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A Cauchy Inequality for the Boltzmann Equation

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The distance to a set of Maxwellians is determined for a family of functions with bounded mass, energy and a small entropy production term. Functions with small masses are close to the null Maxwellian. Functions with masses bounded from below by a constant are approached by functions proportional to the gain term of the Boltzmann operator, taking advantage of its regularity. For these regularizations, the integrands of the entropy production term are small everywhere. Hence classical arguments can be used to obtain the closedness to the set of Maxwellians. Copyright © 2000 John Wiley & Sons, Ltd.

1. Introduction

In kinetic theory for rarefied gas dynamics, described by the Boltzmann equation, the entropy production term is

$$-\int_{\mathbb{R}^3} Q(f,f)\log f(v)\,\mathrm{d}v$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} \tilde{B}(|v-v_*|,\omega)\,(ff_* - f'f'_*)\log\frac{ff_*}{f'f'_*}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}w, \tag{1.1}$$

where $Q(f,f)$ is the collision operator and $\tilde{B}$ a collision kernel. Here, f, f_*, f' and f'_* denote, respectively,

$$f := f(v),\quad f_* := f(v_*),\quad f' := f(v'),\quad f'_* := f(v'_*),$$

where v and v_* are the precollisional velocities, and v' and v'_* the postcollisional velocities, given by

$$v' = v - (v - v_*, \omega)\omega, \qquad v'_* = v_* + (v - v_*, \omega)\omega.$$

The entropy production term is equal to zero if and only if f is a Maxwellian. In some circumstances, like the asymptotics for infinite times or hydrodynamic limits

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of solutions to the Boltzmann equation, (1.1) is known to be small, so that its integrand

$$|ff_* - f'f'_*|$$

is small outside of an arbitrarily small set in (v, v_*, ω). The purpose of this paper is to show that f will then be close to a Maxwellian outside of an arbitrarily small set. Since previous results on functional stability in several variables [6] cannot be applied straightforwardly, a specific study is required and performed here. More precisely, for a family F of functions f satisfying

$$\int f(1 + |v|^\alpha + |\log f|)\, dv \leqslant c_1 \tag{1.2}$$

for some constant $\alpha > 1$, as well as the smallness of the associated entropy production terms, its uniform closedness with respect to a family of Maxwellians is established. This result has already been stated and proven in the frame of non-standard analysis [1]. In the present paper, it is proven by approaching the distribution function f by a function proportional to the gain term of the Boltzmann operator and taking advantage of the regularity of the gain term [7]. It implies previous results on lower bounds of the entropy dissipation term

$$\int \int_{|v| < R} |\log f - m|\, dv \leqslant c \int Q(f, f) \log f(v)\, dv$$

for some negative constant c [5, 10]. There, the supplementary condition on f to be far from vacuum, i.e.

$$f(v) \geqslant A_R, \quad |v| \leqslant R$$

is required. Maxwellian lower bounds for solutions to the homogeneous Boltzmann equation are proven in [4, 8]. However, the condition of being far from vacuum is difficult to obtain in a non-homogeneous context, as soon as infinite times or hydrodynamic limits are involved. Indeed, in the evolutionary setting, the estimate for a lower bound on the distribution function obtained from the initial data vanishes for infinite times, whereas in the setting of hydrodynamic limits, the lower bound on f obtained from the boundary conditions vanishes with the mean free path.

This result is used in the study of the Milne problem and the hydrodynamic limit of solutions to the stationary Boltzmann equation when the mean free path tends to zero [3].

2. Extension and regularization

Recall a result from [7].

Lemma 2.1. *Let r_* and s_* be numbers smaller than $\frac{1}{4}$ to be chosen later. Let $B(r, \theta)$ be a function in $C_0^\infty((0, \infty) \times (0, \pi/2))$ such that*

$$B(r, \theta) = 0, \quad \textit{if } r \leqslant \frac{r_*}{2} \textit{ or } \cos\theta \leqslant \frac{s_*}{2} \textit{ or } \cos\theta \geqslant 1 - \frac{s_*}{2},$$

$$B(r,\theta)=1, \quad \textit{if } r \geqslant r_* \textit{ and } \cos\theta \in [s_*, 1-s_*],$$

$$B(r,\theta) \in [0,1] \quad \textit{else.}$$

Here θ *is defined by* $\cos\theta = |(v-v_*,\omega)|/|v-v_*|$.

