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The Stationary Boltzmann Equation in the Slab with Given Weighted Mass for Hard and Soft Forces

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Abstract. The stationary Boltzmann equation for hard and soft forces is considered in the slab. An L^1 existence theorem is proven in a given indata context with fixed total weighted mass. In the proof a new direct approach is introduced, which uses a certain coupling between mass and boundary flow. Compactness properties are extracted from entropy production estimates and from the boundary behaviour.

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1. – Introduction

Consider the stationary Boltzmann equation in the slab,

$$\xi \frac{\partial}{\partial x} f(x, v) = Q(f, f)(x, v), \; x \in [-1, 1], \quad v \in \mathbb{R}^3 . \tag{1.1}$$

The nonnegative function $f(x, v)$ represents the density of a rarefied gas at position x and velocity v. The collision operator Q is the classical Boltzmann operator

$$\begin{aligned} Q(f, f)(x, v) &= \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \omega)[f' f'^* - f f^*] \, d\omega dv_* \\ &= Q^+(f, f) - Q^-(f, f), \end{aligned}$$

where $Q^+ - Q^-$ is the splitting into gain and loss terms,

$$\begin{aligned} &f^* = f(x, v_*), \quad f' = f(x, v'), \quad f'^* = f(x, v'_*), \\ &v' = v - (v - v_*, \omega)\omega, \quad v'_* = v_* + (v - v_*, \omega)\omega . \end{aligned}$$

The velocity component in the x-direction is denoted by ξ, and $(v - v_*, \omega)$ denotes the Euclidean inner product in $\mathbb{R}^3$. Let ω be represented by the polar

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angle (with polar axis along $v - v_*$) and the azimuthal angle ϕ. The function $B(v - v_*, \omega)$ is the collision kernel of the collision operator Q, and is taken as $| v - v_* |^\beta \, b(\theta)$, with

$$-3 < \beta < 2, \quad b \in L^1_+(0, 2\pi), \quad b(\theta) \geq c > 0, \text{ a.e.}$$

Given positive indata f_b bounded away from zero on compacts and a constant $M > 0$, solutions f to (0.1) are studied with

$$\int_{-1}^{1} \int_{\mathbb{R}^3} (1+ | v |)^\beta f(x, v) \, dx \, dv = M \,, \tag{1.2}$$

$$f(-1, v) = k f_b(-1, v), \; \xi > 0, \quad f(1, v) = k f_b(1, v), \xi < 0 \,, \tag{1.3}$$

for some constant $k > 0$. The constant k is determined from the value M of the β-norm (1.2).

For a general introduction to the Boltzmann equation and the problem area see [5], [6]. We refer to [3] for a review about earlier results concerning the linearized Boltzmann equation as well as the non-linear Boltzmann equation in specific cases like close to equilibrium, small domains etc..

The existence of L^1 solutions to the stationary Boltzmann equation in a slab for maxwellian and hard forces was the main result in [3]. The approach in that paper starts by a classical transformation of the space variable resulting in a homogeneous equation of degree one, compatible with the boundary conditions. That transformation is not well adapted for generalizations to several space variables in the Boltzmann equation case. To avoid it, in the present paper an alternative straightforward approach is taken, which here delivers existence results for maxwellian and hard forces as in [3]. The new approach is also well suited to the mild solution concept used for soft forces, and this paper includes the first general existence proof for the nonlinear stationary Boltzmann equation with soft forces in a slab. The weighted mass is kept constant during the whole sequence of approximations. The constancy of the weighted mass is important for connecting the distribution function inside the slab to its values on the boundary via the exponential form of the equation.

We have chosen to present our approach in [3] for the case of boundary conditions of diffuse reflection type, and in this paper for given indata boundary conditions, but both approaches can be used for both types of boundary conditions. Those parts of the proofs that rely on the boundary behaviour can with advantage be treated differently depending on the boundary conditions, the present given indata paper containing a number of simplifications in comparison with the diffuse reflection paper [3]. In both papers all results hold with analogous proofs when the velocities v are in $\mathbb{R}^n$, $n \geq 2$. Also a number of generalizations of B can be analyzed straightforwardly by the same approach (see [3] for more details). As for the multiplicative constants appearing in the boundary values (1.3), they are proven to belong to a compact subset of $\{k \in \mathbb{R}; k > 0\}$. Here and in several other proofs of this paper, the boundedness of the entropy production term and of the second moment $\int \xi^2 f(v)dv$ in the slab direction play an important role. Similar arguments have earlier been used in [1] for deriving a compactness lemma in weak L^1 as well as an existence theorem for an L^1 solution of the stationary Boltzmann equation in L^1 under a truncation for small velocities.

Denote the collision frequency by

$$\nu(x, v) := \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \omega) f(x, v_*) dv_* \, d\omega .$$

Assuming that $\frac{Q^+(f,f)}{1+f} \in L^1_{\text{loc}}$, $\frac{Q^-(f,f)}{1+f} \in L^1_{\text{loc}}$, the exponential, mild and weak solution concepts in the stationary context (1.1-3) can be formulated as follows.

Definition 1.1. f is an exponential solution to the stationary Boltzmann problem (1.1-3) with β-norm M, if $f \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, $\nu \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, $\int (1+ \mid v \mid)^\beta f(x, v) \, dx \, dv = M$, and there is a constant $k > 0$ such that for almost all v in $\mathbb{R}^3$,

$$\begin{aligned} f(1 + s\xi, v) &= k f_b(-1, v) e^{-\int_{-\frac{2}{\xi}}^{s} \nu(1+\tau\xi, v) d\tau} + \int_{-\frac{2}{\xi}}^{s} e^{-\int_\tau^s \nu(1+\sigma\xi, v) d\sigma} Q^+(f, f) \\ &\quad \times (1 + \tau\xi, v) d\tau, \ \xi > 0, \ \ s \in \left(-\frac{2}{\xi}, 0\right), \\ f(-1 + s\xi, v) &= k f_b(1, v) e^{-\int_{\frac{2}{\xi}}^{s} \nu(-1+\tau\xi, v) d\tau} + \int_{\frac{2}{\xi}}^{s} e^{-\int_\tau^s \nu(-1+\sigma\xi, v) d\sigma} Q^+(f, f) \\ &\quad \times (-1 + \tau\xi, v) d\tau, \ \xi < 0, \ \ s \in \left(\frac{2}{\xi}, 0\right). \end{aligned}$$

Definition 1.2. f is a mild solution to the stationary Boltzmann problem (1.1-3) with β-norm M, if $f \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, $\int (1+ \mid v \mid)^\beta f(x, v) \, dx \, dv = M$, and there is a constant $k > 0$ such that for almost all v in $\mathbb{R}^3$,

$$\begin{aligned} f(1 + s\xi, v) &= k f_b(-1, v) + \int_{-\frac{2}{\xi}}^{s} Q(f, f)(1 + \tau\xi, v) d\tau, \ \xi > 0, \ s \in \left(-\frac{2}{\xi}, 0\right), \\ f(-1 + s\xi, v) &= k f_b(1, v) + \int_{\frac{2}{\xi}}^{s} Q(f, f)(-1 + \tau\xi, v) d\tau, \ \xi < 0, \ s \in \left(\frac{2}{\xi}, 0\right). \end{aligned}$$

Here the integrals for Q^+ and Q^- are assumed to exist separately.

Definition 1.3. f is a weak solution of the stationary Boltzmann problem (1.1-3) with β-norm M, if $f \in L^1_{\text{loc}}((-1, 1) \times \mathbb{R}^3)$, $\int (1+ \mid v \mid)^\beta f(x, v) \, dx \, dv = M$, and there is a constant $k > 0$ such that

$$\begin{aligned} &\int_{-1}^{1} \int_{\mathbb{R}^3} \left(\xi f \frac{\partial \varphi}{\partial x} + Q(f, f) \varphi \right) (x, v) \, dx \, dv \\ &= k \int_{v \in \mathbb{R}^3; \xi < 0} \xi f_b \varphi(1, v) dv - k \int_{v \in \mathbb{R}^3; \xi > 0} \xi f_b \varphi(-1, v) dv , \end{aligned} \tag{1.4}$$

for every $\varphi \in C_c^1([-1,1] \times \mathbb{R}^3)$ with $supp\varphi \subset [-1,1] \times \{v \in \mathbb{R}^3; \mid \xi \mid \geq \delta\}$ for some $\delta > 0$ and φ vanishing on $\{(-1,v); \xi < 0\} \cup \{(1,v); \xi > 0\}$. In (1.4) the integrals for Q^+ and for Q^- are assumed to exist separately.

REMARK. This weak form is somewhat stronger than the mild and exponential ones.

