Bose-Einstein condensates at very low temperatures. A mathematical result in the isotropic case.

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Abstract. A system coupling the condensate density to the non-condensate distribution function of a gas at very low temperature is considered. A global existence in time of a solution to the Cauchy problem is proven for an initial datum with finite mass and energy.

1 Introduction.

Since the recent discovery of Bose-Einstein condensation in ultracold trapped atomic gases [1] [3], that makes possible to observe fundamental properties of quantum statistics, the interest in the quantum framework of the

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Boltzmann equation has increased. In the 1920's, Bose and Einstein theoretically predicted the existence of Bose-Einstein condensates. A fundamental result of quantum statistics stated that above a certain critical density all added bosons enter the ground state, so that Bose-Einstein condensates form. Since then, the presence of Bose-Einstein condensates has been inferred rather than observed in a number of phenomena, like superconductivity and supraconductivity in helium. It is in 1995 only that they were produced in a very low temperature context for a gas of rubidium in a trapped potential. Mathematically, the quantum Boltzmann equation presents formal analogies to the classical Boltzmann equation, but its solutions present quite different features. In particular, the boundedness of the classical entropy provides L^1 compactness for the distribution function, whereas the boundedness of the quantum entropy does not. Indeed, the quantum entropy is bounded from above by a multiple of the mass. Hence the a priori bounds of mass, energy and entropy reduce to bounds on mass and energy. Therefore, concentrations of the distribution function are expected. Splitting the gas distribution function into its Lebesgue absolutely continuous part and its singular part enables to distinguish the non-condensate from the condensate parts of the gas. For a mathematical analysis of the quantum Boltzmann equation we refer to [8], [2], [4]. In [8], global existence and time asymptotics of isotropic solutions to a modified quantum Boltzmann equation are studied in a space-homogeneous frame, under a cut-off condition on the collision kernel. This cut-off prevents Dirac measures to form in finite time. For an initial mass bigger than the mass of the Planckian distribution function, some velocity concentration is proven to occur at infinite time. In [9], distributional isotropic solutions to the homogeneous quantum Boltzmann equation are determined in a hard sphere frame. In [2], some modelling and numerical aspects in quantum kinetic theory for a gas of interacting bosons are reviewed. In order to study the evolution of the condensates, a system is presented, coupling the Gross-Pitaevskii equation for the condensate wave function and a quantum Boltzmann equation for the non-condensate distribution function. In [4], the questions of well-posedness, i.e. existence, uniqueness, stability of solutions, and long time behaviour of the solutions are treated in some particular cases.

In this paper we consider a system of equations coupling the non-condensate and the condensate parts evolutions. This results in a quantum kinetic equation for the non-condensate distribution function, coupled to a Gross-Pitaevskii equation for the condensate wave function. In a very low temperature setting, only the coupling source terms remain in the quantum kinetic equation. Isotropic non-condensate distribution functions are considered in a space-homogeneous frame. Existence of solutions to the coupled system is proven, with bounded condensate densities and measure non-condensate distribution functions. The boundedness of mass and energy allows to give a weak sense to the collision term of the non-condensate distribution function, as the derivative of a bounded measure.

2 The model.

For the derivation of kinetic quantum models and the use of the Gross-Pitaevskii equation for the condensate wave function as well as their physical study, we refer to [6], [10], [11], [12], [13]. The observation of Bose-Einstein condensation in some atomic gases motivates a description of the evolution of the condensates that takes full account of the microscopic nature of atomic interactions in a trap, both close to and far from equilibrium. The conventional description relies on the well-known Gross-Pitaevskii equation, also known as a nonlinear Schrödinger equation. In this equation, one assumes that the atoms are all effectively condensed and the atomic interactions can be accurately modeled by a pseudopotential, expressed in terms of the s-wave scattering length. The resulting equation of motion for the condensate wave function ψ is

$$i\bar{h}\frac{\partial\psi}{\partial t} = (-\frac{\bar{h}^2}{2m}\Delta_x + V + g \mid \psi \mid^2)\psi.$$

Here $\bar{h}$ is the Planck constant, m the mass of the atoms, V an external potential, and $g = \frac{4\pi\bar{h}a_s}{m}$ is the interaction strength determined by the s-wave scattering length a_s. If the atoms are in the dilute gas, they can be studied by a kinetic quantum equation of Boltzmann type,

$$\frac{\partial F}{\partial t} + p \cdot \bigtriangledown_x F = \int B(p-p_*, p'-p)\Big(F'F'_*(1+F)(1+F_*)$$

$$-FF_*(1+F')(1+F'_*)\Big)\delta(p+p_* = p'+p'_*,\, p^2+p_*^2 = p'^2+p'^2_*)dp_*dp'dp'_*, \tag{2.1}$$

where B is a given collision kernel and $F' = F(p')$, $F'_* = F(p'_*)$, $F = F(p)$, $F_* = F(p_*)$.