There is a constant c *such that for any functions* $f \in L^2$ *and* $g \in L^1$,

$$\left\| \int B(|v-v_*|,\theta) f(v') g(v'_*) \,\mathrm{d}v_* \,\mathrm{d}\omega \right\|_{H^1(\mathbb{R}^3_v)} \leqslant c \,\| f \|_{L^2} \| g \|_{L^1}.$$

For any $r > 0$, denote by

$$V_r := \left\{ v \in \mathbb{R}^3;\ |v| \leqslant \frac{1}{r} \right\}, \quad S_r := V_r \times V_r \times S^2.$$

Define

$$\chi_\varepsilon(v) := 1,\ |v| \leqslant \frac{6}{\varepsilon}, \quad \chi_\varepsilon(v) := 0, \text{ else.}$$

For any function g, define

$$\nu(g)(v) := \int_{V_{\varepsilon/6} \times S^2} B(|v-v_*|,\theta) g(v_*) \,\mathrm{d}v_* \,\mathrm{d}\omega.$$

Let $\varepsilon > 0$ be given. Let F_ε be a family of functions satisfying (1.2) as well as $\int f(v)\,\mathrm{d}v > \varepsilon$. For any function f in the family F_ε, denote by

$$g_0 := \sqrt{f}, \quad g_{k+1}(v) := \frac{1}{\nu(g_k)(v)} \int_{V_{\varepsilon/6} \times S^2} B(|v-v_*|,\theta) g_k(v') g_k(v'_*) \,\mathrm{d}v_* \,\mathrm{d}\omega. \tag{2.1}$$

Lemma 2.2. *Let* $K \in \mathbb{N}^*$ *be given. There is a number* r_* *small enough and positive constants* $(i_k(\varepsilon))_{0 \leqslant k \leqslant K}$, *such that*

$$\nu(g_k)(v) \geqslant i_k(\varepsilon), \quad v \in V_\varepsilon, \quad k \leqslant K. \tag{2.2}$$

Proof. By the definition of B,

$$\nu(g_0)(v) \geqslant \int_{|v-v_* \geqslant r_*, \cos\theta \in [s_*, 1-s_*]} \sqrt{(f(v_*))} \,\mathrm{d}v_* \,\mathrm{d}\omega$$

$$\geqslant c \left(\int_{V_\varepsilon} \sqrt{(f(v_*))} \,\mathrm{d}v_* - \int_{|v-v_*| \leqslant r_*} \sqrt{(f(v_*))} \,\mathrm{d}v_* \right).$$

Then, by (1.2),

$$\int_{|v| > 1/\varepsilon} f(v)\,\mathrm{d}v \leqslant c_1 \varepsilon^\alpha, \qquad \int_{|v| \leqslant 1/\varepsilon, f(v) < \varepsilon^5} f(v)\,\mathrm{d}v \leqslant \frac{4}{3}\pi\varepsilon^2, \int_{f(v) > \mathrm{e}^{1/\varepsilon^2}} f(v)\,\mathrm{d}v \leqslant c_1 \varepsilon^2,$$

so that, for ε small enough,

$$\int_{|v| \leqslant 1/\varepsilon,\ \varepsilon^5 \leqslant f \leqslant e^{1/\varepsilon^2}} f(v)\,dv > \frac{\varepsilon}{2}.$$

Hence,

$$\int_{|v| \leqslant 1/\varepsilon} \sqrt{(f(v))}\,dv > e^{-1/2\varepsilon^2} \int_{|v| \leqslant 1/\varepsilon,\ f(v) < e^{1/\varepsilon^2}} f(v)\,dv > \frac{\varepsilon}{2} e^{-1/2\varepsilon^2}$$

and so,

$$v(g_0)(v) \geqslant 2\pi \left(\frac{\varepsilon}{2} e^{-1/2\varepsilon^2} - \left(\frac{4}{3} \pi r_*^3 c_1 \right)^{1/2} \right) \geqslant \frac{\pi}{2} \varepsilon e^{-1/2\varepsilon^2}$$

for r_* small enough. Moreover, by the Cauchy–Schwartz inequality, $v(g_0)$ is bounded in L^∞. Hence g_1 is well defined, and

$$v(g_1)(v) \geqslant c_2 \int B(|v - v_*|, \theta) B(|v_* - u|, \varphi) \times g_0(V'(v_*, u, \mu)) g_0(V'_*(v_*, u, \mu))\,dv_*\,d\mu\,du\,d\omega,$$

where

$$\cos \varphi := \frac{|(v_* - u, \mu)|}{|v_* - u|},$$

$$V'(v_*, u, \mu) := v_* - (v_* - u, \mu)\mu, \qquad V'_*(v_*, u, \mu) := u + (v_* - u, \mu)\mu.$$