Suppose

$$
(1.5)\qquad
\begin{aligned}
&\int_{\xi>0} \left[\xi(1+\mid v\mid^2+\mid \log f_b \mid) + (1+\mid v \mid)^\beta\right] f_b(-1,v)dv < \infty\,,\\
&\int_{\xi<0} \left[\,\mid \xi \mid (1+\mid v\mid^2+\mid \log f_b \mid) + (1+\mid v \mid)^\beta\right] f_b(1,v)dv < \infty\,.
\end{aligned}
$$

The main result of this paper is the proof by a direct method of the following theorems.

THEOREM 1.1. *Given β with $0 \leq \beta < 2$, indata f_b satisfying* (1.5), *and $M > 0$, there is a weak solution to the stationary problem* $(1.1-3)$ *with β-norm M.*

THEOREM 1.2. *Given β with $-3 < \beta < 0$, indata f_b satisfying* (1.5), *and $M > 0$, there is a mild solution to the stationary problem* $(1.1-3)$ *with β-norm M.*

In Section 2 approximate solutions are obtained for this existence problem. Based on those approximate solutions, Theorem 1.1 is proven in Section 3, and Theorem 1.2 in Section 4.

Starting from the slab solutions, results on the small mean free path limit and the half-space problem can be obtained. Such questions, however, are left for a following paper [4].

2. – Approximations with fixed total mass.

The first approximation below of (1.1-3) is of the same type as in the Povzner paper [2] and in the transform based paper on the Boltzmann equation in a slab [3]. In contrast to those papers, here the approximation is carried out directly on the quadratic collision operator. Its main characteristics are truncations bounding domains of integration and integrands in the collision operators. Necessary compactness properties are inserted "by hand" through convolution with mollifiers. Solutions are obtained via strong L^1 fixed point techniques. We give the main steps and refer to [3] for easily adaptable details.

For convenience, take $M = 1$ and in this and the following section $0 \leq \beta < 2$. With $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $j \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, $n \in \mathbb{N}^*$, $l > 0$, $r > 0$, $\mu > 0$, and $\alpha \in]0,1[$, define the map T on the closed and convex subset of $L^1([-1,1] \times \mathbb{R}^3)$

$$
K := \left\{ f \in L^1_+\left([-1,1] \times \mathbb{R}^3\right); \int \min\left(\mu, (1+\mid v \mid)^\beta\right) f(x,v)\, dx\, dv = 1 \right\}\,,
$$

by

$$T(f) = \frac{F}{\int \min(\mu, (1+ \mid v \mid)^{\beta}) F(x,v)\, dx\, dv},$$

where F is the solution to

$$\alpha F + \xi \frac{\partial F}{\partial x} = \int \chi^r B_{m,n,\mu}(v, v_*, \omega) \left(\frac{F}{1+\frac{F}{j}}(x, v') \frac{f * \varphi_l}{1+\frac{f*\varphi_l}{j}}(x, v'_*) \right.$$

$$\left. - F \frac{f * \varphi_l}{1+\frac{f*\varphi_l}{j}}(x, v_*) \right) dv_*\, d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3, \tag{2.1}$$

$$F(-1, v) = \lambda f_b^n(-1, v), \quad \xi > 0, \qquad F(1, v) = \lambda f_b^n(1, v), \quad \xi < 0.$$

The function $\chi^r(v, v_*, \omega)$ is invariant under the collision transformation defined by $J(v, v_*, \omega) = (v', v'_*, -\omega)$, invariant under an exchange of v and v_*, and satisfies $\chi^r \in C^\infty$, $0 \le \chi^r \le 1$,

$$\chi^r(v, v_*, \omega) = 1 \text{ if } \mid \xi \mid > r, \quad \mid \xi_* \mid > r, \quad \mid \xi' \mid > r, \quad \mid \xi'_* \mid > r,$$
$$\chi^r(v, v_*, \omega) = 0 \text{ if } \mid \xi \mid < \frac{r}{2}, \text{ or } \mid \xi_* \mid < \frac{r}{2}, \text{ or } \mid \xi' \mid < \frac{r}{2}, \text{ or } \mid \xi'_* \mid < \frac{r}{2}.$$

$B_{m,n,\mu}$ is a positive C^∞ function approximating $\min(B, \mu)$ when

$$v^2 + v_*^2 < \frac{\sqrt{n}}{2},$$
$$\text{and } \left| \frac{v - v_*}{\mid v - v_* \mid} \cdot \omega \right| > \frac{1}{m}, \text{ and } \left| \frac{v - v_*}{\mid v - v_* \mid} \cdot \omega \right| < 1 - \frac{1}{m},$$
$$B_{m,n,\mu}(v, v_*, \omega) = 0, \text{ if } v^2 + v_*^2 > \sqrt{n}$$
$$\text{or } \left| \frac{v - v_*}{\mid v - v_* \mid} \cdot \omega \right| < \frac{1}{2m}, \text{ or } \left| \frac{v - v_*}{\mid v - v_* \mid} \cdot \omega \right| > 1 - \frac{1}{2m}.$$

The functions φ_l are mollifiers in the x-variable defined by $\varphi_l(x) := l\varphi(lx)$, where

$$\varphi \in C_0^\infty(\mathbb{R}), \text{ support } (\varphi) \subset (-1, 1), \varphi \ge 0, \int_{-1}^{1} \varphi(x)\, dx = 1.$$

Notice that by the χ^r–truncation

$$F(x, v) = \lambda f_b^n(-1, v) e^{-\alpha \frac{1+x}{\xi}} \ge \lambda f_b^n(-1, v) e^{-\frac{2}{\xi}}, \quad 0 < \xi \le \frac{r}{2},$$

and that by the exponential form

$$F(x,v) \geq \lambda f_b^n(-1,v) e^{-\alpha\frac{1+x}{\xi} - \int_{-\frac{1+x}{\xi}}^{0} \int \chi^r B_{m,n,\mu} f*\varphi_l(x+\tau\xi, v_*) dv_* \, d\omega d\tau}$$
$$\geq e^{-\frac{c_*}{\xi}} \lambda f_b^n(-1,v), \quad \xi > \frac{r}{2}.$$

Here $c_* \geq 2$ depends only on μ and r, since for $f \in K$

$$\int \min(\mu, (1+\mid v\mid)^\beta) f(x,v)\, dx\, dv = 1\,.$$

Analogously

$$F(x,v) \geq \lambda f_b^n(1,v) e^{-\frac{c_*}{|\xi|}}, \quad \xi < 0\,.$$

λ is defined by

$$\lambda := \frac{1}{\int_{\xi>0} \min(\mu,(1+|v|)^\beta) f_b^n(-1,v) e^{-\frac{c_*}{\xi}} dv + \int_{\xi<0} \min(\mu,(1+|v|)^\beta) f_b^n(1,v) e^{-\frac{c_*}{|\xi|}} dv},$$

so that, by the above estimates of F from below by ingoing boundary values,

$$\int \min(\mu, (1+\mid v\mid)^\beta) F(x,v)\, dx\, dv \geq 1\,.$$

With $f_b^n := f_b \wedge n$, then (for j large) $f_b^n \leq j$, and $\lim_{n\to+\infty} \| f_b^n - f_b \|_{L^1} = 0$. Notice that c_* can be taken so that for r, μ fixed, λ decreases when n is increasing, with the infimum λ_i strictly positive. Denote by λ_s the corresponding value of λ for $n = 1$. Following the lines of the proofs in Section 2 of [3], one can show that the map T is continuous and compact in K with the strong L^1 topology. By the Schauder fixed point theorem, there are integrable functions f and F, solutions to

$$\alpha f + \xi \frac{\partial f}{\partial x} = \int \chi^r B_{m,n,\mu} \left(\frac{f}{1+\frac{F}{j}}(x,v') \frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v'_*) - f \frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*) \right) dv_*\, d\omega, \quad (x,v) \in (-1,1)\times\mathbb{R}^3, \tag{2.2}$$

$$f(-1,v) = \lambda k f_b^n(-1,v),\ \xi > 0, \quad f(1,v) = \lambda k f_b^n(1,v),\ \xi < 0,$$

$$\alpha F+\xi\frac{\partial F}{\partial x}=\int\chi^r B_{m,n,\mu}\left(\frac{F}{1+\frac{F}{j}}(x,v')\frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v'_*)\right.$$

(2.3)

$$\left.-F\frac{f*\varphi_l}{1+\frac{f*\varphi_l}{j}}(x,v_*)\right)dv_*\,d\omega,\quad (x,v)\in(-1,1)\times\mathbb{R}^3,$$

$$F(-1,v)=\lambda f_b^n(-1,v),\ \xi>0,\quad F(1,v)=\lambda f_b^n(1,v),\ \xi<0,$$

with

$$\int\min(\mu,(1+\mid v\mid)^\beta)f(x,v)\,dx\,dv=1,\quad f=kF,$$

i.e.

