A problem related to microwaves propagating in the atmosphere.

A. Nouri *

Abstract. A model is given for describing the formation of small plasmas while a microwave field propagates in the atmosphere. An existence theorem is derived.

Introduction. When a microwave pulse propagates in the atmosphere, it can meet free electrons, being there because of photodetachment for instance [7]. The equations of electrodynamics are linear with respect to the electric and magnetic fields E, H and the velocity v_e of the electron. For this reason, the superposition principle holds. Any periodical field can be resolved into harmonic components, so that it is sufficient to consider only sinusoidal fields, all the more so because one deals with monochromatic fields and waves. Depending on the physical situation we are trying to model, where the dimensions of the plasmoid are small with respect to the wave length and the skin depth [1], it is relevant to simplify the Maxwell equations, reducing them to

$$curl E = 0, \quad div(\epsilon_0 E) = \rho = e(n_p - n_e), \tag{0.1}$$

where ϵ_0, ρ, n_p and n_e respectively denote the permittivity of the medium, the density, the ion and the electron densities. Hence, E derives from a potential

$$E = - \bigtriangledown V. \tag{0.2}$$

Moreover, in the atmosphere, the electrons are few and far from each other. Hence, at the wave passage, they generate electron avalanches, that are not

*Laboratoire J. A. Dieudonné, URA 168 du CNRS, UNSA, Parc Valrose 06108 Nice Cédex 02 France.

extending much spatially. Denote l the plasma size, assumed small compared to the wave lenghth L of the field. An electric field of the following type

$$E(t,x) = E_0(t,x)e^{i(\frac{t}{\epsilon}+\beta(\epsilon x))},$$

is looked for, where ϵ is a small parameter. We let B be a circular cylinder, with radius r and axis in the x-direction. Moreover, the electron and ion motions are described by the transport equations

$$\frac{\partial n_e}{\partial t} + div\varphi_e = S, \quad \frac{\partial n_p}{\partial t} + div\varphi_p = S, \tag{0.3}$$

where φ_e and φ_p are the electron and ion fluxes, and S the production rate of charged particles. Combining (0.1)-(0.3) leads to

$$div(\epsilon_0 \frac{\partial E}{\partial t} + e(\varphi_e - \varphi_p)) = 0.$$

The ion current density $e\varphi_p$ is negligible with respect to the electron current density, because the ion mobility is much smaller than the electron one. Hence it remains

$$div(\epsilon_0 \frac{\partial E}{\partial t} - en_e v_e) = 0, \tag{0.4}$$

where v_e is the mean electron speed, obtained from

$$\frac{\partial v_e}{\partial t} = -\frac{e}{m}E - \alpha v_e, \tag{0.5}$$

and α is the electron-molecule collision frequency. To close the previous set of equations, it remains to describe the electron density growth by the transport equation

$$\frac{\partial n_e}{\partial t} - div(D \bigtriangledown n_e) - \nu(\mid E \mid)n_e + rn_e n_p = 0.$$

D is the electron diffusion coefficient, ν is the ionisation frequency and r is the ion-electron recombination coefficient. Quasi-neutrality is assumed, i.e. $n_e - n_p << n_e$, so that the last equation is replaced by

$$\frac{\partial n_e}{\partial t} - div(D \bigtriangledown n_e) - \nu(\mid E \mid)n_e + rn_e^2 = 0. \tag{0.6}$$

This paper focuses on the ionisation phenomenon, keeping aside the photo ionisation of the gas that occurs once the plasma has arised. Indeed, there is not a large amount of electrons created by photo ionisation. Rather, the

main point with photo ionisation is that electrons can be created outside the plasma, because of the large speed of propagation of photons, which accelerates the plasma propagation.
As long as there is no plasma, the diffusion coefficient D equals the diffusion for the free electrons D_e. But as soon as a plasma develops, $D = D_a$, where D_a is the ambipolar diffusion coefficient, smaller than D_e, which takes into account that ions prevent electrons from diffusing quickly. D_a is defined by

$$D_a = \frac{\mu_e D_p + \mu_p D_e}{\mu_e + \mu_p},$$

where μ_e, and μ_p are the electron and ion mobilities and D_p is the ion diffusion coefficient. Since μ_p is much smaller than μ_e and the electron temperature $T_e = \frac{e}{k}\frac{D_e}{\mu_e}$ is much bigger than the ion temperature $T_p = \frac{e}{k}\frac{D_p}{\mu_p}$, where k is the Boltzmann constant, it follows that D_a can be approximated by

$$D_a = \frac{\mu_p}{\mu_e} D_e.$$

For estimating the transition from free diffusion to ambipolar diffusion, the parameter $\xi = \frac{\lambda_D}{L}$ is used, where λ_D is the plasma Debye length defined by

$$\lambda_D = \sqrt{\frac{\epsilon_0 k T_e}{e^2 n_e}},$$

and L is the largest size of the plasma. If $\xi >> 1$, the diffusion is free, whereas if $\xi << 1$, it is ambipolar. Classically, the diffusion coefficient D is defined by

