consistency of the mesh (item *(iv)* of Definition 2.1) and the Cauchy-Schwarz inequality yields

$$|\sum_{K\in\mathcal{T}}\frac{1}{2}\sum_{L\in\mathcal{N}_K}(u_L-u_K)\int_{K|L}\tilde{g}(x)\nabla w(x)\cdot\mathbf{n}_{K,L}d\gamma(x) - Q_{\mathcal{D}}(\tilde{g}_{\mathcal{D}}^{\diamond})| \le C_{\tilde{g},w,\Omega}\mathcal{N}_{\mathcal{D}}\text{size}(\mathcal{D}), \tag{81}$$

where $C_{\tilde{g},w,\Omega}\in\mathbb{R}_+$ depends only on $\tilde{g}, w$ and Ω. Using the regularity of w and $\tilde{g}$, the convergence of $u_{\mathcal{D}}$ to u in $L^2(\Omega)$ as $\text{size}(\mathcal{D})\longrightarrow 0$ and using (79), (80) and (81), one gets

$$\lim_{\text{size}(\mathcal{D})\longrightarrow 0} Q_{\mathcal{D}}(\tilde{g}_{\mathcal{D}}^{\diamond}) = Q(\tilde{g}). \tag{82}$$

One has

$$\begin{aligned} |Q(g)-Q_{\mathcal{D}}(g_{\mathcal{D}}^{\diamond})| \le\ & |Q(\tilde{g})-Q_{\mathcal{D}}(\tilde{g}_{\mathcal{D}}^{\diamond})| \\ & +\|u\|_{H^1(\Omega)}\|\nabla w\|_{L^\infty(\Omega)}\|g-\tilde{g}\|_{L^2(\Omega)} \\ & +\mathcal{N}_{\mathcal{D}}\|\nabla w\|_{L^\infty(\Omega)}\|g_{\mathcal{D}}^{\diamond}-\tilde{g}_{\mathcal{D}}^{\diamond}\|_{L^2(\Omega)} \\ \le\ & |Q(\tilde{g})-Q_{\mathcal{D}}(\tilde{g}_{\mathcal{D}}^{\diamond})| \\ & +\|u\|_{H^1(\Omega)}\|\nabla w\|_{L^\infty(\Omega)}\|g-\tilde{g}\|_{L^2(\Omega)} \\ & +M\|\nabla w\|_{L^\infty(\Omega)}(\|g_{\mathcal{D}}^{\diamond}-g\|_{L^2(\Omega)}+\|g-\tilde{g}\|_{L^2(\Omega)}+\|\tilde{g}-\tilde{g}_{\mathcal{D}}^{\diamond}\|_{L^2(\Omega)}). \end{aligned} \tag{83}$$

Thanks to (82) and (83) one gets

$$\limsup_{\text{size}(\mathcal{D})\longrightarrow 0} |Q(g) - Q_{\mathcal{D}}(g_{\mathcal{D}}^{\diamond})| \quad \leq \quad (\|u\|_{H^1(\Omega)} + M)\|\nabla w\|_{L^\infty(\Omega)}\|g - \tilde{g}\|_{L^2(\Omega)}. \tag{84}$$

Now one can let $\tilde{g} \longrightarrow g$ in (84). One then gets

$$\limsup_{\text{size}(\mathcal{D})\longrightarrow 0} |Q(g) - Q_{\mathcal{D}}(g_{\mathcal{D}}^{\diamond})| = 0. \tag{85}$$

which proves that

$$\lim_{\text{size}(\mathcal{D})\longrightarrow 0} Q_{\mathcal{D}}(g_{\mathcal{D}}^{\diamond}) = Q(g). \tag{86}$$

By the same proof, replacing u by w, one also has

$$\lim_{\text{size}(\mathcal{D})\longrightarrow 0} \sum_{K\in\mathcal{T}} \frac{1}{2} \sum_{L\in\mathcal{N}_K} \tau_{K|L}(W_L - W_K)^2 G_{K|L} \quad = \quad \int_\Omega (\nabla w(x))^2 g(x)dx. \tag{87}$$

Thanks to the Cauchy-Schwarz inequality, we may write

$$(Q_{\mathcal{D}}(g_{\mathcal{D}}^{\diamond}))^2 \leq D_{\mathcal{D},u_{\mathcal{D}},G_{\mathcal{D}}} \sum_{K\in\mathcal{T}} \frac{1}{2} \sum_{L\in\mathcal{N}_K} \tau_{K|L}(W_L - W_K)^2 G_{K|L}. \tag{88}$$

Passing to the limit in (88) when size($\mathcal{D}$) $\longrightarrow 0$ yields

$$(\int_\Omega g(x)\nabla u(x)\nabla w(x)dx)^2 \leq \int_\Omega (\nabla w(x))^2 g(x)dx \quad \liminf_{\text{size}(\mathcal{D})\longrightarrow 0} D_{\mathcal{D},u_{\mathcal{D}},G_{\mathcal{D}}}. \tag{89}$$

Since $C^\infty(\bar{\Omega},\mathbb{R})$ is dense in $H^1(\Omega)$, one can let w tend to u in (89), which gives (76). □

6 Uniqueness of the entropy process solution.

One proves in this section the following theorem.

Theorem 6.1 (Uniqueness of the entropy process solution) *Under hypotheses (H), let u and v be two entropy process solutions to Problem (1)-(3) in the sense of Definition 5.1. Then there exists a unique function $w \in L^\infty(\Omega \times (0,T))$ such that $u(x,t,\alpha) = v(x,t,\beta) = w(x,t)$, for almost every $(x,t,\alpha,\beta) \in \Omega \times (0,T) \times (0,1) \times (0,1)$.*

Proof.
This proof uses on the one hand Carrillo's handling of Krushkov entropies, on the other hand the concept of entropy process solution, which allows the use of the theorem of continuity in means, necessary to pass to the limit on mollifiers. Note that the hypothesis (4) makes it easier to handle the boundary conditions.

In order to prove Theorem 6.1, one defines for all $\varepsilon > 0$ a regularization $S_\varepsilon \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ of the function sign given by

$$\begin{array}{ll} S_\varepsilon(a) = -1, & \forall a \in (-\infty, -\varepsilon], \\ S_\varepsilon(a) = \frac{3\varepsilon^2 a - a^3}{2\ \varepsilon^3}, & \forall a \in [-\varepsilon, \varepsilon], \\ S_\varepsilon(a) = 1, & \forall a \in [\varepsilon, +\infty). \end{array} \tag{90}$$

One defines $\mathbb{R}_\varphi = \{a \in \mathbb{R}, \forall b \in \mathbb{R}\setminus\{a\},\ \varphi(b) \neq \varphi(a)\}$. Note that $\varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)$ is countable, because for all $s \in \varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)$, there exists $(a,b) \in \mathbb{R}^2$ with $a < b$ and $\varphi((a,b)) = \{s\}$, and therefore there exists at least one $r \in \mathbb{Q}$ with $r \in (a,b)$ satisfying $\varphi(r) = s$.