$$k:=\frac{1}{\int\min(\mu,(1+\mid v\mid)^\beta)F(x,v)\,dx\,dv}.$$

Let us pass to the limit when $l\to\infty$. Again as in Section 2 of [3] the gain terms in (2.2-3) are strongly compact in L^1. By the exponential form of F and using the truncation of the collision operator by χ^r,

$$F(x,v)\le F(1,v)e^{\frac{c*}{r}},\ \xi>\frac{r}{2},\qquad F(x,v)\le F(-1,v)e^{\frac{c*}{r}},\ \xi<-\frac{r}{2},$$
$$F(x,v)\le\lambda f_b^n(-1,v),\ 0<\xi<\frac{r}{2},\quad F(x,v)\le\lambda f_b^n(1,v),\ -\frac{r}{2}<\xi<0,$$

so that

$$\int\min(\mu,(1+\mid v\mid)^\beta)F(x,v)\,dx\,dv\le c\int(1+\mid v\mid^2)F(x,v)\,dx\,dv$$
$$\le ce^{\frac{c*}{r}}\left(\int_{\xi>\frac{r}{2}}(1+\mid v\mid^2)F(1,v)dv+\int_{\xi<-\frac{r}{2}}(1+\mid v\mid^2)F(-1,v)dv\right)$$
$$+c\lambda\left(\int_{0<\xi<\frac{r}{2}}(1+\mid v\mid^2)f_b^n(-1,v)dv+\int_{-\frac{r}{2}<\xi<0}(1+\mid v\mid^2)f_b^n(1,v)dv\right).$$

Multiplying (2.3) by 1 and $\mid v\mid^2$ and integrating it on $(-1,1)\times\mathbb{R}^3$ implies that

$$\int_{\xi>0}(1+\mid v\mid^2)F(1,v)dv+\int_{\xi<0}(1+\mid v\mid^2)F(-1,v)dv\le c_r.$$

Hence k belongs to some interval $[k_*,1]$, with k_* independent of j, m, n, l, and α, and so as above strong compactness in $L^1\times[k_*,1]$ can be used to get a non trivial limit in (2.2-3) when $l\to+\infty$. The passage to the limit when

α tends to zero is performed analogously. So there are functions f^j and F^j solutions to

$$
\xi \frac{\partial f^j}{\partial x} = \int \chi^r B_{m,n,\mu} \left(\frac{f^j}{1+\frac{F^j}{j}}(x,v') \frac{f^j}{1+\frac{f^j}{j}}(x,v'_*) \right.
$$

(2.4)

$$
\left. - f^j(x,v) \frac{f^j}{1+\frac{f^j}{j}}(x,v_*) \right) dv_* \, d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3,
$$

$$
f^j(-1,v) = \lambda k_j f_b^n(-1,v),\ \xi > 0, \quad f^j(1,v) = \lambda k_j f_b^n(1,v),\ \xi < 0,
$$

$$
\xi \frac{\partial F^j}{\partial x} = \int \chi^r B_{m,n,\mu} \left(\frac{F^j}{1+\frac{F^j}{j}}(x,v') \frac{f^j}{1+\frac{f^j}{j}}(x,v'_*) \right.
$$

$$
\left. - F^j(x,v) \frac{f^j}{1+\frac{f^j}{j}}(x,v_*) \right) dv_* \, d\omega, \quad (x,v) \in (-1,1) \times \mathbb{R}^3,
$$

$$
F^j(-1,v) = \lambda f_b^n(-1,v),\ \xi > 0, \quad F^j(1,v) = \lambda f_b^n(1,v),\ \xi < 0,
$$

with

$$
\int \min(\mu, (1+|v|)^\beta) f^j(x,v) \, dx \, dv = 1, \quad f^j = k_j F^j, \tag{2.5}
$$

$$
k_j = \frac{1}{\int \min(\mu, (1+|v|)^\beta) F^j \, dx \, dv}.
$$

By the exponential form of (2.4), and (2.5),

$$
f^j(x,v) \le e^{\frac{c*}{r}} f^j(1,v), \quad \xi > \frac{r}{2}, \qquad f^j(x,v) \le e^{\frac{c*}{r}} f^j(-1,v), \quad \xi < -\frac{r}{2}.
$$

Hence

$$
\int (1+|v|^2) f^j(x,v) \, dx \, dv < \infty,
$$

uniformly with respect to j. Multiply (2.4) by $\log \frac{f^j}{1+\frac{f^j}{j}} + 1$, and notice that $F^j \geq f^j$, since $F^j = \frac{1}{k_j} f^j$ and $k_j \leq 1$. Then

$$\int \xi \left(f^j \log f^j - j \left(1 + \frac{f^j}{j}\right) \log \left(1 + \frac{f^j}{j}\right)\right)(1, v) dv$$

$$- \int \xi \left(f^j \log f^j - j \left(1 + \frac{f^j}{j}\right) \log \left(1 + \frac{f^j}{j}\right)\right)(-1, v) dv$$

$$\leq \int \chi^r B_{m,n,\mu} \left(\frac{f^{j'}}{1+\frac{f^{j'}}{j}} \frac{f^{j'}_*}{1+\frac{f^{j'}_*}{j}} - \frac{f^j}{1+\frac{f^j}{j}} \frac{f^j_*}{1+\frac{f^j_*}{j}} \right) \log \frac{f^j}{1+\frac{f^j}{j}} \, dx\, dv\, dv_*\, d\omega$$

$$+ \int \chi^r B_{m,n,\mu} \frac{f^{j'}(f^{j'} - F^{j'})}{j\left(1+\frac{F^{j'}}{j}\right)\left(1+\frac{f^{j'}}{j}\right)} \frac{f^{j'}_*}{1+\frac{f^{j'}_*}{j}} \log \frac{f^j}{1+\frac{f^j}{j}} \, dx\, dv\, dv_*\, d\omega$$

$$- \int \chi^r B_{m,n,\mu} \frac{(f^j)^2}{j\left(1+\frac{f^j}{j}\right)} \frac{f^j_*}{\left(1+\frac{f^j_*}{j}\right)} \log \frac{f^j}{1+\frac{f^j}{j}} \, dx\, dv\, dv_*\, d\omega \leq -\frac{1}{4} e(f^j, f^j)$$

$$+ \int_{\frac{f^j}{1+\frac{f^j}{j}} < 1} \chi^r B_{m,n,\mu} \frac{f^{j'}(f^{j'} - F^{j'})}{j\left(1+\frac{F^{j'}}{j}\right)\left(1+\frac{f^{j'}}{j}\right)} \frac{f^{j'}_*}{1+\frac{f^{j'}_*}{j}} \left| \log \frac{f^j}{1+\frac{f^j}{j}} \right| dx\, dv\, dv_*\, d\omega + c\,.$$

Here $e(f^j, f^j)$ is the non-negative term defined by

$$e(f^j, f^j) := \int \chi^r B_{m,n,\mu} \left(\frac{f^{j'}}{1+\frac{f^{j'}}{j}} \frac{f^{j'}_*}{1+\frac{f^{j'}_*}{j}} - \frac{f^j}{1+\frac{f^j}{j}} \frac{f^j_*}{1+\frac{f^j_*}{j}} \right)$$

$$\log \frac{\frac{f^{j'}}{1+\frac{f^{j'}}{j}} \frac{f^{j'}_*}{1+\frac{f^{j'}_*}{j}}}{\frac{f^j}{1+\frac{f^j}{j}} \frac{f^j_*}{1+\frac{f^j_*}{j}}} \, dx\, dv\, dv_*\, d\omega\,.$$

Moreover,

$$f^j(x, v) \geq c'_r f^n_b(-1, v) \geq c_r, \quad \xi > 0, \quad |v|^2 \leq \sqrt{n}\,,$$
$$f^j(x, v) \geq c'_r f^n_b(1, v) \geq c_r, \quad \xi < 0, \quad |v|^2 \leq \sqrt{n}\,,$$

so that

$$\int_{\frac{f^j}{1+\frac{f^j}{j}}<1} \chi^r B_{m,n,\mu} \frac{f^{j\prime} \mid f^{j\prime} - F^{j\prime} \mid}{j\left(1+\frac{F^{j\prime}}{j}\right)\left(1+\frac{f^{j\prime}}{j}\right)} \frac{f_*^{j\prime}}{1+\frac{f_*^{j\prime}}{j}} \left| \log \frac{f^j}{1+\frac{f^j}{j}} \right| dx\, dv\, dv_*\, d\omega$$

$$\le \int_{\frac{c}{1+\frac{c}{j}} \le \frac{f^j}{1+\frac{f^j}{j}} < 1} \chi^r B_{m,n,\mu} f^{j\prime} f_*^{j\prime} \left| \log \frac{f^j}{1+\frac{f^j}{j}} \right| dx\, dv\, dv_*\, d\omega$$

$$\le \left| \log \frac{c}{1+\frac{c}{j}} \right| \int \chi^r B_{m,n,\mu} f^{j\prime} f_*^{j\prime}\, dx\, dv\, dv_*\, d\omega \le c\,.$$

