C. R. Acad. Sci. Paris, t. xxx, Série I, p. xx–xx, 2007 - PXMA????.TEX -
Rubrique/*Numerical Analysis*

A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis

R. Eymard, T. Gallouët and R. Herbin

Université de Marne-la-Vallée, France, (Robert.Eymard@univ-mlv.fr)
Université de Provence, 39 rue Joliot-Curie, 13453 Marseille (gallouet, herbin@cmi.univ-mrs.fr)

(Reçu le jour mois année, accepté après révision le jour mois année)

Abstract. We introduce here a new finite volume scheme which was developed for the discretization of anisotropic diffusion problems; the originality of this scheme lies in the fact that we are able to prove its convergence under very weak assumptions on the discretization mesh.

Un nouveau schéma volumes finis pour les problèmes de diffusion anisotrope : analyse de convergence

Résumé. *On introduit ici un nouveau schéma volumes finis, construit pour la discrétisation de problèmes de diffusion anisotrope sur des maillages généraux ; l'originalité de ce travail réside dans sa preuve de convergence, qui ne nécessite que des hypothèses faibles sur le maillage.*

1. Introduction

The scope of this work is the discretization by a finite volume method of anisotropic diffusion problems on general meshes. Let Ω be a polygonal (or polyhedral) open subset of $\mathbb{R}^d$ ($d = 2$ or 3); let $\mathcal{M}_d(\mathbb{R})$ be the set of $d \times d$ symmetric matrices. We consider the following elliptic conservation equation:

$$-\text{div}(\Lambda \nabla u) = f \text{ in } \Omega, \tag{1}$$

with boundary condition

$$u = 0 \text{ on } \partial\Omega \tag{2}$$

with the following hypotheses on the data:

$$\begin{array}{c}\Lambda \text{ is a measurable function from } \Omega \text{ to } \mathcal{M}_d(\mathbb{R}), \text{ and there exist } \underline{\lambda} \text{ and } \overline{\lambda} \text{ such that} \\ 0 < \underline{\lambda} \leq \overline{\lambda} \text{ and } \text{Sp}(\Lambda(x)) \subset [\underline{\lambda}, \overline{\lambda}] \text{ for a.e. } x \in \Omega. \text{ The function } f \text{ is such that } f \in L^2(\Omega).\end{array} \tag{3}$$

Note présentée par Philippe G. CIARLET

S0764-4442(00)0????-?/FLA

In (4), $\mathrm{Sp}(B)$ denotes for all $B \in \mathcal{M}_d(\mathbb{R})$ the set of the eigenvalues of B. We consider the following weak formulation of problem (2):

$$\left\{ \begin{array}{l} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \Lambda(x)\nabla u(x) \cdot \nabla v(x) \mathrm{d}x = \int_\Omega f(x)v(x)\mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \end{array} \right. \tag{4}$$

2. Discrete functional tools

A finite volume discretization of Ω is a triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:

- $\mathcal{M}$ is a finite family of non empty connex open disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $\mathrm{m}_K > 0$ denote the measure of K.
- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non empty closed subset of a hyperplane of $\mathbb{R}^d$, which has a measure $\mathrm{m}_\sigma > 0$ for the $(d-1)$-dimensional measure of σ. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. We then denote by $\mathcal{M}_\sigma = \{K \in \mathcal{M}, \sigma \in \mathcal{E}_K\}$. We then assume that, for all $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (boundary edge) or $\mathcal{M}_\sigma$ has exactly two elements (interior edge). For all $\sigma \in \mathcal{E}$, we denote by x_σ the barycenter of σ.
- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$, such that $x_K \in K$ and K is star-shaped with respect to x_K.

The following notations are used. The size of the discretization is defined by: $h_\mathcal{D} = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}$. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote for a.e. $x \in \sigma$ by $\boldsymbol{n}_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$). The regularity of the mesh is measured through the parameter $\theta_\mathcal{D} = \min\left\{\frac{\min(d_{K,\sigma}, d_{L,\sigma})}{\max(d_{K,\sigma}, d_{L,\sigma})}, \sigma \in \mathcal{E}_{\mathrm{int}}, \mathcal{M}_\sigma = \{K, L\}\right\}$.