After the time of condensate formation, the kinetic equation (2.1) is inappropriate, and the finite number of particles in the condensate corresponds to the infinite value of the distribution function at energy zero. In order to describe the system of particles interacting with the condensates, the simultaneous treatment of both condensate and non-condensate parts has been developed in [8]. The resulting equations of motion reduce to a generalized Gross-Pitaevskii equation for the condensate wave function, coupled with a quantum Boltzmann equation for the thermal cloud,

$$i\bar{h}\frac{\partial \psi}{\partial t}(x,t) = -\frac{\bar{h}^2}{2m}\Delta_x \psi(x,t) + V(x)\psi(x,t)$$

$$+[U_0(n_c(x,t)+2n(x,t)) - iR(x,t)]\psi(x,t), \tag{2.2}$$

$$\frac{\partial F}{\partial t} + \frac{p}{m} \cdot \bigtriangledown_x U \cdot \bigtriangledown_p F = \bar{Q}(F) + Q_c(F). \tag{2.3}$$

Here, $n_c(t,x) = \mid \psi(x,t) \mid^2$ is the condensate density and $V(x)$ the confining potential. The collision integral $\bar{Q}(F)$ is the quantum operator defined in (2.1), whereas $Q_c(F)$ describes the collisions between condensate and non-condensate particles and is given by

$$\frac{8a_s^2 n_c}{m^2} \int \delta(p_c + p_* = p' + {p'}_*, \epsilon_c + \epsilon_* = \epsilon' + \epsilon'_*)[\delta(p = p_*) - \delta(p = p') - \delta(p = {p'}_*)]$$
$$\Big(F' F'_*(1 + F_*) - F_*(1 + F')(1 + F'_*)\Big) dp_* dp' d{p'}_*. \quad (2.4)$$

Here, $\epsilon = \frac{1}{2}p^2 + U(x,t)$, where $U = V + 2U_0(n_c + n)$ is the mean field potential, and n denotes the non-condensate density

$$n(x,t) = \frac{1}{(2\pi\bar{h})^3} \int F(x,p,t)dp.$$

F_* (resp. F', F'_*) denotes $F(p_*)$ (resp. $F(p')$, $F(p'_*)$). The source term R is given by

$$R(x,t) = \frac{\bar{h}}{2n_c(2\pi\bar{h})^3} \int Q_c(F)dp.$$

In the space-homogeneous case, the system (2.2-3) becomes

$$i\bar{h}\frac{\partial\psi}{\partial t} = (V + U_0(n_c + 2n) - iR)\psi, \quad (2.5)$$
$$\frac{\partial F}{\partial t} = \bar{Q} + Q_c(F), \quad (2.6)$$

so that the condensate density n_c and the non-condensate gas density F evolutions are given by $\bar{h}n'_c = -2Rn_c$, i.e.

$$n'_c = -\frac{1}{(2\pi\bar{h})^3} \int Q_c(F)dp, \quad (2.7)$$

and equation (2.6). Solving the system (2.5-6) comes back to solve (2.6-7) first, where the unknowns are F and n_c, then easily compute the wave function ψ from (2.5). Therefore, we aim at solving (2.6-7). Notice that this system can also be formally obtained by starting from the quantum kinetic equation (2.1) with collision kernel identically equal to one for the total - condensate and non-condensate - gas distribution function f,

$$\frac{\partial f}{\partial t} = \int \delta(p + p_* = p' + p'_*, p^2 + p_*^2 = p'^2 + p_*'^2)$$
$$\Big(f'f'_*(1 + f + f_*) - ff_*(1 + f' + f'_*)\Big) dp_* dp' dp'_*,$$

then splitting f into its condensate part $n_c(t)\delta_{p=p_c}$ and its non-condensate part F, ([10], [12])

$$f(t, p) = n_c(t)\delta_{p=p_c} + F(t, p).$$

It means that n_c and F should respectively satisfy

$$n_c'(t) = n_c(t) \int \delta(p_c + p_* = p' + p'_*, p_c^2 + p_*^2 = p'^2 + p'^2_*)(F'F'_*$$
$$-F_*(1 + F' + F'_*))dp_* dp' dp'_* + n_c^2 B_1 + n_c^3 C_1,$$
$$\frac{\partial F}{\partial t} = \int \delta(p + p_* = p' + p'_*, p^2 + p_*^2 = p'^2 + p'^2_*)\Big(F'F'_*(1 + F + F_*)$$
$$-FF_*(1 + F' + F'_*)\Big) dp_* dp' dp'_* + n_c A + n_c^2 B_2 + n_c^3 C_2,$$

where

$$B_1 = \int \delta(p_c + p_* = p' + p'_*, p_c^2 + p_*^2 = p'^2 + p'^2_*)\Big(F'\delta(p'_* = p_c) + F'_*\delta(p' = p_c)$$
$$-F_*\delta(p' = p_c) - F_*\delta(p'_* = p_c) - (1 + F' + F'_*)\delta(p_* = p_c)\Big) dp_* dp' dp'_*,$$
$$C_1 = \int \delta(p_c + p_* = p' + p'_*, p_c^2 + p_*^2 = p'^2 + p'^2_*)\Big(\delta(p' = p'_* = p_c)$$
$$+\delta(p_* = p' = p'_* = p_c) - \delta(p_* = p' = p_c) - \delta(p_* = p'_* = p_c)\Big) dp_* dp' dp'_*,$$