Hence for all j

$$\int \xi \left(f^j \log f^j - j\left(1+\frac{f^j}{j}\right) \log\left(1+\frac{f^j}{j}\right)\right)(1,v)dv$$
$$- \int \xi \left(f^j \log f^j - j\left(1+\frac{f^j}{j}\right) \log\left(1+\frac{f^j}{j}\right)\right)(-1,v)dv \le c < \infty\,.$$

As in [3], it follows from here that (f^j) is weakly compact in L^1, that we can pass to the limit in (2.4) when $m = \frac{1}{j}$, $j \to \infty$, and that the entropy production term in the limit is bounded uniformly with respect to n. The passage to the limit when $n \to \infty$ is performed via weak L^1 compactness arguments. The uniform boundedness from above of $\int \xi^2 f^n(x,v)dv$ is there used to obtain weak L^1 compactness of $(Q^-(f^n, f^n))$ from the weak L^1 compactness of (f^n). And the weak L^1 compactness of $(Q^+(f^n, f^n))$ is then obtained from the weak L^1 compactness of $(Q^-(f^n, f^n))$ together with the boundedness of the entropy production term. So there is a solution $F^{r,\mu} = w \lim f^n$ to

$$\xi \frac{\partial F^{r,\mu}}{\partial x} = \int \chi^r \min(B,\mu)(F^{r,\mu}(x,v')F^{r,\mu}(x,v'_*)$$
$$- F^{r,\mu}(x,v)F^{r,\mu}(x,v_*))dv_*\, d\omega\,, \qquad (x,v) \in (-1,1)\times\mathbb{R}^3\,,$$

(2.6)
$$F^{r,\mu}(-1,v) = k_{r,\mu} f_b(-1,v), \quad \xi > 0\,,$$
$$F^{r,\mu}(1,v) = k_{r,\mu} f_b(1,v), \quad \xi < 0\,,$$
$$\int \min(\mu, (1+\mid v \mid)^\beta) F^{r,\mu}(x,v)\, dx\, dv = 1\,,$$

for some constant $k_{r,\mu}$ with $0 < \lambda_i k_* \le k_{r,\mu} \le \lambda_s$.

3. – The slab solution for $0 \le \beta < 2$.

For proving the existence Theorem 1.1, it remains to pass to the limit in (2.6) when r tends to zero and μ tends to infinity.

LEMMA 3.1. *There are $c > 0$, $\bar{c} > 0$, and for $\delta > 0$ constants $c_\delta > 0$ and $\bar{c}_\delta > 0$, such that*

$$\begin{aligned}
&\int \xi^2 F^{r,\mu}(x,v)dv < ck_{r,\mu}, && x \in (-1,1),\\
&F^{r,\mu}(x,v) > \bar{c}_\delta k_{r,\mu} f_b(-1,v), && x \in (-1,1),\ \xi > \delta,\ | v | \le \frac{1}{\delta},\\
&F^{r,\mu}(x,v) > \bar{c}_\delta k_{r,\mu} f_b(1,v), && x \in (-1,1),\ \xi < -\delta,\ | v | \le \frac{1}{\delta},\\
&\int_{|\xi|>\delta, |v|\le\frac{1}{\delta}} | v |^2\, F^{r,\mu}(x,v)\, dx\, dv < c_\delta k_{r,\mu}\,.
\end{aligned}$$

PROOF OF LEMMA 3.1. In the first inequality, the left-hand side is a constant of the motion, so it is enough to consider $x = -1$. But there the inequality follows from Green's formula since $| \xi | \le 1 + | v |^2$. By the exponential form of (2.6) and using ingoing boundary values,

$$\begin{aligned}
F^{r,\mu}(x,v) &\ge F^{r,\mu}(-1,v)e^{-\int_{-\frac{1+x}{\xi}}^0 \int \chi^r \min(B,\mu) F^{r,\mu}(x+\tau\xi, v_*)dv_*\, d\omega d\tau}\\
&\ge e^{-c(\delta)} k_{r,\mu} f_b(-1,v),\ \xi > \delta,\ | v | \le \frac{1}{\delta},\\
F^{r,\mu}(x,v) &\ge F^{r,\mu}(1,v)e^{-\int_{\frac{1-x}{\xi}}^0 \int \chi^r \min(B,\mu) F^{r,\mu}(x+\tau\xi, v_*)dv_*\, d\omega d\tau}\\
&\ge e^{-c(\delta)} k_{r,\mu} f_b(1,v),\ \xi < -\delta,\ | v | \le \frac{1}{\delta}\,.
\end{aligned}$$

Similarly using outgoing boundary values,

$$\begin{aligned}
&\int_{|\xi|>\delta, |v|\le\frac{1}{\delta}} | v |^2\, F^{r,\mu}(x,v)\, dx\, dv\\
&= \int_{\xi>\delta, |v|\le\frac{1}{\delta}} \xi\, | v |^2 \int_{-\frac{2}{\xi}}^0 F^{r,\mu}(1+s\xi, v)dsdv\\
&\quad + \int_{\xi<-\delta, |v|\le\frac{1}{\delta}} | \xi || v |^2 \int_{\frac{2}{\xi}}^0 F^{r,\mu}(-1+s\xi, v)dsdv\\
&\le e^{\bar{c}_\delta} k_{r,\mu} \left(\int_{\xi>\delta} \xi\, | v |^2\, F^{r,\mu}(1,v)dv + \int_{\xi<-\delta} | \xi || v |^2\, F^{r,\mu}(-1,v)dv \right)\\
&\le c_\delta k_{r,\mu}\,. \qquad \square
\end{aligned}$$

LEMMA 3.2.
$$\sup_{r\leq r_0,\mu\geq\mu_0} k_{r,\mu} = k_0 < \infty .$$

PROOF OF LEMMA 3.2. It follows from Lemma 3.1 that

$$F^{r,\mu}(x,v) \geq ck_{r,\mu} f_b(-1,v), \quad \xi > \frac{1}{2}, \mid v \mid\leq 2,$$
$$F^{r,\mu}(x,v) \geq ck_{r,\mu} f_b(1,v), \quad \xi < -\frac{1}{2}, \mid v \mid\leq 2,$$

so that
$$1 = \int \min(\mu, (1+\mid v \mid)^{\beta}) F^{r,\mu}(x,v)\, dx\, dv \geq ck_{r,\mu},$$

and
$$k_{r,\mu} \leq k_0 := \frac{1}{c}. \qquad \square$$

It follows by Lemma 3.2 that the entropy dissipation for $F^{r,\mu}$ is bounded by $c_e k_{r,\mu}$, where c_e is independent of r and μ.

LEMMA 3.3. *If* $0 \leq \beta < 2$ *in the collision kernel* B, *then*

$$\underline{\lim}_{r\to 0,\mu\to\infty} k_{r,\mu} > 0 .$$

PROOF OF LEMMA 3.3. Let us prove Lemma 3.3 by contradiction.
If $\underline{\lim}_{r\to 0,\mu\to\infty} k_{r,\mu} = 0$, there are sequences (r_j) and (μ_j) with $\lim_{j\to\infty} r_j = 0$ and $\lim_{j\to\infty} \mu_j = +\infty$, such that $k_j := k_{r_j,\mu_j}$ tends to zero when j tends to infinity. But that leads to the following contradiction with the entropy dissipation bound $c_e k_{r,\mu}$. Denote by $F^j := F^{r_j,\mu_j}$. Write $v = (\xi,\eta,\zeta)$ and $\rho = \sqrt{\eta^2+\zeta^2}$. By Lemma 3.1

$$\int_{10|\xi|\geq\rho} \mid v \mid^2 F^j(x,v)dv \leq c\int \xi^2 F^j(x,v)dv \leq ck_j .$$

Again by Lemma 3.1, there is a constant $\bar{c}$ independent of j and x, such that

$$\frac{F^j}{k_j}(x,v_*) \geq \bar{c}, \quad \rho_* \leq 10, \quad \frac{1}{10} \leq\mid \xi_* \mid\leq 1 .$$

For these v_* and for v with $\mid \xi \mid\leq \frac{\lambda}{10}$, $\rho \geq \lambda \gg 10$, we can for some $c'' > 0$, take a set of $\omega \in S^2$ depending on x, v, v_* of measure c'', so that

$$\mid \xi' \mid\geq c \mid v \mid, \quad \rho' \geq \frac{1}{4}\rho, \quad \mid \xi'_* \mid\geq c \mid v \mid, \quad \rho'_* \geq \frac{1}{4}\rho .$$

Denote by $B_j := \chi^{r_j}\min(\mu_j, B)$ and by $\bar{F}^j = \frac{F^j}{k_j}$. Fix $\epsilon \ll 1$ and take $|\xi| \geq r_j$. For these x, v, v_*, ω, r_j, μ_j, and for $L > 2$,

$$\begin{aligned}\bar{c}b(\theta)F^j(x,v)\min(\mu_j,(1+|v|)^\beta) &\leq b(\theta)\min(\mu_j,(1+|v|)^\beta)F^j(x,v)\bar{F}^j(x,v_*)\\ &\leq cLb(\theta)(|\xi'|^\beta + |\xi'_*|^\beta)k_j\bar{F}^{j'}\bar{F}^{j'}_* + \frac{c}{k_j\log L}B_j(F^jF^j_* - F^{j'}F^{j'}_*)\log\frac{F^jF^j_*}{F^{j'}F^{j'}_*}\,.\end{aligned}$$