A family $\mathcal{F}$ of discretizations is regular if there exists $\theta > 0$ such that for any $\mathcal{D} \in \mathcal{F}$, $\theta_\mathcal{D} \geq \theta$.

Let $X_\mathcal{D} = \mathbb{R}^\mathcal{M} \times \mathbb{R}^\mathcal{E}$ be the set of all $u := ((u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}})$, and let $X_{\mathcal{D},0} \subset X_\mathcal{D}$ be defined as the set of all $u \in X_\mathcal{D}$ such that $u_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$. The space $X_{\mathcal{D},0}$ is equipped with a Euclidean structure, defined by the following inner product:

$$\forall (v, w) \in (X_{\mathcal{D},0})^2,\ [v, w]_\mathcal{D} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}_\sigma}{d_{K,\sigma}} (v_\sigma - v_K)(w_\sigma - w_K). \tag{5}$$

and the associated norm: $\|u\|_{1,\mathcal{D}} = ([u, u]_\mathcal{D})^{1/2}$. Let $H_\mathcal{M}(\Omega) \subset L^2(\Omega)$ be the set of piecewise constant functions on the control volumes on the mesh $\mathcal{M}$ which is equipped with the following inner norm: $\|u\|_{1,\mathcal{M}} = \inf\{\|v\|_{1,\mathcal{D}},\ v \in X_{\mathcal{D},0},\ P_\mathcal{M} v = u\}$, where for all $u \in X_\mathcal{D}$, we denote by $P_\mathcal{M} u \in H_\mathcal{M}(\Omega)$ the element defined by the values $(u_K)_{K\in\mathcal{M}}$ (we then easily see that this definition of $\|\cdot\|_{1,\mathcal{M}}$ coincides with that given in [1] in the case where we set $d_{KL} = d_{K,\sigma} + d_{L,\sigma}$ for all $\sigma \in \mathcal{E}_{\mathrm{int}}$ with $\mathcal{M}_\sigma = \{K, L\}$). For all $\varphi \in C(\Omega, \mathbb{R})$, we denote by $P_\mathcal{D}(\varphi)$ the element of $X_\mathcal{D}$ defined by $((\varphi(x_K))_{K\in\mathcal{M}}, (\varphi(x_\sigma))_{\sigma\in\mathcal{E}})$.

3. The finite volume scheme and its convergence analysis

The finite volume method is based on the discretization of the balance equation associated to equation (2) on cell K. It requires the definition of consistent numerical fluxes $(F_{K,\sigma}^\mathcal{D})_{K\in\mathcal{M},\sigma\in\mathcal{E}_K}$ on the edges of the cells, meant to approximate the diffusion fluxes $-\Lambda\nabla u \cdot \mathbf{n}_K$, where $\mathbf{n}_K$ is the unit outward normal to ∂K.

Let $\mathcal{F}$ be a family of finite volume discretizations; for $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}) \in \mathcal{F}$, $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $F_{K,\sigma}^\mathcal{D}$ a linear mapping from $X_\mathcal{D}$ to $\mathbb{R}^\mathcal{E}$. The family $((F_{K,\sigma}^\mathcal{D})_{\substack{K\in\mathcal{M}\\ \sigma\in\mathcal{E}}})_{\mathcal{D}\in\mathcal{F}}$ is said to be a consistent family of fluxes if for any function $\varphi \in C^2(\mathbb{R}^d,\mathbb{R})$,

$$\lim_{h_{\mathcal{D}}\to 0} \max_{\substack{K\in\mathcal{M}\\ \sigma\in\mathcal{E}_K}} \frac{1}{\mathrm{m}_\sigma}\left|F^{\mathcal{D}}_{K,\sigma}(P_{\mathcal{D}}(\varphi)) + \int_\sigma \Lambda_K\nabla\varphi\cdot\boldsymbol{n}_{K,\sigma}\mathrm{d}\gamma\right| = 0, \tag{6}$$