Let $\kappa \in \mathbb{R}_\varphi$. Let $\varepsilon > 0$ and let u be an entropy processus solution. One introduces in (59) the function $\eta_{\varepsilon,\kappa}(a) = \int_{\varphi(\kappa)}^a S_\varepsilon(s - \varphi(\kappa))ds$. One defines $\mu_{\varepsilon,\kappa}(a) = \int_\kappa^a \eta'_{\varepsilon,\kappa}(\varphi(s))ds$ and $\nu_{\varepsilon,\kappa}(a) = \int_\kappa^a \eta'_{\varepsilon,\kappa}(\varphi(s))f'(s)ds$, for all $a \in \mathbb{R}$. Using the dominated convergence theorem, one gets for all $a \in \mathbb{R}$ that $\lim_{\varepsilon \longrightarrow 0} \eta_{\varepsilon,\kappa}(a) = |a - \varphi(\kappa)|$,
and, since $\kappa \in \mathbb{R}_\varphi$, $\lim_{\varepsilon \longrightarrow 0} \mu_{\varepsilon,\kappa}(a) = |a - \kappa|$ and $\lim_{\varepsilon \longrightarrow 0} \nu_{\varepsilon,\kappa}(a) = f(a \top \kappa) - f(a \bot \kappa)$. One gets for all $\psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\begin{array}{l} \displaystyle\int_{\Omega\times(0,T)} \left[\begin{array}{l} \int_0^1 |u(x,t,\alpha) - \kappa| d\alpha\ \psi_t(x,t) \\ + \int_0^1 (f(u(x,t,\alpha)\top\kappa) - f(u(x,t,\alpha)\bot\kappa)) d\alpha\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t) \\ -S_\varepsilon(\varphi(u)(x,t) - \varphi(\kappa))\nabla\varphi(u)(x,t)\cdot\nabla\psi(x,t) \end{array} \right] dxdt \\ \displaystyle -\int_{\Omega\times(0,T)} \left[S'_\varepsilon(\varphi(u)(x,t) - \varphi(\kappa))(\nabla\varphi(u))^2(x,t)\psi(x,t)\right] dxdt \\ \displaystyle +\int_\Omega |u_0(x) - \kappa|\psi(x,0)dx \qquad\qquad \geq A(\varepsilon,u,\kappa,\psi), \end{array} \tag{91}$$

where for any entropy process solution u, any $\psi \in \mathcal{D}^+(\Omega \times [0,T))$, any $\kappa \in \mathbb{R}_\varphi$ and any $\varepsilon > 0$, $A(\varepsilon,u,\kappa,\psi)$ is defined by

$$\begin{array}{rcl} A(\varepsilon,u,\kappa,\psi) & = & \displaystyle\int_{\Omega\times(0,T)} \left[\begin{array}{l} \int_0^1 \Big(|u(x,t,\alpha) - \kappa| - \mu_{\varepsilon,\kappa}(u(x,t,\alpha))\Big) d\alpha\ \psi_t(x,t) + \\ \int_0^1 \Big((f(u(x,t,\alpha)\top\kappa) - f(u(x,t,\alpha)\bot\kappa)) - \nu_{\varepsilon,\kappa}(u(x,t,\alpha))\Big) d\alpha \\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t) \end{array} \right] dxdt \\ & & \displaystyle + \int_\Omega \Big(|u_0(x) - \kappa| - \mu_{\varepsilon,\kappa}(u_0(x))\Big)\psi(x,0)dx. \end{array} \tag{92}$$

Thanks to the dominated convergence theorem, one has

$$\lim_{\varepsilon \longrightarrow 0} A(\varepsilon,u,\kappa,\psi) = 0. \tag{93}$$

This convergence is not uniform w.r.t. κ (even if κ remains bounded), but $A(\varepsilon,u,\kappa,\psi)$ remains bounded (for a given u) if κ, ψ, ψ_t and $\nabla\psi$ remain bounded and if the support of ψ remains in a fixed compact set of $\mathbb{R}^d \times [0,T)$.
Using (60), one now remarks that, for all $\kappa \in \mathbb{R}$, one has for all $\psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\begin{array}{l} \displaystyle\int_{\Omega\times(0,T)} \left[\begin{array}{l} \int_0^1 |u(x,t,\alpha) - \kappa| d\alpha\ \psi_t(x,t) + \\ \int_0^1 (f(u(x,t,\alpha)\top\kappa) - f(u(x,t,\alpha)\bot\kappa)) d\alpha \\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t) \\ -S_\varepsilon(\varphi(u)(x,t) - \varphi(\kappa))\nabla\varphi(u)(x,t)\cdot\nabla\psi(x,t) \end{array} \right] dxdt \\ + \int_\Omega |u_0(x) - \kappa|\psi(x,0)dx \qquad\qquad \geq B(\varepsilon,u,\kappa,\psi), \end{array} \tag{94}$$

where for an entropy process solution u, all $\psi \in \mathcal{D}^+(\Omega \times [0,T))$, all $\kappa \in \mathbb{R}$ and all $\varepsilon > 0$, $B(\varepsilon,u,\kappa,\psi)$ is defined by

$$B(\varepsilon,u,\kappa,\psi)=\int_{\Omega\times(0,T)}\Big[\nabla\Big(|\varphi(u)(x,t)-\varphi(\kappa)|-\eta_{\varepsilon,\kappa}(\varphi(u)(x,t))\Big)\cdot\nabla\psi(x,t)\Big]\,dxdt. \tag{95}$$

For all $\psi\in\mathcal{D}^+(\Omega\times[0,T))$, one has

$$B(\varepsilon,u,\kappa,\psi)=-\int_{\Omega\times(0,T)}\Big[\Big(|\varphi(u)(x,t)-\varphi(\kappa)|-\eta_{\varepsilon,\kappa}(\varphi(u)(x,t))\Big)\Delta\psi(x,t)\Big]\,dxdt, \tag{96}$$

and

$$\lim_{\varepsilon\longrightarrow 0}B(\varepsilon,u,\kappa,\psi)=0, \tag{97}$$

for all $\psi\in\mathcal{D}^+(\Omega\times[0,T))$, $\varepsilon>0$ and $\kappa\in\mathbb{R}$.
As for the study of A, the quantity $B(\varepsilon,u,\kappa,\psi)$ remains bounded (for a given u) if κ and $\Delta\psi$ remain bounded and if the support of ψ remains in a fixed compact set of $\mathbb{R}^d\times[0,T)$.