Consequently,

$$c\int_{\frac{\lambda}{10}\geq|\xi|\geq r_j,\rho>\lambda}\min(\mu_j,(1+|v|)^\beta)F^j(x,v)\,dx\,dv \leq \frac{cL}{\lambda^{4-\beta}} + \frac{cc_e}{\log L} < \epsilon\,,$$

for L large enough and a suitable $\lambda(L)$. In the same way

$$c\int_{\lambda\leq 10|\xi|\leq\rho}\min(\mu_j,(1+|v|)^\beta)F^j(x,v)\,dx\,dv \leq \frac{cL}{\lambda^{4-\beta}} + \frac{cc_e}{\log L} < \epsilon\,.$$

Similarly for $\rho \leq \lambda$,

$$c\int_{\frac{\lambda}{10}\geq|\xi|\geq r_j,\rho\leq\lambda}\min(\mu_j,(1+|v|)^\beta)F^j(x,v)\,dx\,dv \leq Lck_j + \frac{cc_e}{\log L} < \epsilon\,,$$

for j large enough. Finally there exists j_0, such that

$$\begin{aligned}&\int_{|\xi|\leq r_j}F^j(x,v)(1+|v|)^\beta dv\\ &= k_j\int_{0<\xi<r_j}f_b(-1,v)(1+|v|)^\beta dv + k_j\int_{-r_j<\xi<0}f_b(1,v)(1+|v|)^\beta dv < \epsilon\,,\end{aligned}$$

for $j > j_0$, since $k_j \leq k_0$ and $\lim_{j\to\infty} r_j = 0$. And so, if $\lim_{j\to\infty} k_j = 0$, then for j large enough,

$$1 > 5\epsilon > \int\min(\mu_j,(1+|v|)^\beta)F^j(x,v)\,dx\,dv = 1\,.$$

This contradiction implies that $\underline{\lim}_{r\to 0,\mu\to\infty}k_{r,\mu} > 0$. □

We may now choose r_1, μ_0, and k_1, so that $k_1 \leq k_{r,\mu} \leq k_0$ for $0 < r \leq r_1$, $\mu \geq \mu_0$.

LEMMA 3.4. *For $\delta > 0$, the family $(F^{r,\mu})_{0<r\leq r_1,\mu\geq\mu_0}$ is weakly precompact in* $L^1((-1,1)\times\{v\in I\!R^3; |\xi|\geq\delta, |v|\leq\frac{1}{\delta}\})$.

Proof of Lemma 3.4. From Lemma 3.1 it follows that for any $\delta > 0$,

$$\sup_{x\in(-1,1)} \int_{|\xi|>\delta} F^{r,\mu}(x,v)dv \le \frac{c}{\delta^2},$$

and

$$\begin{aligned}
&\int_{-1}^{1}\int_{|\xi|>\delta,|v|\le\frac{1}{\delta}} F^{r,\mu}\log F^{r,\mu}(x,v)\,dx\,dv\\
&=\int_{\xi>\delta,|v|\le\frac{1}{\delta}} \xi\int_{-\frac{2}{\xi}}^{0} F^{r,\mu}\log F^{r,\mu}(1+s\xi,v)dsdv\\
&\quad+\int_{\xi<-\delta,|v|\le\frac{1}{\delta}} |\xi| \int_{\frac{2}{\xi}}^{0} F^{r,\mu}\log F^{r,\mu}(-1+s\xi,v)dsdv\\
&\le \tilde{c}_\delta\left[\int_{\xi>\delta,|v|\le\frac{1}{\delta}} \xi F^{r,\mu}(1,v)dv+\int_{\xi<-\delta,|v|\le\frac{1}{\delta}} |\xi| F^{r,\mu}(-1,v)dv\right]\\
&\quad+\bar{c}_\delta\left[\int_{\xi>\delta,|v|\le\frac{1}{\delta}} \xi F^{r,\mu}\log F^{r,\mu}(1,v)dv+\int_{\xi<-\delta,|v|\le\frac{1}{\delta}} |\xi| F^{r,\mu}\log F^{r,\mu}(-1,v)dv\right]\\
&\le c_\delta. \qquad \square
\end{aligned}$$

Proof of Theorem 1.1. Denote by $F^j = F^{r_j,\mu_j}$, $k_j = k_{r_j,\mu_j}$. Let $(k_j)_{j\in\mathbb{N}^*}$ be a converging sequence, where

$$\lim_{j\to\infty} r_j = 0, \quad \lim_{j\to\infty} \mu_j = \infty$$

(or $\mu_j = \mu \ge \| B \|_\infty$ in the pseudo-maxwellian case). By Lemma 3.4 there is a subsequence, still denoted (F^j) with $\lim_{j\to\infty} F^j = F$ in weak $L^1([-1,1]\times\{v\in\mathbb{R}^3;\ |\xi|\ge\delta,\ |v|\le\frac{1}{\delta}\})$ for all $\delta>0$. In order to prove that $\int Q_j^{\pm}(F^j,F^j)\varphi(x,v)\,dx\,dv$ have the limits $\int Q^{\pm}(F,F)\varphi(x,v)\,dx\,dv$, we first prove the three following lemmas.

Lemma 3.5.

$$\lim_{\epsilon\to 0} \sup_{S\subset(-1,1);|S|\le\epsilon, j>\frac{1}{\epsilon}} \int_{S\times\mathbb{R}^3} K_j(v)F^j(x,v)\,dx\,dv = 0\,.$$

Here $K_j(v) := \min(\mu_j, (1+|v|)^\beta)$.

Proof of Lemma 3.5. By the exponential form, there is $\tilde{c}>0$, such that

$$\begin{aligned}
F^j(x,v) &\le F^j(1,v)e^{j\tilde{c}}, & \xi &\ge \frac{1}{j}, & |v| &\le 101\,,\\
F^j(x,v) &\le F^j(-1,v)e^{j\tilde{c}}, & \xi &\le -\frac{1}{j}, & |v| &\le 101\,.
\end{aligned}\tag{3.1}$$

By the exponential estimate (3.1),

$$\int_{|\xi|\geq \frac{1}{j},\rho\leq 100} K_j(v)F^j(x,v)dv \leq cje^{j\tilde{c}},$$

and by analogous estimates from below

$$F^j(x,v_*) \geq cF^j(-1,v_*) = ck_j f_b(-1,v_*), \quad \xi_* \geq \frac{1}{10}, \mid v_* \mid\leq 10,$$
$$F^j(x,v_*) \geq cF^j(1,v_*) = ck_j f_b(1,v_*), \quad \xi_* \leq -\frac{1}{10}, \mid v_* \mid\leq 10.$$

For some $c > 0$, $c'' > 0$, for all $\rho \geq 100$ and for half of the v_* as above (depending on v), there is a (v, v_*)-dependent set of $\omega \in S^2$ of measure c'', where independently of j, x,

$$\mid \xi' \mid \sim \mid \xi'_* \mid \sim \mid v \mid, \quad F^j(x,v') \leq c, \quad F^j(x,v'_*) \leq c.$$

If the lemma does not hold, then there are $\eta > 0$ and a subsequence, still denoted (F^j), such that for each j there is $\bar{S}_j \subset (-1,1)$ with $\mid \bar{S}_j \mid\leq \frac{\eta}{4j}e^{-\tilde{c}j-\frac{10c}{\eta}}$, and

$$\int_{\bar{S}_j}\int K_j(v)F^j(x,v)dv\,dx \geq 2\eta.$$

Then

$$K_j(v)F^j(x,v) \leq cK_j(v)F^j(x,v)F^j(x,v_*) \leq ck(|\xi'|^\beta + |\xi'_*|^\beta)F^j(x,v')F^j(x,v'_*)$$
$$+\frac{c}{\log k}B_j(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))\log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}.$$

It follows that, for $k = e^{\frac{10c}{\eta}}$

$$\int_{\bar{S}_j}\int_{\rho\geq 100} K_j(v)F^j(x,v)dv\,dx$$
$$\leq ck\int_{\bar{S}_j}\left(\int_{|\xi|\geq 1} F^j(x,v)dv\right)\left(\int_{|\xi|\geq 1} \mid \xi \mid^\beta F^j(x,v)dv\right)dx + \frac{c}{\log k} \leq \frac{\eta}{5}.$$

It follows for j large, that at least half the value of the integral over $\bar{S}_j$ comes from

$$\int_{\bar{S}_j}\int_{r_j\leq|\xi|\leq\frac{1}{j},\rho\leq 100} K_j(v)F^j(x,v)\,dv\,dx \geq \frac{\eta}{2}, \quad j \in \mathbb{N}^*.$$