$$A=\int \delta(p+p_*=p'+{p'}_*,p^2+p_*^2=p'^2+{p'}_*^2)\Big(F'F'_*\delta(p_*=p_c)$$

$$+F'(1+F+F_*)\delta({p'}_*=p_c)+F'_*(1+F+F_*)\delta(p'=p_c)-FF_*\delta(p'=p_c)$$

$$-FF_*\delta({p'}_*=p_c)-F(1+F'+F'_*)\delta(p_*=p_c)\Big)dp_*dp'd{p'}_*,$$

$$B_2=\int \delta(p+p_*=p'+{p'}_*,p^2+p_*^2=p'^2+{p'}_*^2)\Big(F'\delta({p'}_*=p_*=p_c)$$

$$+F'_*\delta(p'=p_*=p_c)-F\delta(p_*=p'=p_c)-F\delta(p_*={p'}_*=p_c)$$

$$+(1+F+F_*)\delta(p'={p'}_*=p_c)\Big)dp_*dp'd{p'}_*,$$

$$C_2=\int \delta(p+p_*=p'+{p'}_*,p^2+p_*^2=p'^2+{p'}_*^2)\delta(p'={p'}_*=p_*=p_c)dp_*dp'd{p'}_*.$$

Moreover, the four first terms in B_1 cancel each other. The set defined by $2p_c=p'+{p'}_*, 2p_c^2=p'^2+p_*'^2$ reduces to $p=p_*=p_c$, so that the integration on it of the measure F which support does not contain p_c is zero. And so, $B_1=0$. The term C_1, equal to

$$\int \delta(p_c+p_*=p'+p'_*,p_c^2+p_*^2=p'^2+p_*'^2)(2\delta(p_*=p_c)$$

$$-\delta({p'}_*=p_c)-\delta(p'=p_c))dp_*dp'd{p'}_*,$$

vanishes. The term A can also be written as

$$\int \delta(p_c+p_*=p'+p'_*,p_c^2+p_*^2=p'^2+p_*'^2)\Big(\delta(p=p_*)-\delta(p=p')-\delta(p={p'}_*))$$

$$(F'F'_*-F_*(1+F'+F'_*)\Big)dp_*dp'd{p'}_*.$$

It follows from the same arguments as for B_1 that $B_2=0$. And so, the system (2.6-7) is recovered for n_c and F.

In a way similar to the procedure used by Lee and Yang ([7]) for the equilibrium properties of a condensed Bose gas, two regions can be distinguished, namely

- a moderately low temperature region,

and

- a very low temperature region.

In this paper, we restrict to the second region of very low temperature. Moreover, if the number of particles in the condensate is sufficiently large, the interactions with the condensate will dominate the dynamics of the system, so that $\bar{Q}$ is negligible compared to Q_c ([5]). If we finally consider a space-homogeneous frame and isotropic distribution functions, and denote by ϵ and $F(t,\epsilon)$, respectively $\frac{1}{2}p^2$ and the distribution function of the dilute gas, the collision operator Q_c writes $Q_c(F) = n_c(X - 2Y)$, with

$$X = \int \delta(p_* = p' + p'_*)\delta(p_*^2 = p'^2 + p'^2_*)\delta(p = p_*)\Big(F'F'_* - F_*(1 + F' + F'_*)\Big)dp_* dp' dp'_*,$$

$$Y = \int \delta(p_* = p' + p'_*, p_*^2 = p'^2 + p'^2_*)\delta(p = p'_*)\Big(F'F'_* - F_*(1 + F' + F'_*)\Big)dp_* dp' dp'_*.$$

Then, if $\varphi_0 = Arcos\sqrt{\frac{\epsilon'}{\epsilon}}$,

$$X = \int_0^\epsilon \sqrt{2\epsilon'} \int_0^{2\pi} \int_0^\pi sin\varphi' \, \delta\Big(- 4\sqrt{\epsilon\epsilon'}cos\varphi' + 4\epsilon' = 0\Big)$$

$$\Big(F'F(\epsilon-\epsilon')-F(1+F'+F(\epsilon-\epsilon')\Big)d\varphi' d\theta' d\epsilon'$$

$$=2\pi\int_0^\epsilon\sqrt{2\epsilon'}\Big(\int_0^\pi sin\varphi'\,\delta(4\sqrt{\epsilon\epsilon'}(sin\varphi_0)(\varphi'-\varphi_0)=0)d\varphi'\Big)$$

$$\Big(F(\epsilon')F(\epsilon-\epsilon')-F(\epsilon)(1+F(\epsilon')+F(\epsilon-\epsilon'))\Big)d\epsilon'$$

$$=\frac{\pi}{\sqrt{2\epsilon}}\int_0^\epsilon\Big(F(\epsilon')F(\epsilon-\epsilon')-F(\epsilon)(1+F(\epsilon')+F(\epsilon-\epsilon'))\Big)d\epsilon'.$$

