On the Boltzmann equation: global solutions in one spatial dimension

Walter Craig

Department of Mathematics & Statistics

McMaster University

Colloque de mathématiques de Montréal
Centre de Recherches Mathématiques
November 11, 2005

Collaborators

Andrei Biryuk McMaster University

Vlad Panferov McMaster University

Summary

The Boltzmann equation for $f(x,v,t) \geq 0$ with $x \in \mathbb{R}^1/\mathbb{Z}^1$, $v \in \mathbb{R}^3$
The macroscopic density

$$\rho(x,t) = \int_{v\in\mathbb{R}^3} f(x,v,t)\,dv$$

The entropy relative to a Maxwellian $M(v) = m\left(\frac{a}{\pi}\right)^{3/2} e^{-a|v-u|^2}$ is

$$H(f|M) = \int_{x\in\mathbb{T}^1}\int_{v\in\mathbb{R}^3}\left(f\log(\frac{f}{M}) - f + M\right) dvdx$$

Main result: There exists a constant C_0 such that if $\rho_0(x) \in L^\infty_x$ and

for some M_0, $H(f_0|M_0) \leq C_0$ then global strong solutions exist.

Outline

Outline

Outline

Outline

Outline

Boltzmann equation

- The classical Boltzmann equation

$$\partial_t f + v \cdot \partial_x f = Q(f,f) \,, \qquad f(x,v,0) = f_0(x,v) \qquad (1)$$

Phase space coordinates $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$.
Phase space distribution function $f(x,v,t)$.

- In case $Q = 0$ the solution is

$$f(x,v,t) = f_0(x - tv, v) = \Phi_t(f_0)(x,v)$$

Free streaming flow of the equation (1) linearized about $f = 0$

- $Q(f,f)$ is the collision operator

Boltzmann equation

- The classical Boltzmann equation

$$\partial_t f + v \cdot \partial_x f = Q(f,f) \;, \qquad f(x,v,0) = f_0(x,v) \qquad (1)$$

Phase space coordinates $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$.
Phase space distribution function $f(x,v,t)$.

- In case $Q = 0$ the solution is

$$f(x,v,t) = f_0(x - tv, v) = \Phi_t(f_0)(x,v)$$

Free streaming flow of the equation (1) linearized about $f = 0$

- $Q(f,f)$ is the collision operator

Boltzmann equation

- The classical Boltzmann equation

$$\partial_t f + v \cdot \partial_x f = Q(f,f) \ , \qquad f(x,v,0) = f_0(x,v) \qquad (1)$$

Phase space coordinates $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$.
Phase space distribution function $f(x,v,t)$.

- In case $Q = 0$ the solution is

$$f(x,v,t) = f_0(x - tv, v) = \Phi_t(f_0)(x,v)$$

Free streaming flow of the equation (1) linearized about $f = 0$

- $Q(f,f)$ is the collision operator

portrait of Ludwig Boltzmann (1844 - 1906)

29/09/05 http://www.sil.si.edu/digitalcollectio...-identity/thumbnails/TNSIL14-B5-06.jpg #1

portrait of J. C. Maxwell (1831 - 1879)

03 / 1 0/ 05 h t t p :/ / w w w . e g r . ms u . e d u / ~ b o h n s a c3 / n a n o / fig u r e s / Ma x#1e

Collision operator

- The collision operator of Maxwell and Boltzmann

$$\begin{aligned} Q(f,f)(x,v) &= \int_{\mathbb{R}^3_{v_*}} \int_{\mathbb{S}^2_\sigma} \big(f(x,v')f(x,v'_*) - f(x,v)f(x,v_*)\big) \\ &\qquad \times K \, dS_\sigma dv_* \\ &= Q^+(f,f)(x,v) - Q^-(f,f)(x,v) \end{aligned} \tag{2}$$

- Velocities before v', v'_* and after v, v_* a binary collision satisfy

$$v + v_* = v' + v'_* \qquad \text{and} \qquad (v' - v'_*)/|v - v_*| = \sigma \in \mathbb{S}^2$$