By the change of variables $(v_*, u) \to (V'(v_*, u, \mu), V'_*(v_*, u, \mu))$, it follows that

$$\begin{aligned} v(g_1)(v) &\geqslant c_3 \int B(|v - V'(v_*, u, \mu)|, \omega) B(|v_* - u|, \mu) g_0(v_*) g_0(u)\,dv_*\,d\omega\,du\,d\mu \\ &\geqslant c_4 \left[\left(\int g_0(v_*) dv_* \right)^2 - \int_{|v - V'(v_*, u, \mu)| \leqslant r_* \text{ or } \cos\theta \notin [s_*, 1 - s_*]} g_0(v_*) g_0(u)\,dv_*\,d\omega\,du \right. \\ &\qquad \left. - \int_{|v_* - u| \leqslant r_* \text{ or } \cos\varphi \notin [s_*, 1 - s_*]} \right] \geqslant \frac{c_4}{2} \left(\int g_0(v_*)\,dv_* \right)^2 \end{aligned}$$

for r_* and s_* small enough. The proof of (2.2) for $k \in [2, K]$ is analogous, once noticing that the family of $(g_k)_{f \in F}$ is weakly compact in L^1, by Lemma 2.1.

Theorem 2.3. *Let* $\varepsilon > 0$, $K \geqslant 5$ *and* $a(\delta)$ *a function such that* $\lim_{\delta \to 0} a(\delta) = 0$ *be given. There are functions* $(d_k)_{0 \leqslant k \leqslant K}$ *and* $(e_k)_{5 \leqslant k \leqslant K}$ *with*

$$\lim_{\delta \to 0} d_k(\delta) = 0, \qquad \lim_{\delta \to 0} e_k(\delta) = 0$$

such that for every function f satisfying (1.2) *and*

$$\int f(v)\,\mathrm{d}v > \varepsilon, \quad \textit{and} \quad |ff_* - f'f'_*| < a(\delta), \tag{2.3}$$

outside of a set of measure $a(\delta)$ in $S_{\varepsilon/6}$, the functions (g_k) defined by (2.1) *belong to $H^k(V_{\varepsilon/6})$ and satisfy*

$$|\sqrt{f(v)} - g_k(v)| \leqslant d_k(\delta),$$

outside of a subset of $V_{\varepsilon/6}$ of measure $d_k(\delta)$. Moreover, for $k \geqslant 5$,

$$|g_k g_{k_*} - g'_k g'_{k_*}| \leqslant e_k(\delta), \quad (v, v_*, \omega) \in S_{\varepsilon/6}. \tag{2.4}$$

Remark. The second inequality in (2.3) implies that

$$|\sqrt{(ff_*)} - \sqrt{(f'f'_*)}| \leqslant \sqrt{[a(\delta)]} \tag{2.5}$$

outside of a set of measure $a(\delta)$ in $S_{\varepsilon/6}$. Indeed, either

$$\sqrt{(ff_*)} + \sqrt{(f'f'_*)} < \sqrt{[a(\delta)]},$$

and then $|\sqrt{(ff_*)} - \sqrt{(f'f'_*)}| < \sqrt{[a(\delta)]}$ holds. Or

$$\sqrt{(ff_*)} + \sqrt{(f'f'_*)} \geqslant \sqrt{[a(\delta)]},$$

and then

$$|\sqrt{(ff_*)} - \sqrt{(f'f'_*)}| = \frac{ff_* - f'f'_*}{\sqrt{(ff_*)}(ff_*) + \sqrt{(f'f'_*)}} \leqslant \sqrt{[a(\delta)]}.$$

Proof of Theorem 2.3. For any function f in the family, it follows from (1.2) and the Cauchy–Schwartz inequality that

$$\int_{V_{\varepsilon/6}} g(v)\,\mathrm{d}v \leqslant c_5 \varepsilon^{-3/2}. \tag{2.6}$$

By assumption, there is a subset B_0 of $S_{\varepsilon/6}$, with meas$(B_0^c) < a(\delta)$, such that

$$|g_0 g_{0*} - g'_0 g'_{0*}| < \sqrt{[(a(\delta)]}, \quad (v, v_*, \omega) \in B_0. \tag{2.7}$$

Hence,

$$\int \chi_\varepsilon(v) |g_0(v) - g_1(v)|\,\mathrm{d}v < a_1(\delta),$$

where a_1 is a function tending to zero when δ tends to zero. Indeed,

$$\int \chi_\varepsilon(v) |(g_0 - g_1)(v)|\,\mathrm{d}v \leqslant \frac{1}{i_0(\varepsilon)} (X_1 + X_2 + X_3)$$

with

$$X_1 := \int_{B_0} \chi_\varepsilon(v)\chi_\varepsilon(v_*)|g_0 g_{0*} - g'_0 g'_{0*}|\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega,$$

$$X_2 := \int_{B_0^c} \chi_\varepsilon \chi_{\varepsilon *} g_0 g_{0*}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega, \qquad X_3 := \int_{B_0^c} \chi_\varepsilon \chi_{\varepsilon *} g'_0 g'_{0*}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