Moreover,

$$Y=2\pi\int\sqrt{2\epsilon'}\Big(\int_0^\pi sin\varphi'\,\delta(4\sqrt{\epsilon\epsilon'}cos\varphi'=0)d\varphi'\Big)\Big(F'F-F(\epsilon+\epsilon')(1+F'+F)\Big)d\epsilon'$$

$$=2\pi\int\sqrt{2\epsilon'}\int_0^\pi sin\varphi'\,\delta\Big(4\sqrt{\epsilon\epsilon'}(\varphi'-\frac{\pi}{2})=0\Big)\Big(F'F-F(\epsilon+\epsilon')(1+F'+F)\Big)d\varphi' d\epsilon'$$

$$=\frac{\pi}{\sqrt{2\epsilon}}\int\Big(F'F-F(\epsilon+\epsilon')(1+F(\epsilon')+F(\epsilon)\Big)d\epsilon'.$$

Forgetting the constant $\frac{\pi}{\sqrt{2}}$ for the sake of clarity, $Q_c(F)=\frac{n_c}{\sqrt{\epsilon}}Q(F)$, with

$$Q(F)(t,\epsilon)=\int_0^\epsilon\Big(F(\epsilon')F(\epsilon-\epsilon')-4F(\epsilon)F(\epsilon')\Big)d\epsilon'-2F(\epsilon)\int_\epsilon^{+\infty}F(\epsilon')d\epsilon'$$
$$+2\int F(\epsilon+\epsilon')(F(\epsilon')+F(\epsilon))d\epsilon'-\epsilon F(\epsilon)+2\int_\epsilon^{+\infty}F(\epsilon')d\epsilon'.$$

And so, the system to be studied is

$$n_c'(t)=-n_c(t)\int Q(F)d\epsilon, \tag{2.8}$$

$$\frac{\partial}{\partial t}(\sqrt{\epsilon}F)=n_cQ(F),\quad F(0,\epsilon)=F_i(\epsilon), \tag{2.9}$$

with the initial data $n_c(0)$ and F_i given. The total mass and energy are assumed to be bounded, i.e.

$$n_c(0)+\int\sqrt{\epsilon}F_i(\epsilon)d\epsilon<+\infty,\quad \int\epsilon^{\frac{3}{2}}F_i(\epsilon)d\epsilon<+\infty. \tag{2.10}$$

3 A priori estimates.

Lemma 3.1 *For any function* φ,

$$\int Q(F)(\epsilon)\varphi(\epsilon)d\epsilon = 2\int F(\epsilon)\int_0^\epsilon F(\epsilon')\Big(\varphi(\epsilon+\epsilon')+\varphi(\epsilon-\epsilon')-2\varphi(\epsilon)\Big)d\epsilon' d\epsilon$$
$$+\int F(\epsilon)\Big(2\int_0^\epsilon \varphi(\epsilon')d\epsilon' - \epsilon\varphi(\epsilon)\Big)d\epsilon \quad (3.1)$$

Proof of Lemma 3.1. For any function φ defined on $I\!R_+$,

$$\int \varphi(\epsilon)\int_0^\epsilon F(\epsilon')F(\epsilon-\epsilon')d\epsilon' d\epsilon = \int F(\epsilon)F(\epsilon')\varphi(\epsilon+\epsilon')d\epsilon d\epsilon'$$
$$= 2\int F(\epsilon)\Big(\int_0^\epsilon F(\epsilon')\varphi(\epsilon+\epsilon')d\epsilon'\Big)d\epsilon,$$
$$\int \varphi(\epsilon)F(\epsilon+\epsilon')(F(\epsilon')+F(\epsilon))d\epsilon' d\epsilon = \int F(\epsilon)\int_0^\epsilon F(\epsilon')(\varphi(\epsilon-\epsilon')+\varphi(\epsilon'))d\epsilon' d\epsilon.$$

And so,

$$\int Q(F)(\epsilon)\varphi(\epsilon)d\epsilon = 2\int F(\epsilon)\int_0^\epsilon F(\epsilon')(\varphi(\epsilon+\epsilon')+\varphi(\epsilon-\epsilon')-2\varphi(\epsilon))d\epsilon' d\epsilon$$
$$+\int F(\epsilon)\Big(2\int_0^\epsilon \varphi(\epsilon')d\epsilon' - \varphi(\epsilon)\Big)d\epsilon.$$

Lemma 3.2

$$n_c(t) + \int \sqrt{\epsilon}F(t,\epsilon)d\epsilon = n_c(0) + \int \sqrt{\epsilon}F_i(\epsilon)d\epsilon, \quad (3.2)$$
$$\int \epsilon^{\frac{3}{2}}F(t,\epsilon)d\epsilon = \int \epsilon^{\frac{3}{2}}F_i(\epsilon)d\epsilon, \quad a.a.t \in [0,T]. \quad (3.3)$$

Proof of Lemma 3.2. (3.2) follows from adding (2.8) integrated from 0 to t and (2.9) integrated on $(0,t)\times I\!R_+$. (3.3) follows from Lemma 3.1 with $\varphi(\epsilon)=\epsilon$.

Lemma 3.3 *Under (2.10), the bilinear part of $Q(F)$,*

$$\int_0^\epsilon \Big(F(\epsilon')F(\epsilon-\epsilon') - 4F(\epsilon)F(\epsilon')\Big)d\epsilon' - 2F(\epsilon)\int_\epsilon^{+\infty} F(\epsilon')d\epsilon'$$
$$+2\int F(\epsilon+\epsilon')(F(\epsilon')+F(\epsilon))d\epsilon',$$

is the derivative of a bounded measure.