- The collision kernel $K = K(|v - v_*|, \frac{(v-v_*)}{|v-v_*|} \cdot \sigma)$

Collision operator

- The collision operator of Maxwell and Boltzmann

$$\begin{aligned} Q(f,f)(x,v) &= \int_{\mathbb{R}^3_{v_*}} \int_{\mathbb{S}^2_\sigma} (f(x,v')f(x,v'_*) - f(x,v)f(x,v_*)) \\ &\qquad \times K \, dS_\sigma dv_* \\ &= Q^+(f,f)(x,v) - Q^-(f,f)(x,v) \end{aligned} \tag{2}$$

- Velocities before v', v'_* and after v, v_* a binary collision satisfy

$$v + v_* = v' + v'_* \qquad \text{and} \qquad (v' - v'_*)/|v - v_*| = \sigma \in \mathbb{S}^2$$

- The collision kernel $K = K(|v - v_*|, \frac{(v-v_*)}{|v-v_*|} \cdot \sigma)$

Collision operator

- The collision operator of Maxwell and Boltzmann

$$\begin{aligned} Q(f,f)(x,v) &= \int_{\mathbb{R}^3_{v_*}} \int_{\mathbb{S}^2_\sigma} (f(x,v')f(x,v'_*) - f(x,v)f(x,v_*)) \\ &\qquad \times K\, dS_\sigma dv_* \\ &= Q^+(f,f)(x,v) - Q^-(f,f)(x,v) \end{aligned} \tag{2}$$

- Velocities before v', v'_* and after v, v_* a binary collision satisfy

$$v + v_* = v' + v'_* \qquad \text{and} \qquad (v' - v'_*)/|v - v_*| = \sigma \in \mathbb{S}^2$$

- The collision kernel $\quad K = K(|v - v_*|, \frac{(v-v_*)}{|v-v_*|} \cdot \sigma)$

Macroscopic quantities

- macroscopic mass density

$$\rho(x,t) = \int_{v\in\mathbb{R}^3} f(x,v,t)\,dv$$

- macroscopic momentum density

$$\rho u(x,t) = \int_{v\in\mathbb{R}^3} v f(x,v,t)\,dv$$

- macroscopic energy density

$$\rho e(x,t) = \int_{v\in\mathbb{R}^3} |v|^2 f(x,v,t)\,dv$$

Macroscopic conservation laws

- Conservation of mass:

$$\partial_t \rho(x,t) + \nabla_x \cdot F_\rho(x,t) = 0$$

with density flux $F_\rho = \int_{v\in\mathbb{R}^3} v f(x,v,t)\,dv$

- Conservation of momentum:

$$\partial_t \rho u(x,t) + \nabla_x \cdot F_{\rho u}(x,t) = 0$$

with momentum flux $F_{\rho u} = \int_{v\in\mathbb{R}^3} v\otimes v f(x,v,t)\,dv$

- Conservation of energy:

$$\partial_t \rho e(x,t) + \nabla_x \cdot F_{\rho e}(x,t) = 0$$

with energy flux $F_{\rho e} = \int_{v\in\mathbb{R}^3} |v|^2 v f(x,v,t)\,dv$

NB The closure problem

One dimensional geometry

We take $x \in \mathbb{R}^1$ and $v \in \mathbb{R}^3$, which is known as the slab geometry. It is the setting of a narrow shock tube, or a situation with spanwise constant macroscopic quantities.

Furthermore, we ask for periodic spatial boundary conditions; that is $x \in \mathbb{R}^1$ and $v \in \mathbb{R}^3$.

NB: Periodic boundary conditions, or perturbations of a state of thermodynamic equilibrium (a constant Maxwellian background) are more difficult problems than setting $f \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ in the background of a vacuum.