First,

$$X_1 \leqslant \frac{64\pi^3}{9}\left(\frac{6}{\varepsilon}\right)^6 \sqrt{a(\delta)}.$$

Then, define

$$\Omega := \{\omega; \text{meas}\,\{(v, v_*); (v, v_*, \omega) \in B_0^c\} > \sqrt{[a(\delta)]}\}.$$

Then $\text{meas}(\Omega) < \sqrt{a(\delta)}$, and, by (2.6),

$$X_2 \leqslant c_6 \sqrt{[a(\delta)]}\,\varepsilon^{-3} + Y,$$

where

$$Y := \int_{\Omega^c} \int_{(v,v_*);(v,v_*,\omega)\in B_0^c} \chi_\varepsilon(v)\chi_\varepsilon(v_*) g_0 g_{0*}\,\mathrm{d}v\,\mathrm{d}v_*\mathrm{d}\omega.$$

For $\omega \in \Omega^c$, define

$$W_\omega := \{(v, v_*); (v, v_*, \omega) \in B_0^c\}.$$

Then $\text{meas}\,(W_\omega) \leqslant \sqrt{a(\delta)}$, and so,

$$Y \leqslant \int_{\Omega^c} \int_{v;\,\text{meas}(\{v_*;(v,v_*)\in W_\omega\}) > a(\delta)^{1/4}} \chi_\varepsilon \chi_{\varepsilon *} g_0 g_{0*}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega$$
$$+ \int_{\Omega^c} \int_{v;\,\text{meas}(\{v_*;(v,v_*)\in W_\omega\}) < a(\delta)^{1/4}} \chi_\varepsilon \chi_{\varepsilon *} g_0 g_{0*}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega.$$

Since

$$\text{meas}\,(\{v; \text{meas}\,(\{v_*;(v, v_*) \in W_\omega\}) > a(\delta)^{1/4}\}) < a(\delta)^{1/4}$$

and for any real number K,

$$\int_{v;\,\text{meas}(\{v_*;(v,v_*)\in W_\omega\}) > a(\delta)^{1/4}} g_0(v)\,\mathrm{d}v \leqslant K a(\delta)^{1/4} + \frac{1}{\ln K}\int g_0\,|\ln g_0|\,\mathrm{d}v,$$

taking $K = a(\delta)^{-1/8}$ leads to

$$\int_{v;\,\text{meas}(\{v_*;(v,v_*)\in W_\omega\}) > a(\delta)^{1/4}} \chi_\varepsilon g(v)\,\mathrm{d}v \leqslant a(\delta)^{1/8} + \frac{8c_1}{|\ln a(\delta)|}$$

and so,

$$Y \leqslant 4\pi\left(a(\delta)^{1/8} + \frac{8c_1}{|\ln a(\delta)|}\right),$$

The term X_3 can be handled analogously. Hence,

$$\int \chi_\varepsilon(v)|(g_0 - g_1)(v)|\,\mathrm{d}v \leqslant a_1(\delta),$$

where $\lim_{\delta\to 0} a_1(\delta) = 0$. Consequently,

$$R(v) =: |(g_0 - g_1)(v)| \leqslant \sqrt{a_1(\delta)}, \tag{2.8}$$

outside of a subset W_1 of V_δ of measure smaller than $\sqrt{a_1(\delta)}$. Moreover,

$$\left\| D_v \int B(|v - v_*|, \theta)\,\chi_\varepsilon(v_*)\,g(v_*)\,\mathrm{d}v_*\,\mathrm{d}\omega \right\|_{L^\infty} \leqslant c_7 \varepsilon^{-3/2},$$

so that, together with Lemma 2.1,

$$\|g_1\|_{H^1} \leqslant c_8 (i_0^{-2}\varepsilon^{-3} + i_0^{-4}\varepsilon^{-6}).$$

Hence, by the Sobolev imbedding of $H^1(\mathbb{R}^3)$ in $L^2(\mathbb{R}^3)$ and the Cauchy–Schwartz inequality, for any subset P of $\mathbb{R}^3$,

$$\int_P g_1(v)\,\mathrm{d}v \leqslant \left(\int_P g_1^2(v)\mathrm{d}v \right)^{1/2} |P|^{1/2} \leqslant c_8 (i_0^{-2}\varepsilon^{-3} + i_0^{-4}\varepsilon^{-6})|P|^{1/2}, \tag{2.9}$$