Let u and v be two entropy process solutions in the sense of Definition 5.1. One defines the sets $E_u=\{(x,t)\in\Omega\times(0,T),\ u(x,t,\alpha)\in\mathbb{R}_\varphi$, for a.e. $\alpha\in(0,1)\}$ and $E_v=\{(x,t)\in\Omega\times(0,T),\ v(x,t,\alpha)\in\mathbb{R}_\varphi$, for a.e. $\alpha\in(0,1)\}$. Indeed, recall that $\varphi(u)$ and $\varphi(v)$ do not depend of $\alpha\in(0,1)$. Then, $\Omega\times(0,T)\setminus E_u=\cup_{s\in\varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)}E_{s,u}$ with $E_{s,u}=\{(x,t)\in\Omega\times(0,T),\varphi(u)(x,t)=s\}$ (the same property is available for v). Let $\xi\in\mathcal{C}_c^\infty(\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R},\mathbb{R}_+)$ such that, for all $(x,t)\in\Omega\times[0,T)$, $\xi(x,t,\cdot,\cdot)\in\mathcal{D}^+(\Omega\times[0,T))$ and for all $(y,s)\in\Omega\times[0,T)$, $\xi(\cdot,\cdot,y,s)\in\mathcal{D}^+(\Omega\times[0,T))$. One introduces in (91), for $(y,s)\in E_v$, and a.e. $\beta\in(0,1)$, $\kappa=v(y,s,\beta)$ and $\psi=\xi(\cdot,\cdot,y,s)$. One integrates the result on $E_v\times(0,1)$. One then gets

$$\begin{aligned}
&\int_{E_v}\int_{\Omega\times(0,T)}\left[\begin{array}{l}\int_0^1\int_0^1|u(x,t,\alpha)-v(y,s,\beta)|d\alpha d\beta\ \xi_t(x,t,y,s)+\\ \int_0^1\int_0^1(f(u(x,t,\alpha)\top v(y,s,\beta))-f(u(x,t,\alpha)\bot v(y,s,\beta)))d\alpha d\beta\\ \mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s)\\ -S_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))\nabla\varphi(u)(x,t)\cdot\nabla_x\xi(x,t,y,s)\end{array}\right]dxdtdyds\\
&-\int_{E_v}\int_{\Omega\times(0,T)}\left[S'_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))(\nabla\varphi(u))^2(x,t)\xi(x,t,y,s)\right]dxdtdyds\\
&+\int_{E_v}\int_\Omega\int_0^1|u_0(x)-v(y,s,\beta)|\xi(x,0,y,s)d\beta dxdyds\\
&\geq\int_0^1\int_{E_v}A(\varepsilon,u,v(y,s,\beta),\xi(\cdot,\cdot,y,s))dydsd\beta.
\end{aligned}\tag{98}$$

One introduces in (94), for $(y,s)\in\Omega\times(0,T)\setminus E_v$, and any $\beta\in(0,1)$, $\kappa=v(y,s,\beta)$ and $\psi=\xi(\cdot,\cdot,y,s)$. One integrates the result on $(\Omega\times(0,T)\setminus E_v)\times(0,1)$. One then gets

$$\begin{aligned}
&\int_{\Omega\times(0,T)\setminus E_v}\int_{\Omega\times(0,T)}\left[\begin{array}{l}\int_0^1\int_0^1|u(x,t,\alpha)-v(y,s,\beta)|d\alpha d\beta\ \xi_t(x,t,y,s)\\ +\int_0^1\int_0^1(f(u(x,t,\alpha)\top v(y,s,\beta))-f(u(x,t,\alpha)\bot v(y,s,\beta)))d\alpha d\beta\\ \mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s)\\ -S_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))\nabla\varphi(u)(x,t)\cdot\nabla_x\xi(x,t,y,s)\end{array}\right]dxdtdyds\\
&+\int_{\Omega\times(0,T)\setminus E_v}\int_\Omega\int_0^1|u_0(x)-v(y,s,\beta)|\xi(x,0,y,s)d\beta dxdyds\\
&\geq\int_0^1\int_{\Omega\times(0,T)\setminus E_v}B(\varepsilon,u,v(y,s,\beta),\xi(\cdot,\cdot,y,s))dydsd\beta.
\end{aligned}\tag{99}$$

Adding (98) and (99) gives

$$\int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[\begin{array}{l}\int_0^1\int_0^1|u(x,t,\alpha)-v(y,s,\beta)|d\alpha d\beta\ \xi_t(x,t,y,s)\\ +\int_0^1\int_0^1(f(u(x,t,\alpha)\top v(y,s,\beta))-f(u(x,t,\alpha)\bot v(y,s,\beta)))d\alpha d\beta\\ \mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s)\\ -S_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))\nabla\varphi(u)(x,t)\cdot\nabla_x\xi(x,t,y,s)\end{array}\right]dxdtdyds$$
$$-\int_{E_v}\int_{\Omega\times(0,T)}\left[S'_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))(\nabla\varphi(u))^2(x,t)\xi(x,t,y,s)\right]dxdtdyds \tag{100}$$
$$+\int_{\Omega\times(0,T)}\int_\Omega\int_0^1|u_0(x)-v(y,s,\beta)|\xi(x,0,y,s)d\beta dxdyds$$
$$\geq\int_0^1\int_{E_v}A(\varepsilon,u,v(y,s,\beta),\xi(\cdot,\cdot,y,s))dydsd\beta+\int_0^1\int_{\Omega\times(0,T)\setminus E_v}B(\varepsilon,u,v(y,s,\beta),\xi(\cdot,\cdot,y,s))dydsd\beta$$

One now exchanges the roles of u and v, and add the resulting equations. It gives

$$T_1+T_2+T_3(\varepsilon)+T_4(\varepsilon)+T_5(\varepsilon)\geq T_6(\varepsilon), \tag{101}$$

where

$$T_1 = \int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[\begin{array}{l}\int_0^1\int_0^1|u(x,t,\alpha)-v(y,s,\beta)|d\alpha d\beta\ (\xi_t(x,t,y,s)+\xi_s(x,t,y,s))\\ +\int_0^1\int_0^1(f(u(x,t,\alpha)\top v(y,s,\beta))-f(u(x,t,\alpha)\bot v(y,s,\beta)))d\alpha d\beta\\ \Big(\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s)+\mathbf{q}(y,s)\cdot\nabla_y\xi(x,t,y,s)\Big)\end{array}\right]dxdtdyds, \tag{102}$$

$$\begin{aligned}T_2 &= \int_{\Omega\times(0,T)}\int_\Omega\int_0^1|u_0(x)-v(y,s,\beta)|\xi(x,0,y,s)d\beta dxdyds\\ &\quad+\int_{\Omega\times(0,T)}\int_\Omega\int_0^1|u_0(y)-u(x,t,\alpha)|\xi(x,t,y,0)d\alpha dydxdt,\end{aligned} \tag{103}$$

$$\begin{aligned}T_3(\varepsilon) &= -\int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[\begin{array}{l}S_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))\nabla\varphi(u)(x,t)\cdot\\ (\nabla_x\xi(x,t,y,s)+\nabla_y\xi(x,t,y,s))\end{array}\right]dxdtdyds\\ &\quad-\int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[\begin{array}{l}S_\varepsilon(\varphi(v)(y,s)-\varphi(u)(x,t))\nabla\varphi(v)(y,s)\cdot\\ (\nabla_x\xi(x,t,y,s)+\nabla_y\xi(x,t,y,s))\end{array}\right]dxdtdyds,\end{aligned} \tag{104}$$