Conserved quantities

- Total Mass

$$M(f) = \int_{\mathbb{T}^1} \rho(x,t)\,dx = \int_{\mathbb{T}^1}\int_{v\in\mathbb{R}^3} f(x,v,t)\,dvdx$$

- Total Momentum

$$I(f) = \int_{\mathbb{T}^1} \rho u(x,t)\,dx = \int_{\mathbb{T}^1}\int_{v\in\mathbb{R}^3} v f(x,v,t)\,dvdx$$

- Total Energy

$$E(f) = \int_{\mathbb{T}^1} \rho e(x,t)\,dx = \int_{\mathbb{T}^1}\int_{v\in\mathbb{R}^3} |v|^2 f(x,v,t)\,dvdx$$

These quantities are constants of motion for the Boltzmann equation.

Relative entropy

Define the relative entropy of a phase space distribution function $f(x,v)$ with respect to a constant Maxwellian distribution by

$$H(f|M) = \int_{\mathbb{T}^1} \int_{v\in\mathbb{R}^3} \Big(f \log(\frac{f}{M}) - f + M \Big) \, dvdx$$

Proposition (1)

Positivity properties of the relative entropy

1. $$H(f|M) \geq 0$$
2. *Let* $m := \int_{v\in\mathbb{R}^3} M \, dv$. *The Csiszár–Kullback–Pinsker inequality is equivalent to*

$$\int_{\mathbb{T}^1} \int_{v\in\mathbb{R}^3} |f(x,v) - M(v)| \, dvdx \leq \sqrt{4mH(f|M)} + H(f|M)$$

Relative entropy

Define the relative entropy of a phase space distribution function $f(x,v)$ with respect to a constant Maxwellian distribution by

$$H(f|M) = \int_{\mathbb{T}^1} \int_{v\in\mathbb{R}^3} \Big(f \log(\frac{f}{M}) - f + M \Big)\, dv dx$$

Proposition (1)

Positivity properties of the relative entropy

1. $$H(f|M) \geq 0$$
2. *Let* $m := \int_{v\in\mathbb{R}^3} M\, dv$. *The Csiszár–Kullback–Pinsker inequality is equivalent to*

$$\int_{\mathbb{T}^1} \int_{v\in\mathbb{R}^3} |f(x,v) - M(v)|\, dv dx \leq \sqrt{4mH(f|M)} + H(f|M)$$

The H-theorem

Proposition (2)

The relative entropy $H(f|M)$ decreases along the flow of the Boltzmann equation

$$\partial_t H(f|M) \leq 0$$

Remarks: The following are equivalent:

1. $f \in L\log(L)$ and $\int_x \int_v (1+|v|^2)|f|\,dvdx < +\infty$
2. $f \in L\log\left(\frac{|f|}{M}\right)$ for some Maxwellian $M(v)$

The $\|\!|f|\!\|$ norm

Definition (3)

For functions $f(x,v)$ on phase space define the norm

$$|\!|\!|f|\!|\!| := \sup_{\substack{x\in\mathbb{T}^1 \\ p\in\mathbb{R}^+}} \int_{\mathbb{R}^3_v} |f|(x-pv,v)\,dv \tag{3}$$

Denote by X the space of functions for which this is finite.

The function space $X \subseteq L^\infty_x(L^1_v)$.

NB: The streaming flow $\Phi_t(f)(x,v) = f(x-tv,v)$ is not continuous on $L^\infty_x(L^1_v)$ but it is so on X

The $|||f|||$ norm

Definition (3)

For functions $f(x,v)$ on phase space define the norm

$$|||f||| := \sup_{\substack{x\in\mathbb{T}^1 \\ p\in\mathbb{R}^+}} \int_{\mathbb{R}^3_v} |f|(x-pv,v)\,dv \tag{3}$$

Denote by X the space of functions for which this is finite.

The function space $X \subseteq L^\infty_x(L^1_v)$.

NB: The streaming flow $\Phi_t(f)(x,v) = f(x-tv,v)$ is not continuous on $L^\infty_x(L^1_v)$ but it is so on X

First global result

Theorem (4)

Assume that the collision kernel K satisfies hypothesis (H1). Then there is a constant K_0 such that if initial data f_0 for the Boltzmann equation (1) *satisfies*

$$\|\!|f_0\|\!| < +\infty \qquad \text{and} \qquad H(f_0|M_0) \leq \frac{1}{4\pi K_0} \tag{4}$$

for some Maxwellian M_0, then the solution $f(\cdot, t) \in X$ for all $t \in \mathbb{R}^+$.