Here at least half the integral comes from the set of (x, v) with $F^j(x, v) \geq c_1 j$. For each v such that $F^j(x, v) \geq c_1 j$, let

$$V_* := \left\{ v_* \in \mathbb{R}^3; \frac{1}{10} \leq \mid \xi_* \mid \leq 1, \rho_* \leq 100, \mid \rho - \rho_* \mid > 10 \right\}.$$

By Lemmas 3.1 and 3.3, $F^j(x, v_*) \geq c$, $v_* \in V_*$. Then, from the geometry of the velocities involved, and from $\int_{|\xi|\geq 1} F^j(x, v)dv \leq c$, for some $c'' > 0$, given v it holds for v_* in a subset of V_* of measure (say) $\frac{|V_*|}{2}$, and for $\omega \in S^2$ in a (v, v_*)-dependent subset of measure c'', that

$$\mid \xi' \mid \geq 1, \mid \xi'_* \mid \geq 1, \; F^j(x, v') \leq \tilde{c}, \; F^j(x, v'_*) \leq \tilde{c}.$$

It follows that, for some $c > 0$ independent of such v, $v_* \in V_*$, ω and for j large,

$$cF^j(x, v) \leq F^j(x, v)F^j(x, v_*) - F^j(x, v')F^j(x, v'_*),$$

$$\frac{F^j(x, v)F^j(x, v_*)}{F^j(x, v')F^j(x, v'_*)} \geq cj.$$

And so using the entropy dissipation estimate,

$$\int_{\bar{S}_j} \int_{r_j \leq |\xi| \leq \frac{1}{j}, \rho \leq 100} K_j(v) F^j(x, v)\, dv\, dx \leq \frac{c_2}{\log j} \leq \frac{\eta}{4}$$

for j large. The lemma follows from this contradiction. □

LEMMA 3.6. *Given $\eta > 0$, there is j_0 such that for $j > j_0$ and outside a j-dependent set in x of measure less than η,*

$$\lim_{\lambda \to \infty} \int_{\rho \geq \lambda} K_j(v) F^j(x, v) dv = 0,$$

uniformly with respect to x and j.

PROOF OF LEMMA 3.6. It follows from the geometry of the velocities involved, and from the inequality

$$\int_{|\xi| \geq \bar{c}\lambda} \mid \xi \mid^{\beta} F^j(x, v) dv \leq \frac{c}{\lambda^{2-\beta}},$$

that for some $c'' > 0$, for each (v, v_*) with $\rho \geq \lambda \gg 10$, and v_* in a suitable subset (depending on v) of

$$V_* := \left\{ v \in \mathbb{R}^3; \mid \xi_* \mid \geq \frac{1}{10}, \mid v_* \mid \leq 10 \right\},$$

and of measure exceeding $\frac{1}{2} \mid V_* \mid$, there is a (v, v_*)-dependent subset of $\omega \in S^2$ with measure c'', such that

$$F^j(x, v') \leq 1, \qquad F^j(x, v'_*) \leq 1,$$
$$\bar{c}\rho \leq \mid v' \mid \leq c \mid \xi' \mid, \quad \bar{c}\rho \leq \mid v'_* \mid \leq c \mid \xi'_* \mid .$$

Moreover, $F^j(x, v_*) \geq c$, for $\mid \xi_* \mid \geq \frac{1}{10}, \mid v_* \mid \leq 10$ with c independent of j. Hence for $|\xi| \geq r_j$,

$$c_2 K_j(v) F^j(x, v) \leq c_1 K_j(v) F^j(x, v) F^j(x, v_*) \leq K \mid \xi' \mid^\beta F^j(x, v') F^j(x, v'_*)$$
$$+ \frac{2}{\log K} B_j (F^j(x, v) F^j(x, v_*) - F^j(x, v') F^j(x, v'_*)) \log \frac{F^j(x, v) F^j(x, v_*)}{F^j(x, v') F^j(x, v'_*)} .$$

Multiplying (2.6) by $\log \frac{F^j}{k_j} + 1$ and integrating it on $(-1, 1) \times \mathbb{R}^3$ implies that

$$\int \chi^j B_j \left(F^j(x, v) \frac{F^j}{k_j}(x, v_*) - F^j(x, v') \frac{F^j}{k_j}(x, v'_*) \right)$$
$$\log \frac{F^j(x, v) F^j(x, v_*)}{F^j(x, v') F^j(x, v'_*)} dv\, dv_*\, d\omega \leq c_\eta \tag{3.2}$$

outside of a set $S'_j \subset [-1, 1]$ of measure η. Let us integrate this inequality on the above set of (v, v_*, ω), obtaining

$$\int_{|\rho| > \lambda} K_j(v) F^j(x, v) dv \leq \frac{c}{\lambda^{2-\beta}} + \frac{cK}{\lambda^{4-\beta}}$$
$$+ \frac{c}{\log K} \int \chi^j B_j (F^j(x, v) F^j(x, v_*) - F^j(x, v') F^j(x, v'_*))$$
$$\log \frac{F^j(x, v) F^j(x, v_*)}{F^j(x, v') F^j(x, v'_*)} dv\, dv_*\, d\omega \leq \frac{cK}{\lambda^{2-\beta}} + \frac{c}{\log K}, \quad x \in S_j^{\prime c} ,$$

by (3.2). K may be chosen so that $\frac{1}{\log K}$ is small, and then λ so that $\frac{cK}{\lambda^{2-\beta}}$ is small, which implies that $\int_{|\rho| > \lambda} F^j(x, v) dv$ tends to zero uniformly outside of a j-dependent set of measure bounded by η. □

LEMMA 3.7. *Given $\lambda > 0$ and $\epsilon > 0$, there is j_0 such that for $j > j_0$ and outside a j-dependent set in x of measure less than ϵ,*

$$\int_{\rho \leq \lambda, |\xi| \leq \frac{1}{i}} K_j(v) F^j(x, v) dv$$

tends to zero when $i \to +\infty$, uniformly with respect to x and j.

Proof of Lemma 3.7. Given $0 < \eta \ll 1$ and x, j, either

$$\int_{\rho\leq\lambda,|\xi|\leq\frac{1}{i}} K_j(v)F^j(x,v)dv \leq \eta^2 < \eta\,,$$

or

$$\int_{\rho\leq\lambda,|\xi|\leq\frac{1}{i}} K_j(v)F^j(x,v)dv > \eta^2\,.$$

In the latter case

$$\int_{\rho\leq\lambda,|\xi|\leq\frac{1}{i},F^j(x,v)\leq\frac{\eta^2}{4\lambda^{2+\beta}\pi}i} K_j(v)F^j(x,v)dv \leq \frac{\eta^2}{2} < \eta\,,$$

and

$$\int_{\rho\leq\lambda,|\xi|\leq\frac{1}{i},F^j(x,v)\geq\frac{\eta^2}{4\lambda^{2+\beta}\pi}i} K_j(v)F^j(x,v)dv \geq \frac{\eta^2}{2}\,.$$

For each (x, v) such that $F^j(x,v) \geq \frac{\eta^2}{4\lambda^{2+\beta}\pi}i$, take v_* in

$$V_* := \left\{ v_* \in \mathbb{R}^3;\ \frac{1}{10} \leq\mid \xi_* \mid\leq 1,\ \rho_* \leq 100,\ \mid \rho-\rho_* \mid> 10 \right\}\,.$$

Then $F^j(x,v_*) \geq c > 0$ for $v \in V_*$, and with c independent of j. For some $c'' > 0$, given v with $|\xi| \geq r_j$, we may take v_* in a half volume of V_* and ω in a subset of S^2 of measure c'', so that

$$\mid \xi' \mid\geq 1,\ \mid \xi'_* \mid\geq 1,\ F^j(x,v') \leq \tilde{c},\ F^j(x,v'_*) \leq \tilde{c}\,,$$

with $\tilde{c}$ independent of j. Hence, for such x, v, v_* and ω,

$$\begin{aligned} K_j(v)F^j(x,v) &\leq cK_j(v)F^j(x,v)F^j(x,v_*) \\ &\leq \frac{\bar{c}}{\log i}B_j(v-v_*,\omega)(F^j(x,v)F^j(x,v_*) \\ &\qquad - F^j(x,v')F^j(x,v'_*))\log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}\,. \end{aligned}$$