and so,

$$\begin{aligned}
&\int \chi_\varepsilon \chi_{\varepsilon *} |g_1 g_{1*} - g_1' g_{1*}'|\,\mathrm{d}v\,\mathrm{d}v_*\mathrm{d}\omega \\
&\quad \leqslant \int_{(v,v_*,\omega)\in B_0 \cap (W_1^c \times W_1^c \times S^2);(v',v_*')\in W_1^c \times W_1^c} \chi_\varepsilon \chi_{\varepsilon *} \\
&\qquad \times |(g+R)(g_* + R_*) - (g' + R')(g_*' + R_*')|\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\qquad + \int_{(v,v_*,\omega); v\in W_1 \, or \, v_* \in W_1 \, or \, v' \in W_1 \, or \, v_*' \in W_1} \chi_\varepsilon \chi_{\varepsilon *} (g_1 g_{1*} + g_1' g_{1*}')\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega \\
&\qquad + \int_{B_0^c} \chi_\varepsilon \chi_{\varepsilon *} |g_1 g_{1*} - g_1' g_{1*}'|\,\mathrm{d}v\,\mathrm{d}v_*\mathrm{d}\omega.
\end{aligned}$$

By (2.5), (2.6) and (2.8), the first term of the right-hand side in bounded from above by

$$4\pi\left(\frac{4\pi}{3}\left(\frac{6}{\varepsilon}\right)^3\right)^2 \sqrt{a(\delta)} + c_9 \sqrt{(a_1(\delta))}\,(\varepsilon^{-9/2} + \sqrt{(a_1(\delta))}\,\varepsilon^{-6}).$$

By (2.9), the second term of the right-hand side is bounded from above by $c_{10}(i_0^{-2}\varepsilon^{-3} + i_0^{-4}\varepsilon^{-6})^{1/2}a_1(\delta)^{1/4}$. By (2.9), the third term of the right-hand side is bounded from above by a function tending to zero when δ tends to zero. Hence,

$$\int \chi_\varepsilon \chi_{\varepsilon *} |g_1 g_{1*} - g'_1 g'_{1*}| \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\omega \leqslant b_1(\delta)$$

with $\lim_{\delta \to 0} b_1(\delta) = 0$, and so,

$$|g_1 g_{1*} - g'_1 g'_{1*}| \leqslant \sqrt{[b_1(\delta)]} =: d_1(\delta),$$

outside of a set B_1^c of measure $d_1(\delta)$. The procedure used to construct g_1 can be repeated from g_1 instead of g_0 and $d_1(\delta)$ instead of $\sqrt{\delta}$. When repeated k times, $1 \leqslant k \leqslant K$ it leads to functions $(g_k)_{1 \leqslant k \leqslant K}$, such that

$$g_k \in H^k, \quad v(g_k)(v) \geqslant i_k(\varepsilon), a.a.\ v \in V_{\varepsilon/6},$$

$$|(g_0 - g_k)(v)| \leqslant d_k(\delta),$$

except on a set B_k^c of measure smaller than $d_k(\delta)$, and

$$|g_k g_{k*} - g'_k g'_{k*}| \leqslant a_k(\delta),$$

outside of a set of measure $a_k(\delta)$, with $a_k(\delta)$ tending to zero when δ tends to zero. Moreover, $\|g_k\|_{H^k}$ is bounded from above by a polynomial expression in $\|f\|_{L^1}$ and $\|g\|_{L^1}$, and so by a constant depending on ε and independent of the family of functions f. Hence, by a Sobolev imbedding, there is for $k \geqslant 5$ a constant $\lambda_k(\varepsilon)$ such that

$$\|g_k\|_{W^{1,\infty}} \leqslant \lambda_k(\varepsilon).$$

Moreover, for any $(v, v_*, \omega) \in B_k^c$, there is $(\bar{v}, \bar{v}_*, \bar{\omega}) \in B_k$ such that

$$|v - \bar{v}| < d_k^{1/3}(\delta),\ |v_* - \bar{v}_*| < d_k^{1/3}(\delta),\ |\omega - \omega_*| < d_k^{1/3}(\delta).$$

Expressing $g_k(v) = (g_k(v) - g_k(\bar{v})) + g_k(\bar{v}), \ldots$, allows to bound $|g_k g_{k*} - g'_k g'_{k*}|$ from above by terms like

$$|g_k(v) - g_k(\bar{v})|\, g_k(v_*) \leqslant |Dg_k|_\infty |v - v_*| |g_k|_\infty \leqslant \lambda_k^2(\varepsilon)\, d_k^{1/3}(\delta),$$

that tend to zero when δ tends to zero.