More specifically the theorem asserts that there is an estimate on the L_x^∞ norm of the density $\rho(x,t)$. When strict inequality holds in (4)

$$\exists \beta \quad \text{such that} \quad \|\rho(x,t)\|_{L_x^\infty} \leq \|\!|f(x,v,t)\|\!| \leq C_0 \exp(\sqrt{t/\beta})$$

First global result

Theorem (4)

Assume that the collision kernel K satisfies hypothesis (H1). Then there is a constant K_0 such that if initial data f_0 for the Boltzmann equation (1) *satisfies*

$$\|\!|f_0\|\!| < +\infty \qquad \text{and} \qquad H(f_0|M_0) \leq \frac{1}{4\pi K_0} \tag{4}$$

for some Maxwellian M_0, then the solution $f(\cdot, t) \in X$ for all $t \in \mathbb{R}^+$.

More specifically the theorem asserts that there is an estimate on the L^∞_x norm of the density $\rho(x,t)$. When strict inequality holds in (4)

$$\exists\beta \quad \text{such that} \quad \|\rho(x,t)\|_{L^\infty_x} \leq \|\!|f(x,v,t)\|\!| \leq C_0 \exp(\sqrt{t/\beta})$$

Hypotheses on the collision kernel

The result depends upon a newly described smoothing property of the collision kernel K.

$$K(r,\xi) = K\left(|v - v_*|, \frac{(v - v_*)\cdot\sigma}{|v - v_*|}\right)$$

Hypothesis (**H1**)

Large relative velocity interactions are soft, while small relative velocity interactions are hard:

$$\textbf{(H1)} \qquad 0 \leq K(r,\xi) \leq \frac{K_0 w}{1 + w\log^{1+\varepsilon}(w+1)}$$

NB This hypothesis is related to one appearing in Cercignani's work.

Compare with Boltzmann's collision kernel

Power law potential interactions between molecules for $1 < p \leq \infty$ are in the form

$$V(|q_1 - q_2|) = \frac{\gamma}{|q_1 - q_2|^p}$$

In this case, the classical Boltzmann collision kernel has the form

$$K(r, \xi) = b(\xi)|v - v_*|^\beta$$

where $-\infty < \beta = \beta(p) \leq 1$.
For hard spheres $\beta = 1$, For the Maxwell molecule case $p = 5$ it turns out that $\beta = 0$.

NB: For all but the hard spheres case, $b(\xi)$ diverges nonintegrably as $\xi \to \pm 1$. Grad cutoffs appear here, where they truncate grazing collisions.

A stronger hypothesis on the collision kernel gives a better result

Hypothesis (H2)

Small relative velocity interactions are absent.

(H2) *(i)* $K(r,\xi) = 0$ *for* $r < R_1$

(ii) $K(r,\xi) \geq \beta(\xi) \sup_{\xi \in (-1,1)} \big(K(r,\xi)\big)$ *for* $R_1 < r$,

where $\beta(\xi)$ *is positive on a set of positive measure.*

Theorem (5)

Assume that hypotheses (H1) *and* (H2) *hold, and that*

$$\|f_0\| < +\infty \quad \text{and} \quad M(f_0) + E(f_0) < +\infty \qquad (5)$$

then for all $t \in \mathbb{R}^+$ *solutions to* (1) *exist and have* L^∞_x *macroscopic density* $\rho(x,t)$.

NB These solutions could have infinite entropy

A stronger hypothesis on the collision kernel gives a better result

Hypothesis (H2)

Small relative velocity interactions are absent.

(H2) (i) $K(r,\xi) = 0$ *for* $r < R_1$

(ii) $K(r,\xi) \geq \beta(\xi) \sup_{\xi \in (-1,1)} \big(K(r,\xi)\big)$ *for* $R_1 < r$,

where $\beta(\xi)$ *is positive on a set of positive measure.*

Theorem (5)

Assume that hypotheses (H1) *and* (H2) *hold, and that*

$$\|f_0\| < +\infty \quad \text{and} \quad M(f_0) + E(f_0) < +\infty \tag{5}$$

then for all $t \in \mathbb{R}^+$ *solutions to* (1) *exist and have* L^∞_x *macroscopic density* $\rho(x,t)$.