<u>Proof of Lemma 3.3.</u> By Lemma 3.1 and (2.10), for any function $\varphi \in C^1(I\!R_+)$ such that φ and φ' are bounded,

$$\mid \int Q(F)(\epsilon)\varphi(\epsilon)d\epsilon \mid$$
$$\leq 2\int \sqrt{\epsilon}F(\epsilon)\int_0^\epsilon \sqrt{\epsilon'}F(\epsilon') \mid \sqrt{\frac{\epsilon'}{\epsilon}}\int_0^1 \Big(\varphi'(\epsilon+\lambda\epsilon') - \varphi'(\epsilon-\lambda\epsilon')\Big)d\lambda \mid d\epsilon' d\epsilon$$
$$\leq 4 \mid \varphi' \mid_\infty \Big(\int \sqrt{\epsilon}F(\epsilon)d\epsilon\Big)^2$$
$$\leq c \mid \varphi' \mid_\infty .$$

It follows from Lemmas 3.1 and 3.3 that weak solutions of the Cauchy problem (2.8-9) can be defined.

Definition 3.1 *A weak solution to the Cauchy problem (2.8-9) on the interval of time $[0,T]$ is $(n_c,F) \in C^1([0,T]) \times L^\infty(0,T,M_{\sqrt{\epsilon}}(I\!R_+))$ such that for any function $\varphi \in C^1(I\!R_+)$ such that φ and φ' are bounded,*

$$n_c(t) = n_c(0)e^{-\int_0^t \int \epsilon F(s,\epsilon)d\epsilon ds},$$

$$\int \sqrt{\epsilon}F(t,\epsilon)\varphi(\epsilon)d\epsilon - \int \sqrt{\epsilon}F_i(\epsilon)\varphi(\epsilon)d\epsilon = n_c(0)\int_0^t e^{-\int_0^s \int \epsilon F(\tau,\epsilon)d\epsilon d\tau}$$
$$\Big(\int \sqrt{\epsilon}F(s,\epsilon)\int_0^\epsilon \sqrt{\epsilon'}F(s,\epsilon')\sqrt{\frac{\epsilon'}{\epsilon}}\int_0^1 (\varphi'(\epsilon+\lambda\epsilon') - \varphi'(\epsilon-\lambda\epsilon'))d\lambda d\epsilon' d\epsilon$$
$$+\int F(s,\epsilon)(2\int_0^\epsilon \varphi(\epsilon')d\epsilon' - \epsilon\varphi(\epsilon))d\epsilon\Big)ds.$$

4 The existence theorem for the Cauchy problem.

Theorem 4.1 *Under assumption (2.10) of bounded initial mass and energy, there exists a weak solution* $(n_c, F) \in C^1([0,T]) \times L^\infty(0,T; M_{\sqrt{\epsilon}}(I\!\!R^+))$ *to the Cauchy problem (2.8-9) in the sense of Definition 3.1.*

The proof of Theorem 4.1 splits into two parts. An approximation procedure first leads to a sequence (F^j), solution to a Cauchy problem with approximated collision operators behaving smoothly close to the energy zero. Then the passage to the limit in the equation satisfied by F^j when $j \to +\infty$ provides a weak solution to the Cauchy problem (2.8-9).

Lemma 4.4 *For any* $j \in I\!\!N^*$, *there is a unique solution* $(n_c^j, F^j) \in C^1([0,T]) \times C^1([0,T], L^1_{\sqrt{\epsilon}}(I\!\!R_+))$ *to*

$$n_c^{j\prime} = -n_c^j \int Q_j(F^j) d\epsilon, \quad n_c^j(0) = n_c(0), \tag{4.1}$$

$$\sqrt{\epsilon}\frac{\partial F^j}{\partial t} = n_c^j Q_j(F^j), \quad F^j(0,\epsilon) = F_i(\epsilon), \tag{4.2}$$

where

$$Q_j(F)(\epsilon) = \int_{\frac{1}{j}}^{\epsilon - \frac{1}{j}} F' F(\epsilon - \epsilon') d\epsilon' - 4\chi_{\epsilon > \frac{1}{j}} F(\epsilon) \int_{\frac{1}{j}}^{\epsilon} F' d\epsilon'$$
$$-2\chi_{\epsilon > \frac{1}{j}} F(\epsilon) \int_{\epsilon}^{+\infty} F' d\epsilon' + 2\int_{\frac{1}{j}}^{+\infty} F' F(\epsilon + \epsilon') d\epsilon'$$
$$+2\chi_{\epsilon > \frac{1}{j}} F(\epsilon) \int F(\epsilon + \epsilon') d\epsilon' - \chi_{\epsilon < j} \epsilon F(\epsilon) + 2\chi_{\frac{1}{j} < \epsilon < j} \int_{\epsilon}^{+\infty} F' d\epsilon'.$$

