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Abstract

We consider the Boltzmann equation for a gas in a horizontal slab, subject to a gravitational force. The boundary conditions are of diffusive type, specifying the wall temperatures, so that the top temperature is lower than the bottom one (Benard setup). We consider a 1-dimensional stationary solution, which is close for small Knudsen number to the laminar purely conductive stationary solution of the Oberbeck-Boussinesq equations, and prove its stability under small 1-dimensional perturbations and for small Knudsen number.

1. Introduction

The bifurcation theory allows to study phenomena of spontaneous formation of patterns from a homogeneous state. The Rayleigh-Benard problem is a prototype of pattern formation: when a layer of viscous fluid under the action of a gravitational force is heated from below, convective instabilities set in when the vertical temperature gradient exceeds a certain critical value. Below this value the motionless state is stable, while beyond the critical value it becomes unstable and various pattern flows appear, for example roll patterns. This phenomenon is called Rayleigh-Benard convection and has been studied within the scheme of the Oberbeck-Boussinesq approximation. In recent years Y. Sone and the team in Kyoto have made extensive numerical

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studies of the kinetic aspects of the problem using an asymptotic approach, see e.g. [12] and the monographs [11]. In this paper we start a study of stability properties for this system on the level of the Boltzmann equation. Our starting point is the stationary Boltzmann equation in a slab of width $2d$ with diffusive boundary conditions.

A stationary solution F_s of the Boltzmann equation in the laminar regime has been constructed in [7] for small Knudsen number ε. It is given in terms of a truncated expansion in ε whose leading term in the bulk is a global Maxwellian. The term of order ε determines the hydrodynamic quantities which are close, up to order ε^2, to the density, velocity and temperature of the hydrodynamic laminar flow, stationary solution of the OBE. This purely conducting solution of the Boltzmann equation was constructed only for small difference of temperatures, corresponding to having a small Rayleigh number. Here, (Sections 2 and 3), we construct this solution for any value of the Rayleigh number, provided that the force is small enough. This solution can be proved to be positive, by using the approach in [2]. Moreover, we prove the stability of F_s for the Boltzmann dynamics under suitable small 1-dimensional initial perturbations and for small Knudsen number.

At the hydrodynamical level the 1-dimensional perturbations, i.e. perturbations depending only on the z coordinate, in the direction orthogonal to the walls are easy to study. The component of the velocity field u_z orthogonal to the walls has to be zero by the incompressibility condition and the impermeability condition, while the other components u_x, u_y and the temperature are solutions of a set of decoupled parabolic equations so that general results on parabolic equations provide their decay in time for any Rayleigh number. On the contrary, more general perturbations (depending on x, y) decay to zero, proving the stability of the laminar flow, only up to a critical value of the Rayleigh number [9]. Above this value the purely conductive solution becomes unstable and the roll solutions appear and are stable, at least for values of the Rayleigh number close to the critical one [8].

At the kinetic level even the 1-dimensional perturbations require some effort to be studied. We study the Boltzmann equation for the perturbation $\Phi = F - F_s$ and write the solution again in terms of a truncated expansion in ε.

$$\Phi(t,z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(t,z,v) + \varepsilon R(t,z,v), \quad z \in (-d,d) \ . \tag{1.1}$$

The expansion start with a term of first order

$$\Phi^{(1)} = M\Big(\rho^1 + u^1 \cdot v + T^1 \frac{|v|^2-3}{2}\Big) \ ,$$

where M is the standard Maxwellian and the fields $\rho^1(t,z), u^1(t,z), T^1(t,z)$ are solutions of the hydrodynamic equations for the perturbation.

The next orders involve boundary layer corrections as well as kinetic corrections in the bulk. We give in Section 4 a procedure to compute these terms and show that they decay to zero exponentially in time. The main difficulty is the control of the remainder R asymptotically in time. Starting from the technical frame of [10], new techniques were introduced and developed in [1]–[4] to prove also the bifurcation for the Taylor-Couette flow [1]. The extension of this method to the present setting is not trivial. The added difficulties in comparison with the Taylor-Couette problem are the force term and the diffuse reflection boundary conditions which require a number of additional ideas in the proofs. The result is summarized in the following main theorem.

Theorem 1.1. *Assume that the gravitational force is small enough. Assume that the perturbation at time zero depends only on z. More, assume that the initial perturbation matches the expansion up to order ε^4 (as detailed in Section 4 below). Then there exists a steady solution F_s in $L^2_{M^{-1}}$ for the Boltzmann equation in the slab with exterior gravitational force field. Here M is the standard Maxwellian. This solution is stable. The stability is uniform in ε for $\varepsilon \le \varepsilon_0$ small enough.*

Here, stable means that the perturbation vanishes asymptotically in time. In the context, it follows from

$$\int_0^{+\infty} \int_{[-d,d]} \int_{\mathbb{R}^3} |\Phi(t,z,v)|^2 M^{-1} dt dz dv < \infty \ ,$$

which is proved in Section 5. We remark that the method presented here strongly relies on the fact that the problem we are dealing with has suitable stability properties at the fluid dynamic level, which we show to be preserved in the kinetic setup by means of a perturbative analysis starting from an Hilbert-type asymptotic expansion plus boundary layer corrections. The preservation of the fluid dynamic stability at kinetic level also occurs in the Taylor-Couette case discussed in [1], where the bifurcation phenomenon also arises. Our approach can, in particular, be extended to prove the stability of the purely conductive solution for $3d$ initial perturbations as well as the stability of the roll solution beyond and close to the critical Rayleigh number.

2. Asymptotic Expansion for the Stationary Solution

In this section we recall the basic elements of the expansion for the stationary purely conductive solution presented in [7] for sake of completeness and introduce the setup and the relevant notations.

The starting point is the stationary Boltzmann equation in a slab of width $2d$ with diffusive boundary conditions

$$\begin{aligned} &v_z\frac{\partial f}{\partial z}-g\frac{\partial f}{\partial v_z}=Q(f,f),\\ &f(-d,v)=\bar M_-(v)\int_{w_z<0}\mid w_z\mid f(-d,w)dw,\quad v_z>0,\\ &f(d,v)=\bar M_+(v)\int_{w_z>0}w_zf(d,w)dw,\quad v_z<0,\end{aligned}$$

where $\bar M_\pm(v)=\frac{\bar\rho}{2\pi T_\pm^2}e^{-\frac{v^2}{2T_\pm}}$, with $T_+<T_-$.

$$Q(f,g)(z,v,t)=\frac12\int_{\mathbb R^3}dv_*\int_{S_2}d\omega B(\omega,|v-v_*|)\{f'_*g'+f'g'_*-f_*g-g_*f\}$$

where h',h'_*,h,h_* stand for $h(z,v',t),h(z,v'_*,t),h(z,v,t),h(z,v_*,t)$ respectively, $S_2=\{\omega\in\mathbb R^3\,|\omega^2=1\}$, B is the differential cross section and v',v'_* are the incoming velocities of a collision with outgoing velocities v, v_* and impact parameter ω. We confine ourselves to the collision cross section $B(\omega,V)=|V\cdot\omega|$ corresponding to hard spheres.

We put the equation in dimensionless form by using d, T_-, and $\bar\rho$ as reference length, reference temperature and reference density respectively. We use $\sqrt{T_-}$ as reference velocity. We also redefine the collision cross section to make explicit its dependence on ℓ_0, the mean free path of the gas in equilibrium at temperature T_- and density $\bar\rho$. We get

$$\begin{aligned} &v_z\frac{\partial f}{\partial z}-\frac{1}{Fr}\frac{\partial f}{\partial v_z}=\frac1\varepsilon Q(f,f),\\ &f(-1,v)=M_-(v)\int_{w_z<0}\mid w_z\mid f(-1,w)dw,\quad v_z>0,\end{aligned}$$

$$f(1,v)=M_+(v)\int_{w_z>0}w_zf(1,w)dw,\quad v_z<0,$$

where $Fr=\frac{2T_-}{dg}$, $\varepsilon=\frac{\ell_0}{d}$ and $\delta\hat T=\frac{T_+-T_-}{2T_-}$ are the dimensionless parameters called Froude number, Knudsen number, and rescaled temperature difference respectively. Moreover,

$$M_-(v)=\frac{1}{2\pi}e^{-\frac{v^2}{2}},\quad M_+(v)=\frac{1}{2\pi(1+2\delta\hat T)^2}e^{-\frac{v^2}{2(1+2\delta\hat T)}} \tag{2.1}$$

The Rayleigh number, which is relevant at the hydrodynamical level, is defined as (see [11])

$$Ra=\frac{16\delta\hat T}{\pi Fr\varepsilon^2}$$

Since we will be interested in the behavior of the system in the vanishing ε limit, we rescale the Froude number and the difference temperature in such a way that the Rayleigh number is independent of ε. Define $G=\frac{1}{\varepsilon Fr}$ and $\lambda=\frac{\delta\hat T}{\varepsilon}$. Then $Ra=\frac{G16\lambda}{\pi}$.

The rescaled Boltzmann equation is

$$\begin{aligned} &v_z\frac{\partial f^\varepsilon}{\partial z}-\varepsilon G\frac{\partial f^\varepsilon}{\partial v_z}=\frac1\varepsilon Q(f^\varepsilon,f^\varepsilon),\\ &f^\varepsilon(-1,v)=M_-(v)\int_{w_z<0}\mid w_z\mid f^\varepsilon(-1,w)dw,\quad v_z>0,\\ &f^\varepsilon(1,v)=M_+(v)\int_{w_z>0}w_zf(1,w)dw,\quad v_z<0\ ,\end{aligned} \tag{2.2}$$

with M_- given in (2.1) and $M_+(v)=\frac{1}{2\pi(1+2\lambda\varepsilon)^2}e^{-\frac{v^2}{2(1+2\lambda\varepsilon)}}$.

A solution to this equation was constructed in [7] by means of an ϵ expansion. We refer to that paper for the detailed construction of the terms in the expansions. Here, we report only the main results. We write the solution of (2.2) in the form

$$f^\varepsilon=M+\varepsilon f^1+\sum_{n=2}^5\varepsilon^nf^n+\varepsilon R \tag{2.3}$$

where M is the standard Maxwellian ($M=(2\pi)^{-1/2}M_-$) and

$$f^1=M\Big(r+u\cdot v+\theta\frac{|v|^2-3}{2}\Big),$$

with r, u, θ solutions of the Boussinesq equations. In the present setup the stationary solution reduces to

$$u = 0, \quad \theta = \lambda(1+z), \quad r = -(G+\lambda)z. \tag{2.4}$$

The higher terms of the expansion $f^n, n = 2, \dots, 5$, are decomposed into two parts B_n and $b_n^\pm$, representing the bulk and boundary layer corrections. The B_n have to satisfy for $n = 2, \dots, 5$

$$v_z \partial_z B_{n-1} - G \partial_{v_z} B_{n-2} = 2Q(M, B_n) + \sum_{i+j=n} Q(B_i, B_j) \tag{2.5}$$

where $B_0 \equiv M$ and $B_1 \equiv f^1$.

The boundary layer corrections relative to the wall $z = \pm 1$, $b_n^\pm$, are chosen to satisfy, for $n = 2, \dots, 5$, the equations

$$\begin{aligned} & v_z \frac{\partial b_n^\pm}{\partial z^\pm} \mp \varepsilon^2 G^\pm \frac{\partial}{\partial v_z} b_n^\pm \\ & = \mathcal{L}^\pm b_n^\pm + 2Q(\Delta M, b_{n-1}^-)\chi^\pm \\ & \quad \mp (-G^0 + G^\mp) \frac{\partial}{\partial v_z} b_{n-2}^\pm + \sum_{\substack{i,j \geq 1 \\ i+j=n}} \Big[2Q(B_i, b_j^\pm) + Q(b_i^\pm, b_j^\pm) + Q(b_i^\mp, b_j^\mp) \Big], \end{aligned} \tag{2.6}$$

where we have put

$$z^\pm = \varepsilon^{-1}(1 \mp z), \quad z^\pm \in [0, 2\varepsilon^{-1}] \tag{2.7}$$

and

$$\begin{aligned} & b_0^\pm = b_1^\pm = 0, \qquad \chi^+ = 1, \chi^- = 0, \qquad \mathcal{L}^\pm = 2Q(M^\pm, \cdot), \quad M^- = M, \\ & \qquad \Delta M = \varepsilon^{-1}[M - M^+], \qquad M^+ = (1 + \varepsilon r(1)) M [(1 + 2\lambda\varepsilon)]^{-1/2}. \end{aligned}$$

The constant gravity force G has been decomposed into three parts: a bulk part G_0 and two boundary parts $G^\pm$ which are different from zero in the bulk and near the walls respectively. Their definition is

$$G = G^+ + G^0 + G^- \tag{2.8}$$

with G^0 and $G^\pm$ smooth functions such that for some $\delta > 0$

$$G^+(z) = \begin{cases} G, & 1 - \delta\,\varepsilon \leq z \leq 1 \\ 0, & -1 \leq z \leq 1 - 2\delta\,\varepsilon \end{cases}, \tag{2.9}$$

$$\begin{aligned} G^-(z) &= \begin{cases} G, & -1 \leq z \leq -1 + \delta\,\varepsilon \\ 0, & -1 + 2\delta\,\varepsilon \leq z \leq 1 \end{cases} \\ G^0(z) &= \begin{cases} G, & -1 + 2\delta\,\varepsilon \leq z \leq 1 - 2\delta\,\varepsilon \\ 0, & |z| \geq 1 - \delta\,\varepsilon \end{cases}. \end{aligned} \tag{2.10}$$

Note that $G^\pm(z^\pm)$ is zero for $z^\pm \in [2\delta, 2\varepsilon^{-1}]$.

We remark that, as discussed in [7], we need to include the gravity in the Milne problems (2.6), although it is of order ε^2, to fix the following difficulty: the solution to the Milne problem cannot have bounded derivative with respect to v_z near the boundary, because it is discontinuous in v_z at $v_z = 0$ on the boundary. If the gravity were not included in (2.6), one should keep in the expansion the v_z-derivative of the boundary layer corrections, which then would be singular at the boundaries. Therefore we are forced to look at the Milne problem with a force term along the direction orthogonal to the boundary. On the other hand the results on [5] only apply to suitably decaying forces and this motivates the splitting of G into $G^+ + G^- + G^0$, with G^0 supported far away from the boundary, so that the v_z-derivative of the boundary layer corrections need to be computed only far from the boundary where they are bounded. We also note that most of our treatment works also for hard potentials. The only point where we need the hard spheres assumption is to use the decay results in [5], which were obtained only in the hard sphere case.

Finally the equation for the remainder is

$$v_z \frac{\partial}{\partial z} R - \varepsilon G \frac{\partial}{\partial v_z} R = \frac{1}{\varepsilon} \mathcal{L} R + \sum_{i=1}^{5} \varepsilon^j J(f^j, R) + \varepsilon Q(R, R) + A \tag{2.11}$$

where

$$\mathcal{L} R = Q(M, R) + Q(R, M), \quad J(h, g) = 2Q(h, g)$$

and A is given by

$$\begin{aligned} \varepsilon^{-4} A = & -v_z \tfrac{\partial}{\partial z} B_5 + G \tfrac{\partial}{\partial v_z}(B_4 + \varepsilon B_5) + (G^0 + G^-) \tfrac{\partial}{\partial v_z}[(b_4^+ + \varepsilon b_5^+)] \\ & + (G^0 + G^+) \tfrac{\partial}{\partial v_z}[(b_4^- + \varepsilon b_5^-)] + \textstyle\sum_{\substack{k,m \geq 1 \\ k+m \geq 6}} \varepsilon^{k+m-6} Q(f_k, f_m). \end{aligned} \tag{2.12}$$

We impose on the f^j the following boundary conditions:

$$\begin{aligned} f^j(z,v) &= M_-(v)\int_{w_z<0}|w_z|f^j(z,w)dw+\gamma^-_{j,\varepsilon}(v), \quad z=-1, \quad v_z>0,\\ f^j(z,v) &= M_+(v)\int_{w_z>0}|w_z|f^j(z,w)dw+\gamma^+_{j,\varepsilon}(v), \quad z=1, \quad v_z<0, \end{aligned} \tag{2.13}$$

with the functions $\gamma^\pm_{n,\varepsilon}(v)$ exponentially small in ε^{-1} and such that

$$\langle\gamma^\pm_{n,\varepsilon}v_z\rangle=\int_{\mathbb{R}^3}dv\gamma^\pm_{n,\varepsilon}(v)v_z=0,$$

specified by the expansion. Here and below we use the short notation $\langle f\rangle$ to denote the integration on the velocities of a function f. Finally, we impose the following conditions on R:

$$\begin{aligned} R(z,v) &= M_-(v)\int_{w_z<0}|w_z|R(z,w)dw-\sum_{n=2}^5\varepsilon^{n-3}\gamma^-_{n,\varepsilon}, \ z=-1, \ v_z>0,\\ R(z,v) &= M_+(v)\int_{w_z>0}|w_z|R(z,w)dw-\sum_{n=2}^5\varepsilon^{n-3}\gamma^+_{n,\varepsilon}, \ z=\ 1, \ v_z<0. \end{aligned} \tag{2.14}$$

Set

$$\tilde L^q:=\Big\{f:[-1,1]\times\mathbb{R}^3\to\mathbb{R};|f|_q:=\Big(\int M(v)\Big(\int|f(z,v)|^q dz\Big)^{\frac{2}{q}}dv\Big)^{\frac12}<+\infty\Big\}.$$

Theorem 2.1. *It is possible to uniquely determine functions f^n, $n=1,\dots,5$ satisfying the conditions* (2.13). *Moreover, for $q=2,\infty$*

$$|M^{-1}f^n|_q<+\infty.$$

3. Linear Estimates in the Stationary Case

In this section we give the main relevant a priori estimates for the linearized equation for the remainder R with a given source term D:

$$v_z\frac{\partial R}{\partial z}-\varepsilon G\frac{\partial R}{\partial v_z}=\frac{1}{\varepsilon}\Big(\mathcal{L}R+2\sum_{j=1}^{j_1}\varepsilon^j J(R,f^j))+D\Big) \tag{3.1}$$

satisfying the boundary conditions (2.14).