NB These solutions could have infinite entropy

Uniqueness in X

- Solutions $f(x, v, t)$ of the Boltzmann equation (1) such that

$$\rho(x, t) = \int_{\mathbb{R}^3_v} f(x, v, t)\, dv \in C([0, T] : L^\infty_x)$$

  are known as strong solutions.
- Solutions in the class X are unique.
- However the more general property of weak/strong uniqueness holds.

Uniqueness in X

- Solutions $f(x, v, t)$ of the Boltzmann equation (1) such that

$$\rho(x, t) = \int_{\mathbb{R}^3_v} f(x, v, t)\, dv \in C([0, T] : L^\infty_x)$$

  are known as strong solutions.
- Solutions in the class X are unique.
- However the more general property of weak/strong uniqueness holds.

Uniqueness in X

- Solutions $f(x, v, t)$ of the Boltzmann equation (1) such that

$$\rho(x, t) = \int_{\mathbb{R}^3_v} f(x, v, t)\, dv \in C([0, T] : L^\infty_x)$$

  are known as strong solutions.
- Solutions in the class X are unique.
- However the more general property of weak/strong uniqueness holds.

Dissipative solutions

Definition (P.-L. Lions (1994))

A nonnegative $f(x,v,t) \in C([0,T]; L^1(\mathbb{T}^1 \times \mathbb{R}^3))$ is a dissipative solution of the Boltzmann equation (1) if for all

$$g(x,v,t) \in L^\infty([0,T]; L^\infty_x(\mathbb{T}^1; L^1_v(\mathbb{R}^3)))$$

satisfying

$$\partial_t g + v \cdot \nabla_x g - Q(g,g) = \mathcal{E}(g) ,$$

in the sense of distributions, then f obeys the inequality

$$\begin{aligned} \partial_t \int |f-g|\, dv \quad + \quad \nabla_x \cdot \int v|f-g|\, dv \\ \leq \qquad \int Q(g,f-g)\, \mathrm{sgn}(f-g)\, dv - \int \mathcal{E}(g)\, \mathrm{sgn}(f-g)\, dv \end{aligned}$$

Weak/strong uniqueness

Theorem (6)

Given initial data satisfying (4), *that is*

$$\|f_0\| < +\infty \quad \textit{and} \quad H(f_0|M_0) \leq \frac{1}{4\pi K_0} \tag{6}$$

any other dissipative solution starting with the same initial data f_0 must coincide with the strong solution for all $t \in \mathbb{R}^+$

In the case where hypothesis *(H2)* also holds, for uniqueness we only require

$$\|f_0\| \quad \text{and} \quad M(f_0) + E(f_0) < +\infty$$

Weak/strong uniqueness

Theorem (6)

Given initial data satisfying (4), *that is*

$$\|f_0\| < +\infty \quad \textit{and} \quad H(f_0|M_0) \leq \frac{1}{4\pi K_0} \tag{6}$$

any other dissipative solution starting with the same initial data f_0 *must coincide with the strong solution for all* $t \in \mathbb{R}^+$

In the case where hypothesis *(H2)* also holds, for uniqueness we only require

$$\|f_0\| \quad \text{and} \quad M(f_0) + E(f_0) < +\infty$$

Moments and derivatives

This question has to do with the evolution of smoothness and moment properties of the phase space distribution function $f(x, v, t)$, once the basic property of being a strong solution is established.