<u>Proof of Lemma 4.1</u> Denote by $c_1 = n_c(0) + \int \sqrt{\epsilon} F_i(\epsilon) d\epsilon$. Starting from a nonnegative function $f(t,\epsilon)$ such that $\int \sqrt{\epsilon} f(t,\epsilon) d\epsilon \le c_1, \quad t \ge 0$, there are

functions $(N(t), F(t,\epsilon))$ solutions to

$$N' = -N \int \tilde{Q}_j(f,F)d\epsilon, \quad N(0) = n_c(0), \tag{4.3}$$

$$\sqrt{\epsilon}\frac{\partial F}{\partial t} = N\tilde{Q}_j(f,F), \quad F(0,\epsilon) = F_i(\epsilon), \tag{4.4}$$

where $\tilde{Q}_j(f,F)$ is defined by

$$\tilde{Q}_j(f,F)(t,\epsilon) = \int_{\frac{1}{j}}^{\epsilon-\frac{1}{j}} f'f(\epsilon-\epsilon')d\epsilon' - 4\chi_{\epsilon>\frac{1}{j}}F(\epsilon)\int_{\frac{1}{j}}^{\epsilon} f'd\epsilon'$$
$$-2\chi_{\epsilon>\frac{1}{j}}F(\epsilon)\int_{\epsilon}^{+\infty} f'd\epsilon' + 2\int_{\frac{1}{j}}^{+\infty} f'f(\epsilon+\epsilon')d\epsilon' + 2\chi_{\epsilon>\frac{1}{j}}f(\epsilon)\int f(\epsilon+\epsilon')d\epsilon'$$
$$-\chi_{\epsilon<j}\epsilon F(\epsilon) + 2\chi_{\frac{1}{j}<\epsilon<j}\int_{\epsilon}^{+\infty} f'd\epsilon'.$$

Indeed, consider the sequence (F^j) defined by $F^0 = 0$, and

$$\sqrt{\epsilon}\frac{\partial F^{j+1}}{\partial t} = N^j\tilde{Q}_j(f,F^{j+1}), \quad F^{j+1}(0,\epsilon) = F_i(\epsilon),$$

where N^j is the solution to

$$N^{j\prime} = -N^j \int \tilde{Q}_j(f,F^j)d\epsilon, \quad N^j(0) = n_c(0).$$

From N^j and F^{j+1} written in exponential form, it follows that $N^j \geq 0$ and $F^{j+1} \geq 0$. Then a contraction argument is used in $C^0([0,T^*], L^1_{\sqrt{\epsilon}})$ for T^* small enough, to prove that (F^j) converges. The time T^* is chosen so that uniformly in j,

$$\int \sqrt{\epsilon}F^j(t,\epsilon)d\epsilon \leq 2c_1, \quad t \in [0,T^*].$$

It can be done in the following way. Since

$$N^j(t) = n_c(0)e^{-\int_0^t \int \tilde{Q}_j(f,F^j)d\epsilon ds},$$

$$\mid \int \tilde{Q}_j(f,F^j)d\epsilon \mid \leq 5j\Big(\int \sqrt{\epsilon} f d\epsilon\Big)^2 + 4j\Big(\int \sqrt{\epsilon} f d\epsilon\Big)\Big(\int \sqrt{\epsilon} F d\epsilon\Big)$$
$$+\sqrt{j}\int \sqrt{\epsilon} F d\epsilon + 2j^{\frac{3}{2}} \int \sqrt{\epsilon} f d\epsilon,$$

and

$$\int \sqrt{\epsilon} f d\epsilon \leq c_1, \quad \int \sqrt{\epsilon} F^j d\epsilon \leq 2c_1,$$

it holds that

$$N^j(t) \leq n_c(0)e^{20c_1^2 j^2 T^*}, \quad t \in [0,T^*], \quad j \in I\!N^*,$$

and

$$\int \sqrt{\epsilon} F^{j+1}(t,\epsilon)d\epsilon \leq \int \sqrt{\epsilon} F_i(\epsilon)d\epsilon + 20c_1^2 j^2 n_c(0)T^* e^{20c_1^2 j^2 T^*}$$
$$\leq c_1 + 20c_1^2 j^2 n_c(0)T^* e^{20c_1^2 j^2 T^*} \leq 2c_1,$$

for T^* small enough. Let us prove that

$$\sup_{t\in[0,T^*]} \int \sqrt{\epsilon} \mid (F^{j+2} - F^{j+1})(t,\epsilon) \mid d\epsilon \leq k \sup_{t\in[0,T^*]} \int \sqrt{\epsilon} \mid (F^{j+1} - F^j)(t,\epsilon) \mid d\epsilon,$$

for some $k < 1$, uniformly with respect to j.