Lemma 2.4. *There are an integer $k \geqslant 8$ and a positive constant m_k such that for every function f satisfying* (1.2), *the associated function g_k defined by* (2.1) *satisfies*

$$g_k(v) \geqslant m_k, \quad v \in V_{\varepsilon/6}.$$

Proof. By Lemma 2.2 and Theorem 2.3, the function g_8 belongs to H^8 and satisfies

$$2\pi \int_{|v - v_*| \geqslant r_*} g_8(v_*)\, \mathrm{d}v_* \geqslant i_8(\varepsilon), \quad v \in V_{\varepsilon/6},$$

and so, by the Lipschitz regularity of g_8, there are a vector v_0, a number $R > 0$ and a constant $c_{11}(\varepsilon)$ such that

$$g_8(v) > c_{11}(\varepsilon), \quad |v - v_0| \leqslant R.$$

Then

$$g_9(v) = \frac{1}{\nu(g_8)(v)} \int B(|v - v_*|, \theta) g_8(v') g_8(v'_*) \, \mathrm{d}v_* \, \mathrm{d}\omega$$

$$\geqslant (c_{11}(\varepsilon))^2 \int_{(v_*, \omega); |v' - v_0| \leqslant R, |v'_* - v_0| \leqslant R} B(|v - v_*|, \theta) \, \mathrm{d}v_* \, \mathrm{d}\omega$$

$$\geqslant c_{12}(\varepsilon), \quad |v - v_0| \leqslant R.$$

The end of the proof of Lemma 2.4 will use the following lemma.

Lemma 2.5. *There is a constant c_{13} and, for each v satisfying*

$$|v - v_0| \in \left] R, \frac{\sqrt{10}}{3} R \right],$$

a subset $S(v)$ of

$$\{(v_*, \omega) \in \mathbb{R}^3 \times S^2; \quad |v - v_*| \geqslant r_*, \quad \cos\theta \in [s_*, 1 - s_*]\}$$

with $|S(v)| \geqslant c_{13}$, such that

$$|v' - v_0| \leqslant R, \quad |v'_* - v_0| \leqslant R, \quad (v_*, \omega) \in S(v).$$

End of the proof of Lemma 2.4. By Lemma 2.5,

$$g_9(v) \geqslant \frac{1}{\|\nu(g_8)\|_{L^\infty}} \int_{S(v)} g_8(v') g_8(v'_*) \, \mathrm{d}v_* \, \mathrm{d}\omega$$

$$\geqslant \frac{c_{12}^2(\varepsilon) c_{13}}{\|\nu(g_8)\|_{L^\infty}}, \quad |v - v_0| \leqslant \frac{\sqrt{10}}{3} R.$$

Repeating the same procedure from g_9, a finite number of iterations leads to some $k \geqslant 8$ and positive constant m_k, such that

$$g_k(v) \geqslant m_k, \quad v \in V_{\varepsilon/6}.$$

Proof of Lemma 2.5. Take $v_0 = 0$ for the sake of clarity. Denote by

$$e := 1 - \frac{R^2}{v^2}.$$

Note that $e \in]0, \frac{1}{9}]$. In an orthonormal basis with $v/|v|$ as third vector, let $v - v_*$ and ω be respectively parametrized by

$$v - v_* = A(\cos\alpha\cos\beta, \sin\alpha\cos\beta, \sin\beta), \quad A \geqslant 0,$$

$$\omega = (\cos\lambda\cos\mu, \sin\lambda\cos\mu, \sin\mu).$$

Consider those v_* and ω such that

$$(\beta, \mu) \in \left[0, \frac{\pi}{2}\right[^2, \quad \beta + \mu \in \left[0, \frac{\pi}{2}\right[.$$

Then,

$$v'^2 = |v - (v - v_*, \omega)\omega|^2 \leqslant R^2$$

is equivalent to

$$(v - v_*, \omega)^2 - 2|v|(\sin\mu)(v - v_*, \omega) + v^2 - R^2 \leqslant 0,$$

i.e.

$$\sin^2\mu > e$$

and

$$A(\cos(\alpha - \lambda)\cos\beta\cos\mu + \sin\beta\sin\mu) = (v - v_*, \omega)$$

$$\in [|v|(\sin\mu - \sqrt{(\sin^2\mu - e)}), |v|(\sin\mu + \sqrt{(\sin^2\mu - e)})].$$

Moreover,

$$v_*'^2 = |v - (v - v_*) + (v - v_*, \omega)\omega|^2 \leqslant R^2$$

is equivalent to

$$(v - v_*, \omega)^2 - 2|v|(\sin\mu)(v - v_*, \omega) - A^2 + 2|v|(\sin\beta)A - (v^2 - R^2) \geqslant 0,$$

which follows from the sufficient condition

$$A^2 - 2|v|(\sin\beta)A + v^2 - R^2 + v^2\sin^2\mu \leqslant 0.$$

This last inequality holds if and only if

$$\sin^2\beta \geqslant \sin^2\mu + e$$

and

$$A \in [|v|(\sin\beta - \sqrt{(\sin^2\beta - (\sin^2\mu + e))}), |v|(\sin\beta + \sqrt{(\sin^2\beta - (\sin^2\mu + e))})]. \tag{2.10}$$