We put $R=M\mathrm{e}^{-\varepsilon G(z+1)}\tilde R:=\tilde M\tilde R$. Then, since

$$\Big(v_z\frac{\partial}{\partial z}-\varepsilon G\frac{\partial}{\partial v_z}\Big)[M\mathrm{e}^{-\varepsilon G(z+1)}]=0, \tag{3.2}$$

$\tilde R$ has to solve the following boundary value problem:

$$v_z\frac{\partial\tilde R}{\partial z}-\varepsilon G\frac{\partial\tilde R}{\partial v_z}=\frac{1}{\varepsilon}\Big(\tilde L\tilde R+2\sum_{j=1}^{j_1}\varepsilon^j\tilde J(\tilde R,\tilde M^{-1}f^j))+\tilde D\Big), \tag{3.3}$$

$$\begin{aligned} \tilde R(-1,v) &= \int_{w_z<0}|w_z|M_-(w)\tilde R(-1,w)dw-\tilde M^{-1}\sum_{n=2}^5\varepsilon^{n-3}\gamma^-_{n,\varepsilon}, \quad v_z>0,\\ \tilde R(1,v) &= \frac{M_+}{\tilde M}(v)\int_{w_z>0}|w_z|M(w)\tilde R(1,w)dw-\tilde M^{-1}\sum_{n=2}^5\varepsilon^{n-3}\gamma^+_{n,\varepsilon}, \ v_z<0, \end{aligned}$$

where

$$\tilde Lh=\frac{1}{\tilde M}\mathcal{L}(\tilde Mh), \quad \tilde J(h,g)=\frac{1}{\tilde M}J(\tilde Mh,\tilde Mg), \quad \tilde D=\frac{1}{\tilde M}D.$$

Note that, since the factor $\mathrm{e}^{-\varepsilon G(z+1)}$ does not depend on v, $\tilde Lh=\frac{1}{\tilde M}\mathcal{L}(\tilde Mh)=\frac{1}{M}\mathcal{L}(Mh)=Lh$. Remind that $L=K-\nu$ with K a compact operator and ν a positive function of v which, in the case of the hard spheres cross section, satisfies the estimate

$$\nu_0(1+|v|)\le\nu(v)\le\nu_1(1+|v|) \tag{3.4}$$

with $0<\nu_0<\nu_1$.

Recall that $\psi_0=1,\psi_1=v_x,\psi_2=v_y,\psi_3=v_z,\psi_4=\frac{1}{\sqrt6}(v^2-3)$ form an orthonormal basis for the kernel of L in $L^2_M(\mathbb{R}^3)=L^2(\mathbb{R}^3,Mdv)$. Introduce an orthogonal splitting of functions $f\in L^2_M([-1,1]\times\mathbb{R}^3)$ into $f=f_\|+f_\perp=Pf+(I-P)f$, where $f_\perp$ is the non hydrodynamic part

$$\int M(v)(1,v,v^2)f_\perp(z,v)dv=0$$

and the fluid dynamic part is given explicitly by

$$f_\|(z,v)=f_0(z)+f_1(z)v_x+f_2(z)v_y+f_3(z)v_z+f_4(z)\frac{1}{\sqrt6}(v^2-3),$$

with

$$\int M\psi_0 f(z,v)dv = f_0(z), \quad \int M\psi_4 f(z,v)dv = f_4(z),$$
$$\int M\psi_1 f(z,v)dv = f_1(z), \quad \int M\psi_2 f(z,v)dv = f_2(z),$$
$$\int M\psi_3 f(z,v)dv = f_3(z).$$

The relevant ingoing boundary space is

$$L^+ := \Big\{f; \mid f \mid_\sim := \Big(\int_{v_z>0} v_z M(v) \mid f(-1,v) \mid^2 dv\Big)^{\frac{1}{2}} + \Big(\int_{v_z<0} \mid v_z \mid M(v) \mid f(1,v) \mid^2 dv\Big)^{\frac{1}{2}} < +\infty\Big\}.$$

We have already introduced the space

$$\tilde{L}^q := \Big\{f; |f|_q := \Big(\int M(v)\Big(\int |f(z,v)|^q dz\Big)^{\frac{2}{q}} dv\Big)^{\frac{1}{2}} < +\infty\Big\}.$$

We also need the space

$$\mathcal{W}^{q-}([-1,1]\times\mathbb{R}^3) = \mathcal{W}^{q-} := \{f; \nu^{\frac{1}{2}} f \in \tilde{L}^q, \nu^{-\frac{1}{2}} Df \in \tilde{L}^q, \gamma^- f \in L^+\},$$

where $Df = v_z\frac{\partial F}{\partial z} - \varepsilon G\frac{\partial F}{\partial v_z}$ and the operator γ^- denotes the ingoing trace operator on the set

$$\Gamma^- = \{-1\}\times\{v_z > 0\} \cap \{1\}\times\{v_z < 0\}.$$

The first result of this section is an existence and uniqueness theorem for eq. (3.3) with given indata for $j_1 = 5$ and with $f^j \in \tilde{L}^\infty$ for $j = 1, \ldots, 5$.

Lemma 3.1. *Let $\nu^{-\frac{1}{2}} g \in \tilde{L}^q$, $F_b \in L^+$, $2 \le q < \infty$, be given. There exists a unique solution $F \in \mathcal{W}^{q-}$ to*

$$v_z\frac{\partial F}{\partial z} - \varepsilon G\frac{\partial F}{\partial v_z} = \frac{1}{\varepsilon}\Big(\tilde{L}F + 2\sum_{j=1}^{5}\varepsilon^j \tilde{J}(F, \tilde{M}^{-1}f^j) + g\Big), \tag{3.5}$$
$$F(-1,v) = F_b(-1,v),\ v_z > 0, \quad F(1,v) = F_b(1,v),\ v_z < 0.$$

Proof of Lemma 3.1. The proof of Lemma 3.1 follows the lines of [10], pp 68-70, where $G = 0$. Notice first that the a priori estimates (3.9) and (3.10) below imply uniqueness in L^2. Then use the solution formula from the proof

of Lemma 3.3 below, and consider the case when the asymptotic expansion $\tilde{M}\varphi = \sum_1^5 \varepsilon^j f^j$ with ε-orders up to 5, equals zero. Now UK is compact in $\tilde{L}^2$ (e.g by first proving the compactness of UE for $EF := \int MFdv$ and then using the splitting $K = K' + K''$ from Lemma 3.3 below), so the L^2 case follows from Fredholm's alternative. The $\tilde{L}^\infty$ case then follows from (3.8), and the intermediate cases hold similarly. Finally the addition of the perturbation $\tilde{J}(F,\varphi)$ does not change the result. □

A similar result holds in the case of diffuse reflection boundary conditions. The following proof uses the method in [10] to extend results from given indata to more general boundary data.

Let ε_0 be such that

$$\int_{v_z>0}\Big(\frac{M_+}{M_-}(v) - 1\Big)^2 v_z M(v)dv < \frac{1}{16}, \quad \varepsilon < \varepsilon_0.$$

Lemma 3.2. *Let $\varepsilon < \varepsilon_0$ and $\nu^{-\frac{1}{2}} g \in \tilde{L}^q$, $2 \le q < \infty$, be given. There exists a solution $F \in \mathcal{W}^{q-}$ to*

$$v_z\frac{\partial F}{\partial z} - \varepsilon G\frac{\partial F}{\partial v_z} = \frac{1}{\varepsilon}(\tilde{L}F + 2\sum_{j=1}^{5}\varepsilon^j \tilde{J}(F, \tilde{M}^{-1}f^j)) + g, \tag{3.6}$$
$$F(-1,v) = \int_{w_z<0} \mid w_z \mid F(-1,w)M_-(w)dw, \quad v_z > 0,$$
$$F(1,v) = \frac{M_+}{M}(v)\int_{w_z>0} w_z F(1,w)M(w)dw, \quad v_z < 0.$$

Proof of Lemma 3.2. A proof of Lemma 3.2 in the case without $\tilde{J}$ and force term can be found in [10]. First, there is uniqueness of the solution of (3.6) in the case without $\tilde{J}$ term under the condition that the total mass is fixed. Indeed, if there were two solutions, multiplying the equation satisfied by their difference F by $\tilde{M}F$, integrating on $(-1,1)\times I\!R^3$ and using the spectral inequality would imply, for ε small enough, that $(I-P)F = 0$. Hence the fluid dynamic part of the difference PF and F satisfy

$$v_z\frac{\partial F}{\partial z} - \varepsilon G\frac{\partial F}{\partial v_z} = 0.$$

The solution h of the previous equation can be represented as

$$h(z,v)=h(-1,v_x,v_y,\sqrt{v_z^2+2\varepsilon G(z+1)}),\ v_z<0,\ v_z^2+2\varepsilon G(z+1)<4\varepsilon G,$$
$$h(z,v)=h(1,v_x,v_y,-\sqrt{v_z^2+2\varepsilon G(z+1)-4\varepsilon G},\ v_z<0, v_z^2+2\varepsilon G(z+1)>4\varepsilon G,$$
$$h(z,v)=h(-1,v_x,v_y,\sqrt{v_z^2+2\varepsilon G(z+1)},\quad v_z>0.$$

Then,

$$\begin{aligned} F(-1,v) &= F(-1,v_x,v_y,-v_z), \quad -2\sqrt{\varepsilon G}<v_z<0,\\ F(-1,v) &= F(1,v_x,v_y,-\sqrt{v_z^2-4\varepsilon G}), \quad v_z<-2\sqrt{\varepsilon G},\\ F(1,v) &= F(-1,v_x,v_y,\sqrt{v_z^2+4\varepsilon G}), \quad v_z>0. \end{aligned}$$

But F is a fluid polynomial, so the preceeding three relations for the boundary values together with the normalization $\int F\tilde{M}dxdv=0$ condition give that

$$F(-1,v)=0, \quad v_z>0,$$

so that F is identically equal to zero.

An existence proof in the case without the $\tilde{J}$ term is as follows. Denote by V the solution operator for

$$\begin{aligned} &v_z\frac{\partial F}{\partial z}-\varepsilon G\frac{\partial F}{\partial v_z}=\frac{1}{\varepsilon}\tilde{L}F+g, \\ &F(-1,v)=F_b(-1,v),\ v_z>0, \quad F(1,v)=F_b(1,v),\ v_z<0. \end{aligned} \tag{3.7}$$

V can be split into

$$V=V_0g+V_1F_b,$$

where V_0 (resp. V_1) is the solution operator for (3.7) with $F_b=0$ (resp. $g=0$). The function $F=V_0g+V_1\gamma^-F$ solves (3.6) if and only if it solves the equation

$$\gamma^-F=R\gamma^+(V_0g+V_1\gamma^-F).$$

Here, γ^-F (resp. γ^+F) denotes the trace of F on the ingoing (resp. outgoing) boundary

$$\begin{aligned} \Gamma^- &= \{(-1,v);v_z>0\}\cup\{(1,v);v_z<0\},\\ (\text{resp. } \Gamma^+ &= \{(-1,v);v_z<0\}\cup\{(1,v);v_z>0\}), \end{aligned}$$

and $R=R_0+R_1$, where R_0 (resp. R_1) is defined by

$$\begin{aligned} R_0F(-1,v) &= \int_{w_z<0}F(-1,w)\mid w_z\mid M_-(w)dw, \quad v_z>0,\\ R_0F(1,v) &= \int_{w_z>0}F(1,w)w_zM_-(w)dw, \quad v_z<0,\\ R_1(-1,v) &= 0,\ v_z<0,\\ R_1F(1,v) &= (\frac{M_+}{M_-}(v)-1)\int_{w_z>0}F(1,w)w_zM_-(w)dw, \quad v_z<0. \end{aligned}$$

Write the operator V_1 as $V_1=U\bar{K}V_1+W$, where $\tilde{L}=\bar{K}-\nu$ and U is the solution operator of

$$\begin{aligned} &v_z\frac{\partial}{\partial z}(Ug)-\varepsilon G\frac{\partial(Ug)}{\partial v_z}+\frac{\nu}{\varepsilon}Ug=\frac{g}{\varepsilon},\\ &Ug(-1,v)=0,\ v_z>0, \quad Ug(1,v)=0,\ v_z<0, \end{aligned}$$

while W is the solution operator of

$$\begin{aligned} &v_z\frac{\partial}{\partial z}(WF_b)-\varepsilon G\frac{\partial(WF_b)}{\partial v_z}+\frac{\nu}{\varepsilon}WF_b=0,\\ &WF_b(-1,v)=F_b(-1,v),\ v_z>0, \quad WF_b(1,v)=F_b(1,v),\ v_z<0. \end{aligned}$$

Let S (resp. P^+) be the specular reflexion (resp. averaging) operator

$$SF(\pm1,v_x,v_y,v_z)=F(\pm1,v_x,v_y,-v_z),$$
$$P^+F(-1,v)=\int_{w_z<0}F(-1,w)\mid w_z\mid M_-(w)dw,\ v_z<0,\ P^+F(-1,v)=0,\ v_z>0$$
$$P^+F(1,v)=\int_{w_z>0}F(1,w)\mid w_z\mid M_-(w)dw,\ v_z>0,\ P^+F(1,v)=0,\ v_z<0.$$

Setting $F^+=S\gamma^-F$, the problem to be solved is equivalent to $F^+=BF^++Z^+$, where B is the sum of three terms $B=B_0+B_1+B_2$, defined as

$$\begin{aligned} B_0 &= SR_0\gamma^+WSP^+,\\ B_1 &= SR_0\gamma^+WS(I-P^+)+SR_1\gamma^+WS,\\ B_2 &= S(R_0+R_1)\gamma^+UKV_1S, \end{aligned}$$

and $Z^+=S(R_0+R_1)\gamma^+V_0g$. First, B_0 is compact in $L^{2+}=SL^+$ since $B_0F^+(-1,v)$ (resp. $B_0F^+(1,v)$) are linear combinations (with v-dependent given coefficients) of

$$\int_{w_z<0} F^+(-1,w) \mid w_z \mid M_-(w)dw \text{ and } \int_{w_z>0} F^+(1,w) \mid w_z \mid M_-(w)dw.$$

Then the operator B_2 is compact in L^{2+} since $\gamma^+ U\tilde{K}$ is compact from L^2 into L^{2+} by its averaging construction. Finally, the operator $I-B_1$ is invertible. Indeed $\| B_1 \|_{L^{2+}} \leq 1$, so that for any $Z^+ \in L^{2+}$ and any $\beta \in [0,1[$, there is a unique f^+_β solution to $(I-\beta B_1)f^+_\beta = Z^+$. Then,

$$\| B_1 f^+_\beta \|_{L^{2+}} \leq \| (I-P^+)f^+_\beta \|_{L^{2+}} + \| SR_1 \|_{L^{2+}} \| f^+_\beta \|_{L^{2+}} .$$

It follows from $f^+_\beta = \beta B_1 f^+_\beta + Z^+$ that

$$\begin{aligned} \| (I-P^+)f^+_\beta \|_{L^{2+}} &\leq \| (I-P^+)B_1 f^+_\beta \|_{L^{2+}} + \| Z^+ \|_{L^{2+}} \\ &= \| (I-P^+)SR_1\gamma^+ WSf^+_\beta \|_{L^{2+}} + \| Z^+ \|_{L^{2+}} \\ &\leq \| SR_1 \|_{L^{2+}} \| f^+_\beta \|_{L^{2+}} + \| Z^+ \|_{L^{2+}} . \end{aligned}$$

And so,

$$\begin{aligned} &\| f^+_\beta \|_{L^{2+}} \\ &\leq \| B_1 f^+_\beta \|_{L^{2+}} + \| Z^+ \|_{L^{2+}} \leq 2 \| SR_1 \|_{L^{2+}} \| f^+_\beta \|_{L^{2+}} + 2 \| Z^+ \|_{L^{2+}} . \end{aligned}$$

For $\varepsilon < \varepsilon_0$, $\| SR_1 \|_{L^{2+}} < \frac{1}{4}$, so that $\| f^+_\beta \|_{L^{2+}} < 4 \| Z^+ \|_{L^{2+}}$.

This proves that $(I-B_1)^{-1}$ exists and is continuous. The Fredholm alternative then proves the existence of F^+. From here adding the small perturbation $\tilde{J}$, the problem can be solved by a standard iterative procedure, and the result still holds. □

Remark. By using the a priori estimates (5.20) in [7] we can conclude that, if λ is small enough, then the solution is unique also including the term $\tilde{J}$. By using Corollary 3.6 instead we have uniqueness for G small.

We shall use the rest of this section to obtain the already mentioned a priori estimates for the linear problem (3.5) with force term. For the non-fluid-dynamic part $F_\perp$ of the solution and for the comparison of the solution in different $\tilde{L}^q$-spaces, we may use explicit computations.