First, writing $\tilde{Q}_j(f,F)$ as $\alpha(f) - F\nu(f)$, the difference $\sqrt{\epsilon}\frac{\partial}{\partial t}(F^{j+2} - F^{j+1})$ can be split into

$$\sqrt{\epsilon}\frac{\partial}{\partial t}(F^{j+2} - F^{j+1}) = N^{j+1}(\alpha(f) - F^{j+2}\nu(f)) - N^j(\alpha(f) - F^{j+1}\nu(f))$$
$$= \alpha(f)(N^{j+1} - N^j) - N^{j+1}(F^{j+2} - F^{j+1})\nu(f) + (N^{j+1} - N^j)F^{j+1}\nu(f).$$

Then,

$$| (N^{j+1} - N^j)(t) | \leq n_c(0) e^{-\int_0^t \int \alpha(f) d\epsilon ds} \times$$

$$\times | e^{\int_0^t \int \nu(f) F^{j+1}(s,\epsilon) d\epsilon ds} - e^{\int_0^t \int \nu(f) F^j(s,\epsilon) d\epsilon ds} |$$

$$\leq n_c(0) | \int_0^t \int \nu(f)(F^{j+1} - F^j)(s,\epsilon) d\epsilon ds | e^{cT^*}$$

$$\leq c \int_0^t \int \sqrt{\epsilon} | (F^{j+1} - F^j)(s,\epsilon) | d\epsilon ds.$$

Consequently,

$$\frac{\partial}{\partial t} \int \sqrt{\epsilon} | (F^{j+2} - F^{j+1})(t,\epsilon) | d\epsilon \leq c \int_0^t \int \sqrt{\epsilon} | (F^{j+1} - F^j)(s,\epsilon) d\epsilon ds$$

$$+ c \int \sqrt{\epsilon} | (F^{j+2} - F^{j+1})(t,\epsilon) d\epsilon.$$

Hence,

$$\int \sqrt{\epsilon} | (F^{j+2} - F^{j+1})(t,\epsilon) | d\epsilon \leq c \int_0^t \int \sqrt{\epsilon} | (F^{j+1} - F^j)(s,\epsilon) d\epsilon ds.$$

And so,

$$\sup_{t \in [0,T^*]} \int \sqrt{\epsilon} | (F^{j+2} - F^{j+1})(t,\epsilon) | d\epsilon \leq \tilde{c} T^* \sup_{t \in [0,T^*]} \int | (F^{j+1} - F^j)(t,\epsilon) | d\epsilon.$$

It is sufficient to choose $T^* < \frac{1}{2\tilde{c}}$ to end the contraction argument. And so, there are (N, F) solutions to (4.3-4) on $[0, T^*]$. But adding equations (4.3) and (4.4) implies that

$$N(T^*) + \int \sqrt{\epsilon} F(T^*, \epsilon) d\epsilon = N(0) + \int \sqrt{\epsilon} F(0, \epsilon) d\epsilon \leq c_1.$$

This means that the whole argument for defining (N, F) solution to (4.3-4) on $[0, T^*]$ also holds on $[T^*, 2T^*]$, ...finally on the whole interval $[0, T]$.

Consider the map $\mathcal{T}$ that maps $(n, f) \in C^0([0,T]) \times C^0([0,T], L^1(\mathbb{R}^+))$ such that

$$n_c(t) + \int \sqrt{\epsilon} f(t,\epsilon) d\epsilon \leq c_1, \quad t \in \mathbb{R}^+,$$

into (N, F) solution to (4.3-4). It follows from the expressions of $N(t)$ and $F(t, \epsilon)$ written in exponential form that they stay nonnegative like their initial data. Then,

$$N(t) + \int \sqrt{\epsilon} F(t,\epsilon) d\epsilon \leq c_1. \tag{4.5}$$

For $\tilde{T}$ small enough, $\mathcal{T}$ is a contraction in $C^0([0,\tilde{T}]) \times C^0([0,\tilde{T}], L^1_{\sqrt{\epsilon}})$. Indeed, consider (n_1, f_1) and (n_2, f_2) such that (4.5) holds, and

$(N_1, F_1) = \mathcal{T}(n_1, f_1)$, $(N_2, F_2) = \mathcal{T}(n_2, f_2)$. Then,

$$\begin{aligned}\frac{\partial}{\partial t} \int \sqrt{\epsilon} \mid F_1 - F_2 \mid d\epsilon \leq N_1 \int \mid \tilde{Q}_j(f_1, F_1) - \tilde{Q}_j(f_2, F_2) \mid d\epsilon \\ + \mid N_1 - N_2 \mid \int \mid \tilde{Q}_j(f_2, F_2) \mid d\epsilon \\ \leq c_2 N_1 \Big(\int \sqrt{\epsilon} \mid F_1 - F_2 \mid d\epsilon + \int \sqrt{\epsilon} \mid f_1 - f_2 \mid d\epsilon \Big) + c_3 \mid N_1 - N_2 \mid .\end{aligned}$$

Here, and in the following, c_i, $i \geq 2$, denote constants depending on c_1 and j. Moreover,

$$N_i(t) = n_c(0) e^{-\int_0^t \int \tilde{Q}_j(f_i, F_i) d\epsilon ds}, \quad 1 \leq i \leq 2,$$

so that

$$\begin{aligned} N_1(t) \leq n_c(0) e^{c_2 \tilde{T}}, \\ \mid N_1(t) - N_2(t) \mid \leq c_3 e^{c_4 \tilde{T}} \int_0^t \int \sqrt{\epsilon} \mid (F_1 - F_2)(s, \epsilon) \mid d\epsilon ds, \quad t < \tilde{T}.\end{aligned}$$