where $\Lambda_K = \frac{1}{\mathrm{m}_K}\int_K \Lambda dx$. In order to get some estimates on the approximate solutions, we need a coercivity property: the family of numerical fluxes $((F^{\mathcal{D}}_{K,\sigma})_{\substack{K\in\mathcal{M}\\ \sigma\in\mathcal{E}}})_{\mathcal{D}\in\mathcal{F}}$ is said to be coercive if there exists $\alpha > 0$ such that, for any $\mathcal{D} = (\mathcal{M},\mathcal{E},\mathcal{P}) \in \mathcal{F}$ and for any $u \in X_{\mathcal{D},0}$,

$$\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}(u_K - u_\sigma)F^{\mathcal{D}}_{K,\sigma}(u) \geq \alpha\|u\|^2_{1,\mathcal{D}}. \tag{7}$$

Finally the family of numerical fluxes $((F^{\mathcal{D}}_{K,\sigma})_{\substack{K\in\mathcal{M}\\ \sigma\in\mathcal{E}}})_{\mathcal{D}\in\mathcal{F}}$ is said to be symmetric if for any $\mathcal{D} = (\mathcal{M},\mathcal{E},\mathcal{P}) \in \mathcal{F}$, the bilinear form defined by

$$\langle u,v\rangle_{\mathcal{D}} = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}} F^{\mathcal{D}}_{K,\sigma}(u)(v_K - v_\sigma),\ \forall (u,v)\in X^2_{\mathcal{D},0},$$

is such that

$$\langle u,v\rangle_{\mathcal{D}} = \langle v,u\rangle_{\mathcal{D}},\ \forall (u,v)\in X^2_{\mathcal{D},0}.$$

The finite volume scheme may then be written by approximating the integration of (2) in each control volume, and requiring that the scheme be conservative:

$$\text{Find } u^{\mathcal{D}} = ((u^{\mathcal{D}}_K)_{K\in\mathcal{M}}, (u^{\mathcal{D}}_\sigma)_{\sigma\in\mathcal{E}}) \in X_{\mathcal{D},0}; \tag{8}$$

$$\sum_{\sigma\in\mathcal{E}_K} F^{\mathcal{D}}_{K,\sigma}(u^{\mathcal{D}}) = \int_K f(x)\mathrm{d}x,\ \forall K\in\mathcal{M}; \tag{9}$$

$$F^{\mathcal{D}}_{K,\sigma}(u^{\mathcal{D}}) + F^{\mathcal{D}}_{L,\sigma}(u^{\mathcal{D}}) = 0,\ \forall\sigma\in\mathcal{E}_{\mathrm{int}},\ \mathcal{M}_\sigma = \{K,L\}. \tag{10}$$

or, in equivalent form:

$$\text{Find } u^{\mathcal{D}} = ((u^{\mathcal{D}}_K)_{K\in\mathcal{M}}, (u^{\mathcal{D}}_\sigma)_{\sigma\in\mathcal{E}}) \in X_{\mathcal{D},0} \text{ s.t. } \langle u^{\mathcal{D}}, v\rangle_{\mathcal{D}} = \int_\Omega f(x)P_{\mathcal{M}}v(x)\mathrm{d}x, \forall v\in X_{\mathcal{D},0}. \tag{11}$$

THEOREM 3.1. – *Under assumptions (4), let u be the unique solution to (5). Consider a regular family of admissible meshes $\mathcal{F}$, along with a family of consistent, coercive and symmetric fluxes $((F^{\mathcal{D}}_{K,\sigma})_{\substack{K\in\mathcal{M}\\ \sigma\in\mathcal{E}}})_{\mathcal{D}\in\mathcal{F}}$. Then, for all $\mathcal{D}\in\mathcal{F}$, there exists a unique $u^{\mathcal{D}}\in X_{\mathcal{D},0}$ solution to (10) or (12), and $P_{\mathcal{M}}u^{\mathcal{D}}$ converges to u, solution of (5) in $L^q(\Omega)$, for all $q\in[1,+\infty)$ if $d=2$ and all $q\in[1,2d/(d-2))$ if $d>2$, as $h_{\mathcal{D}}\to 0$. Moreover, $\nabla_{\mathcal{D}}u^{\mathcal{D}}\in H_{\mathcal{M}}(\Omega)^d$, defined by $\mathrm{m}_K(\nabla_{\mathcal{D}}u^{\mathcal{D}})_K = \sum_{\sigma\in\mathcal{E}_K}\mathrm{m}_\sigma(u_\sigma - u_K)\boldsymbol{n}_{K,\sigma}$ for all $K\in\mathcal{M}$, converges to ∇u in $L^2(\Omega)^d$.*