Lemma 3.3. *For $q=2,\ \infty$, let F be a solution in $\mathcal{W}^{q-}$ to* (3.5). *The*

following estimate holds for small enough $\varepsilon > 0$;

$$\mid \nu^{\frac{1}{2}} F \mid_\infty \leq c(\mid \nu^{-\frac{1}{2}} g \mid_\infty + \varepsilon^{-\frac{1}{q}} \mid \nu^{\frac{1}{2}} F \mid_q + \mid F_b \mid_\sim). \tag{3.8}$$

Proof of Lemma 3.3. We first turn to the estimate (3.8) in the case where $f^j, j=1,\dots,5$, equal zero. If for some function H, F is solution to

$$\begin{aligned} m \quad & v_z \frac{\partial F}{\partial z} - \varepsilon G \frac{\partial F}{\partial v_z} + \frac{1}{\varepsilon}\nu(v)F = \frac{1}{\varepsilon}H, \\ & F(-1,v)=0,\ v_z>0, \quad F(1,v)=0,\ v_z<0, \end{aligned}$$

then

$$\begin{aligned} F(z,v) &\leq \frac{1}{\varepsilon} \| H \|_\infty \int_{s_1 \text{ or } \tilde{s}_2}^0 \exp\left\{\frac{1}{\varepsilon}\int_0^\tau \nu(v_x,v_y,v_z-\sigma\varepsilon)d\sigma\right\} d\tau \\ &= \frac{1}{\epsilon} \| H \|_\infty \int_{s_1 \text{or} \tilde{s}_2}^0 \exp\Big\{-\frac{1}{\epsilon^2 G}\int_{v_z}^{v_z-\tau\epsilon G} \nu(v_x,v_y,r)dr\Big\}d\tau, \end{aligned}$$

where

$$\begin{aligned} \varepsilon G s_1 &= v_z - \sqrt{v_z^2 + 2\varepsilon G(1+z)}, \quad v_z \geq -\sqrt{2\varepsilon G(1-z)}, \\ \varepsilon G \tilde{s}_2 &= v_z + \sqrt{v_z^2 - 2\varepsilon G(1-z)}, \quad v_z \leq -\sqrt{2\varepsilon G(1-z)}. \end{aligned}$$

Then,

$$F(z,v) \leq \frac{1}{\varepsilon} \| H \|_\infty \int_{-\infty}^0 e^{u(\tau)} d\tau,$$

where

$$u(\tau) = \frac{1}{\varepsilon^2 G}\int_{v_z-\tau\varepsilon G}^{v_z} \nu(v_x,v_y,r)dr.$$

Consequently,

$$F(z,v) \leq \frac{1}{\nu_0} \| H \|_\infty \int_{-\infty}^0 u'(\tau)e^{u(\tau)}d\tau \leq \frac{1}{\nu_0} \| H \|_\infty .$$

Denote by $U_\varepsilon X$ the solution to

$$\begin{aligned} & v_z\frac{\partial}{\partial z}(U_\varepsilon X) - \varepsilon G\frac{\partial}{\partial v_z}(U_\varepsilon X) + \frac{1}{\varepsilon}\nu(v)U_\varepsilon X = X, \\ & U_\varepsilon X(-1,v)=0,\ v_z>0, \quad U_\varepsilon X(1,v)=0,\ v_z<0. \end{aligned}$$

A solution F to (3.5) with $f^j = 0, j = 1, \dots, 5$, satisfies

$$v_z \frac{\partial F}{\partial z} - \varepsilon G \frac{\partial F}{\partial v_z} + \frac{1}{\varepsilon}\nu(v)F = \frac{1}{\varepsilon}(KF + g).$$

Split the kernel k of K into $k_n = \mathrm{sign}k \min(|k|, n)$ and the remaining part $k - k_n$, and denote the corresponding operators by K' and K''. The operator norm of $K - K' = K''$ tends to zero, and K is compact in L^2_M. It immediately follows that F can be written as

$$F = \frac{1}{\varepsilon^2} U_\varepsilon\Big(K' U_\varepsilon(K'F)\Big) + Z_1 F + Z_2 g + Z_3 F_b.$$

The K''-factor makes the operator norm of Z_1 in $\tilde{L}^\infty$ tend to zero (uniformly in ε) when the cut-off $n \to \infty$. Also by straight forward computations

$$|\nu^{\frac{1}{2}} Z_2 g|_\infty \le c\, |\nu^{-\frac{1}{2}} g|_\infty, \qquad |\nu^{\frac{1}{2}} Z_3 F_b|_\infty \le c\, |F_b|_\sim .$$

It remains the term $U_\varepsilon\Big(K' U_\varepsilon(K'F)\Big)$. The first U_ε is (uniformly in ε) bounded in $\tilde{L}^\infty$, so it is enough to consider $K' U_\varepsilon K'$. Setting $EF(x) = \int F(x, v) M(v) dv$, we can estimate $K' U_\varepsilon K'$ by a cut-off dependent multiple of $EU_\varepsilon E$ in the operator norm. For fixed ε the operator $EU_\varepsilon E$ is bounded from L^p into L^q for $p > d$, $q = \infty$, $d \ge 1$, as well as for $1 < p \le d$, $q < dp(d-p)^{-1}$, $d > 1$. Here $d = 1$. For the proof of this estimate of EUE we follow [M Chapter 6]. First,

$$\| \frac{1}{\varepsilon^2} U_\varepsilon\Big(K' U_\varepsilon(K'F)\Big) \|_\infty \le \frac{c}{\varepsilon} \| EU_\varepsilon H \|_\infty,$$

where the norms are the relevant operator norms and

$$H(z) = (EF)(z).$$

Moreover,

$$\begin{aligned} & EU_\varepsilon H(z) \\ = & \int M(v) \int_{s_1 \,\mathrm{or}\, \bar{s}_2}^0 \exp\left\{-\frac{1}{\varepsilon}\int_s^0 \nu(v_x, v_y, v_z - \tau\varepsilon G) d\tau\right\} H(z + s v_z - s^2 \frac{\varepsilon G}{2}) ds dv \\ \le & A(z), \end{aligned}$$

where

$$A(z) := \int \Big(\int_{s_1 \,or\, \bar{s}_2}^0 e^{\nu_0 \frac{s}{\varepsilon}} H(z + s v_z - s^2 \frac{\varepsilon G}{2}) ds \Big) e^{-\frac{1}{2}v_z^2} dv_z$$

$$= A_1(z) + A_2(z) + A_3(z),$$

where

$$\begin{aligned} A_1(z) &:= \int_{v_z < -\sqrt{2\varepsilon G(1-z)}} \Big(\int_{\bar{s}_2}^0 e^{\nu_0 \frac{s}{\varepsilon}} H(z + s v_z - s^2 \frac{\varepsilon G}{2}) ds \Big) e^{-\frac{1}{2}v_z^2} dv_z, \\ A_2(z) &:= \int_{v_z > \sqrt{2\varepsilon G(1-z)}} \Big(\int_{s_1}^0 e^{\nu_0 \frac{s}{\varepsilon}} H(z + s v_z - s^2 \frac{\varepsilon G}{2}) ds \Big) e^{-\frac{1}{2}v_z^2} dv_z, \\ A_3(z) &:= \int_{v_z^2 < 2\varepsilon G(1-z)} \Big(\int_{s_1}^0 e^{\nu_0 \frac{s}{\varepsilon}} H(z + s v_z - s^2 \frac{\varepsilon G}{2}) ds \Big) e^{-\frac{1}{2}v_z^2} dv_z. \end{aligned}$$

Then,

$$\begin{aligned} A_2(z) &= \int_{v_z > \sqrt{2\varepsilon G(1-z)}} \Big(\int_{-1}^z \exp\Big\{ -\frac{2\nu_0}{\varepsilon} \frac{z - r}{v_z + \sqrt{v_z^2 + 2\varepsilon G(z-r)}} \Big\} \\ &\quad \times H(r) \frac{dr}{\sqrt{v_z^2 + 2\varepsilon G(z-r)}} \Big) e^{-\frac{1}{2}v_z^2} dv_z \\ &= \int_{-1}^z H(r) \varphi_2(z - r) dr, \end{aligned}$$

if

$$\varphi_2(s) = \int_{u > \sqrt{2\varepsilon G(1-z)}} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{u^2 + 2\varepsilon G s}} e^{-\frac{2s\nu_0}{\varepsilon(u + \sqrt{u^2 + 2\varepsilon G s})}} du .$$

For any $\alpha > 0$,

$$\begin{aligned} & \frac{e^{-\frac{1}{2}u^2}}{\sqrt{u^2 + 2\varepsilon G s}} e^{-\frac{2\nu_0 s}{\varepsilon(u + \sqrt{u^2 + 2\varepsilon G s})}} \\ \le & \; c\Big(\frac{\varepsilon}{s}\Big)^\alpha e^{-u} \frac{(u + \sqrt{u^2 + 2\varepsilon G s})^\alpha}{\sqrt{u^2 + 2\varepsilon G s}} e^{-\Big(u + \frac{s\nu_0}{\varepsilon(u + \sqrt{u^2 + 2\varepsilon G s})}\Big)} \\ \le & \; c\Big(\frac{\varepsilon}{s}\Big)^\alpha u^{\alpha - 1} e^{-u} e^{-\sqrt{\frac{s}{\varepsilon}}}. \end{aligned}$$

Hence,

$$\varphi_2(s) \le c\Big(\frac{\varepsilon}{s}\Big)^\alpha e^{-\sqrt{\frac{s}{\varepsilon}}}, \qquad \| \varphi_2 \|_{q'} \le c\varepsilon^{\frac{1}{q'}}, \; q' \ge 1.$$

And so,

$$\| A_2 \|_\infty \le \| H \|_q \| \varphi_2 \|_{q'} \le c\varepsilon^{\frac{1}{q'}} \| H \|_q .$$

The terms $A_1(z)$ and $A_3(z)$ can be treated analogously. With this estimate of $EU_\varepsilon E$ and choosing the cut-off n large enough, (3.8) follows when $f^j = 0$, $j = 1, \ldots, 5$. But $M\varphi := \sum_1^5 \varepsilon^j f^j$ is of order ε, and taking ε small enough, it follows that the addition of $\tilde{J}(F, \varphi)$ to g does not change the result in this part of the proof, neither does the addition of a fluid component to g. □

Next lemma provides a priori L^2 estimates of the hydrodynamic component of F, $F_\parallel$, as well as the non fluidodynamic part, $F_\perp$. Analogous estimates are proved in [7] for diffusive boundary conditions. We give here the proof in the case of given indata for sake of completeness.

Lemma 3.4. *Let $g = g_\parallel + g_\perp$, and let F be a solution in $\mathcal{W}^{2-}$ to (3.5). For $\varepsilon > 0$ and small enough,*

$$| \nu^{\frac{1}{2}} F_\perp |_2 \leq c(| \nu^{-\frac{1}{2}} g_\perp |_2 + \sqrt{\varepsilon} | F_b |_\sim + \frac{1}{\varepsilon} | g_\parallel |_2 + \varepsilon\lambda | F_\parallel |_2 . \tag{3.9}$$

Moreover, for λ small enough,

$$| F_\parallel |_2 \leq c(\frac{1}{\varepsilon} | \nu^{-\frac{1}{2}} g_\perp |_2 + \frac{1}{\varepsilon\sqrt{\varepsilon}} | g_\parallel |_2 + | F_b |_\sim). \tag{3.10}$$

Proof of Lemma 3.4. We first prove the estimate (3.9) when $\varphi = 0$. Consider the mapping from $\nu^{-\frac{1}{2}} \tilde{L}^q \times L^+$ into $\mathcal{W}^{q-}$ given by $(g, F_b) \to F$, with F a solution to (3.5) for $\varphi = 0$ and $g = g_\perp$. We multiply (3.5) by $\tilde{M}$ and integrate over space and velocity. Then, identity (3.2), Green's formula and the spectral inequality for the linearized collision operator L, i.e.

$$- \int M f L f dv \geq c \int M \nu f_\perp^2 dv,$$

give

$$\varepsilon | SF |_\sim^2 + | \nu^{\frac{1}{2}} F_\perp |_2^2 \leq c | \nu^{-\frac{1}{2}} g_\perp |_2^2 + \varepsilon | F_b |_\sim^2 .$$

The inclusion of $\tilde{J}(F, \varphi)$ as well a fluid component to g, adds $c\varepsilon|\nu^{\frac{1}{2}} F_\perp|_2^2$, which is incorporated in the left hand side together with a term $\frac{1}{\varepsilon^2}|g_\parallel|_2^2$, and a term $c\lambda^2\varepsilon^2 | F_\parallel |_2^2$. This concludes the proof of (3.9). As a by-product, we have also the following estimate for $| SF |_\sim$

$$| SF |_\sim \leq c \left[\frac{1}{\sqrt{\varepsilon}} | \nu^{-\frac{1}{2}} g_\perp |_2 + \frac{c}{\varepsilon\sqrt{\varepsilon}} | g_\parallel |_2 + \lambda\sqrt{\varepsilon} | F_\parallel |_2 \right] + | F_b |_\sim \tag{3.11}$$

We start the proof of (3.10) by estimating F_z. Multiplying the equation (3.5) by M and integrating over $\mathbb{R}_v^3$, leads to

$\frac{\partial F_z}{\partial z} - \epsilon G F_z = \frac{g_0}{\epsilon}$, i.e.

$$F_z(z) = F_z(-1) e^{\epsilon G(z+1)} + \int_{-1}^{z} \frac{g_0}{\epsilon} e^{\epsilon G(z-s)} ds. \tag{3.12}$$

By definition of $F_z(-1)$,

$$\begin{aligned} | F_z(-1) | &= | \int v_z F(-1, v) M dv | \\ &\leq c\Big(\int | v_z | F^2(-1, v) M dv \Big)^{\frac{1}{2}} \leq c(| SF |_\sim + | F_b |_\sim). \end{aligned}$$

And so by (3.12) with $\| \cdot \|_2$ denoting the L^2-norm in space

$$\| F_z \|_2 \leq c(\frac{1}{\varepsilon} \| g_0 \|_2 + | SF |_\sim + | F_b |_\sim). \tag{3.13}$$

To bound the hydrodynamical part of F, we multiply (3.5) by $\tilde{M} v_z \psi_i$, $i \neq 2$ and integrate over $[-1, z] \times \mathbb{R}^3$.

Denoting $p_i(z) =< v_z^2 \psi_i \tilde{M} F >$, $i = 0, \ldots, 4$, we get

$$p_i(z) = p_i(-1) + \int_{-1}^{z} dz' \int dv \tilde{M} v_z \psi_i \frac{1}{\varepsilon} \Big[\tilde{L} F + \sum_{j=1}^{5} \varepsilon^j \tilde{J}(F, \tilde{M}^{-1} f^j) + \varepsilon^2 G \frac{\partial}{\partial v_z} F + g \Big].$$

As before, we have

$$|p_i(-1)| \leq c(| SF |_\sim + | F_b |_\sim).$$

The relation between p_i and the components $F_i, i \neq 3$, of the fluid dynamic part of F is

$$p_i = \sum_{j \neq 3} A_{ij} F_j + < M v_z^2 \psi_3^2 > F_3 \delta_{i3} + < M v_z^2 \psi_i F_\perp >$$

with $A =< M v_z^2 \psi_i \psi_j >$ a non-singular bounded matrix. By inverting the previous relation we get

$$F_i(z) = \sum_{j \neq 3} A_{ij}^{-1} \Big[p_j - < M v_z^2 \psi_3^2 > F_3 \delta_{j3} + < M v_z^2 \psi_j F_\perp > \Big].$$

This allows to estimate F_i and hence $F_\parallel$ as

$$| F_\parallel |_2 \le C\Big[\frac{1}{\varepsilon} | \sqrt{\nu} F_\perp |_2 + \frac{1}{\varepsilon} | g |_2 + \lambda | F_\parallel |_2 \Big] + c | SF |_\sim + | F_b |_\sim, \quad (3.14)$$

so that for λ small enough we get

$$| F_\parallel |_2 \le C\Big[\frac{1}{\varepsilon} | \sqrt{\nu} F_\perp |_2 + \frac{1}{\varepsilon} | g |_2 \Big] + c | SF |_\sim + | F_b |_\sim . \quad (3.15)$$

Finally, by using (3.11) we get

$$| F_\parallel |_2 \le C\Big[\frac{1}{\varepsilon} | \nu^{\frac{1}{2}} F_\perp |_2 + \frac{1}{\varepsilon\sqrt{\varepsilon}} | g_\parallel |_2 \Big] + | F_b |_\sim .$$

Note that the only point where we need λ small is to pass from (3.14) to (3.15). □

We give now a new stronger estimate for the hydrodynamic part of the solution which is true under the assumption of small G, instead of small λ. The proof requires a careful analysis of suitable moments of higher orders.

Lemma 3.5. *Let $g = g_\parallel + g_\perp$, and let F be a solution in $\mathcal{W}^{2-}$ to (3.5). For $\epsilon > 0$ and G small enough*

$$| F_\parallel |_2 \le c\big(\frac{1}{\epsilon} | \nu^{-\frac{1}{2}} g_\perp |_2 + \frac{1}{\epsilon^2} | g_\parallel |_2 + \frac{1}{\sqrt{\epsilon}} | F_b |_\sim + | F_\perp |_2\big). \quad (3.16)$$

Proof of Lemma 3.5. Define

$$f_{x^i y^j z^k}(z) := \int M v_x^i v_y^j v_z^k f_\perp(z, v) dv, \; i + j + k \ge 2,$$

and $f_{x^i y^j z^k 2}(z)$, $f_{x^i y^j z^k \bar{A}}(z)$, $f_{x^i y^j z^k \bar{B}}(z)$ correspondingly, when there is an extra factor $| v |^2, \bar{A}$, resp. $\bar{B}$ in the integrand.

Here $\bar{A}$ and $\bar{B}$ are non-hydrodynamic solutions to

$$\tilde{L}(v_z \bar{A}) = v_z(v^2 - 5T), \quad \tilde{L}(v_x v_z \bar{B}) = v_x v_z .$$

The estimate for the F_z-moment is given by (3.13).