Since there is $c' > 0$ such that, uniformly with respect to j, the integral

$$\int B_j(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))\log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}dv\,dv_*\,d\omega$$

is bounded by c' outside of a j-dependent set S'_j of measure ϵ in x, it follows that for $x \in S'^c_j$,

$$\int_{|\rho|\leq\lambda,|\xi|\leq\frac{1}{i}} K_j(v)F^j(x,v)dv \leq \frac{c'\bar{c}}{\log i} + \frac{c}{i} + 2\eta < 3\eta\,,$$

for i large enough. □

END OF THE PROOF OF THEOREM 1.1. It follows from Lemma 3.1 and Lemma 3.5 that the sequence of functions of the variable x

$$\left(\int (\sup_{|v|\leq\lambda} B_j(v, v_*, \omega))F^j(x, v_*)\, d\omega\, dv_*\right)_{j\in\mathbb{N}^*}$$

is weakly compact in $L^1([-1, 1])$. Moreover,

$$F^j(x, v) \leq c_\delta F^j(1, v), \qquad \xi > \delta, \mid v \mid\leq \frac{1}{\delta},$$
$$F^j(x, v) \leq c_\delta F^j(-1, v), \quad \xi < -\delta, \mid v \mid\leq \frac{1}{\delta},$$

and the sequences $(F^j(1, v))_{j\in\mathbb{N}^*}$ and $(F^j(-1, v))_{j\in\mathbb{N}^*}$ are respectively weakly compact in

$$L^1\left(\left\{v \in \mathbb{R}^3;\ \mid v \mid\leq \frac{1}{\delta},\ \xi \geq \delta\right\}\right),$$

and

$$L^1\left(\left\{v \in \mathbb{R}^3;\ \mid v \mid\leq \frac{1}{\delta},\ \xi \leq -\delta\right\}\right).$$

Hence $(Q_j^-(F^j, F^j))_{j\in\mathbb{N}^*}$ is weakly compact in $L^1([-1, 1] \times \{v \in \mathbb{R}^3; \mid v \mid\leq \frac{1}{\delta}, \mid \xi \mid\geq \delta\})$. It is a consequence of the weak L^1 compactness of $(Q_j^-(F^j, F^j))_{j\in\mathbb{N}^*}$ and the boundedness of $(e(F^j, F^j))_{j\in\mathbb{N}^*}$, that $(Q_j^+(F^j, F^j))_{j\in\mathbb{N}^*}$ is weakly compact in $L^1([-1, 1] \times \{v \in \mathbb{R}^3; \mid v \mid\leq \frac{1}{\delta}, \mid \xi \mid\geq \delta\})$. This together with Lemma 3.1 implies a (subsequence) limit when $j \to \infty$ in the weak form of equation (2.6) for any test function φ with compact support and vanishing for $\mid \xi \mid\leq \delta$ for some $\delta > 0$. Using the weak L^1 compactness of (F^j), $(Q_j^{\pm}(F^j, F^j))$, and Lemma 3.6-7, we may conclude (cf [7]) that F satisfies the weak form of our boundary value problem (1.1-3) for such functions φ, and that F has β-norm equal to M in the hard force case. That in turn implies that F is a mild solution. On the other hand, the integrability properties of $Q^{\pm}(F, F)$ satisfied by the above weak solutions are stronger than what is required from a mild solution. □

4. – The slab solution for $-3 < \beta < 0$.

Let us consider the approximate solution F^μ of the bounded B_μ case (2.6) for $B_\mu = \max(\frac{1}{\mu}, \min(B, \mu))$. The $k_{r,\mu}$ bounds from above and below in Section 3 still hold with similar proofs.

LEMMA 4.1. *The family* $(F^\mu)_{\mu\geq\mu_0}$ *is weakly compact in* $L^1([-1, 1] \times \{v \in \mathbb{R}^3; \mid v \mid\leq \frac{1}{\delta}, \mid \xi \mid> \delta\})$ *for* $\delta > 0$.

PROOF OF LEMMA 4.1. Given $M > 0$ and B_μ, the proof of the previous two sections imply that there is a solution F^μ satisfying

$$\int_{-1}^{1}\int_{\mathbb{R}^3} K_\mu(v)F^\mu(x,v)\,dx\,dv = M$$

for all $\mu \geq 1$. Here $K_\mu(v) = \max(\frac{1}{\mu}, \min(\mu, (1+ \mid v \mid)^\beta))$. And so, uniformly with respect to $\mu \geq \mu_0$,

$$\int_{-1}^{1}\int_{|v|\leq\frac{1}{\delta}} B_\mu F^\mu(x,v_*)dv\,dv_*\,d\omega\,dx \leq c_\delta < \infty\,.$$

So, given $\epsilon > 0$ and $\mid v \mid\leq \frac{1}{\delta}$, outside of a set $V_{\mu\delta} \subset \{v \in \mathbb{R}^3; \mid v \mid\leq \frac{1}{\delta}\}$ with measure $\mid V_{\mu\delta} \mid< \epsilon$, it holds that

$$\int_{-1}^{1}\int B_\mu F^\mu(x,v_*)dv_*\,d\omega\,dx \leq \frac{c_\delta}{\epsilon}\,.$$

By the exponential form, for $\mid v \mid\leq \frac{1}{\delta}$, $v \notin V_{\mu\delta}$,

$$F^\mu(x,v) \leq c_{\delta\epsilon}F^\mu(1,v),\ \xi > \delta, \qquad F^\mu(x,v) \leq c_{\delta\epsilon}F^\mu(-1,v),\ \xi < -\delta\,.$$

Outside of some set $V'_{\mu\delta} \subset \{v \in \mathbb{R}^3; \mid v \mid\leq \frac{1}{\delta}, \mid \xi \mid\geq \delta\}$ with $\mid V'_{\mu\delta} \mid< \epsilon$, the functions $F^\mu(1,v)$ and $F^\mu(-1,v)$ are bounded by a constant independent of μ. Hence the family $(F^\mu)_{\mu\geq\mu_0}$ satisfies an equiintegrability type condition with respect to $[-1,1] \times \{v \in V^c_{\mu\delta} \cap V'^c_{\mu\delta}; \mid v \mid\leq \frac{1}{\delta}, \mid \xi \mid> \delta\}$. It remains to obtain this, also with respect to $V_{\mu\delta} \cup V'_{\mu\delta}$. Since

$$\int \xi^2 F^\mu(x,v)dv < c, \quad x \in (-1,1)\,,$$

it is enough to prove that, given $\eta_1 > 0$, there exists $\eta_2 > 0$ such that

$$\int_{-1}^{1}\int_A F^\mu(x,v)\,dx\,dv < \eta_1\,,$$

uniformly in $\mu \geq \mu_0$ for $\mid A \mid< \eta_2$, $A \subset \{v \in \mathbb{R}^3; \mid \xi \mid\geq \delta, \mid v \mid\leq \frac{1}{\delta}\}$. If this latter criterium does not hold, then there is a constant η_1, a sequence (μ_j) with $\lim_{j\to\infty}\mu_j = +\infty$ and a sequence of subsets (A_j) of $\{v \in \mathbb{R}^3; \mid v \mid\leq \frac{1}{\delta}, \mid \xi \mid> \delta\}$, with measure $\mid A_j \mid\leq \frac{1}{j^3}$, such that

$$\int_{-1}^{1}\int_{A_j} F^{\mu_j}(x,v)\,dx\,dv > \eta_1\,.$$

Then

$$\int_{(x,v)\in(-1,1)\times A_j;\, F^{\mu_j}(x,v)>\frac{1}{40}\eta_1 j^3} F^{\mu_j}(x,v)\,dx\,dv > \frac{3}{4}\eta_1\,.$$

By the exponential form, uniformly in $\mu \geq \mu_0$,

$$F^{\mu}(x,v_*) \geq c_0',\ x\in(-1,1),\ v_*\notin V_{\mu\delta},\ |\, v_* \,|\leq \frac{1}{\delta},\ |\, \xi_* \,|> \delta\,.$$

Also there is a constant $c>0$ and a set W_j equal

$$\left\{ v\in V^c_{\mu_j\frac{\delta}{2}}\cap V'^c_{\mu\frac{\delta}{2}};\ |\, v \,|\leq \frac{2}{\delta},\ |\, \xi \,|\geq \delta \right\},$$

such that

$$F^{\mu_j}(x,v)\leq c,\quad x\in(-1,1),\quad v\in W_j\,.$$

For some $c''>0$, for any $(x,v)\in(-1,1)\times A_j$ such that $F^{\mu_j}(x,v)>\frac{1}{40}\eta_1 j^3$ and any $v_*\in W_j\cap\{|\, v \,|\leq \frac{1}{\delta}\}$ but outside a v-dependent set of measure (say) one, there is a subset of S^2 with measure c'', such that

$$v' = v-(v-v_*,\omega)\omega\in W_j,\qquad v_*' = v_*+(v-v_*,\omega)\omega\in W_j\,.$$

Consequently for some (j-independent) $c_0>0$ and the above (j-dependent) x, v, v_*, ω,

$$c_0F^{\mu_j}(x,v)\leq B_{\mu_j}[F^{\mu_j}(x,v)F^{\mu_j}(x,v_*)-F^{\mu_j}(x,v')F^{\mu_j}(x,v_*')]\,,$$

$$c_0 j^3 \leq \frac{F^{\mu_j}(x,v)F^{\mu_j}(x,v_*)}{F^{\mu_j}(x,v')F^{\mu_j}(x,v_*')}\,.$$