Sketch of proof Taking $v = u^{\mathcal{D}}$ in (12), we get the following *a priori* estimate on $u^{\mathcal{D}}$:

$$\alpha\|u^{\mathcal{D}}\|^2_{1,\mathcal{D}} \leq \|f\|_{L^2(\Omega)}\|u_{\mathcal{D}}\|_{L^2(\Omega)}.$$

The discrete Sobolev inequality [1] holds thanks to the above definition of $\theta_{\mathcal{D}}$, that is, there exists $C>0$ depending only on q,Ω and θ such that: $\|P_{\mathcal{M}}u^{\mathcal{D}}\|_{L^q(\Omega)} \leq C\|P_{\mathcal{M}}u^{\mathcal{D}}\|_{1,\mathcal{M}}$. Therefore, thanks to the fact that $\|P_{\mathcal{M}}u^{\mathcal{D}}\|_{1,\mathcal{M}} \leq \|u^{\mathcal{D}}\|_{1,\mathcal{D}}$, we obtain that: $\|P_{\mathcal{M}}u^{\mathcal{D}}\|_{1,\mathcal{M}} \leq \|u^{\mathcal{D}}\|_{1,\mathcal{D}} \leq \frac{C}{\alpha}\|f\|_{L^2(\Omega)}$, which yields the existence and uniqueness of $u^{\mathcal{D}}$. Then, prolonging by 0 the function $P_{\mathcal{M}}u^{\mathcal{D}}$ outside of Ω, we get the estimate

$$\|P_{\mathcal{M}}u^{\mathcal{D}}(\cdot+\xi) - P_{\mathcal{M}}u^{\mathcal{D}}\|_{L^1(\mathbb{R}^d)} \leq |\xi|(d\,\mathrm{m}(\Omega))^{1/2}\|u^{\mathcal{D}}\|_{1,\mathcal{D}},\ \forall\xi\in\mathbb{R}^d.$$

We can therefore apply the Fréchet–Kolmogorov theorem, which is a compactness criterion in $L^1(\mathbb{R}^d)$. Again using the discrete Sobolev inequality, we get that, up to a subsequence, $P_{\mathcal{M}} u^{\mathcal{D}}$ converges, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, in $L^q(\mathbb{R}^d)$ to some function $\tilde{u}$, with $\tilde{u}(x) = 0$ for a.e. $x \in \mathbb{R}^d \setminus \Omega$. Furthermore, in the spirit of lemma 2 of [4], we can show that $\nabla_{\mathcal{D}} u^{\mathcal{D}}$ converges to $\nabla \tilde{u}$ weakly in $L^2(\mathbb{R}^d)^d$. Therefore $\tilde{u} \in H_0^1(\Omega)$. To complete the proof of the theorem, we pass to the limit $h_{\mathcal{D}} \to 0$ on the weak form of the scheme: for $\varphi \in C_c^\infty(\Omega)$, we take $v = P_{\mathcal{D}}(\varphi)$ in (12). Using the symmetry and the consistency (7) of the fluxes $F_{K,\sigma}^{\mathcal{D}}(\varphi)$, we obtain that $\tilde{u}$ verifies (5) with $v = \varphi$. Therefore, by uniqueness, $\tilde{u} = u$ and the whole sequence converges. The strong convergence of $\nabla_{\mathcal{D}} u^{\mathcal{D}}$ to ∇u is obtained, using (8), the convergence of $\langle u^{\mathcal{D}}, u^{\mathcal{D}} \rangle_{\mathcal{D}}$ to $\int_\Omega \nabla u \cdot \Lambda \nabla u \mathrm{d}x$ and following the principles of the proof of lemma 2.6 in [5].