We next consider the F_x-moment. Here the $\tilde{J}(F, \varphi)$-term requires some care. The first step is an estimate of F_{xz}. Multiply (3.5) by $M v_x$ and integrate,

$$\frac{\partial}{\partial z} F_{xz} - \epsilon G F_{xz} = \frac{g_x}{\epsilon}. \quad (3.17)$$

A multiplication of (3.5) with $v_x v_z \bar{B} M$ and integration over $\mathbb{R}^3_v$ leads for some real γ to

$$\begin{aligned} &\frac{\partial(\alpha F_x + F_{xz^2\bar{B}})}{\partial z} + \epsilon G \gamma(\alpha F_x + F_{xz^2\bar{B}}) \\ &= \frac{1}{\epsilon}\Big(F_{xz} + g_{xz\bar{B}} + 2\int v_x v_z \bar{B} \tilde{J}(F, \varphi) M dv\Big) + \epsilon G \beta, \end{aligned}$$

where $\alpha < 0$, and β is a multiple of a non-hydrodynamic moment of F. Notice that the definition of $\bar{B}$ implies that $\int M v_x v_z \bar{B} \tilde{J}(F_\parallel, \tilde{M}^{-1} f^1)$ is zero. This is crucial for not getting a term $\lambda | F_\parallel |_2$ in the final estimate.

Set $\tilde{F}_x = \alpha F_x + F_{xz^2\bar{B}} = \int v_z^2 v_x \bar{B} F M dv$. Then

$$\frac{\partial \tilde{F}_x}{\partial z} + \epsilon G \gamma \tilde{F}_x = \frac{1}{\epsilon}\Big(F_{xz} + g_{xz\bar{B}} + 2\int v_x v_z \bar{B} \tilde{J}(F, \varphi) M dv\Big) + \epsilon G \beta.$$

The last equation together with the equation satisfied by F_{xz}, (3.17), give

$$\begin{aligned} &\frac{\partial}{\partial z}(F_{xz} \tilde{F}_x) + \epsilon G(\gamma + 1) F_{xz} \tilde{F}_x \\ &= \frac{F_{xz}^2}{\epsilon} + F_{xz}\Big[\frac{1}{\epsilon}(g_{xz\bar{B}} + 2\int v_x v_z \bar{B} \tilde{J}(F_\perp, \varphi) M dv) + \epsilon G \beta\Big] + \frac{g_x}{\epsilon} \tilde{F}_x . \end{aligned}$$

Integrating over $z \in [-1, 1]$ the previous equation we get an estimate for the L^2 norm of $\frac{F_{xz}}{\sqrt{\varepsilon}}$. For an arbitrary $\eta > 0$

$$\begin{aligned} \frac{\| F_{xz} \|_2^2}{\epsilon} &\le |(F_{xz} \tilde{F}_x)(1)| + |(F_{xz} \tilde{F}_x)(-1)| + \varepsilon c G \| F_{xz} \|_2 \| \tilde{F}_x \|_2 \\ &\quad + \frac{1}{\varepsilon} \| F_{xz} \|_2 \Big[\| g_{xz\bar{B}} \|_2 + c\varepsilon | \nu^{\frac{1}{2}} F_\perp |_2 + \epsilon c G | F_\perp |_2 + \epsilon^2 c | F |_2 \Big] \\ &\quad + \frac{1}{\varepsilon} \| g_x \|_2 \| \tilde{F}_x \|_2 \\ &\le |(F_{xz} \tilde{F}_x)(1)| + |(F_{xz} \tilde{F}_x)(-1)| + \varepsilon \eta \| \tilde{F}_x \|_2{}^2 + c_\eta \frac{1}{\varepsilon^3} \| g_x \|_2{}^2 \\ &\quad + \frac{c_\eta}{\epsilon} | \nu^{-\frac{1}{2}} g_\perp |_2^2 + \epsilon c_\eta | \nu^{\frac{1}{2}} F_\perp |_2^2 + \frac{\eta}{\epsilon} \| F_{xz} \|_2^2 + \epsilon^3 c_\eta | F_\parallel |_2^2 . \end{aligned}$$

And so an estimate of $\tilde{F}_x$ can be obtained,

$$\begin{aligned} \| \tilde{F}_x \|_2 &\le c\Big(\frac{1}{\epsilon}(\| \nu^{-\frac{1}{2}} g_\perp \|_2 + \| F_{xz} \|_2) + |\tilde{F}_x(-1)| + | \nu^{\frac{1}{2}} F_\perp |_2 + \epsilon | F_\parallel |_2 \Big) \\ &\le c\Big(\frac{1}{\epsilon} \| \nu^{-\frac{1}{2}} g_\perp \|_2 + | \nu^{\frac{1}{2}} F_\perp |_2 \Big) \end{aligned}$$

$$+\frac{1}{\sqrt{\varepsilon}}\Big[\sqrt{\varepsilon}\eta\|\tilde{F}_x\|_2+c\frac{1}{\varepsilon\sqrt{\varepsilon}}\|g_x\|_2+\epsilon\mid F_{\parallel}\mid_2\Big]$$
$$+c\frac{1}{\sqrt{\varepsilon}}\Big[\mid\tilde{F}_x(\pm1)\mid+\mid F_{xz}(\pm1)\mid\Big]. \tag{3.18}$$

Now we have to estimate the boundary terms $\tilde{F}_x(\pm1)$ and $\tilde{F}_{xz}(\pm1)$. We cannot use directly (3.11) because this estimate depends on $\lambda\sqrt{\varepsilon}\mid F_{\parallel}\mid_2$. We do instead the following: Multiply (3.5) by $\tilde{M}(1+\mid v\mid^2)$ and integrate,

$$\frac{\partial}{\partial z}F_{(1+|v|^2)v_z}+\epsilon GF_{(1+|v|^2)v_z}=\frac{1}{\epsilon}g_{(1+|v|^2)}-2\epsilon GF_z.$$

It follows that

$$\begin{aligned}&\Big|\int_{v_z<0}F(-1,v)v_xv_zMdv\Big|+\Big|\int_{v_z>0}F(1,v)v_xv_zMdv\Big|\\ &\le\int F^{out}(1+\mid v\mid^2)\mid v_z\mid Mdv\\ &\le c\Big(\int F_b(1+\mid v\mid^2)\mid v_z\mid Mdv+\frac{1}{\epsilon}\parallel g_{(1+|v|^2)}\parallel_2+\epsilon G\parallel F_z\parallel_2\Big)\\ &\le c\mid F_b\mid_\sim+\frac{1}{\epsilon}\mid g_{\parallel}\mid_2+\epsilon G\mid SF\mid_\sim\\ &\le c\mid F_b\mid_\sim+\frac{1}{\epsilon}\mid g_{\parallel}\mid_2+c\sqrt{\varepsilon}G\mid\nu^{-\frac12}g_\perp\mid_2+\varepsilon G\Big[\frac{c}{\varepsilon\sqrt{\varepsilon}}\mid g_{\parallel}\mid_2+\lambda\sqrt{\varepsilon}\mid F_{\parallel}\mid_2\Big].\end{aligned}$$

We have used (3.13) to get the third inequality and (3.11) to get the last one. The same estimate holds for

$$\Big|\int_{v_z<0}F(-1,v)v_xv_z^2Mdv\Big|+\Big|\int_{v_z>0}F(1,v)v_xv_z^2Mdv\Big|.$$

By replacing in (3.18) we get

$$\parallel\tilde{F}_x\parallel_2\le c\Big(\mid\nu^{\frac12}F_\perp\mid_2+\frac{1}{\epsilon}\mid\nu^{-\frac12}g_\perp\mid_2+\frac{1}{\varepsilon^2}\mid g_{\parallel}\mid_2+\frac{1}{\sqrt{\varepsilon}}\mid F_b\mid_\sim+\eta\mid F_{\parallel}\mid_2\Big). \tag{3.19}$$

The result and proof for F_y is analogous.

A multiplication of (3.5) with v_zM and integration over $\mathbb{R}^3_v$ leads to

$$\frac{\partial}{\partial z}\int Mv_z^2Fdv-\epsilon G\int v_zM\frac{\partial F}{\partial v_z}dv=\frac{1}{\epsilon}\int Mv_zgdv \tag{3.20}$$

i.e. with $\bar{F}_0=F_0+\frac{2F_4}{\sqrt6}+F_{z^2}=\int v_z^2FMdv$,

$$\frac{\partial\tilde{F}_0}{\partial z}-\frac{1}{2}\epsilon G\tilde{F}_0=\frac{g_z}{\epsilon}-\epsilon GF_0.$$

And so

$$\parallel\tilde{F}_0\parallel_2\le c(\frac{1}{\epsilon}\parallel g_z\parallel_2+\epsilon\parallel F_0\parallel_2+\mid SF\mid_\sim+\mid F_b\mid_\sim). \tag{3.21}$$

A multiplication of (3.5) with $v_z\tilde{A}M$ and integration over $\mathbb{R}^3_v$ leads to

$$\frac{\partial\int v_z^2\bar{A}FMdv}{\partial z}-\epsilon G\int v_z\bar{A}\frac{\partial F}{\partial v_z}dv=\frac{1}{\epsilon}\int v_z\bar{A}(LF+g+\bar{J}(\varphi,F))Mdv,$$

or with $\gamma=\frac{1}{\sqrt6}\int v_z\bar{A}v_z(v^2-3)Mdv<0$,

$$\frac{\partial}{\partial z}(\gamma F_4+F_{z^2\bar{A}})=\frac{1}{\epsilon}(F_{v_z(v^2-5)}+g_{z\bar{A}}+\int v_z\bar{A}\bar{J}(\varphi,F)Mdv)+\epsilon G\beta',$$

where β' is a moment of F. With the notation $\tilde{F}_4=\gamma F_4+F_{z^2\bar{A}}$ this can be written as

$$\frac{\partial\tilde{F}_4}{\partial z}=\frac{1}{\epsilon}(F_{v_z(v^2-5)}+g_{z\bar{A}}+\int v_z\bar{A}\bar{J}(\varphi,F)Mdv)+\epsilon G\beta'.$$

Here a second step is an estimate of $\frac{F_{v_z(v^2-5)}}{\epsilon}$. Multiply (3.5) with $M(v^2-5)$ and integrate,

$$\frac{\partial}{\partial z}F_{v_z(v^2-5)}=\frac{g_{(v^2-5)}}{\epsilon}+\epsilon G\beta,$$

where β is a moment of F. The last two equations together give

$$\begin{aligned}\frac{\partial}{\partial z}(F_{v_z(v^2-5)}\tilde{F}_4)&=\frac{F^2_{v_z(v^2-5)}}{\epsilon}+\frac{F_{v_z(v^2-5)}}{\epsilon}\Big(g_{z\bar{A}}+\epsilon^2G\beta'\\ &\quad+\int v_z\bar{A}\bar{J}(\varphi,F)Mdv\Big)+\frac{g_{(v^2-5)}}{\epsilon}\tilde{F}_4+\epsilon G\beta\tilde{F}_4.\end{aligned} \tag{3.22}$$

To evaluate the left hand side, again a boundary estimate is needed,

$$\begin{aligned}&\mid\int_{v_z<0}F(-1,v)(v^2-5)v_zMdv\mid+\mid\int_{v_z>0}F(1,v)(v^2-5)v_zMdv\mid\\ &\le 5\int F^{out}(1+\mid v\mid^2)\mid v_z\mid Mdv\\ &\le c\Big(\int F_b(1+\mid v\mid^2)\mid v_z\mid Mdv+\frac{\parallel g_{(1+|v|^2)}\parallel_2}{\epsilon}+\epsilon G\parallel F_z\parallel_2\Big).\end{aligned}$$

It is easy to realize that we have the same situation as for the estimate of $\tilde{F}_x$ but for the term $\epsilon G\beta\tilde{F}_4$ in (3.22), which gives a term $\sqrt{G\varepsilon}\mid F_{\parallel}\mid_2$ in the bound for $\frac{\|F_{v_z(v^2-5)}\|_2}{\sqrt{\epsilon}}$. Hence,

$$\| \tilde{F}_4 \|_2 \le c\Big(| \nu^{\frac{1}{2}} F_\perp |_2 + \frac{1}{\epsilon} | \nu^{-\frac{1}{2}} g_\perp |_2 + \frac{1}{\varepsilon^2} | g_\| |_2 + \frac{1}{\sqrt{\varepsilon}} | F_b |_\sim + \sqrt{G} \, | F_\| |_2 \Big) \tag{3.23}$$

To conclude the proof of the Lemma it is enough to collect all the estimates together, use (3.9) and take G small enough. □

We conclude this section by noticing that Lemma 3.4 and Lemma 3.5 imply

Corollary 3.6. *Let $g = g_\| + g_\perp$, and let F be a solution in $\mathcal{W}^{2-}$ to (3.5). For $\epsilon > 0$ and G small enough,*

$$| F_\| |_2 \le c\big(\frac{1}{\epsilon} | \nu^{-\frac{1}{2}} g_\perp |_2 + \frac{1}{\epsilon^2} | g_\| |_2 + \frac{1}{\sqrt{\epsilon}} | F_b |_\sim\big) \tag{3.24}$$

$$| \nu^{\frac{1}{2}} F_\perp |_2 \le c\big(| \nu^{-\frac{1}{2}} g_\perp |_2 + \sqrt{\varepsilon} \, | F_b |_\sim + \frac{1}{\varepsilon} | g_\| |_2\big) \tag{3.25}$$

As for the existence of the stationary solution F_s, its asymptotic expansion was discussed in Section 2, and the rest term is obtained by using the above estimates as in [10], pp70-72, 99-105, using Maslova's mapping from the diffuse reflection case to the given indata case, also adapting from the force-free case of [10] to the present case with a force term. That analysis is similar in spirit to the time dependent case as treated in the following sections. Hence, we will not give the explicit proof for the control of the remainder. We notice that for hard spheres the solution F_s constructed in this way is positive. That can be proved in the same way as the corresponding positivity in the Taylor-Couette case [2]. In conclusion, we can state

Theorem 3.7. *Put*

$$f^1 = M\Big(r + \frac{v^2-3}{2}\theta\Big)$$

with r and θ the thermal conduction solution of the Boussinesq equations corresponding to the temperatures $T_- = 1$ and $T_+ = (1+2\varepsilon\lambda)$, given by (2.4).

Then there are $G_0 > 0$ and $\varepsilon_0 > 0$ such that, if $G < G_0$ and $\varepsilon < \varepsilon_0$, there exists in $\tilde{L}^2 \cap \tilde{L}^\infty$ a positive stationary solution f^ε to the boundary value problem (2.2) such that

$$\Big| M^{-1}\Big(f^\varepsilon - (M + \varepsilon f^1)\Big)\Big|_\infty \le C\varepsilon^2 \,.$$

4. Stability: the Expansion

We study in this section the behavior in time of a small perturbation to the stationary solution F_s constructed in the previous sections.

Consider the Boltzmann equation

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{\epsilon} v_z \frac{\partial}{\partial z} F - G \frac{\partial F}{\partial v_z} &= \frac{1}{\epsilon^2} Q(F,F), \\ F(0,z,v) &= F_s(z,v) + \zeta_0(z,v), \quad z \in (-1,1),\ v \in \mathbb{R}^3, \\ F(t,-1,v) &= M_-(v) \int_{w_z<0} |w_z| F(t,-1,w)dw, \ t>0,\ v_z>0, \\ F(t,1,v) &= M_+(v) \int_{w_z>0} w_z F(t,1,w)dw, \ t>0,\ v_z<0, \end{aligned}$$

where

$$M_- = \frac{1}{2\pi} e^{-\frac{v^2}{2}}, \quad M_+(v) = \frac{1}{2\pi(1+2\varepsilon\lambda)^2} e^{-\frac{v^2}{2(1+2\varepsilon\lambda)}}$$

and ζ_0 is the initial perturbation of F_s, discussed below.

As seen in the previous sections, the stationary solution can be written as $F_s = M + \Phi_s$. The function $\zeta = F - F_s$ is then a solution to

$$\begin{aligned} &\frac{\partial \zeta}{\partial t} + \frac{1}{\epsilon} v_z \frac{\partial}{\partial z}\zeta - G\frac{\partial \zeta}{\partial v_z} = \frac{1}{\epsilon^2}\Big(\mathcal{L}\zeta + Q(\zeta,\zeta) + 2J(\zeta,\Phi_s)\Big), \qquad (4.1) \\ &\zeta(0,z,v) = \zeta_0(z,v), \quad z \in (-1,1),\ v \in \mathbb{R}^3, \\ &\zeta(t,-1,v) = M_- \int_{w_z<0} |w_z| \zeta(t,-1,w)dw, \ t>0,\ v_z>0, \\ &\zeta(t,1,v) = M_+(v) \int_{w_z>0} w_z \zeta(t,1,w)dw, \ t>0 \ v_z<0 \,. \end{aligned}$$

The following initial perturbations $\zeta(0,z,v) = \zeta_0(z,v)$ are considered,

$$\zeta_0(z,v) = \sum_{n=1}^{5} \varepsilon^n \Phi^{(n)}(0,z,v) + \epsilon^5 p_5 \tag{4.2}$$

where the measurable function $p_5(z,v)$ may depend on ϵ and satisfies

$$\| p_5 \|_{\infty,2} := \sup_{\epsilon>0} \Big(\int \big(\sup_{-1\le z\le 1} p_5^2(z,v)\big) M dv \Big)^{\frac{1}{2}} < c,$$

for some c. The non hydrodynamical part of the functions $\Phi^{(n)}(0,z,v)$ is determined by the expansion as explained below together with part of the hydrodynamical part. We will denote by $I_i^{(n)}(t,z)$ the coefficients of the functions ψ_i in the hydrodynamic part of $\Phi^{(n)}(t,z,v)$. The functions $I_i^{(n)}(t,z)$ are not completely arbitrary. There are constraints due to the expansion rules that will be given later on in this section. For example, we require $I_3^{(1)}=0$ for compatibility with the impermeability boundary condition. Finally, we require

$$\int_{[-1,1]\times\mathbb{R}^3}\zeta_0(z,v)\psi_i(v)dzdv=0,\quad i=0,\dots,4.$$

Since the Boltzmann equation conserves total mass, momentum and energy at any positive time the solution has to satisfy $\int_{[-1,1]\times\mathbb{R}^3}\zeta(t,z,v)\psi_i(v)dvdz=0$.