There is a constant c_1, independent of x and j, such that

$$\int_{|v|\leq\frac{1}{\delta},|\xi|>\delta} F^{\mu_j}(x,v)dv \leq \delta^{-2}\int \xi^2F^{\mu_j}(x,v)dv\leq c_1\,.$$

There is a constant c_2 independent of j, such that outside a j-dependent set T_j of measure bounded by $\frac{\eta_1}{8c_1}$,

$$\int B_{\mu_j}(F^{\mu_j}F_*^{\mu_j}-F^{\mu_j'}F_*^{\mu_j'})\log\frac{F^{\mu_j}F_*^{\mu_j}}{F^{\mu_j'}F_*^{\mu_j'}}dv\,dv_*\,d\omega\leq c_2\,.$$

And so

$$\eta_1\leq\int_{(x,v)\in(-1,1)\times A_j}F^{\mu_j}(x,v)\,dx\,dv\leq I_1+I_2+I_3\,.$$

Here I_1 is the integral over the subset where $F^{\mu_j} \leq \frac{\eta_1 j^3}{40}$, I_2 is the integral over $T_j \times A_j$, and I_3 is the remaining integral. Hence

$$\eta_1 \leq I_1 + I_2 + I_3 \leq \frac{\eta_1}{4} + \frac{\eta_1}{4} + \frac{c_3}{\log j},$$

which leads to a contradiction for j large enough. This ends the proof of Lemma 4.1. □

Proof of Theorem 1.2. It follows from Lemma 4.1 that there is a sequence (F^{μ_j}) converging weakly to some F in $L^1([-1,1]\times\{v \in \mathbb{R}^3;\ |\ v\ | \leq \frac{1}{\delta},\ |\ \xi\ | > \delta\})$, for any $\delta > 0$. From here with minor adaptations in the arguments, Lemma 3.7 from the hard force case holds, and Lemma 3.6 holds if $\lim_{\lambda\to\infty}$ is replaced by $\lim_{\min(j,\lambda)\to\infty}$. As for Lemma 3.5, the following is a proof in the case of soft forces. Since $K_{\mu_j}(v) \leq 1$ and

$$\int \xi^2 F^{\mu_j}(x, v) dv \leq c, \tag{4.1}$$

it holds that

$$\int_{|\xi| \geq \frac{1}{j}, \rho \leq 100} K_j(v) F^{\mu_j}(x, v) dv \leq c j^2.$$

Recall that

$$\int_{-1}^{1} \int (1+\ |\ v\ |)^{\beta} F^{\mu_j}(x, v)\, dx\, dv = 1$$

for all j. And so, uniformly with respect to j,

$$\int_{-1}^{1} \int_{|v| \leq \frac{1}{\delta}} \int_{\mathbb{R}^3 \times S^2} B_{\mu_j}(v - v_*, \omega) F^{\mu_j}(x, v_*) dv\, dv_*\, d\omega\, dx \leq c_\delta < \infty.$$

So, given $\epsilon > 0$, for $|\ v\ | \leq \frac{1}{\delta}$, outside of a set $V_{j\delta} \subset \{v \in \mathbb{R}^3;\ |\ v\ | \leq \frac{1}{\delta}\}$ with measure $|\ V_{j\delta}\ | < \epsilon$, it holds that

$$\int_{-1}^{1} \int B_\mu(v - v_*, \omega) F^{\mu_j}(x, v_*) dv_*\, d\omega\, dx \leq \frac{c_\delta}{\epsilon}.$$

By the exponential estimates,

$$\begin{aligned}
&F^{\mu_j}(x, v_*) \geq c F^{\mu_j}(-1, v_*), \quad \xi_* \geq \frac{1}{10}, \quad |\ v_*\ | \leq 10, \quad v_* \notin V_{j\delta}, \\
&F^{\mu_j}(x, v_*) \geq c F^{\mu_j}(1, v_*), \quad \xi_* \leq -\frac{1}{10}, \quad |\ v_*\ | \leq 10, \quad v_* \notin V_{j\delta}.
\end{aligned}$$

From here the conclusion of Lemma 3.5 in the soft force case follows similarly to the proof of Lemma 3.5 for hard forces.

With the help of Lemma 3.5-7, we shall next prove that the family of loss terms in renormalized form, is weakly compact in the space $L^1([-1,1]\times\{v\in \mathbb{R}^3; \mid v\mid\leq \frac{1}{\delta}, \mid \xi \mid > \delta\})$. It is enough to prove the weak L^1 compactness of $\int \mid v-v_* \mid^{\beta} F^{\mu_j}(x,v_*)dv_*$. This integral can be split into the sum of four terms,

$$I_1^{\mu_j} := \int_{|v_*|\geq\lambda} \mid v-v_*\mid^{\beta} F^{\mu_j}(x,v_*)dv_* \,,$$

$$I_2^{\mu_j} := \int_{|v_*|\leq\lambda, |\xi_*|\leq\frac{1}{i}} \mid v-v_*\mid^{\beta} F^{\mu_j}(x,v_*)dv_* \,,$$

$$I_3^{\mu_j} := \int_{|v_*|\leq\lambda, |\xi_*|\geq\frac{1}{i}, |v-v_*|<\frac{1}{k}} \mid v-v_*\mid^{\beta} F^{\mu_j}(x,v_*)dv_* \,,$$

$$I_4^{\mu_j} := \int_{|v_*|\leq\lambda, |\xi_*|\geq\frac{1}{i}, |v-v_*|>\frac{1}{k}} \mid v-v_*\mid^{\beta} F^{\mu_j}(x,v_*)dv_* \,.$$

Let $\epsilon>0, \eta>0$ be given. By (4.1) and Lemma 3.6, $I_1^{\mu_j} < \frac{\epsilon}{4}$ outside a subset of $[-1,1]$ of measure η, when $\min(\lambda, \mu_j)$ is large enough. Next, given λ, by Lemma 3.7,

$$\lim_{i\to+\infty} I_2^{\mu_j} = 0 \,,$$

in the sense of Lemma 3.7. Hence, given $\eta>0$, the quantities λ and i can be chosen so that outside of a subset of $[-1,1]$ of measure η, the terms $I_1^{\mu_j}, I_2^{\mu_j} < \frac{\epsilon}{4}$, uniformly in μ for μ large. Also,

$$\begin{aligned}\int_{|v|\leq\frac{1}{\delta}} I_3^{\mu_j} dv &\leq \int_{|v_*|<\lambda, |\xi_*|\geq\frac{1}{i}} F^{\mu_j}(x,v_*)\Big(\int_{|v|<\frac{1}{\delta}, |v-v_*|<\frac{1}{k}} |v-v_*\mid^{\beta} dv\Big)dv_* \\ &\leq \frac{ci^2}{k^{3+\beta}} \int \xi_*^2 F^{\mu_j}(x,v_*)dv_* \leq \frac{ci^2}{k^{3+\beta}} \,.\end{aligned} \tag{4.2}$$

So for i given, (4.2) tends to zero when k tends to infinity, uniformly with respect to μ. Hence k can be chosen large enough so that $\int_{|v|\leq\frac{1}{\delta}} I_3^{\mu_j} dv < \frac{\epsilon}{4}$, uniformly with respect to μ. Finally

$$I_4^{\mu_j} \leq \frac{i^2}{k^{\beta}} \int_{|v_*|\leq\lambda, |\xi_*|\geq\frac{1}{i}} \xi_*^2 F^{\mu_j}(x,v_*)dv_* \,,$$

so that, by the uniform boundedness from above of $\int \xi_*^2 F^{\mu_j}(x,v_*)dv_*$, $(I_4^{\mu_j})$ is weakly compact in $L^1([-1,1]\times\{v\in\mathbb{R}^3; \mid v\mid\leq\frac{1}{\delta}, \mid\xi\mid>\delta\})$. Using the above estimates of $I_k^{\mu_j}, k=1, ..., 4$ together with Lemma 3.5, the weak compactness of the loss terms in renormalized form follows from here. By a classical argument using the uniform boundedness of the entropy production terms, it then follows that also the family of gain terms in renormalized form is weakly compact in $L^1([-1,1]\times\{v\in\mathbb{R}^3; \mid v\mid\leq\frac{1}{\delta}, \mid\xi\mid>\delta\})$. From here, the argument in [7] for the time-dependent problem can be used to prove that F satisfies the mild form of the stationary problem (1.1-3). $\square$

REMARK. In the case $-1 \leq \beta < 0$, there are slightly stronger solutions to (1.1-3) with β-norm M, namely in the sense of (1.4) where now

$$\int_{-1}^{1} \int Q(f, f)(x, v)\varphi(x, v)\, dx\, dv$$
$$:= \int_{-1}^{1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{S^2} B(v - v_*, \omega) f(x, v) f(x, v_*)(\varphi(x, v') - \varphi(x, v)) dx\, dv\, dv_*\, d\omega.$$

This can be proved using the ideas of the above proof of mild solutions.

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