$$\begin{aligned}T_4(\varepsilon) &= \int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[S_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))\nabla\varphi(u)(x,t)\cdot\nabla_y\xi(x,t,y,s)\right]dxdtdyds\\ &\quad+\int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[S_\varepsilon(\varphi(v)(y,s)-\varphi(u)(x,t))\nabla\varphi(v)(y,s)\cdot\nabla_x\xi(x,t,y,s)\right]dxdtdyds,\end{aligned} \tag{105}$$

$$\begin{aligned}T_5(\varepsilon) &= -\int_{E_v}\int_{\Omega\times(0,T)}\left[S'_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))(\nabla\varphi(u))^2(x,t)\xi(x,t,y,s)\right]dxdtdyds\\ &\quad-\int_{\Omega\times(0,T)}\int_{E_u}\left[S'_\varepsilon(\varphi(u)(x,t)-\varphi(v)(y,s))(\nabla\varphi(v))^2(y,s)\xi(x,t,y,s)\right]dxdtdyds,\end{aligned} \tag{106}$$

and

$$
\begin{aligned}
T_6(\varepsilon) &= \int_0^1 \int_{E_v} A(\varepsilon, u, v(y,s,\beta), \xi(\cdot,\cdot,y,s)) dyds d\beta \\
&+ \int_0^1 \int_{\Omega\times(0,T)\setminus E_v} B(\varepsilon, u, v(y,s,\beta), \xi(\cdot,\cdot,y,s)) dyds d\beta \\
&+ \int_0^1 \int_{E_u} A(\varepsilon, v, u(x,t,\alpha), \xi(x,t,\cdot,\cdot)) dxdtd\alpha \\
&+ \int_0^1 \int_{\Omega\times(0,T)\setminus E_u} B(\varepsilon, v, u(x,t,\alpha), \xi(x,t,\cdot,\cdot)) dxdtd\alpha.
\end{aligned} \tag{107}
$$

(108)

By an integration by parts in (105) and using the fact that ξ vanishes on $\partial\Omega \times (0,T) \times \Omega \times (0,T)$ and on $\Omega \times (0,T) \times \partial\Omega \times (0,T)$ one gets

$$
\begin{aligned}
T_4(\varepsilon) &= \int_{\Omega\times(0,T)} \int_{\Omega\times(0,T)} \left[S_\varepsilon'(\varphi(u)(x,t) - \varphi(v)(y,s)) \xi(x,t,y,s) \nabla\varphi(u)(x,t) \cdot \nabla\varphi(v)(y,s) \right] dxdtdyds \\
&+ \int_{\Omega\times(0,T)} \int_{\Omega\times(0,T)} \left[S_\varepsilon'(\varphi(v)(y,s) - \varphi(u)(x,t)) \xi(x,t,y,s) \nabla\varphi(v)(y,s) \cdot \nabla\varphi(u)(x,t) \right] dxdtdyds.
\end{aligned} \tag{109}
$$

Recall that $E_{s,u} = \{(x,t) \in \Omega \times (0,T), \varphi(u)(x,t) = s\}$ for all $s \in \mathbb{R}$. One has $\nabla\varphi(u) = 0$ a.e. on $E_{s,u}$ (see [4] for instance). Since $\Omega \times (0,T) \setminus E_u = \cup_{s \in \varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)} E_{s,u}$, and since $\varphi(\mathbb{R} \setminus \mathbb{R}_\varphi)$ is countable, the following equations hold.

$$
\nabla\varphi(u) = 0, \text{ a.e. on } \Omega \times (0,T) \setminus E_u \tag{110}
$$

and

$$
\nabla\varphi(v) = 0, \text{ a.e. on } \Omega \times (0,T) \setminus E_v. \tag{111}
$$

It leads to

$$
\begin{aligned}
T_4(\varepsilon) &= \int_{E_u\times E_v} \left[S_\varepsilon'(\varphi(u)(x,t) - \varphi(v)(y,s)) \xi(x,t,y,s) \nabla\varphi(u)(x,t) \cdot \nabla\varphi(v)(y,s) \right] dxdtdyds \\
&+ \int_{E_u\times E_v} \left[S_\varepsilon'(\varphi(v)(y,s) - \varphi(u)(x,t)) \xi(x,t,y,s) \nabla\varphi(v)(y,s) \cdot \nabla\varphi(u)(x,t) \right] dxdtdyds
\end{aligned} \tag{112}
$$

and

$$
\begin{aligned}
T_5(\varepsilon) &= -\int_{E_u\times E_v} \left[S_\varepsilon'(\varphi(u)(x,t) - \varphi(v)(y,s)) (\nabla\varphi(u))^2(x,t) \xi(x,t,y,s) \right] dxdtdyds \\
&- \int_{E_u\times E_v} \left[S_\varepsilon'(\varphi(u)(x,t) - \varphi(v)(y,s)) (\nabla\varphi(v))^2(y,s) \xi(x,t,y,s) \right] dxdtdyds.
\end{aligned} \tag{113}
$$

Therefore $\forall \varepsilon > 0$,

$$
\begin{aligned}
T_4(\varepsilon) + T_5(\varepsilon) &= -\int_{E_v} \int_{E_u} \left[S_\varepsilon'(\varphi(u)(x,t) - \varphi(v)(y,s)) \xi(x,t,y,s) \Big(\nabla\varphi(u)(x,t) - \nabla\varphi(v)(y,s) \Big)^2 \right] dxdtdyds \\
&\leq 0.
\end{aligned} \tag{114}
$$

One thus gets $\forall \varepsilon > 0$,

$$
T_1 + T_2 + T_3(\varepsilon) \geq T_6(\varepsilon). \tag{115}
$$

One can now let $\varepsilon \longrightarrow 0$ in (115). This gives, since $T_6(\varepsilon) \longrightarrow 0$ (thanks to the dominated convergence theorem),

$$\begin{aligned}
&\int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)} \left[\begin{array}{l} \int_0^1\int_0^1 |u(x,t,\alpha)-v(y,s,\beta)|d\alpha d\beta\ (\xi_t(x,t,y,s)+\xi_s(x,t,y,s))+ \\ \int_0^1\int_0^1 (f(u(x,t,\alpha)\top v(y,s,\beta))-f(u(x,t,\alpha)\bot v(y,s,\beta)))d\alpha d\beta \\ \Big(\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s)+\mathbf{q}(y,s)\cdot\nabla_y\xi(x,t,y,s)\Big) \\ -(\nabla_x|\varphi(u)(x,t)-\varphi(v)(y,s)|+\nabla_y|\varphi(u)(x,t)-\varphi(v)(y,s)|) \\ \cdot(\nabla_x\xi(x,t,y,s)+\nabla_y\xi(x,t,y,s)) \end{array}\right] dxdtdyds \\
&+\int_{\Omega\times(0,T)}\int_\Omega\int_0^1 |u_0(x)-v(y,s,\beta)|\xi(x,0,y,s)d\beta dxdyds \\
&+\int_{\Omega\times(0,T)}\int_\Omega\int_0^1 |u_0(y)-u(x,t,\alpha)|\xi(x,t,y,0)d\alpha dydxdt \qquad \geq 0.
\end{aligned} \tag{116}$$