We write an ε-expansion for ζ in the form

$$\zeta(t,z,v)=\sum_{n=1}^{5}\Phi^{(n)}(t,z,v)\varepsilon^n+\varepsilon R(t,z,v)$$

We proceed as in Section 2 in building up all the terms of the expansions. For the proof of the stability we need to show that $\Phi^{(n)}(t,z,v)$ converge to zero when time tends to infinity in a suitable norm. To this end, we will construct explicitly only the first terms of the expansions. The behavior of the higher order terms will then be evident from this analysis.

If we use the expansion in (4.1) we see immediately that $\Phi^{(1)}$ has to satisfy

$$\mathcal{L}\Phi^{(1)}=0,$$

so that $\Phi^{(1)}$ has to be in Null $\mathcal{L}$, which means that it is a combination of the collision invariants $M\psi_i, i=0,\dots,4$. Hence we have

$$\Phi^{(1)}=M\Big(\rho^1+u^1\cdot v+T^1\frac{|v|^2-3}{2}\Big),$$

so that $\rho^1\equiv I_0^{(1)},\quad u_i^1\equiv I_i^{(1)}, i=1,2,3,\quad T^1\equiv\frac{2}{\sqrt6}I_4^{(1)}$. We notice that the boundary conditions force $u^1(-1,t)=u^1(1,t)=0$, $T^1(-1,t)=0=T^1(1,t)$ for any $t>0$. In consequence, we do not need boundary layer correction to the first order in ε. Indeed, in $z=-1$ the solution is already Maxwellian. On the other hand, $M+\varepsilon\Phi^{(1)}$, when evaluated for $z=1$ is not proportional to the Maxwellian M_+, even with the previous assumptions, but differs from

it for terms of order ε^2 which will appear in the corrections of higher order. Hence, for $n>1$ we have to decompose the higher order corrections in a bulk term $B^{(n)}$ and two boundary layer terms $b_\pm^{(n)}$.

To determine the functions ρ^1, u^1 and T^1 which give $\Phi^{(1)}(=B^{(1)})$, we consider the equation that we get by equating the terms of next order. Note that we know from the previous sections that the stationary solution can also be expanded in ε and we denote by $\Phi_s^{(n)}$ the terms of this expansion. The equation which we get at next order, by ignoring boundary layer corrections, is

$$v_z\frac{\partial}{\partial z}\Phi^{(1)}=\Big(\mathcal{L}B^{(2)}+Q(\Phi^{(1)},\Phi^{(1)})+2J(\Phi^{(1)},\Phi_s^{(1)})\Big)\tag{4.3}$$

It can be seen as an equation in $B^{(2)}$, whose solvability conditions are

$$\langle\psi_iv_z\partial_z\Phi^{(1)}\rangle=0,\quad i=0,\dots,4$$

where $\langle\cdot\rangle$ stands for integration over the velocities. This is equivalent to

$$\partial_zu_z^1=0,\qquad \partial_z(T^1+\rho^1)=0.\tag{4.4}$$

The first equation in (4.4) is the usual incompressibility condition in $d=1$ which implies, together with the boundary conditions, $u_z^1=0$, while the second equation in (4.4) is the Boussinesq condition. The Boussinesq condition fixes $\rho^1=-T^1$, up to a constant. To determine T^1 and the other components of u^1 we look at the solvability condition at next order in ε. Indeed, once (4.4) are satisfied, we can deduce from (4.3) the following expression for $B^{(2)}$, where $\mathcal{L}^{-1}$ denotes the inverse of the restriction of $\mathcal{L}$ to the orthogonal of its null space

$$B^{(2)}=\mathcal{L}^{-1}\Big[v_z\partial_z\Phi^{(1)}-Q(\Phi^{(1)},\Phi^{(1)})-2Q(\Phi^{(1)},\Phi_s^{(1)})\Big]+M\sum_{i=0}^{4}\psi_i\,I_i^{(2)}(t,z)\,.\tag{4.5}$$

The coefficients $I_i^{(2)}$ are undetermined at this point and will be partly fixed by the solvability condition for the equation at next order in ε and the rest of them in some later step:

$$\begin{aligned}&\frac{\partial}{\partial t}B^{(1)}+v_z\frac{\partial}{\partial z}B^{(2)}+G\frac{\partial}{\partial v_z}B^{(1)}\\&=\mathcal{L}B^{(3)}+2J(\Phi^{(2)},\Phi^{(1)})+2J(\Phi^{(2)},\Phi_s^{(1)})+2J(\Phi^{(1)},\Phi_s^{(2)})\end{aligned}\tag{4.6}$$

The solvability conditions for this equation are

$$\langle\psi_i[\frac{\partial}{\partial t}B^{(1)}+v_z\frac{\partial}{\partial z}B^{(2)}+G\frac{\partial}{\partial v_z}B^{(1)}]\rangle=0,\quad i=0,\dots,4 \tag{4.7}$$

and this produces the equations for u^1 and T^1. Let us fix $i=1,2,3$ in (4.7). Then the first term gives the time derivative of u^1. The third one reduces to 0 for $i=1,2$ and to $-G\rho^1$ for $i=3$, after integrating by parts. Finally we write

$$\langle v\otimes vB^{(2)}\rangle=\langle[v\otimes v-\frac{v^2}{3}\mathbb{I}]B^{(2)}\rangle+\langle\frac{v^2}{3}\mathbb{I}B^{(2)}\rangle\ .$$

The first term, as is well known, gives rise to the dissipative and transport terms, while the second one can be interpreted as the second order correction to the pressure P_2. The result is

$$\begin{aligned}\frac{\partial}{\partial t}u_z^1+u_z^1\frac{\partial}{\partial z}u_z^1 &= \nu\frac{\partial^2}{\partial z^2}u_z^1-\frac{\partial}{\partial z}P_2+G\rho^1,\\ \frac{\partial}{\partial t}u_i^1 &= \nu\frac{\partial^2}{\partial z^2}u_i^1,\quad i=1,2,\end{aligned}$$

where $\nu>0$ is the usual kinematic viscosity coefficient. We remark that the term due to $Q(\Phi^{(1)},\Phi_s^{(1)})$ in (4.5), which in general produces the transport along the stationary flow, does not contribute in this case because the stationary velocity field in the z direction, $\langle v_z\Phi_s^{(1)}\rangle$, vanishes. Using the Boussinesq condition we replace the term $G\rho^1$ by $-GT^1+const.$ The constant can be absorbed in the pressure term that we rename p. Since $u_z^1=0$ we are left with

$$\frac{\partial}{\partial z}p=-GT^1\ .$$

This determines p in terms of T^1 up to a constant.

Remark. There are constants (one coming from the Boussinesq condition, another from the pressure condition) at any order which will be determined in the end by the total mass condition. Since we are asking that the total mass of the perturbation is zero we can put to zero all the constants.

To get the equation for the temperature, one has to look at (4.7) for $i=4$. It is actually more convenient to replace ψ_4 with the equivalent $\tilde\psi_4=\frac{1}{2}(v^2-5)$.

An integration by parts yields

$$G\langle\tilde\psi_4\frac{\partial}{\partial v_z}f_1\rangle=-u_z^1G=0,$$

while

$$\langle v\tilde\psi_4B^{(2)}\rangle=-\kappa\frac{\partial T^1}{\partial z},$$

with $k>0$ the thermal conductivity. In conclusion,

$$\frac{5}{2}\frac{\partial}{\partial t}T^1=k\frac{\partial^2}{\partial z^2}T^1\ .$$

This equation has to be solved with boundary conditions $T^1(\pm1,t)=0$, for $t>0$ and initial condition T_0^1, which is completely arbitrary. Since $\int_{-1}^1 dzT_0^1(z)=0$, standard results on the heat equation imply that the L^2-norms of the solution $T^1(z,t)$ and of its derivatives converge to zero exponentially in time. Also the components of the velocity u_i^1, $i=1,2$, solve similar parabolic equations with boundary conditions $u_i^1=0$ and initial condition $(u_0^1)_i$, which are again arbitrary. Since T^1 differs from ρ^1 for a constant, that can be put to zero, and u_z^1 is identically zero, we can conclude that also $\Phi^{(1)}$ converges to zero exponentially in time together with its spatial derivatives.

The second order term in the expansion, $\Phi^{(2)}$, is not yet completely determined. Equation (4.7) with $i=0$ gives

$$\frac{\partial}{\partial t}\rho^1=\frac{\partial}{\partial z}I_3^{(2)},$$

This fixes $I_3^{(2)}$ up to a constant. Moreover, a combination of $I_0^{(2)}$ and $I_4^{(2)}$ contributes to the pressure p which is determined by the previous equations, so that these parameters are not independent.

The non-hydrodynamic part of $B^{(2)}$ is a linear function of the derivatives of ρ^1,T^1 which are in general different from zero on the boundaries. Therefore the non hydrodynamical part of $B^{(2)}$ is completely fixed (even a time $=0$) and violates the boundary conditions. We need to introduce $b_\pm^{(2)}$ to adjust the boundary conditions by compensating the non hydrodynamical part of $B^{(2)}$ which is not Maxwellian. We explain how to find the correction $b_-^{(2)}$. The correction $b_+^{(2)}$ is found in a similar way. The notation is the one introduced in Section 2. We choose $b_-^{(2)}$ by solving, for any $t>0$, the Milne problem for $z^- > 0$:

$$v_z\frac{\partial}{\partial z^-}h - \varepsilon^2 G^- \frac{\partial}{\partial v_z}h = \mathcal{L}^- h, \quad \langle v_z h\rangle = 0, \tag{4.8}$$

with z^- defined in (2.7) as the rescaled z variable near the bottom plate and G^- defined by (2.9). We impose the boundary condition in $z^- = 0$ in such a way that the incoming flux of h in $z=-1$, $v_z > 0$ is given by $(I-P)B^{(2)}(-1,v;t)$. The results in [5] tell us that the solution approaches, as $z^- \to +\infty$ a function $q_-^{(2)}(v,t)$ in Null $\mathcal{L}^-$. Note that in $q_-^{(2)}$ there is no term proportional to ψ_3 because of the vanishing mass flux condition in the direction of the z axis $\langle v_z h\rangle = 0$. Thus we set $b_-^{(2)}(z,v,t) = h(z,v,t) - q_-^{(2)}(v,t)$, which will go to zero at infinity exponentially in z^-. This produces a term $b_-^{(2)}(2\varepsilon^{-1}, v, t) = \gamma_{2,\varepsilon}^-(v,t)$, exponentially small in ε^{-1} on the opposite boundary. The resulting term in the expansion is thus $\Phi^{(2)} = B^{(2)} + b_+^{(2)} + b_-^{(2)}$ and is such that in $z=-1$, for example, it has zero non hydrodynamic part, while the hydrodynamic part is

$$\Phi^{(2)}(-1,v;t) = \sum_{i=0}^{4} I_i^{(2)}(-1;t)M(v)\psi_i(v) + b_+^{(2)}(2\varepsilon^{-1},v,t) - q_-^{(2)}, \; v_z>0, \; t>0$$

We are not yet done since $\Phi^{(2)}(-1,v)$ is not Maxwellian for $v_z > 0$, (as it should, in order to satisfy the boundary conditions) because of the presence of terms proportional to ψ_i, $i=1,2,4$ in $q_-^{(2)}$ and $b_+^{(2)}(2\varepsilon^{-1};t)$. The latter is not important and will be put in the remainder. The former will be compensated by the coefficients $I_i^{(2)}$, $i \neq 0,3$, that can be chosen arbitrarily on the boundaries. To satisfy the impermeability conditions we have to choose $I_3^{(2)} = 0$ on the boundaries. Finally we get

$$\Phi^{(2)}(\pm 1, v_z \gtrless 0; t) = \alpha_2^{\pm} M_{\pm} + \gamma_{2,\varepsilon}^{\pm}, \quad \alpha_2^{\pm} = I_0^{(2)}(\pm 1) - \langle q_{\pm}^{(2)}(0)\rangle,$$

where $\gamma_{2,\varepsilon}^{\pm}$ are terms exponentially small in ε

The coefficients $I_i^{(2)}$, $i=1,2,4$ of the hydrodynamical part of $B^{(2)}$ are determined by the compatibility condition for the equation at next order in ε:

$$\Big\langle \psi_i\Big[\frac{\partial}{\partial t}B^{(2)} + v_z\frac{\partial}{\partial z}B^{(3)} + G\frac{\partial}{\partial v_z}B^{(2)}\Big]\Big\rangle = 0$$

where

$$\begin{aligned} B^{(3)} &= \mathcal{L}^{-1}\Big[\frac{\partial}{\partial t}\Phi^{(1)} + v_z\frac{\partial}{\partial z}B^{(2)} + G\frac{\partial}{\partial v_z}\Phi^{(1)} - 2J(\Phi^{(1)}, B^{(2)}) \\ &\quad -2J(\Phi_s^{(1)}, B^{(2)}) - 2J(\Phi^{(1)}, B_s^{(2)})\Big] + M\sum_{i=0}^{4}\psi_i\, I_i^{(3)} \end{aligned}$$

together with the b. c. $I_i^{(2)} = (q_-^{(2)})_i$, $i=1,2,4$. Then $I_0^{(2)}$ is found up to a constant that is chosen so that the total mass associated to $\Phi^{(2)}$ vanishes. Proceeding as in the determination of the Boussinesq equation, we find now a set of three linear time-dependent non-homogeneous Stokes equations for $I_i^{(2)}$. The non-homogeneous terms depend on the third order spatial derivatives of $\Phi^{(1)}$. General theorems for the Stokes equation assures the existence of a solution for the chosen boundary vanishing exponentially in time.

Once $B^{(2)}$ is completely determined, the last equation gives the non-hydrodynamical part of $B^{(3)}$. As before, we introduce the terms $b_{\pm}^{(3)}$ to compensate $(I-P)B^{(3)}$ on the boundaries $z=\pm 1$. The term $b_{\pm}^{(3)}$ is found as a solution of a Milne problem with a source term, which depends on the previous boundary corrections $b_{\pm}^{(2)}$ and $\Phi^{(1)}$. The procedure can be continued to any order.

We notice that $(I-P)\Phi^{(n)}$ at time zero are not arbitrary, since they depend on $\Phi^{(n-1)}$ and its derivatives. We can instead assign at time zero $I_i^{(n)}, i=1,2,4$. Notice that the rest term R at time zero is of order ε^4. By using the results in [5] and the exponential decay in time of $\Phi^{(n)}$ we can state the following theorem.

Theorem 4.1. *Assume that at time zero, for some s suitable large*

$$\| M\frac{\partial^s}{\partial z^s} I_i^{(n)}(0,z) \|_{L^2} < \infty, \quad i=1,2,4, \quad n=1,\dots 4\ .$$

Then, it is possible to determine the functions $\Phi^{(n)}$, $n=2,\dots,5$ in the asymptotic expansion (4.2) including the boundary conditions

$$\begin{aligned} \Phi^{(n)}(t,-1,v) &= M_-(v)\int_{w_z<0}|w_z|\Phi^{(n)}(t,-1,w)dw + \gamma_{n,\varepsilon}^-, \; t>0, \; v_z>0, \\ \Phi^{(n)}(t,1,v) &= M_+(v)\int_{w_z>0}|w_z|\;\Phi^{(n)}(t,1,w)dw + \gamma_{n,\varepsilon}^+, \; t>0, \; v_z<0, \end{aligned}$$

the normalization condition

$$\int_{\mathbb{R}^3\times[-1,1]} dvdz\Phi^{(n)} = 0$$

and

$$\| \Phi^{(n)} \|_{2,2,2} < \infty, \quad \| \Phi^{(n)} \|_{\infty,\infty,2} < \infty .$$

Here,

$$\begin{aligned}\| f \|_{2,2,2} &= \Big(\int_0^\infty \int_{-1}^1 \int_{\mathbb{R}^3} |f(s,z,v)|^2 M(v) dsdzdv\Big)^{\frac12},\\ \| f \|_{\infty,\infty,2} &= \sup_{t>0}\Big(\int_{\mathbb{R}^3} \sup_{z\in[-1,1]} |f(t,z,v)|^2 M(v) dv\Big)^{\frac12}.\end{aligned}$$

5. Stability: the remainder

We shall now construct the rest term $R = \tilde{M}\bar{R}$ for $\tilde{M}\bar{\zeta} = \zeta$ and prove that

$$\int_0^{+\infty}\int_{[-1,1]}\int_{\mathbb{R}^3} |\bar{R}(t,z,v)|^2 M(v) dtdzdv < c.$$

This in turn implies the L^2-convergence to zero of $\bar{R}(.,.,t)$ when time tends to infinity.