- Moments $M_\kappa(f) = \sup_{x\in\mathbb{T}^1} \int_{\mathbb{R}^3_v} v^\kappa f(x, v, t)\, dv$
- Derivatives in v $Q_\lambda(f) = \sup_{x\in\mathbb{T}^1} \int_{\mathbb{R}^3_v} |\partial_v^\lambda f(x, v, t)|\, dv$
- Derivatives in x $P_\mu(f) = \sup_{x\in\mathbb{T}^1} \int_{\mathbb{R}^3_v} |\partial_x^\mu f(x, v, t)|\, dv$

Propagation of moments and derivatives

Theorem (7)

Given initial data $f_0(x,v)$ *satisfying* (4) *(or in the case where hypothesis (H2) also holds,* (5)*). If in addition for integers* k, ℓ, m *the data satisfies*

$$\sum_{|\kappa|\leq k, |\lambda|\leq \ell, |\mu|\leq m} \sup_{x\in\mathbb{T}^1} \int_{v\in\mathbb{R}^3} |v^\kappa||\partial_v^\lambda \partial_x^\mu f_0(x,v)|\, dvdx < +\infty$$

then for all $t > 0$

$$\sum_{|\kappa|\leq k, |\lambda|\leq \ell, |\mu|\leq m} \sup_{x\in\mathbb{T}^1} \int_{v\in\mathbb{R}^3} |v^\kappa||\partial_v^\lambda \partial_x^\mu f(x,v,t)|\, dvdx \leq \varphi_{k\ell m}(t) < +\infty$$

The growth rate is bounded by $c_{k\ell m} \exp(\exp(\sqrt{t/\beta}))$.

Propagation of moments and derivatives

Theorem (7)

Given initial data $f_0(x,v)$ satisfying (4) *(or in the case where hypothesis (H2) also holds,* (5)*). If in addition for integers k, ℓ, m the data satisfies*

$$\sum_{|\kappa|\le k, |\lambda|\le \ell, |\mu|\le m} \sup_{x\in\mathbb{T}^1} \int_{v\in\mathbb{R}^3} |v^\kappa||\partial_v^\lambda \partial_x^\mu f_0(x,v)|\, dvdx < +\infty$$

then for all $t > 0$

$$\sum_{|\kappa|\le k, |\lambda|\le \ell, |\mu|\le m} \sup_{x\in\mathbb{T}^1} \int_{v\in\mathbb{R}^3} |v^\kappa||\partial_v^\lambda \partial_x^\mu f(x,v,t)|\, dvdx \le \varphi_{k\ell m}(t) < +\infty$$

The growth rate is bounded by $c_{k\ell m} \exp(\exp(\sqrt{t/\beta}))$.

Main ideas of the proof

- Using Duhamel's principle and the streaming flow, rewrite the Boltzmann equation as an integral equation

$$f(x,v,t) = f_0(x - tv, v) + \int_0^t Q\big(x - (t-s)v, v, s\big)\, ds$$

- Solutions satisfy the maximum principle:

$$f_0(x,v) \geq 0 \quad \text{implies} \quad f(x,v,t) \geq 0 \ \text{ for } \ t > 0$$

- Drop the Q^- term for the inequality

$$0 \leq f(x,v,t) \leq f_0(x - tv, v) + \int_0^t Q^+\big(x - (t-s)v, v, s\big)\, ds$$

- Integrate in $v \in \mathbb{R}^3$

$$0 \leq \rho(x,t) \leq \int_v f_0(x - tv, v)\, dv + \int_0^t \int_v Q^+\big(x - (t-s)v, v, s\big)\, dvds \tag{7}$$

The integrand in (7) involves a spherical integral through the part of the collision operator Q^+

$$\begin{aligned}
\int_v & Q^+\big(x-(t-s)v, v, s\big)\, dv \\
= & \int_v \int_{v_*} \int_{\mathbb{S}^2_\sigma} f(x-(t-s)v, v')f(x-(t-s)v, v'_*) \\
& \times K\big(|v-v_*|, \frac{(v-v_*)}{|v-v_*|}\cdot\sigma\big)\, dS_\sigma dv_* dv \\
= & \int_v \int_{v_*} \int_{\mathbb{S}^2_\sigma} f(x-(t-s)v', v)f(x-(t-s)v', v_*) \\
& \times K\big(|v-v_*|, \frac{(v-v_*)}{|v-v_*|}\cdot\sigma\big)\, dS_\sigma dv_* dv
\end{aligned}$$