Now, let us consider the analog of (60) for v instead of u, with $\kappa = u_0(x)$ and $\psi(y,s) = \int_s^T \xi(x,0,y,\tau)d\tau$ and integrate the result on $x \in \Omega$. One then gets

$$\begin{aligned}
&\int_\Omega\int_{\Omega\times(0,T)} \left[\begin{array}{l} -\int_0^1 |v(y,s,\beta)-u_0(x)|d\beta\ \xi(x,0,y,s)+ \\ \int_0^1 (f(v(y,s,\beta)\top u_0(x))-f(v(y,s,\beta)\bot u_0(x)))d\beta\ \mathbf{q}(y,s)\cdot \\ \nabla_y\int_s^T \xi(x,0,y,\tau)d\tau \\ -\nabla_y|\varphi(v)(y,s)-\varphi(u_0(x))|\cdot \\ \int_s^T \nabla_y\xi(x,0,y,\tau)d\tau \end{array}\right] dydsdx \quad + \\
&\int_\Omega\int_\Omega |u_0(x)-u_0(y)|\int_0^T \xi(x,0,y,\tau)d\tau dxdy \qquad \geq 0.
\end{aligned} \tag{117}$$

A sequence of mollifiers in $\mathbb{R}$ and $\mathbb{R}^d$ is now introduced. Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d,\mathbb{R}_+)$ and $\bar\rho \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R}_+)$ be such that

$$\{x \in \mathbb{R}^d;\ \rho(x) \neq 0\} \subset \{x \in \mathbb{R}^d;\ |x| \leq 1\},$$

$$\{x \in \mathbb{R};\ \bar\rho(x) \neq 0\} \subset [-1,0] \tag{118}$$

and

$$\int_{\mathbb{R}^d} \rho(x)dx = 1,\ \int_{\mathbb{R}} \bar\rho(x)dx = 1. \tag{119}$$

For $n \in \mathbb{N}^\star$, define $\rho_n = n^d\rho(nx)$ for all $x \in \mathbb{R}^d$ and $\bar\rho_n = n\bar\rho(nx)$ for all $x \in \mathbb{R}$.

One sets $\xi(x,t,y,s) = \psi(x,t)\rho_n(x-y)\bar\rho_m(t-s)$, where $\psi \in C_c^\infty(\Omega\times[0,T),\mathbb{R}_+)$ and n and m are large enough to ensure, for all $(x,t) \in \Omega\times[0,T)$, $\xi(x,t,\cdot,\cdot) \in \mathcal{D}^+(\Omega\times[0,T))$ and for all $(y,s) \in \Omega\times[0,T)$, $\xi(\cdot,\cdot,y,s) \in \mathcal{D}^+(\Omega\times[0,T))$. This choice is not symmetrical in (x,t) and (y,s), which gives an easier way to take the limit as $n \longrightarrow \infty$ and $m \longrightarrow \infty$. One gets, from (116),

$$\int_{\Omega\times(0,T)}\int_{\Omega\times(0,T)}\left[\begin{array}{l}\rho_n(x-y)\bar\rho_m(t-s)\\ \int_0^1\int_0^1|u(x,t,\alpha)-v(y,s,\beta)|d\alpha d\beta\ \psi_t(x,t)\\ -\int_0^1\int_0^1\left(\begin{array}{c}f(u(x,t,\alpha)\top v(y,s,\beta))\\ -f(u(x,t,\alpha)\bot v(y,s,\beta))\end{array}\right)d\alpha d\beta\\ (\rho_n(x-y)\bar\rho_m(t-s)\mathbf{q}(x,t)\cdot\nabla\psi(x,t)\\ -\psi(x,t)\bar\rho_m(t-s)(\mathbf{q}(x,t)-\mathbf{q}(y,s))\cdot\nabla\rho_n(x-y))\\ -\rho_n(x-y)\bar\rho_m(t-s)(\nabla_x|\varphi(u)(x,t)-\varphi(v)(y,s)|\\ +\nabla_y|\varphi(u)(x,t)-\varphi(v)(y,s)|)\cdot\nabla\psi(x,t)\end{array}\right]dxdtdyds$$
$$+\int_{\Omega\times(0,T)}\int_\Omega\int_0^1|u_0(x)-v(y,s,\beta)|\psi(x,0)\rho_n(x-y)\bar\rho_m(-s)d\beta dxdyds \qquad \geq 0. \tag{120}$$

The second of the two initial terms vanishes because of the asymmetric choice of $\bar\rho_m$. Using the same test function in (117), at $t=0$, i.e. $\xi(x,0,y,s)=\psi(x,0)\rho_n(x-y)\bar\rho_m(-s)$ and (119), we get

$$\int_\Omega\int_{\Omega\times(0,T)}\left[\begin{array}{l}-\int_0^1|v(y,s,\beta)-u_0(x)|d\beta\ \psi(x,0)\rho_n(x-y)\bar\rho_n(-s)\\ -\int_0^1(f(v(y,s,\beta)\top u_0(x))-f(v(y,s,\beta)\bot u_0(x)))d\beta\ \mathbf{q}(y,s)\cdot\\ \psi(x,0)\nabla\rho_n(x-y)\int_s^T\bar\rho_m(-\tau)d\tau\\ +\nabla_y|\varphi(v)(y,s)-\varphi(u_0(x))|\cdot\\ \psi(x,0)\nabla\rho_n(x-y)\int_s^T\bar\rho_m(-\tau)d\tau\end{array}\right]dydsdx$$
$$+\int_\Omega\int_\Omega|u_0(x)-u_0(y)|\psi(x,0)\rho_n(x-y)dxdy \qquad \geq 0. \tag{121}$$

One can now add (120) and (121) let m tend to ∞ and use the theorem of continuity in means. Since the function $s\longrightarrow\int_s^T\bar\rho_m(-\tau)d\tau$ is bounded and tends to zero as $m\longrightarrow\infty$ for all $s\in(0,T)$, one gets

$$\int_\Omega\int_{\Omega\times(0,T)}\left[\begin{array}{l}\rho_n(y-x)\int_0^1\int_0^1|u(x,t,\alpha)-v(y,t,\beta)|d\alpha d\beta\ \psi_t(x,t)\\ +\int_0^1\int_0^1\left(\begin{array}{c}f(u(x,t,\alpha)\top v(y,t,\beta))\\ -f(u(x,t,\alpha)\bot v(y,t,\beta))\end{array}\right)d\alpha d\beta\\ (\rho_n(y-x)\mathbf{q}(x,t)\cdot\nabla\psi(x,t)+\\ \psi(x,t)(\mathbf{q}(y,t)-\mathbf{q}(x,t))\cdot\nabla\rho_n(y-x))\\ -\rho_n(x-y)(\nabla_x|\varphi(u)(x,t)-\varphi(v)(y,t)|\\ +\nabla_y|\varphi(u)(x,t)-\varphi(v)(y,t)|)\cdot\nabla\psi(x,t)\end{array}\right]dxdtdy$$
$$+\int_\Omega\int_\Omega|u_0(x)-u_0(y)|\psi(x,0)\rho_n(x-y)dxdy \qquad \geq 0. \tag{122}$$