The rest term $\bar{R}$ satisfies the following problem,

$$\frac{\partial \bar{R}}{\partial t} + \frac{1}{\epsilon} v_z \frac{\partial \bar{R}}{\partial z} - G\frac{\partial \bar{R}}{\partial v_z} = \frac{1}{\epsilon^2}\tilde{L}\bar{R} + \frac{1}{\epsilon}\tilde{Q}(\bar{R},\bar{R}) + \frac{2}{\epsilon}H(\bar{R}) + \alpha, \tag{5.1}$$

$$\bar{R}(0,z,v) = \bar{R}_0(z,v) = \epsilon^4 p_5(z,v),$$

$$\begin{aligned}\bar{R}(t,-1,v) &= \int_{w_z<0}\Big(\bar{R}(t,-1,w) + \frac{\bar{\psi}(t,-1,w)}{\epsilon}\Big)|w_z| M_- dw \\ &\quad -\frac{1}{\epsilon}\bar{\psi}(t,-1,v), \qquad t>0,\ v_z>0,\end{aligned} \tag{5.2}$$

$$\begin{aligned}\bar{R}(t,1,v) &= M^{-1}(v) M_+(v)\int_{w_z>0}\Big(\bar{R}(t,1,w) + \frac{\bar{\psi}(t,1,w)}{\epsilon}\Big) w_z M dw\\ &\quad -\frac{1}{\epsilon}\bar{\psi}(t,1,v), \qquad t>0,\ v_z<0 .\end{aligned}$$

where $\tilde{Q}(\bar{R},\bar{R}) = \frac{1}{\tilde{M}}Q(\tilde{M}\bar{R},\tilde{M}\bar{R})$. Here $\bar{\psi}(t,\pm1,v)$ are the terms exponentially small in ε coming from the expansions and

$$H(\bar{R}) = \frac{1}{\epsilon}\tilde{J}(\bar{R}, \bar{\Phi} + \bar{\Phi}_s) ,$$

with $\tilde{M}\bar{\Phi} = \sum_{n=1}^5 \Phi^{(n)}\varepsilon^n$. Below from H, we shall in particular consider the influence of the first order terms of $\bar{\Phi} + \bar{\Phi}_s$ which will be denoted $\epsilon\psi_{11}$. The function α contains all terms fully coming from the asymptotic expansion. It is of fourth order in ϵ non-hydrodynamically and fifth order hydrodynamically. Uniformly in ϵ its terms converge to zero when time tends to infinity.

The following norms will be used

$$\begin{aligned}\| f \|_{2t,2,2} &= \Big(\int_0^t \int_{-1}^1 \int_{\mathbb{R}^3} |f(s,z,v)|^2 M(v) dsdzdv\Big)^{\frac12},\\ \| f \|_{\infty,2,2} &= \sup_{t>0}\Big(\int_{-1}^1 \int_{\mathbb{R}^3} |f(t,z,v)|^2 M(v) dzdv\Big)^{\frac12},\\ \| f \|_{\infty,\infty,2} &= \sup_{t>0}\Big(\int_{\mathbb{R}^3} \sup_{z\in[-1,1]} |f(t,z,v)|^2 M(v) dv\Big)^{\frac12},\\ \| f \|_{2t,2,\sim} &= \Big(\int_0^t \int_{v_z>0} v_z M(v) \mid f(s,-1,v)\mid^2 dvds\Big)^{\frac12}\\ &\quad +\Big(\int_0^t \int_{v_z<0} \mid v_z\mid M(v) \mid f(s,1,v)\mid^2 dvds\Big)^{\frac12} < +\infty,\\ \| f \|_{\infty,2,\sim} &= \Big(\sup_{t>0}\int_{v_z>0} v_z M(v) \mid f(t,-1,v)\mid^2 dv\Big)^{\frac12}\\ &\quad +\Big(\sup_{t>0}\int_{v_z<0} \mid v_z\mid M(v) \mid f(t,1,v)\mid^2 dv\Big)^{\frac12} < +\infty.\end{aligned}$$

Some of the a priori bounds for $\bar{R}$ will follow from dual solutions to a linear problem (in the rescaled time variable $\bar{\tau} = \varepsilon^{-1}t$) discussed in the following lemma.

Lemma 5.1. *Let $\varphi(\bar{\tau},z,v)$ be solution to*

$$\frac{\partial \varphi}{\partial \bar{\tau}} + v_z\frac{\partial \varphi}{\partial z} - \epsilon G\frac{\partial \varphi}{\partial v_z} = \frac{1}{\epsilon}\tilde{L}\varphi + g - \tilde{J}^*(\psi_{11},\varphi), \tag{5.3}$$

denoting the adjointed, with initial value and boundary indata equal zero. Then for G small enough,

$$\| \nu^{\frac12}(I-P)\varphi \|_{2,2,2} \le c\Big(\epsilon \| \nu^{-\frac12}(I-P)g \|_{2,2,2} + \| Pg \|_{2,2,2}\Big), \tag{5.4}$$

$$\| P\varphi \|_{2,2,2} \leq c\Big(\| \nu^{-\frac{1}{2}}(I-P)g \|_{2,2,2} + \frac{1}{\epsilon} \| Pg \|_{2,2,2} \Big), \quad (5.5)$$

$$\| \varphi \|_{\infty,2,2} \leq c\Big(\sqrt{\epsilon} \| \nu^{-\frac{1}{2}}(I-P)g \|_{2,2,2} + \frac{1}{\sqrt{\epsilon}} \| Pg \|_{2,2,2} \Big), \quad (5.6)$$

$$\| \varphi^{out} \|_{2,2\sim} \leq c\Big(\sqrt{\epsilon} \| \nu^{-\frac{1}{2}}(I-P)g \|_{2,2,2} + \frac{1}{\sqrt{\epsilon}} \| Pg \|_{2,2,2} \Big). \quad (5.7)$$

Proof of Lemma 5.1. The methods from [10] (a variant of [10] Scn 7.5) may be adapted to the present setting with a force term, to obtain the existence of a solution to (5.3).
For a first a priori estimate, multiply (5.3) by φ and integrate the resulting equation on $[0,\bar{T}] \times [-1,1] \times \mathbb{R}^3$. That leads to

$$\begin{aligned} &\| \varphi \|^2_{\infty\bar{T},2,2} + \| S\varphi \|^2_{2\bar{T},2\sim} + \frac{1}{\epsilon} \| \nu^{\frac{1}{2}}(I-P)\varphi \|^2_{2\bar{T},2,2} \\ &\leq c(\epsilon \| \nu^{-\frac{1}{2}}(I-P)g \|^2_{2\bar{T},2,2} + \eta_1 \| P\varphi \|^2_{2\bar{T},2,2} + \frac{1}{\eta_1} \| Pg \|^2_{2\bar{T},2,2}). \quad (5.8)\end{aligned}$$

Let β be a truncation function belonging to $C^1(\mathbb{R})$ with support $(0,+\infty)$, and such that $\beta(\bar{\tau}) = 1$ for $\bar{\tau} > \delta_1$ for some $\delta_1 > 0$. Let $\tilde{\varphi} = \varphi\beta$. Then

$$\frac{\partial \tilde{\varphi}}{\partial \bar{\tau}} = (\frac{1}{\epsilon}\tilde{L} - v_z\frac{\partial}{\partial z} + \epsilon G\frac{\partial}{\partial v_z})\tilde{\varphi} + \varphi\frac{\partial \beta}{\partial \bar{\tau}} + g\beta - \tilde{J}^*(\psi_{11},\varphi)\beta. \quad (5.9)$$

We shall now consider the equation for the Fourier transform in time of $\tilde{\varphi}$, which looks like the stationary problem of Section 3. Let $\mathcal{F}$ be the Fourier transform in $\bar{\tau}$ with Fourier variable σ, and write $\Phi = \mathcal{F}\tilde{\varphi}$. In Fourier space (5.9) becomes

$$-i\sigma\Phi = (\frac{1}{\epsilon}\tilde{L} - v_z\frac{\partial}{\partial z} + \epsilon G\frac{\partial}{\partial v_z})\Phi + \mathcal{F}(\varphi\frac{\partial\beta}{\partial\bar{\tau}}) + \mathcal{F}((g - \tilde{J}^*(\psi_{11},\varphi))\beta). \quad (5.10)$$

We now estimate the fluid moments of Φ one by one, using their particular couplings to other moments. Our approach will be an elaboration of the one in Lemma 3.5. Define

$$\varphi_{x^iy^jz^k} := \int M v_x^i v_y^j v_z^k \varphi_\perp(v)dv, \quad i+j+k \geq 2,$$

and $\varphi_{x^iy^jz^k2}$ ($\varphi_{x^iy^jz^k\bar{A}}$, $\varphi_{x^iy^jz^k\bar{B}}$) correspondingly when there is an extra factor $| v |^2$ ($\bar{A}$ resp. $\bar{B}$) in the integrand.

For $| \sigma | \leq \sigma_0$, $\sigma_0 > 0$ sufficiently small, and using the z-derivative to express the moments, the steps of Lemma 3.5 can be followed without

change, at the end including an integration in time. The two terms $-i\sigma\Phi$ and $\mathcal{F}\varphi\frac{\partial\beta}{\partial\bar{\tau}}$ do not cause additional complications. With $\chi_{\sigma_0}(\sigma)$ denoting the characteristic function for the interval $[-\sigma_0,\sigma_0]$, this leads to

$$\| \chi_{\sigma_0}\Phi_\| \|^2_{2,2,2} \leq c(\| \nu^{-\frac{1}{2}}g_\perp \|^2_{2,2,2} + \frac{1}{\epsilon^2} \| g_\| \|^2_{2,2,2} + \sigma_0 \| \chi_{\sigma_0}\Phi_\| \|^2_{2,2,2}). \quad (5.11)$$

We illustrate on the v_z-moment. Multiplying the equation with M and integrating over $\mathbb{R}^3_v$, leads to $\frac{\partial\Phi_z}{\partial z} - \epsilon G\Phi_z = \mathcal{F}(g_0\beta) + \mathcal{F}(\varphi_0\frac{\partial\beta}{\partial\bar{\tau}}) + i\sigma\Phi_0$, i.e.

$$\Phi_z(z) = \Phi_z(-1)e^{\epsilon G(z+1)} + \int_{-1}^z (\mathcal{F}(g_0\beta) + \mathcal{F}(\varphi_0\frac{\partial\beta}{\partial\bar{\tau}}) + i\sigma\Phi_0)e^{\epsilon G(z-s)}ds.$$

By definition of $\Phi_z(-1)$,

$$\begin{aligned} | \Phi_z(-1) | &= | \int v_z\Phi(-1,v)Mdv | \\ &\leq c\Big(\int | v_z | \Phi^2(-1,v)Mdv \Big)^{\frac{1}{2}} \leq c | S\Phi |_{2\sim}. \end{aligned}$$

And so

$$\begin{aligned} \| \chi_{\sigma_0}\Phi_z \|^2_2 \leq\ & c(\| \mathcal{F}(g_0\beta) \|^2_2 + | S\Phi |^2_{2\sim} + \sigma_0 \| \Phi_0 \|^2_2) \\ &+ \int_{-1}^1 dz[\int_{-1}^z \mathcal{F}(\varphi_0\frac{\partial\beta}{\partial\bar{\tau}})\Phi_z e^{\epsilon G(z-s)}ds], \end{aligned}$$

thus in the limit $\delta_1 \to +0$, $\bar{T} \to \infty$,

$$\| \chi_{\sigma_0}\Phi_z \|^2_{2,2} \leq c(\| g_0\beta \|^2_{2,2} + | S\Phi |^2_{2,2\sim} + \sigma_0 \| \chi_{\sigma_0}\Phi_0 \|^2_{2,2}).$$

The term which is new with respect to the proof of Lemma 3.4 is the last one and has a small factor σ in front. In the other estimates we have always the same structure and in the end we get (5.11). Remember that in the proof of Lemma 3.4 one of the terms required the condition G small. Hence, in the end we get for σ_0 and G small enough

$$\| \chi_{\sigma_0}\Phi_\| \|^2_{2,2,2} \leq c(\| \nu^{-\frac{1}{2}}g_\perp \|^2_{2,2,2} + \frac{1}{\epsilon^2} \| g_\| \|^2_{2,2,2}\ . \quad (5.12)$$

For $\sigma \geq \sigma_0$ we use a different approach. We multiply by suitable functions and integrate over v in such a way that the $\sigma\Phi$-term gives moments of Φ and then we estimate these moments by means of the other terms in the equation. We start with the $\Phi_0 + \frac{2}{\sqrt{6}}\Phi_4$ and Φ_z moments. Multiply (5.10) with $M(\psi_0 + \frac{2}{\sqrt{6}}\psi_4)$ and integrate,

$$
\begin{aligned}
&-i\sigma(\Phi_0 + \frac{2}{\sqrt{6}}\Phi_4 + \Phi_{z^2}) + \frac{\partial}{\partial z}(\frac{5}{3}\Phi_z + \frac{1}{3}\Phi_{v_z(v^2-5)}) \\
&= -i\sigma\Phi_{z^2} + \epsilon G\beta' + \mathcal{F}(\beta(g_0 + \frac{2}{\sqrt{6}}g_4)) + \int \mathcal{F}(\varphi\frac{\partial\beta}{\partial\bar{\tau}})(1 + \frac{2}{\sqrt{6}}\psi_4)Mdv.
\end{aligned}
$$

Here β' is a moment of Φ. The projection along v_z of (5.10) gives

$$
\begin{aligned}
&-i\sigma(\frac{5}{3}\Phi_z + \frac{1}{3}\Phi_{v_z(v^2-5)}) + \frac{5}{3}\frac{\partial}{\partial z}(\Phi_0 + \frac{2}{\sqrt{6}}\Phi_4 + \Phi_{z^2}) \\
&= \frac{-i\sigma}{3}\Phi_{v_z(v^2-5)} + \epsilon G\beta'' + \frac{5}{3}\mathcal{F}(g_z\beta) + \frac{5}{3}\mathcal{F}(\varphi_z\frac{\partial}{\partial\bar{\tau}}\beta),
\end{aligned}
$$

where β'' is a moment of Φ. These two equations together give

$$
\begin{aligned}
&-i\sigma(\frac{5}{3}(\Phi^a)^2 + (\Phi^b)^2) + \frac{5}{3}\frac{\partial}{\partial z}(\Phi^a\,\Phi^b) \\
&= -i\sigma(\frac{5}{3}\Phi_{z^2}(\Phi^a) + \frac{1}{3}\Phi_{v_z(v^2-5)}(\Phi^b)) + \epsilon G(\frac{5}{3}\beta'(\Phi^a) + \beta''(\Phi^b)) \\
&\quad + \frac{5}{3}\Big(\mathcal{F}(g_z\beta) + \mathcal{F}(\varphi_z\frac{\partial}{\partial\bar{\tau}}\beta)\Big)(\Phi^b) \\
&\quad + \Big(\mathcal{F}(\beta(g_0 + \frac{2}{\sqrt{6}}g_4)) + \int \mathcal{F}(\varphi\frac{\partial\beta}{\partial\bar{\tau}})(1 + \frac{2}{\sqrt{6}}\psi_4)Mdv\Big)\frac{5}{3}(\Phi^a)
\end{aligned}
$$

where $\Phi^a := \Phi_0 + \frac{2}{\sqrt{6}}\Phi_4 + \Phi_{z^2}$ and $\Phi^b := \frac{5}{3}\Phi_z + \frac{1}{3}\Phi_{v_z(v^2-5)}$. We conclude that

$$
\begin{aligned}
&\| \Phi^a \|_2^2 + \| \Phi^b \|_2^2 \\
&\le c(| S\Phi |_{2\sim}^2 + \| \nu^{\frac{1}{2}}(I-P)\Phi \|_2^2 + \epsilon^2 G^2 \| \Phi \|_2^2 + \| \mathcal{F}g_{\|} \|_2^2) \\
&\quad -\frac{1}{i\sigma}((\int \mathcal{F}(\varphi\frac{\partial\beta}{\partial\bar{\tau}})(1 + \frac{2}{\sqrt{6}}\psi_4)Mdv)\frac{5}{3}(\Phi^a) + \frac{5}{3}\mathcal{F}(\varphi_z\frac{\partial}{\partial\bar{\tau}}\beta)(\Phi^b)) .
\end{aligned}
$$

Now we consider $(1-\chi_{\sigma_0})(\Phi^a + \Phi^b)$, so that we work in a region $\frac{1}{\sigma} \le \frac{1}{\sigma_0}$. Thus in the limit $\delta_1 \to +0$, $\bar{T} \to \infty$, the terms containing $\frac{\partial\beta}{\partial\bar{\tau}}$ go to zero for any finite σ_0 and we get, uniformly in σ_0,

$$
\begin{aligned}
&\| (1-\chi_{\sigma_0})(\frac{5}{3}(\Phi_0 + \frac{2}{\sqrt{6}}\Phi_4 + \Phi_{z^2}) \|_{2,2}^2 + \| (1-\chi_{\sigma_0})(\frac{5}{3}\Phi_z + \frac{1}{3}\Phi_{v_z(v^2-5)}) \|_{2,2}^2 \\
&\le c(| S\Phi |_{2,2\sim}^2 + \| \nu^{\frac{1}{2}}(I-P)\Phi \|_{2,2,2}^2 + \epsilon^2 G^2 \| \Phi \|_{2,2,2}^2 + \| \mathcal{F}g_{\|} \|_{2,2,2}^2). \quad (5.13)
\end{aligned}
$$