Hence the inequalities $v'^2 \leqslant R^2$ and $v_*'^2 \leqslant R^2$, with $|v - v_*| \geqslant r_*, \cos\theta \in [s_*, 1 - s_*]$ follow from the sufficient condition (2.10) together with

$$(\beta, \mu) \in \left[0, \frac{\pi}{2}\right[^2, \ (\beta - \mu, \beta + \mu) \in \left[0, \frac{\pi}{2}\right[^2,$$

$$\sin^2\mu > e, \quad \sin^2\beta > \sin^2\mu + e, \quad A \geqslant r_* - \tan\beta\tan\mu$$

$$+ \frac{1}{\cos\beta\cos\mu} \max\left\{s_*, |v|\,\frac{\sin\mu - \sqrt{(\sin^2\mu - e)}}{A}\right\} < \cos(\alpha - \lambda)$$

$$< -\tan\beta\tan\mu + \frac{1}{\cos\beta\cos\mu} \min\left\{1 - s_*, |v|\,\frac{\sin\mu + \sqrt{(\sin^2\mu - e)}}{A}\right\}. \quad (2.11)$$

There is a subset $S(v)$ of positive measure satisfying

$$|v'| \leqslant R, \quad |v_*'| \leqslant R, \quad (v - v_*, \omega) \in S(v),$$

as soon as

$$\mu \in \left]\mathrm{Arcsin}\,\frac{1}{2\sqrt{2}}, \frac{\pi}{6}\right[, \ \beta \in \left]\mathrm{Arcsin}\,\frac{1}{2}\sqrt{\left(\frac{3}{2}\right)}, \frac{\pi}{3}\right[,$$

and the following compatibility conditions for (2.11) hold:

$$(1 - s_*)\sin\beta > \sin\mu, \cos(\beta - \mu) > s_*, r_*\, s_* < \frac{R}{2\sqrt{2}}, r_* < \frac{R}{2}\sqrt{\left(\frac{3}{2}\right)},$$

which is true for r_* and s_* small enough.

3. A functional inequality

Theorem 3.1. *Let a family F of functions from $\mathbb{R}^3$ with values in $\mathbb{R}$, satisfying* (1.2) *and such that $f > 0$ almost everywhere. Given $\varepsilon > 0$, there are $\delta \in]0, \frac{\varepsilon}{6}[$ and a function $a(\delta)$ tending to zero when δ tends to zero, such that if*

$$|f(v)f(v_*) - f(v')f(v_*') < a(\delta)$$

in S_δ outside of a subset of measure bounded by $a(\delta)$, then there is a Maxwellian M_f such that $|f|v| - M_f(v)| < \varepsilon$ outside of a subset of V_ε of measure bounded by ε.

Proof. The theorem follows from Section 2 together with arguments that can be found in [2]. Let $\varepsilon \in]0, 1[$ and $\delta > 0$ be given. Either

$$\int_{V_\varepsilon} f(v)\,\mathrm{d}v < \varepsilon^2$$

and then $M_f = 0$ is suitable. Or $\int_{V_\varepsilon} f(v)\,dv > \varepsilon^2$. Then, let g_k be the function associated to f in Theorem 2.3 and Lemma 2.4, that satisfies

$$|(\sqrt{f} - g_k)(v)| \leqslant d_k(\delta),$$

outside of a subset of $V_{\varepsilon/6}$ of measure $d_k(\delta)$,

$$|g_k g_{k*} - g'_k g'_{k*}| \leqslant e_k(\delta), \quad (v, v_*, \omega) \in S_{\varepsilon/6}$$

and

$$g_k(v) \geqslant m_k, \quad v \in V_{\varepsilon/6}.$$

Define

$$\tilde{S} := \{(v, v_*, \omega) = (v' - (v' - v'_*, \omega)\omega, v'_* + (v' - v'_*, \omega)\omega, \omega),$$

$$(v, v_*, \omega) \in S_{\varepsilon/6} \text{ and } (v', v'_*, \omega) \in S_{\varepsilon/6}\}.$$

Define $\varphi := \log g_k$. Then

$$|\varphi(v) + \varphi(v_*) - \varphi(v') - \varphi(v'_*)| < c_{14} e_k(\delta), \quad (v, v_*, \omega) \in \tilde{S}.$$

Taking $v = 0$, $v_* = U + V$, $v' = U$, $v'_* = V$, with $(U, V) \in V_{\varepsilon/2} \times V_{\varepsilon/2}$ and $(U, V) = 0$, it holds that

$$|\varphi(0) + \varphi(U + V) - \varphi(U) - \varphi(V)| < c_{14} e_k(\delta).$$

Define

$$G(U) := \varphi(U) - \varphi(0), \quad K(U) := \tfrac{1}{2}(G(U) + G(-U)),$$

$$B(U) := \tfrac{1}{2}(G(U) - G(-U))$$

and by

$$M_{\varepsilon/2} := \{(U, V) \in V_{\varepsilon/2} \times V_{\varepsilon/2};\ (U, V) = 0\}.$$