Remarking that

$$\begin{array}{l}\displaystyle\int_\Omega\int_{\Omega\times(0,T)}\left[\begin{array}{l}\rho_n(x-y)(\nabla_x|\varphi(u)(x,t)-\varphi(v)(y,t)|\\ +\nabla_y|\varphi(u)(x,t)-\varphi(v)(y,t)|)\cdot\nabla\psi(x,t)\end{array}\right]dxdtdy\\ \displaystyle=-\int_\Omega\int_{\Omega\times(0,T)}\left[\ \rho_n(x-y)|\varphi(u)(x,t)-\varphi(v)(y,t)|\Delta\psi(x,t)\ \right]dxdtdy,\end{array} \tag{123}$$

it is possible to let $n\longrightarrow\infty$ in (122). Using $\mathrm{div}\mathbf{q}=0$ and the theorem of continuity in means again, one gets

$$\int_{\Omega\times(0,T)}\left[\begin{array}{l}\int_0^1\int_0^1|u(x,t,\alpha)-v(x,t,\beta)|d\alpha d\beta\ \psi_t(x,t)\\ +\int_0^1\int_0^1(f(u(x,t,\alpha)\top v(x,t,\beta))-f(u(x,t,\alpha)\bot v(x,t,\beta)))d\alpha d\beta\\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t)\\ -\nabla|\varphi(u)(x,t)-\varphi(v)(x,t)|\cdot\nabla\psi(x,t)\end{array}\right]dxdt \quad \geq 0. \tag{124}$$

One notices that (124) holds for any $\psi \in H^1(\Omega \times (0,T))$, with $\psi \geq 0$ and $\psi(.,T) = 0$, using a density argument. Therefore one can now take, in (124), for ψ the functions $\psi_\varepsilon(x,t) = (T-t)\min(\frac{d(x,\partial\Omega)}{\varepsilon}, 1)$, for $\varepsilon > 0$.
Assume momentarily that for all $w \in H_0^1(\Omega)$ with $w \geq 0$,

$$\liminf_{\varepsilon \longrightarrow 0} \int_\Omega \nabla w(x) \cdot \nabla \min(\frac{d(x,\partial\Omega)}{\varepsilon}, 1) dx \geq 0 \tag{125}$$

(The proof of (125) is given below).

The expression $\mathbf{q}(x,t) \cdot \nabla \min(\frac{d(x,\partial\Omega)}{\varepsilon}, 1)$ satisfies

$$\lim_{\varepsilon \longrightarrow 0} \mathbf{q}(x,t) \cdot \nabla \min(\frac{d(x,\partial\Omega)}{\varepsilon}, 1) = 0, \text{ for a.e. } (x,t) \in \Omega \times (0,T),$$

and under condition (4) (and (H5)) remains bounded independently of ε for a.e. $(x,t) \in \Omega \times (0,T)$. Letting $\varepsilon \longrightarrow 0$, (124), with $\psi = \psi_\varepsilon$, gives

$$-\int_{\Omega \times (0,T)} \left[\int_0^1 \int_0^1 |u(x,t,\alpha) - v(x,t,\beta)| d\alpha d\beta \right] dxdt \quad \geq 0,$$

which finally proves that $u = v$ and that u is a classical function of space and time (it does not depend on α).

Proof of (125)

Let $\varepsilon > 0$. Let $(\partial\Omega_i)_{i=1,\ldots,N}$ be the faces of Ω, $\mathbf{n}_i$ their normal vector outward to Ω, and for $i = 1, \ldots N$, let Ω_i be the subset of Ω such that, for all $x \in \Omega_i$, $d(x,\partial\Omega_i) < \varepsilon$ and $d(x,\partial\Omega_i) < d(x,\partial\Omega_j)$ for all $j \neq i$. One has

$$\int_{\cup_{i=1}^N \Omega_i} \nabla w(x) \cdot \nabla \min(d(x,\partial\Omega)/\varepsilon, 1) dx = \sum_{i=1}^N \int_{\Omega_i} \frac{\nabla w(x) \cdot \mathbf{n}_i}{\varepsilon} dx.$$

For each Ω_i, let $\tilde{\Omega}_i$ be the largest cylinder generated by $\mathbf{n}_i$ included in Ω_i. One denotes by $\partial\Omega_i'$ the face of $\tilde{\Omega}_i$ parallel to $\partial\Omega_i$. Let Ω_ε be defined by $\Omega_\varepsilon = \Omega \setminus \cup_{i=1}^N \tilde{\Omega}_i$. One has $meas(\Omega_\varepsilon) \leq C(\Omega)\varepsilon^2$ and

$$\int_\Omega \nabla w(x) \cdot \nabla \min(d(x,\partial\Omega)/\varepsilon, 1) dx \geq \sum_{i=1}^N \int_{\partial\Omega_i'} \frac{w(x)}{\varepsilon} d\gamma(x) - \int_{\Omega_\varepsilon} \frac{|\nabla w(x)|}{\varepsilon} dx.$$

Thanks to the Cauchy-Schwarz inequality, one gets

$$(\int_{\Omega_\varepsilon} |\nabla w(x)| dx)^2 \leq \text{meas}(\Omega_\varepsilon) \int_{\Omega_\varepsilon} (\nabla w(x))^2 dx.$$

One concludes, using $\lim_{\varepsilon \longrightarrow 0} \int_{\Omega_\varepsilon} (\nabla w(x))^2 dx = 0$.