where $v' = \frac{1}{2}\big((v+v_*) + |v-v_*|\sigma\big)$

A smoothing property

In one dimensional geometries, the integral over $\sigma \in \mathbb{S}^2$ converts to a spatial integral over the interval $[v_{min}, v_{max}]$ where

$$v_{min} = \tfrac{1}{2}((v + v_*) \cdot e_1 - |v - v_*|)$$
$$v_{max} = \tfrac{1}{2}((v + v_*) \cdot e_1 + |v - v_*|)$$

Changing variables in the integral (assume for simplicity that $K(r, \xi) = 0$ for $r > R$)

$$\int_v Q^+ \big(x - (t - s)v, v, s\big)\, dv \leq \frac{1}{t - s} \int_{-R(t-s)}^{R(t-s)} 4\pi K_0 \rho^2 \big(x + z, s\big)\, dz$$

Two estimates of the integrand

- The first estimate of the integrand is simply through $\|\rho\|_{L^\infty}$

$$\frac{1}{t-s}\int_{-R(t-s)}^{R(t-s)} 4\pi K_0 \rho^2(x+z,s)\,dz \leq 8\pi K_0 R\|\rho\|_{L^\infty}^2$$

- The second estimate uses the relative entropy $H(f|M)$

$$\frac{1}{t-s}\int_{-R(t-s)}^{R(t-s)} 4\pi K_0 \rho^2(x+z,s)\,dz \leq \frac{4\pi K_0 H(f|M)}{t-s}\,\frac{\|\rho\|_{L^\infty}}{\log(\|\rho\|_{L^\infty})}$$

Each one individually gives rise to an estimate which doesn't forbid blowup in finite time.

The integral inequality

Estimate the integrand by optimizing the two estimates
Denote $\|f(x,v,t)\| = \varphi(t)$ and $\alpha = C_0 K_0 H(f_0|M)$

$$\varphi(t) \leq \varphi(0) + \int_0^t \min\left\{c_1\varphi^2(s), \left(\frac{\alpha}{t-s} + c_2\right)\frac{\varphi(s)}{\log(\varphi(s))}\right\} ds \quad (8)$$

Theorem (8)

Global solutions of (8) *depend only upon* $\alpha \leq 1$.

The constant c_1 depends upon ε, K_0 and $H(f_0|M)$, while c_2 depends upon the initial data.

The Bony functional

- The proof of the second theorem under hypothesis *(H2)* is based on the Bony functional

$$b(t) = \int\int\int\int f(x,v,t) f(x,v_*,t) \times K(|v-v_*|, \sigma\cdot(v-v_*)/|v-v_*|)|v-v_*|^2 \, dS_\sigma dv dv_* dx$$

which has the property that $\int_0^\infty b(t)\,dt < +\infty$

- Using similar ideas in the case where *(H2)* holds, the integral inequality is

$$\varphi(t) \le \varphi(0) + \int_0^t \min\left\{ c_1\varphi^2(s), \left(\frac{1}{t-s}+1\right) c_3 b(s) \right\} ds \quad (9)$$

global existence

Proposition (9)

Given $b(t)$, the maximal solution $\varphi(t)$ of the integral inequality (9) *is a locally bounded function of t.*

Furthermore the quantity $|||f(\cdot, t)||| \leq \varphi(t)$. This implies global existence of a strong solution $f(x, v, t)$

However no rate of growth is available from this method, and indeed there may not be any quantitative rate.

Future directions

- Theorem (5) is restricted to one dimensional geometries. It is an open question whether Theorem (4) is so constrained.
- Can we relax the conditions *(H1)* on the collision kernel, possibly by using energy conservation.
- Does a similar theorem hold for nonelastic collisions (P. Degond). One of the difficult tendencies is for particles to coagulate (this is avoided precisely through hypothesis *(H2)* in our second category of results).
- Use Boltzmann-like kinetic equations to study homogeneous forms of dispersive nonlinear Hamiltonian partial differential equations