The projection along $v_x v_z \bar{B}$ gives,

$$
\begin{aligned}
&-i\sigma\Phi_{xz\bar{B}} + \frac{\partial}{\partial z}(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) + \epsilon G\alpha'(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) \\
&= \frac{\Phi_{xz}}{\epsilon} + \mathcal{F}(g_{xz\bar{B}}\beta) + \epsilon G\beta' + \mathcal{F}\left(\int -v_x v_z \bar{B}\tilde{J}^*(\psi_{11},\varphi)Mdv\beta + \varphi\frac{\partial}{\partial\bar{\tau}}\beta\right),
\end{aligned}
$$

where $\alpha < 0$ and β' is a non-hydrodynamic moment of Φ. A projection along v_x gives

$$
-i\sigma(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) + \alpha\frac{\partial}{\partial z}\Phi_{xz} + \frac{\alpha\epsilon G}{2}\Phi_{xz} = -i\sigma\Phi_{xz^2\bar{B}} + \alpha\mathcal{F}(g_x\beta + \varphi_x\frac{\partial}{\partial\bar{\tau}}\beta).
$$

The last two equations together give

$$
\begin{aligned}
&-i\sigma(\alpha\Phi_{xz\bar{B}}\Phi_{xz} + (\alpha\Phi_x + \Phi_{xz^2\bar{B}})^2) + \alpha\frac{\partial}{\partial z}(\Phi_{xz}(\alpha\Phi_x + \Phi_{xz^2\bar{B}})) \\
&\quad + \alpha\epsilon G(\alpha' + \frac{1}{2})\Phi_{xz}(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) \\
&= \alpha\frac{\Phi_{xz}^2}{\epsilon} + \alpha\Phi_{xz}(\mathcal{F}(g_{xz\bar{B}}\beta) + \epsilon G\beta') \\
&\quad + \mathcal{F}(\alpha g_x\beta + \varphi_x\frac{\partial}{\partial\bar{\tau}}\beta)(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) - i\sigma\Phi_{xz^2\bar{B}}(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) \\
&\quad + \alpha\Phi_{xz}\mathcal{F}\left(\int -v_x v_z\bar{B}\tilde{J}^*(\psi_{11},\varphi)Mdv\beta + \varphi\frac{\partial}{\partial\bar{\tau}}\beta\right).
\end{aligned}
$$

This equation has almost the same structure as the equation for $\bar{F}_x$ in the proof of Lemma 3.5. By repeating the steps in that proof we conclude that, for arbitrary η,

$$
\begin{aligned}
\| (1-\chi_{\sigma_0})(\alpha\Phi_x + \Phi_{xz^2\bar{B}}) \|_{2,2} \le\ & c(\| \nu^{\frac{1}{2}}(I-P)\Phi \|_{2,2,2} + \eta \| P\Phi \|_{2,2,2} \\
& + \| \nu^{-\frac{1}{2}}\mathcal{F}g_{\perp} \|_{2,2,2} + \frac{1}{\epsilon} \| \mathcal{F}g_{\|} \|_{2,2,2}). \quad (5.14)
\end{aligned}
$$

Notice that Lemma 3.5 is stated for equation (3.5) where the known term is $\frac{g}{\epsilon}$. Instead in (5.3) the known term is g. This explain the difference in the factors ϵ.

The result and proof are analogous for Φ_y.

There remains an estimate for Φ_0. The projection along $v_z - v_z\bar{A}\frac{2}{\sqrt{6}\gamma}$, where $\gamma = \frac{1}{\sqrt{6}}\int v_z^2\tilde{A}(v^2-3)Mdv < 0$, gives

$$\begin{aligned}
&-i\sigma\Phi_z + \frac{\partial}{\partial z}(\Phi_0+\beta') \\
&= -i\sigma\frac{2}{\sqrt{6}}\Phi_{z\tilde{A}} - \frac{2}{\sqrt{6}\gamma\epsilon}\Phi_{v_z(v^2-5)} + \epsilon G\beta'' + \mathcal{F}(g_z\beta - \frac{2}{\sqrt{6}\gamma}g_{z\tilde{A}}\beta) \\
&\quad +\mathcal{F}\Big(\varphi_z\frac{\partial}{\partial\bar{\tau}}\beta - \frac{2}{\sqrt{6}\gamma}\varphi_{z\tilde{A}}\frac{\partial}{\partial\bar{\tau}}\beta\Big) - \frac{2}{\sqrt{6}\gamma}\mathcal{F}\Big(\int v_z\tilde{A}\tilde{J}^*(\psi_{11},\varphi)Mdv\beta\Big).
\end{aligned}$$

Here β' is a non-hydrodynamic and β'' a general moment of Φ. The projection along ψ_0 gives

$$-i\sigma(\Phi_0+\beta') + \frac{\partial}{\partial z}\Phi_z - \frac{\epsilon G}{2}\Phi_z = -i\sigma\beta' + \mathcal{F}(\varphi_0\frac{\partial\beta}{\partial\bar{\tau}}) + \mathcal{F}(g_0\beta).$$

Again these two equations give

$$\begin{aligned}
&-i\sigma((\Phi_0+\beta')^2+\Phi_z^2) + \frac{\partial}{\partial z}((\Phi_0+\beta')\Phi_z) \\
&= \epsilon G(\beta'' + \frac{1}{2}(\Phi_0+\beta'))\Phi_z + (\Phi_0+\beta')\Big(-i\sigma\beta' + \mathcal{F}(\varphi_0\frac{\partial\beta}{\partial\bar{\tau}}) + \mathcal{F}(g_0\beta)\Big) \\
&\quad +\Phi_z\Big(-i\sigma\frac{2}{\sqrt{6}}\Phi_{z\tilde{A}} - \frac{2}{\sqrt{6}\gamma\epsilon}\Phi_{v_z(v^2-5)} + \mathcal{F}\Big(g_z\beta - \frac{2}{\sqrt{6}\gamma}g_{z\tilde{A}}\beta\Big) \\
&\quad +\mathcal{F}\Big(\varphi_z\frac{\partial}{\partial\bar{\tau}}\beta - \frac{2}{\sqrt{6}\gamma}\varphi_{z\tilde{A}}\frac{\partial}{\partial\bar{\tau}}\beta\Big) - \frac{2}{\sqrt{6}\gamma}\mathcal{F}\Big(\int v_z\tilde{A}\tilde{J}^*(\psi_{11},\varphi)Mdv\beta\Big)\Big).
\end{aligned}$$

Using also (5.13) and (5.14) we conclude that

$$\begin{aligned}
&\| (1-\chi_{\sigma_0})(\Phi_0+\beta') \|_{2,2,2}^2 \\
&\le c(\| \Phi_z \|_{2,2,2}^2 + \| \nu^{\frac{1}{2}}(I-P)\Phi \|_{2,2,2}^2 + \eta \| \Phi_\| \|_{2,2,2}^2 + \| \nu^{-\frac{1}{2}}\mathcal{F}g_\perp \|_{2,2,2}^2 \\
&\quad +\frac{1}{\epsilon^2} \| \mathcal{F}g_\| \|_{2,2,2}^2) \\
&\le c(\| \nu^{\frac{1}{2}}(I-P)\Phi \|_{2,2,2}^2 + \eta \| \Phi_\| \|_{2,2,2}^2 + \| \nu^{-\frac{1}{2}}\mathcal{F}g_\perp \|_{2,2,2}^2 \\
&\quad +\frac{1}{\epsilon^2} \| \mathcal{F}g_\| \|_{2,2,2}^2). \qquad (5.15)
\end{aligned}$$

In particular $(1-\chi_{\sigma_0})P\Phi$ is bounded by the right hand side of (5.15). This together with (5.12) gives the estimate (5.5) for $P\varphi$ of the lemma. Finally, by choosing $\eta_1 = \sqrt{\epsilon}$ in (5.8) and using (5.5), we get the last two estimates (5.6) and (5.7). This ends the proof of the lemma. □

For the iteration procedure in the construction of the rest term below,

we shall be using a sum of two systems, the first one being

$$\begin{aligned}
&\frac{\partial R_1}{\partial t} + \frac{1}{\epsilon}v_z\frac{\partial R_1}{\partial z} - G\frac{\partial R_1}{\partial v_z} = \frac{1}{\epsilon^2}LR_1 + \frac{1}{\epsilon}g + \frac{1}{\epsilon}\tilde{J}(\psi_{11},R_1), \qquad (5.16)\\
&R_1(0,z,v) = R_0(z,v),\\
&R_1(t,-1,v) = -\frac{1}{\epsilon}\bar{\psi}(t,-1,v), \quad t>0,\ v_z>0,\\
&R_1(t,1,v) = -\frac{1}{\epsilon}\bar{\psi}(t,1,v), \quad t>0,\ v_z<0.
\end{aligned}$$

A proof of existence for (5.16) can be adapted from [10] to the present case with an additional force term.

To estimate the non-hydrodynamic part of R_1, multiply (5.16) with $2R_1M$, integrate over $[0,t]\times[-1,1]\times\mathbb{R}^3$, and use the spectral inequality, to obtain

$$\begin{aligned}
&\frac{1}{\epsilon} \| R_1^{out} \|_{2t,2,\sim}^2 + \| R_1(t) \|_{2,2}^2 + \frac{1}{\epsilon^2} \| \nu^{\frac{1}{2}}(I-P)R_1 \|_{2t,2,2}^2 \\
&\le c\Big(\| R_0 \|_{2,2}^2 + \| \nu^{-\frac{1}{2}}(I-P)g \|_{2t,2,2}^2 + (1+\frac{\eta_1}{2\epsilon}) \| PR_1 \|_{2t,2,2}^2 \\
&\quad + \frac{1}{2\eta_1\epsilon} \| Pg \|_{2t,2,2}^2 + \frac{1}{\epsilon^3} \| \bar{\psi} \|_{2t,2,\sim}^2 \Big), \qquad (5.17)
\end{aligned}$$

for every $\eta_1 > 0$.

The a priori bounds on PR_1 are discussed in the following lemma. They are more involved and based on dual techniques using the problem (5.3).

Denote by

$$h(t,z,v) := PR_1.$$

Lemma 5.2. *There is ϵ_0 such that for $0<\epsilon<\epsilon_0$ and G small*

$$\| h \|_{2,2,2}^2 \le \frac{c}{\epsilon}(\| R_0 \|_{2,2}^2 + \| \nu^{-\frac{1}{2}}(I-P)g \|_{2,2,2}^2 + \frac{1}{\epsilon^3} \| Pg \|_{2,2,2}^2 + \frac{1}{\epsilon^3} \| \bar{\psi} \|_{2,2,\sim}^2).$$

Proof of Lemma 5.2. In the variables $(\bar{\tau},z,v)$, the function R_1 is a solution to

$$\begin{aligned}
&\frac{\partial R_1}{\partial\bar{\tau}} + v_z\cdot\frac{\partial R_1}{\partial z} - \epsilon G\frac{\partial R_1}{\partial v_z} = \frac{1}{\epsilon}\bar{L}R_1 + g + \bar{J}(\psi_{11},R_1),\\
&R_1(0,z,v) = R_0(z,v),\\
&R_1(\bar{\tau},-1,v) = -\frac{1}{\epsilon}\bar{\psi}(\bar{\tau},-1,v), \quad \bar{\tau}>0,\ v_z>0,
\end{aligned}$$

$$R_1(\bar\tau,1,v) = -\frac{1}{\epsilon}\bar\psi(\bar\tau,1,v), \quad \bar\tau > 0,\ v_z < 0.$$

Let φ be the solution to

$$\frac{\partial\varphi}{\partial\bar\tau} + v_z\cdot\frac{\partial\varphi}{\partial z} - \epsilon G\frac{\partial\varphi}{\partial v_z} = \frac{1}{\epsilon}\tilde L\varphi + h - \tilde J^*(\psi_{11},\varphi)$$

with initial and ingoing boundary values zero. Denote by

$$(f,g)_H = \int f(v)g(v)\tilde M(v)dv, \quad \tilde M := M(v)e^{-\epsilon G(z+1)}.$$

Multiply the equation for R_1 by $\tilde M\varphi$ and the equation for φ by $R_1\tilde M$ and integrate over v. We have

$$\frac{\partial}{\partial\bar\tau}(R_1,\varphi)_H + \int v_z\tilde M[\varphi\frac{\partial}{\partial z}R_1 + R_1\frac{\partial}{\partial z}\varphi]dv - \epsilon G\int\tilde M[\varphi\frac{\partial}{\partial v_z}R_1 + R_1\frac{\partial}{\partial v_z}\varphi]dv$$
$$= \frac{2}{\epsilon}(\tilde L R_1,(I-P)\varphi)_H + (g,\varphi)_H + (h,PR_1)_H.$$

After integrating on z, by integration by part, the second and third integral give

$$-\int(R_1\varphi)[v_z\frac{\partial}{\partial z}\tilde M - \epsilon G\frac{\partial}{\partial v_z}\tilde M]dvdz + (v_z\varphi,R_1)_H(1) - (v_z\varphi,R_1)_H(-1)$$
$$= (v_z\varphi,R_1)_H(1) - (v_z\varphi,R_1)_H(-1)\ .$$

We integrate also with respect to $\bar\tau$. This gives

$$\begin{aligned}\| h\|^2_{2\bar\tau,2,2} \le{}& \frac{K_1}{2}\| R_1(\bar\tau,\cdot,\cdot)\|^2_{2,2} + \frac{1}{2K_1}\|\varphi(\bar\tau,\cdot,\cdot)\|^2_{2,2} + \frac{K_2}{2}\| R_1^{out}\|^2_{2\bar\tau,2\sim}\\ &+\frac{1}{2K_2}\|\varphi^{out}\|^2_{2\bar\tau,2\sim} + \frac{K_3}{2\epsilon}\|\nu^{\frac12}(I-P)R_1\|^2_{2\bar\tau,2,2}\\ &+\frac{1}{2K_3\epsilon}\|\nu^{\frac12}(I-P)\varphi\|^2_{2\bar\tau,2,2} + \frac{K_4}{2}\|\nu^{-\frac12}(I-P)g\|^2_{2\bar\tau,2,2}\\ &+\frac{1}{2K_4}\|\nu^{\frac12}(I-P)\varphi\|^2_{2\bar\tau,2,2} + \frac{K_2}{2}\| Pg\|^2_{2\bar\tau,2,2} + \frac{1}{2K_2}\| P\varphi\|^2_{2\bar\tau,2,2},\end{aligned}$$

for any positive constants K_j, $j = 1,\dots,4$.

It then follows from Lemma 5.1 and (5.17) that

$$\begin{aligned}\| h\|^2_{2,2,2} \le{}& c[(K_1+K_2+K_3)\| R_0\|^2_{2,2} + (\frac{K_1}{\epsilon^2}+\frac{K_2}{\epsilon^2}+\frac{K_3}{\epsilon^2})\|\bar\psi\|^2_{2,2,\sim}\\ &+(\epsilon K_1 + K_3 + K_4 + \epsilon K_2)\|\nu^{-\frac12}(I-P)g\|^2_{2,2,2})\end{aligned}$$

$$\begin{aligned}&+(\frac{1}{\epsilon K_1}+\frac{1}{\epsilon K_3}+\frac{1}{K_4}+\frac{\eta_1}{\epsilon^2K_1}+\frac{1}{K_2\epsilon^2})\| h\|^2_{2,2,2}\\ &+(\frac{K_1}{\eta_2}+\frac{K_3}{\eta_1}+\frac{K_2}{\eta_2}+K_2)\| Pg\|^2_{2,2,2}\\ &+\| PR_1\|^2_{2,2,2}(\eta_2K_1+\eta_1K_3+\eta_2K_2)].\end{aligned}$$

Choosing $\epsilon < 1$, then K_1 and K_3 (resp.K_2) of order ϵ^{-1} (resp. ϵ^{-2}) and η_1 (resp. η_2) of order ϵ (resp. ϵ^2), leads to

$$\begin{aligned}&\| h\|^2_{2,2,2}\\ &\le c\Big(\frac{1}{\epsilon^2}\| R_0\|^2_{2,2} + \frac{1}{\epsilon}\|\nu^{-\frac12}(I-P)g\|^2_{2,2,2} + \frac{1}{\epsilon^4}\| Pg\|^2_{2,2,2} + \frac{1}{\epsilon^4}\|\bar\psi\|^2_{2,2,\sim}\Big).\end{aligned}$$

This ends the proof of Lemma 5.2 when coming back to the t-variable. □

Lemma 5.3. *Any solution R_1 to the system*

$$\begin{aligned}&\frac{\partial R_1}{\partial t} + \frac{1}{\epsilon}v_z\cdot\frac{\partial R_1}{\partial z} - G\frac{\partial R_1}{\partial v_z} = \frac{1}{\epsilon^2}\tilde L R_1 + \frac{2}{\epsilon}H(R_1) + \frac{1}{\epsilon}g, &&(5.18)\\ &R_1(0,z,v) = R_0(z,v),\\ &R_1(t,-1,v) = -\frac{1}{\epsilon}\bar\psi(t,-1,v), \quad t>0,\ v_z>0,\\ &R_1(t,1,v) = -\frac{1}{\epsilon}\bar\psi(t,1,v), \quad t>0,\ v_z<0,\end{aligned}$$

satisfies

$$\begin{aligned}\|\nu^{\frac12}R_1\|_{2,2,2} \le{}& c\Big(\frac{1}{\sqrt\epsilon}\| R_0\|_{2,2} + \frac{1}{\sqrt\epsilon}\|\nu^{-\frac12}(I-P)g\|_{2,2,2} + \frac{1}{\epsilon^2}\| Pg\|_{2,2,2}\\ &+\frac{1}{\epsilon^2}\|\bar\psi\|_{2,2\sim}\Big),\\ \| R_1\|_{\infty,2,2} \le{}& \frac{c}{\sqrt\epsilon}\Big(\| R_0\|_{2,2} + \|\nu^{-\frac12}(I-P)g\|_{2,2,2} + \frac{1}{\epsilon\sqrt\epsilon}\| Pg\|_{2,2,2}\\ &+\frac{1}{\epsilon\sqrt\epsilon}\|\bar\psi\|_{2,2\sim}\Big),\\ \|\nu^{\frac12}R_1\|_{\infty,\infty,2} \le{}& c\Big(\frac{1}{\epsilon}\| R_0\|_{2,2} + \frac{1}{\epsilon}\|\nu^{-\frac12}(I-P)g\|_{2,2,2} + \frac{1}{\epsilon^{\frac52}}\| Pg\|_{2,2,2}\\ &+\frac{1}{\epsilon^{\frac52}}\|\bar\psi\|_{2,2,\sim} + \epsilon\|\nu^{-\frac12}g\|_{\infty,\infty,2} + \| R_0\|_{\infty,2} + \frac{1}{\epsilon}\|\bar\psi\|_{\infty,2\sim}\Big).\end{aligned}$$