Remark 6.1 *Inequation (125) could also be proved in the case where Ω is regular instead of polygonal, with a slightly different method. Let $\Omega_\varepsilon = \{x \in \Omega, d(x,\partial\Omega) < \varepsilon\}$ and let $\partial\Omega_\varepsilon'$ be the other face of Ω_ε. The normal vector to $\partial\Omega_\varepsilon'$ at any point x is equal to $\nabla d(x,\partial\Omega)$. Therefore one has*

$$\int_{\Omega} \nabla w(x) \cdot \nabla \min(d(x,\partial\Omega)/\varepsilon, 1) dx = \int_{\partial\Omega'_\varepsilon} \frac{w(x)}{\varepsilon} d\gamma(x) - \int_{\Omega_\varepsilon} w(x) \frac{\Delta d(x,\partial\Omega)}{\varepsilon} dx.$$

Since Hardy's inequality leads to

$$\int_{\Omega_\varepsilon} \left(\frac{w(x)}{d(x,\partial\Omega)} \right)^2 dx \leq C(\Omega) \int_{\Omega_\varepsilon} (\nabla w(x))^2 dx,$$

one concludes using $m_{\varepsilon \longrightarrow 0} \int_{\Omega_\varepsilon} (\nabla w(x))^2 dx = 0.$

□

7 Conclusion

Let us finally prove the convergence theorem by way of contradiction:
Assume that the convergence stated in the Theorem 2.1 does not hold. Then there exist $\varepsilon > 0$, $p \in [1, +\infty)$ and a sequence $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ such that $\|u_{\mathcal{D}_m} - u\|_{L^p(\Omega\times(0,T)} \geq \varepsilon$, for any $m \in \mathbb{N}$. Then by Theorem 5.1, there exists a subsequence of the sequence $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$, still denoted by $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ which converges to an entropy process solution of Problem (1)-(3) . By Theorem 6.1 this entropy process solution is the unique entropy weak solution to Problem (1)-(3) , and from Lemma 7.1 which is stated below, the convergence of $(u_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is strong in any $L^q(\Omega \times (0,T))$. This is in contradiction with the fact that $\|u_{\mathcal{D}_m} - u\|_{L^p(\Omega\times(0,T)} \geq \varepsilon$, for any $m \in \mathbb{N}$.

Lemma 7.1 *Let Q be a Borelian subset of $\mathbb{R}^k$ and let $(u_n)_{n\in\mathbb{N}} \subset L^\infty(Q)$ be such that u_n converges to $u \in L^\infty(Q \times (0,1))$ in the nonlinear weak star sense where u does not depend on α, then $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^p_{loc}(Q)$ for any $p \in [1, \infty)$.*

Proof. Let K be a compact subset of Q, since u_n converges to u in the nonlinear weak star sense , one has

$$\int_K |u_n(x) - u(x)|^2 dx = \int_K u_n^2(x) dx - 2\int_K u_n(x)u(x) dx + \int_K u(x)^2 dx \longrightarrow 0 \text{ as } n \longrightarrow +\infty;$$

since K is bounded, one also has:

$$\int_K |u_n(x) - u(x)|^p dx \longrightarrow 0 \text{ as } n \longrightarrow +\infty, \forall p \in [1,2]$$

and since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(Q)$,

$$\int_K |u_n(x) - u(x)|^p dx \longrightarrow 0 \text{ as } n \longrightarrow +\infty, \forall p > 2.$$

□

Remark 7.1 *An interesting (and open to our knowledge) question is to find the convergence rate of the finite volume approximations. In the case of a pure hyperbolic equation, i.e. $\varphi = 0$, it was proven by several authors (under varying assumptions, see e.g. [9], [22], [11], [6]) that the error between the approximate finite volume solution and the entropy weak solution is of order less than $h^{1/4}$ where h is the size of the mesh, under a usual CFL condition for the explicit schemes which are considered in [9], [22], [11], [6], and of order less than $h^{1/4} + k^{1/2}$ where k is the time step in the* *case of the implicit scheme considered in [11]. However, it is also known that these estimates are not sharp, since numerically the order of the error behaves as* $1/2$.

In the case of a pure linear parabolic equation, estimates of order 1 were obtained in [17] (see also [13])

We made a first attempt in the direction of an error estimate in the case of the present degenerate parabolic equation by looking at the analogous continuous problem [15]: let u_ε *be the unique solution to*

$$u_t(x,t) + \mathrm{div}\Big(\mathbf{q}\ f(u)\Big)(x,t) - \Delta\varphi(u)(x,t) - \varepsilon\Delta u(x,t) = 0,\ \textit{for}\ (x,t) \in \Omega \times (0,T), \tag{126}$$

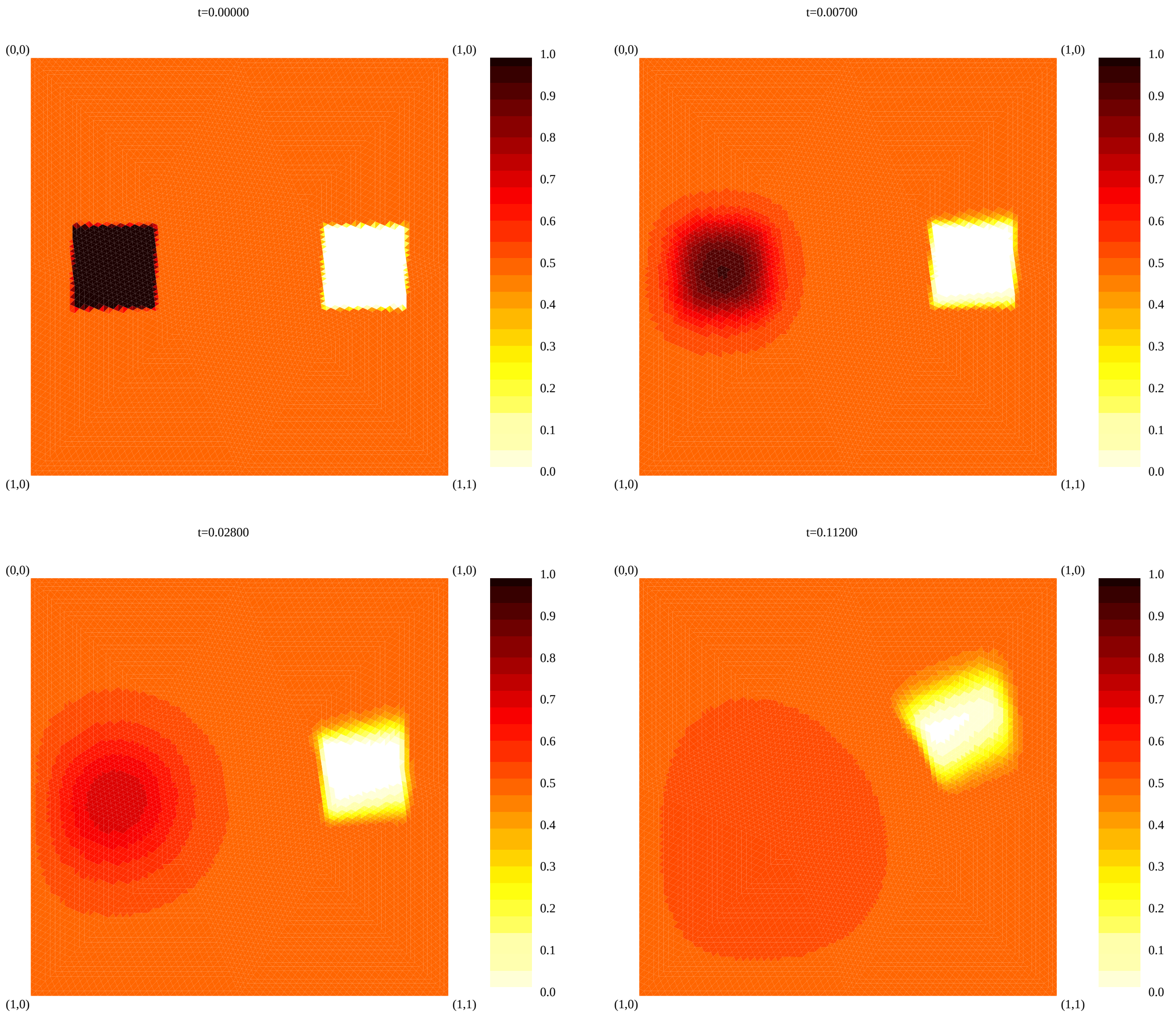
with initial condition (2) and boundary condition (3) and let u *be the unique entropy weak solution solution of Problem (1)-(3) , then under assumptions (H), we are able to prove that* $\|u_\varepsilon - u\|_{L^1(Q_T)} \leq C\varepsilon^{1/5}$ *where* $C \in \mathbb{R}_+$ *depends only on the data. This estimate is however probably not optimal and we have not yet been able to transcribe its proof to the discrete setting (the term* $-\varepsilon\Delta u$ *being the continuous diffusive representation of the diffusive perturbation introduced by the finite volume scheme).*