And so,

$$\frac{\partial}{\partial t}\int\sqrt{\epsilon}\mid(F_1-F_2)(s,\epsilon)\mid\leq c_5\int\sqrt{\epsilon}\mid(F_1-F_2)(t,\epsilon)d\epsilon$$

$$+c_6e^{c_7\tilde{T}}\int_0^t\int\sqrt{\epsilon}\mid(F_1-F_2)(s,\epsilon)\mid d\epsilon+c_8\int\sqrt{\epsilon}\mid(f_1-f_2)(t,\epsilon)\mid d\epsilon,\quad t<\tilde{T}.$$

Hence, $x(t):=\int_0^t\int\sqrt{\epsilon}\mid(F_1-F_2)(s,\epsilon)d\epsilon ds$ satisfies a second-order linear differential equation with a source term $h(t)\leq c_8\int\sqrt{\epsilon}\mid(f_1-f_2)(t,\epsilon)\mid d\epsilon$.

Hence,

$$x(t)=\frac{1}{c_9-c_{10}}\int_0^t h(s)(e^{c_9(t-s)}-e^{c_{10}(t-s)})ds.$$

Consequently,

$$\int\sqrt{\epsilon}\mid(F_1-F_2)(t,\epsilon)\mid d\epsilon\leq\frac{c_8}{\mid c_9-c_{10}\mid}\Big(c_9(e^{c_9t}-1)+c_{10}(e^{c_{10}t}-1)\Big)$$
$$\times\sup_{s\leq\tilde{T}}\int\sqrt{\epsilon}\mid(f_1-f_2)(s,\epsilon)\mid d\epsilon,\quad t\leq\tilde{T}.$$

Hence,

$$\sup_{t\leq\tilde{T}}\int\sqrt{\epsilon}\mid(F_1-F_2)(t,\epsilon)\mid d\epsilon\leq c_{11}(e^{c_{12}\tilde{T}}-1)\sup_{t\leq\tilde{T}}\int\sqrt{\epsilon}\mid(f_1-f_2)(t,\epsilon)\mid d\epsilon,$$
$$\sup_{t\leq\tilde{T}}\mid(N_1-N_2)(t)\mid\leq c_{13}\tilde{T}e^{c_{14}\tilde{T}}(e^{c_{12}\tilde{T}}-1)\sup_{t\leq\tilde{T}}\int\sqrt{\epsilon}\mid(f_1-f_2)(t,\epsilon)\mid d\epsilon.$$

And so, a Banach fixed point argument can be applied to $\mathcal{T}$ in $C^0([0,\tilde{T}])\times C^0([0,\tilde{T}],L^1_{\sqrt{\epsilon}})$ for $\tilde{T}$ small enough. It follows from (4.5), holding on $[0,T]$, that the previous procedure can be applied on $[\tilde{T},2\tilde{T}]$, ..., up to T. By (4.1-2), (n^j,F^j) belongs to $C^1([0,T])\times C^1([0,T],L^1_{\sqrt{\epsilon}}(I\!R_+))$.

End of the proof of Theorem 4.1

It remains to pass to the limit in (4.1-2) when j tends to $+\infty$. Let a subsequence of $(\sqrt{\epsilon}F^j)$, still denoted by $(\sqrt{\epsilon}F^j)$, weakly-* converging to some bounded measure $\sqrt{\epsilon}F$ in M. Integrate (4.2) multiplied by φ on $(0,t)$, so that

$$\int \sqrt{\epsilon}F^j(t,\epsilon)\varphi(\epsilon)d\epsilon - \int \sqrt{\epsilon}F_i(\epsilon)\varphi(\epsilon)d\epsilon$$

$$= \int_0^t n_c^j(s) \int_{\frac{1}{j}}^{+\infty} \sqrt{\epsilon}F^j(s,\epsilon) \int_{\frac{1}{j}}^{\epsilon} \sqrt{\epsilon'}F^j(s,\epsilon')\sqrt{\frac{\epsilon'}{\epsilon}} \int_0^1 (\varphi'(\epsilon+\lambda\epsilon') - \varphi'(\epsilon-\lambda\epsilon'))d\lambda d\epsilon' d\epsilon ds$$

$$+ \int_0^t n_c^j(s)\Big(2\int_{\frac{1}{j}}^{+\infty} (\int_{\frac{1}{j}}^{min(\epsilon,j)} \varphi(\epsilon')d\epsilon')F(s,\epsilon)d\epsilon - \int_0^j \epsilon F(s,\epsilon)\varphi(\epsilon)d\epsilon\Big)ds,$$

where

$$n_c^j(s) = n_c(0)e^{-\int_0^s \int \epsilon F^j(\tau,\epsilon)d\epsilon d\tau}.$$

In this weak formulation, it is possible to pass to the limit when $j \to +\infty$.

BY A DIAGONAL PROCESS???

The existence of a solution $(n,F) \in C^1([0,T]) \times L^\infty(0,T,M(I\!R_+))$ to the Cauchy problem (2.8-9) is therefore proven.

5 Conclusion.

In this paper, we have proven the existence of a global solution to an homogeneous quantum coupled system describing the evolution of a gas at very low temperature, for an initial datum with finite mass and energy.

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