Proof of Lemma 5.3. Consider the solution R_1 to

$$\begin{aligned}&\frac{\partial R_1}{\partial t} + \frac{1}{\epsilon}v_z\cdot\frac{\partial R_1}{\partial z} - G\frac{\partial R_1}{\partial v_z} = \frac{1}{\epsilon^2}\tilde L R_1 + \frac{1}{\epsilon}g + \frac{2}{\epsilon}H(R_1),\\ &R_1(0,z,v) = R_0(z,v),\end{aligned}$$

$$R_1(t,-1,v) = -\frac{1}{\epsilon}\bar{\psi}(t,-1,v), \quad t>0,\ v_z>0,$$
$$R_1(t,1,v) = -\frac{1}{\epsilon}\bar{\psi}(t,1,v), \quad t>0,\ v_z<0.$$

By Green's formula and the spectral inequality

$$\begin{aligned}&\frac{1}{\sqrt{\epsilon}} \parallel R_1^{out} \parallel_{2,2\sim} + \sup_{t\geq 0} \parallel R_1(t) \parallel_{2,2} + \frac{1}{\epsilon} \parallel \nu^{\frac{1}{2}}(I-P)R_1 \parallel_{2,2,2} \\ &\leq c\Big(\parallel R_0 \parallel_{2,2} + \frac{1}{\epsilon^{\frac{3}{2}}} \parallel \bar{\psi} \parallel_{2,2,\sim} + \parallel \nu^{-\frac{1}{2}}(I-P)g \parallel_{2,2,2} \\ &\quad +(1+\frac{\eta}{\sqrt{\epsilon}}) \parallel PR_1 \parallel_{2,2,2} + \frac{1}{\eta\sqrt{\epsilon}} \parallel Pg \parallel_{2,2,2} \Big),\end{aligned}$$

for any $\eta > 0$. Moreover, it follows from Lemma 5.2 that

$$\begin{aligned}&\parallel PR_1 \parallel_{2,2,2} \\ &\leq c\Big(\frac{1}{\sqrt{\epsilon}} \parallel R_0 \parallel_{2,2} + \frac{1}{\sqrt{\epsilon}} \parallel (I-P)g \parallel_{2,2,2} + \frac{1}{\epsilon^2} \parallel Pg \parallel_{2,2,2} + \frac{1}{\epsilon^2} \parallel \bar{\psi} \parallel_{2,2\sim} \Big),\end{aligned}$$

since the higher order terms in H compared to $\tilde{J}(\psi_{11}, .)$ do not change that estimate. Choosing $\eta = \sqrt{\epsilon}$ leads to the first and second inequalities. Then, some additional computations using the solution formula, imply

$$\begin{aligned}&\parallel \nu^{\frac{1}{2}} R_1 \parallel_{\infty,\infty,2} \\ &\leq c\Big(\frac{1}{\epsilon^{\frac{1}{2}}} \parallel R_1 \parallel_{\infty,2,2} + \parallel R_0 \parallel_{\infty,2} + \epsilon \parallel \nu^{-\frac{1}{2}} g \parallel_{\infty,\infty,2} + \parallel R_1^{in} \parallel_{\infty,2\sim} \Big),\end{aligned}$$

which leads to the last inequality of Lemma 5.3. □

There remains a second part to obtain the full equation for $\tilde{R}$,

$$\epsilon\frac{\partial R_2}{\partial t} + v_z\frac{\partial R_2}{\partial z} - \epsilon G\frac{\partial R_2}{\partial v_z} = \frac{1}{\epsilon}\bar{L}R_2 + 2H(R_2), \tag{5.19}$$

$$\begin{aligned}R_2(0,z,v) &= 0\\ R_2(t,-1,v) &= \int_{w_z<0}\Big(R_1(t,-1,w)+R_2(t,-1,w)+\frac{\bar{\psi}(t,-1,w)}{\epsilon}\Big)|w_z|M_-dw,\\ &\quad t>0,\ v_z>0,\\ R_2(t,1,v) &= M^{-1}(v)M_+(v)\\ &\quad \times\int_{w_z>0}\Big(R_1(t,1,w)+R_2(t,1,w)+\frac{\bar{\psi}(t,1,w)}{\epsilon}\Big)w_zMdw,\\ &\quad t>0,\ v_z<0.\end{aligned}$$

The existence theory for the solution R_2 is presented in [10], (p. 150). There the case of no exterior force is discussed, but an extension including the present exterior force term is easily carried out, if one includes the corresponding characteristics into the proofs in [10].

The following a priori estimates hold for R_2. By Green's formula

$$\begin{aligned}&\epsilon \mid R_2(t) \mid_{2,2}^2 + \parallel R_2^{out} \parallel_{2t,2,\sim}^2 + \frac{c_1}{\epsilon} \parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel_{2t,2,2}^2 \\ &\leq \parallel R_2^{in} \parallel_{2t,2\sim}^2 + c_2\epsilon \parallel PR_2 \parallel_{2t,2,2}^2 .\end{aligned}$$

Also, by using the bounds in the proof of Lemma 5.3,

$$\parallel R_1^{out} \parallel_{2,2,\sim}^2 \leq c(\parallel R_0 \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}}(I-P)g \parallel_{2,2,2}^2 + \frac{1}{\epsilon^3} \parallel Pg \parallel_{2,2,2}^2 + \frac{1}{\epsilon^3} \parallel \bar{\psi} \parallel_{2,2\sim}^2).$$

Similarly to the earlier R_1-case, PR_2 satisfies

$$\parallel PR_2 \parallel_{2t,2,2} \leq \frac{c}{\epsilon} \parallel R_2^{in} \parallel_{2t,2\sim} .$$

For R_2^{in} we get, by [10], (p101-102,150-151), the special properties of the operator $L + 2\epsilon H$, which will be presented in a forth caming paper and a control of the ϵ-dependence,

$$\parallel R_2^{in} \parallel_{2t,2\sim} \leq c(\parallel R_1^{out} \parallel_{2t,2\sim} + \parallel \frac{\bar{\psi}}{\epsilon} \parallel_{2t,2\sim} .$$

Using the solution formula similarly to Lemma 5.3,

$$\parallel \nu^{\frac{1}{2}} R_2 \parallel_{\infty,\infty,2} \leq c(\frac{1}{\sqrt{\epsilon}} \parallel R_2 \parallel_{\infty,2,2} + \parallel R_2^{in} \parallel_{\infty,2\sim}).$$

From [10] and the above discussion, we also obtain

$$\parallel R_2^{in} \parallel_{\infty,2\sim} \leq c(\parallel R_1^{out} \parallel_{\infty,2\sim} + \frac{1}{\epsilon} \parallel \bar{\psi} \parallel_{\infty,2\sim}).$$

These estimates together give

Lemma 5.4. *A solution to the R_2-problem* (5.19) *satisfies*

$$\begin{aligned}&\parallel \nu^{\frac{1}{2}}(I-P)R_2 \parallel_{2,2,2}^2 \\ &\leq c\Big(\parallel R_0 \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}}(I-P)g \parallel_{2,2,2}^2 + \frac{1}{\epsilon^3} \parallel Pg \parallel_{2,2,2}^2 + \frac{1}{\epsilon^3} \parallel \bar{\psi} \parallel_{2,2\sim}^2 \Big),\\ &\parallel PR_2 \parallel_{2,2,2}^2 \\ &\leq \frac{c}{\epsilon^2}\Big(\parallel R_0 \parallel_{2,2}^2 + \parallel \nu^{-\frac{1}{2}}(I-P)g \parallel_{2,2,2}^2 + \frac{1}{\epsilon^3} \parallel Pg \parallel_{2,2,2}^2 + \frac{1}{\epsilon^3} \parallel \bar{\psi} \parallel_{2,2\sim}^2 \Big),\end{aligned}$$

$$
\begin{aligned}
&\| \nu^{\frac{1}{2}} R_2 \|^2_{\infty,\infty,2}\\
&\quad \le c\Big(\frac{1}{\epsilon}\| R_2 \|^2_{\infty,2,2} + \| R_1^{out} \|^2_{\infty,2\sim} + \frac{1}{\epsilon^2}\| \bar\psi \|^2_{\infty,2\sim}\Big)\\
&\quad \le c\Big(\frac{1}{\epsilon^3}\| R_0 \|^2_{2,2} + \frac{1}{\epsilon^3}\| \nu^{-\frac{1}{2}}(I-P)g \|^2_{2,2,2} + \frac{1}{\epsilon^6}\| Pg \|^2_{2,2,2}\\
&\qquad + \frac{1}{\epsilon^6}\| \bar\psi \|^2_{2,2\sim} + \frac{1}{\epsilon^2}\| \bar\psi \|^2_{\infty,2\sim} + \| R_0 \|^2_{\infty,2} + \epsilon^2 \| \nu^{-\frac{1}{2}} g \|^2_{\infty,\infty,2}\Big).
\end{aligned}
$$

We are now ready to discuss the iteration procedure for the existence of the rest term R and its convergence to zero, when $t \to +\infty$. We shall prove that R can be obtained as the limit of an approximating sequence, and that

$$
\int_0^{+\infty} \int_{[-1,1]} \int_{\mathbb{R}^3} R^2(t,x,v) M(v) dt dx dv < \infty. \tag{5.20}
$$

This in turn implies the L^2-convergence to zero of $R(.,.,t)$, when time tends to infinity.

Theorem 5.5. There exists a solution $R = \tilde M \bar R$ to the rest term problem (5.1), and it holds that

$$
\int_0^{+\infty} \int_{[-1,1]} \int_{\mathbb{R}^3} R^2(t,z,v) M(v) dt dz dv < \infty. \tag{5.21}
$$

Moreover,

$$
\lim_{t\to\infty} \bar R(t,z,v) = 0 \ .
$$

Proof of Theorem 5.5. Let the approximating sequence $(R^n)_0^\infty$ be defined by $R^0 = 0$, and

$$
\begin{aligned}
&\frac{\partial R^{n+1}}{\partial t} + \frac{1}{\epsilon} v_z \cdot \frac{\partial R^{n+1}}{\partial z} - G \frac{\partial R^{n+1}}{\partial v_z}\\
&\qquad = \frac{1}{\epsilon^2} \tilde L R^{n+1} + \frac{2}{\epsilon} H(R^{n+1}) + \frac{1}{\epsilon} \tilde Q(R^n, R^n) + \alpha,\\
&R^{n+1}(0,z,v) = R_0(z,v),\\
&R^{n+1}(t,-1,v) = \int_{w_z<0} \big(R^{n+1}(t,-1,w) + \bar\psi(t,-1,w)\big) |w_z| M_- dw\\
&\qquad\qquad -\frac{1}{\epsilon}\bar\psi(t,-1,v), \quad t>0,\ v_z>0,\\
&R^{n+1}(t,1,v) = M^{-1} M_+(v) \int_{w_z>0} \big(R^{n+1}(t,1,w) + \bar\psi(t,1,w)\big) w_z M dw\\
&\qquad\qquad -\frac{1}{\epsilon}\bar\psi(t,1,v), \quad t>0,\ v_z<0.
\end{aligned}
$$

We use that R_0 is of ϵ-order four, $\epsilon\alpha_\perp$ of order four, and $\epsilon\alpha_\parallel$ of order five. In particular, the function R^1 is solution to

$$
\begin{aligned}
&\frac{\partial R^1}{\partial t} + \frac{1}{\epsilon} v_z \cdot \frac{\partial R^1}{\partial z} - G \frac{\partial R^1}{\partial v_z} = \frac{1}{\epsilon^2} \tilde L R^1 + \frac{2}{\epsilon} H(R^1) + \alpha,\\
&R^1(0,z,v) = R_0(z,v),\\
&R^1(t,-1,v) = \int_{w_z<0} \big(R^1(t,-1,w) + \bar\psi(t,-1,w)\big) |w_z| M_- dw - \frac{1}{\epsilon}\bar\psi(t,-1,v),\\
&\qquad t>0,\ v_z>0,\\
&R^1(t,1,v) = M^{-1} M_+(v) \int_{w_z>0} \big(R^1(t,1,w) + \bar\psi(t,1,w)\big) w_z M dw - \frac{1}{\epsilon}\bar\psi(t,1,v),\\
&\qquad t>0,\ v_z<0.
\end{aligned}
$$

Now we split R^1 into two parts R_1 and R_2 solutions of (5.18) and (5.19) above, with $g = \varepsilon\alpha$. Then, by using the corresponding a priori estimates Lemma 5.3 resp. Lemma 5.4, together with the subexponential decrease of $\bar\psi$, and the conditions on R_0 and α we obtain

$$
\| \nu^{\frac{1}{2}} R^1 \|_{\infty,\infty,2} \le c_1 \epsilon^2, \quad \| \nu^{\frac{1}{2}} R^1 \|_{2,2,2} \le c_1 \epsilon^{\frac{5}{2}},
$$

for some constant c_1.

By induction, suppose that up to a given n

$$
\begin{aligned}
\| \nu^{\frac{1}{2}} R^j \|_{\infty,\infty,2} &\le 2c_1 \epsilon^2, \quad j \le n+1,\\
\| \nu^{\frac{1}{2}} (R^{n+1} - R^n) \|_{2,2,2} &\le c_2 \epsilon \| \nu^{\frac{1}{2}} (R^n - R^{n-1}) \|_{2,2,2}, \quad n \ge 0,
\end{aligned}
$$

for some constant c_2. To complete the induction,

$$
\begin{aligned}
&\frac{\partial}{\partial t}(R^{n+2} - R^{n+1}) + \frac{1}{\epsilon} v_z \frac{\partial}{\partial z}(R^{n+2} - R^{n+1}) - G \frac{\partial}{\partial v_z}(R^{n+2} - R^{n+1})\\
&\qquad = \frac{1}{\epsilon^2} \tilde L (R^{n+2} - R^{n+1}) + \frac{2}{\epsilon} H(R^{n+2} - R^{n+1}) + \frac{1}{\epsilon} G^{n+1},\\
&(R^{n+2} - R^{n+1})(0,z,v) = 0,\\
&(R^{n+2} - R^{n+1})(t,-1,v) = \int_{w_z<0} (R^{n+2} - R^{n+1})(t,-1,w) |w_z| M dw,\\
&\qquad\qquad t>0,\ v_z>0,\\
&(R^{n+2} - R^{n+1})(t,1,v) = \frac{M_+}{M} \int_{w_z>0} (R^{n+2} - R^{n+1})(t,1,w) |w_z| M dw,\\
&\qquad\qquad t>0,\ v_z<0.
\end{aligned}
$$

Here

$$G^{n+1} = (I-P)G^{n+1} = \bar{J}(R^{n+1}+R^n, R^{n+1}-R^n).$$

It follows that

$$\begin{aligned}\| \nu^{\frac{1}{2}}(R^{n+2}-R^{n+1}) \|_{2,2,2} &\leq \frac{c}{\epsilon} \| \nu^{-\frac{1}{2}} G^{n+1} \|_{2,2,2} \\ &\leq \frac{c}{\epsilon}\Big(\| \nu^{\frac{1}{2}} R^{n+1} \|_{\infty,\infty,2} + \| \nu^{\frac{1}{2}} R^n \|_{\infty,\infty,2} \Big) \| \nu^{\frac{1}{2}}(R^{n+1}-R^n) \|_{2,2,2} \\ &\leq c_2\epsilon \| \nu^{\frac{1}{2}}(R^{n+1}-R^n) \|_{2,2,2} .\end{aligned}$$

Consequently,

$$\begin{aligned}&\| \nu^{\frac{1}{2}} R^{n+2} \|_{2,2,2} \\ &\leq \| \nu^{\frac{1}{2}}(R^{n+2}-R^{n+1}) \|_{2,2,2} + \ldots + \| \nu^{\frac{1}{2}}(R^2-R^1) \|_{2,2,2} + \| \nu^{\frac{1}{2}} R^1 \|_{2,2,2} \\ &\leq 2c_1\epsilon^{\frac{5}{2}},\end{aligned}$$

for ϵ small enough. Similarly $\| \nu^{\frac{1}{2}} R^{n+2} \|_{\infty,\infty,2} \leq 2c_1\epsilon^2$. In particular (R^n) is a Cauchy sequence in $L^2_M([0,+\infty[\times\Omega\times\mathbb{R}^3)$. The existence of $\bar{R}$ follows, and the estimate (5.21) holds. This means that there is a sequence of Lebesgue points in time, $(t_j)_{j=1}^{\infty}$ with t_j tending to infinity with j, where the $L^2_M([-1,1]\times\mathbb{R}^3)$- norm of the solution $\bar{R}$ tends to zero. But the L^2_M-norm of $\bar{R}$ for fixed $t \geq t_j$ is uniformly bounded by the norm at t_j plus some tail integrals from t_j to ∞, hence tends to zero when time tends to infinity. This completes the study of the R-term and Theorem 5.5. follows. □

Finally, the stability follows from Theorem 5.5, hence Theorem 4.1 and Theorem 5.5 imply Theorem 1.1.

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