8 A numerical example

We finally present some numerical results which we obtained by implementing the scheme which was studied above in a prototype code.
The domain Ω is the unit square $(0,1) \times (0,1)$. We define two subregions $\Omega_1 = (0.1, 0.3) \times (0.4, 0.6)$ and $\Omega_2 = (0.7, 0.9) \times (0.4, 0.6)$. The initial data is given by 0.5 in $\Omega \setminus (\Omega_1 \cup \Omega_2)$, 1 in Ω_1 and 0 in Ω_2. It is represented on upper left corner of the figure below. The boundary value is the constant 0.5.
The function φ is defined by $\varphi(s) = 0$ if $s \in [0, 0.5]$ and $\varphi(s) = 0.2(s - 0.5)$ if $s \in [0.5, 1]$, so that the diffusion effect only takes place in the areas where the saturation u is greater than .5. The function f is defined by $f(s) = s$ and the field $\mathbf{q}$ is defined by $\mathbf{q}(x, y) = (10(x - x^2)(1 - 2y), -10(y - y^2)(1 - 2x))$. Hence there is a linear rotating convective transport.

We define a coarse mesh of 14 admissible triangles on the unit square, from which we obtain a fine mesh of 12 600 triangles by refining these 14 triangles uniformly 30 times. This fine mesh is used for the computations.

The figure below presents the obtained results at times 0.000, 0.007, 0.028 and 0.112. The black points correspond to the value 1, the white ones to the value 0, with a continuous scale of greys between these values. One observes that the initial value 0 is transported, only modified by the numerical diffusion due to the convective upstream weighting, and that, on the contrary, the initial value 1 is rapidly smoothed, due to the effect of the parabolic term which is active on the range $[0.5, 1]$.



Computed solution at time $t = 0$ (initial condition), $t = 0.007$, $t = 0.028$ and $t = 0.112$.

References

[1] C. Bardos, A.Y. Leroux, J.C. Nédélec, First order quasilinear equations with boundary conditions. *Comm. P. D. E.*, **4** (1979), 9, 1017-1034.

[2] Ph. Bénilan, Equations d'évolution dans un espace de Banach quelconque et applications. Thèse d'état, Université d'Orsay, France (1972).

[3] F. Bouchut, F. Guarguaglini, R. Natalini, Diffusive BGK approximations for nonlinear multidimensional parabolic equations. *preprint* (1999).

[4] H. Brezis, Analyse Fonctionnelle: Théorie et Applications. (Masson, Paris) (1983).

[5] J. Carrillo, Entropy solutions for nonlinear degenerate problems. *Arch. Rat. Mech. Anal.*, **147** (1999), 269-361.

[6] C. Chainais-Hillairet, Finite volume schemes for a nonlinear hyperbolic equation. Convergence towards the entropy solution and error estimate. *Model. Math. Anal. Numer.*, **33** (1999), 1, 129-156.

[7] S. Champier, T. Gallouet, R. Herbin, Convergence of an upstream finite volume scheme for a nonlinear hyperbolic equation on a triangular mesh. *Numer. Math.* **66** (1993), 2, 139-157.

[8] B. Cockburn, G. Gripenberg, Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. *J. Differential Equations*, **151** (1999), 2, 231-251.

[9] B.Cockburn, F. Coquel, P. LeFloch, An error estimate for finite volume methods for multidimensional conservation laws. *Math. Comput.*, **63** (1994), 207, 77-103.

[10] R. DiPerna, Measure-valued solutions to conservation laws. *Arch. Rat. Mech. Anal.*, **88** (1985), 223-270.

[11] R. Eymard, T. Gallouët, M. Ghilani, R. Herbin, Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes. *IMA Journal of Numerical Analysis*, (1997) 18, 563-594.

[12] R. Eymard, T. Gallouët, R. Herbin, Existence and Uniqueness of the Entropy Solution to a Nonlinear Hyperbolic Equation. *Chi.Ann.of Math* **16** Ser B 1(1995), 1-14.

[13] R. Eymard, T. Gallouët, R. Herbin, Finite volume methods, prépublication LATP 97-19 6632, to appear in *Handbook of numerical analysis, Ph. Ciarlet & J.L. Lions ed.* .

[14] R. Eymard, T. Gallouët, R. Herbin, Convergence of finite volume schemes for semilinear convection diffusion equations. *Numer. Math.* **82** (1999), 1, 91-116.

[15] R. Eymard, T. Gallouët, R. Herbin, Convergence of the solutions to perturbed degenerate parabolic-hyperbolic equations on bounded domains. *in preparation.*

[16] R. Eymard, T. Gallouët, D. Hilhorst, Y. Nait Slimane, Finite volumes and nonlinear diffusion equations. *RAIRO Model. Math. Anal. Numer.* **32** (1998), no. 6, 747-761.

[17] R. Herbin, Finite volume methods for diffusion convection equations on general meshes. *in Finite volumes for complex applications, Problems and Perspectives*, F. Benkhaldoun and R. Vilsmeier eds, Hermes (1996), 153-160.

[18] R. Herbin, E. Marchand, Finite volume approximation of a class of variational inequalities, *accepted for publication in IMA Journal of Numerical Analysis.*

[19] S.N. Krushkov, First Order quasilinear equations with several space variables. *Math. USSR. Sb.* **10** (1970), 217-243.

[20] A. Lagha-Benabdallah, F. Smadhi, Existence de solutions faibles pour un problème aux limites associé à une équation parabolique dégénérée. *Maghreb Math. Rev.* **2** (1993), 201-222.

[21] M. Ohlberger, A posteriori error estimates for vertex centered finite volume approximations of convection-diffusion-reaction equations. *University of Freiburg, Germany* (2000) preprint 12.

[22] J.P. Vila, Convergence and error estimate in finite volume schemes for general multidimensional conservation laws, I. explicit monotone schemes. *Model. Math. Anal. Numer.* **28** (1994), 3, 267-285.