4. An example of consistent, coercive and symmetric family of fluxes

Let us first note that the case of the classical four point finite volume schemes on triangles (also based on a consistent coercive and symmetric family of fluxes, see [6]) is included in the framework presented here. However, for general meshes or anistropic diffusion operators, the construction of an approximation to the normal flux is more strenuous [2, 3, 7]; it is often performed by the reconstruction of a discrete gradient, either in the edges of the cell, or in the cell itself. We propose the following numerical fluxes, defined for $u \in X_{\mathcal{D}}$ by

$$F_{K,\sigma}(u) = -\mathrm{m}_\sigma \left(\nabla_{\mathcal{D}} u_K \cdot \Lambda_K \boldsymbol{n}_{K,\sigma} + \alpha_K \left(\frac{R_{K,\sigma}(u)}{d_{K,\sigma}} - \sum_{\sigma' \in \mathcal{E}_K} m_{\sigma'} \frac{R_{K,\sigma'}(u)}{d_{K,\sigma'}} (x_{\sigma'} - x_K) \cdot \frac{\boldsymbol{n}_{K,\sigma}}{\mathrm{m}_K} \right) \right)$$

where Λ_K is the mean value of the matrix $\Lambda(x)$ for $x \in K$, $\nabla_{\mathcal{D}} u_K$ is defined in Theorem 3.1, $R_{K,\sigma}(u) = u_\sigma - u_K - \nabla_{\mathcal{D}} u_K.(x_\sigma - x_K)$, and $(\alpha_K)_{K \in \mathcal{M}}$ is any family of strictly positive real numbers, bounded by above and below. We thus get a consistent, coercive and symmetric family of fluxes, in the above stated sense. In fact, in the same spirit as in the scheme derived in [5] for meshes satisfying an orthogonality condition, the above expression for $F_{K,\sigma}(u)$ is deduced from the variational form of the scheme, which is based on the following inner product:

$$\langle u, v \rangle_{\mathcal{D}} = \sum_{K \in \mathcal{M}} \left[m_K \nabla_{\mathcal{D}} u_K \cdot \Lambda_K \nabla_{\mathcal{D}} v_K \ + \alpha_K \sum_{\sigma \in \mathcal{E}_K} \frac{\mathrm{m}_\sigma}{d_{K,\sigma}} R_{K,\sigma}(u) R_{K,\sigma}(v) \right], \ \forall u, v \in X_{\mathcal{D},0}.$$

References

[1] Y. Coudière, T. Gallouët, and R. Herbin. Discrete Sobolev inequalities and L^p error estimates for finite volume solutions of convection diffusion equations. *M2AN Math. Model. Numer. Anal.*, 35(4):767–778, 2001.

[2] Yves Coudière, Jean-Paul Vila, and Philippe Villedieu. Convergence rate of a finite volume scheme for a two-dimensional convection-diffusion problem. *M2AN Math. Model. Numer. Anal.*, 33(3):493–516, 1999.

[3] K. Domelevo and P. Omnes. A finite volume method for the laplace equation on almost arbitrary two-dimensional grids. *M2AN Math. Model. Numer. Anal.*, 39(6):1203–1249, 2005.

[4] R. Eymard and T. Gallouët. H-convergence and numerical schemes for elliptic equations. *SIAM Journal on Numerical Analysis*, 41(2):539–562, 2000.

[5] R. Eymard, T. Gallouët, and R. Herbin. A cell-centered finite-volume approximation for anisotropic diffusion operators on unstructured meshes in any space dimension. *IMA J. Numer. Anal.*, 26(2):326–353, 2006.

[6] R. Herbin. An error estimate for a finite volume scheme for a diffusion-convection problem on a triangular mesh. *Numer. Methods Partial Differential Equations*, 11(2):165–173, 1995.

[7] C. Le Potier. Schéma volumes finis pour des opérateurs de diffusion fortement anisotropes sur des maillages non structurés. *C. R. Math. Acad. Sci. Paris*, 340(12):921–926, 2005.