Study of K. For any $(U, V) \in M_{\varepsilon/2}$,

$$|G(U + V) - G(U) - G(V)| < c_{14} e_k(\delta),$$

$$|G(-U - V) - G(-U) - G(-V)| < c_{14} e_k(\delta),$$

$$|G(-U + V) - G(-U) - G(V)| < c_{14} e_k(\delta),$$

$$|G(U - V) - G(U) - G(-V)| < c_{14} e_k(\delta),$$

so that

$$\left|K(p_1) - K\left(\frac{p_1 + p_2}{2}\right) - K\left(\frac{p_1 - p_2}{2}\right)\right| < c_{14}e_k(\delta),$$

$$\left|K(p_2) - K\left(\frac{p_1 + p_2}{2}\right) - K\left(\frac{p_1 - p_2}{2}\right)\right| < c_{14}e_k(\delta)$$

for any (p_1, p_2) such that $p_1^2 = p_2^2$. Hence,

$$|K(p_1) - K(p_2)| < 2c_{14}e_k(\delta), \quad p_1^2 = p_2^2.$$

Hence there is a function $\bar{K}$ defined on $(0, 1/\varepsilon^2)$ such that

$$K(p) = \bar{K}(p^2) + \tilde{K}(p)$$

with $|\tilde{K}(p)| < 2c_{14}e_k(\delta)$, for all $p \in V_\varepsilon$, and so,

$$|\bar{K}(U^2 + V^2) - \bar{K}(U^2) - \bar{K}(V^2)| < 7c_{14}e_k(\delta), \quad (U, V) \in M_{\varepsilon/2}.$$

It follows from [9] that there is a constant d such that for every $x \in [0, 1/\varepsilon^2]$,

$$|\bar{K}(x) - \mathrm{d}x| < c_{15}e_k(\delta).$$

Hence,

$$|K(p) - \mathrm{d}p^2| \leqslant |K(p) - \bar{K}(p^2)| + |\bar{K}(p^2) - \mathrm{d}p^2|$$
$$< c_{16}e_k(\delta), \quad p \in V_\varepsilon.$$

Study of B. Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. Then, for any $(U, V) = (u_1e_1 + u_2e_2 + u_3e_3, v_1e_1 + v_2e_2 + v_3e_3) \in M_{\varepsilon/2}$,

$$|B((u_1 + v_1)e_1) - B(u_1e_1) - B(v_1e_1)$$
$$- \{ - B((u_2 + v_2)e_2) - B((u_3 + v_3)e_3) + B(u_2e_2) + B(v_2e_2) + B(u_3e_3)$$
$$+ B(v_3e_3)\}| < 7c_{14}e_k(\delta)$$

and

$$|B((u_1 + v_1)e_1) - B(u_1e_1) - B(v_1e_1)$$
$$+ \{ - B((u_2 + v_2)e_2) - B((u_3 + v_3)e_3) + B(u_2e_2) + B(v_2e_2) + B(u_3e_3)$$
$$+ B(v_3e_3)\}| < 7c_{14}e_k(\delta).$$

Hence,

$$|B((u_1 + v_1)e_1) - B(u_1e_1) - B(v_1e_1) < 14c_{14}e_k(\delta), \quad (U, V) \in M_{\varepsilon/2}.$$

Define $b_1(x) := B(xe_1)$, $x \in [-1/\varepsilon, 1/\varepsilon]$. Then

$$|b_1(x+y) - b_1(x) - b_1(y)| < 14c_{14}e_k(\delta), \quad (x,y) \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right]^2, \quad x+y \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right].$$

It follows from [9] that there is a constant $\tilde{b}_1$ such that

$$|b_1(x) - \tilde{b}_1 x| < c_{17}e_k(\delta), \quad x \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right].$$

Analogous results are obtained for $B(u_2e_2)$ and $B(u_3e_3)$, and so, there is a constant vector $\tilde{b}$ such that

$$|B(U) - (\tilde{b}, U)| < c_{18}e_k(\delta), \quad U \in V_{\varepsilon/4}.$$

Finally, there are constants $a := \varphi(\xi_0) - (\tilde{b}, \xi_0) - d\xi_0^2$, $\bar{b} := \tilde{b} - 2d\xi_0$ and d such that

$$|\varphi(v) - \{a + (\bar{b}, v) + dv^2\}| < c_{19}e_k(\delta), \quad v \in V_\varepsilon.$$

Choosing δ small enough so that $c_{19}e_k(\delta) < \varepsilon$ and $d_k(\delta) < \varepsilon$, ends the proof of Theorem 3.1.

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