$$D = D_a \frac{1 + \xi^2}{1 + \frac{D_a}{D_e}\xi^2}. \tag{0.7}$$

Since quasi-harmonic solutions $E_0 e^{i(\frac{t}{\epsilon} + \beta(\epsilon x))}$ are investigated for the electric field, (0.4-5) become

$$div(\epsilon_r E_0 e^{i\beta}) = 0, \tag{0.8}$$

where

$$\epsilon_r = 1 - i\frac{e^2}{m\epsilon_0\omega(\alpha + i\omega)} n_e.$$

Here we restrict to frequencies ω such that $\omega << \alpha$, so that

$$\epsilon_r = 1 - i\frac{n_e}{n_c}\frac{\omega}{\alpha},$$

where $n_c = \frac{m\epsilon_0\omega^2}{e^2}$. Taking the modulus of (0.8) leads to

$$div[\sqrt{1 + (\frac{n_e}{n_c}\frac{\omega}{\alpha})^2}E_0] = 0. \tag{0.9}$$

The initial condition is

$$n(t=0,x) = n_0(x),$$

whereas the boundary condition expresses that ouside of the given circular cylinder B, and far enough from the small plasma inside B, the electric field equals the given microwave field

$$E_0(t,x,y,z) = E_\infty, \quad y^2 + z^2 \geq r^2, \tag{0.10}$$

$$\lim_{(x,y,z)\in B; |x|\to\infty} E_0 = E_\infty. \tag{0.11}$$

Since we neglect the spatial variations of φ in the present discussion, E_0 can be derived from a potential. With the help of (0.2), define ψ by

$$\bigtriangledown\psi = E_0 - E_\infty.$$

For the sake of simplicity, the constants $\frac{\mu_p}{\mu_e}D_e$, $(\frac{\omega}{n_c\alpha})^2$, $\frac{\epsilon_0 kT_e}{e^2}$ and r appearing in (0.7), (0.9) and (0.6) are set equal to unity. Then the model describing the formation and the expansion of a small plasma are

$$div(f(n)\bigtriangledown\psi) = -div(f(n)E_\infty), \tag{0.12}$$

and

$$n_t - div(D(n)\bigtriangledown n) - \nu(|\bigtriangledown\psi|)n + n^2 = 0, \quad n(t=0) = n_0, \tag{0.13}$$

where

$$f(n) = (1+n^2)^{\frac{1}{2}}, \quad D(n) = \frac{n+1}{n+c}, c << 1,$$

and ν, n_0 and E_∞ respectively are a given function from $\mathbb{R}_+$ into $\mathbb{R}_+$, a given function from $\mathbb{R}^3$ into $\mathbb{R}_+$ and a vector of $\mathbb{R}^3$.

1 The existence theorem.

First, the problem (0.12-13) will be solved with space variables in

$$B_j := \{x \in B; \mid x \mid \leq j\}.$$

The decoupled problems, i.e (0.12) for a given n and (0.13) for a given ψ are solved. Then the whole problem (0.12-13) is solved, by passing to the limit when j tends to ∞.

Lemma 1.1 *Let $n \in L^\infty((0,T) \times B_j)$. Then there is a unique weak solution $\psi \in L^2(0,T; H_0^1(B_j))$ of (0.12).*

Proof of Lemma 1.1.
Restricting to B_j, (0.12) has the weak formulation

$$\int_0^t \int_{B_j} f(n) \bigtriangledown \psi \cdot \bigtriangledown \eta = - \int_0^t \int_{B_j} f(n) E_\infty \cdot \bigtriangledown \eta, \quad \eta \in L^2(0,T; H_0^1(B_j)).$$

$(\psi, \eta) \to \int_0^T \int_{B_j} f(n) \bigtriangledown \psi \cdot \bigtriangledown \eta$ is a continuous, coercive (since $f(n) \geq 1$), bilinear form defined on $(L^2(0,T; H_0^1(B_j))^2$, and
$\eta \to - \int_0^T \int_{B_j} f(n) E_\infty \cdot \bigtriangledown \eta$ is a linear continuous form on $L^2(0,T; H_0^1(B_j))$, since $f(n) \in L^\infty(I\!R^3)$ when $n \in L^\infty((0,T) \times I\!R^3$. Hence Lax-Milgram's theorem yields the result of existence and uniqueness of $\psi \in L^2(0,T; H_0^1(B_j))$.

Definition 1.2 *Let $\psi \in L^\infty(0,T; H_0^1(B_j))$. A weak solution of (0.13) is $n \in L^2(0,T; H_0^1(B_j))$ such that*

$$\int_0^T \int_{B_j} n\varphi_t - D(n) \bigtriangledown n \cdot \bigtriangledown \varphi + [\nu(\mid \bigtriangledown \psi \mid) - n] n\varphi$$
$$= \int_{B_j} n_0(x) \varphi(0,x) dx,$$

for any $\varphi \in H^1((0,T) \times B_j)$, compactly supported in $[0,T) \times B_j$.

Lemma 1.3 *Let $\psi \in L^\infty(0,T; H_0^1(B_j))$. In the class of functions in $L^2(0,T; H_0^1(B_j))$ satisfying*

$$0 \leq n(t,x) \leq \mid n_0 \mid_\infty e^{(1+|\nu|_\infty)t},$$

there is a unique weak solution of (0.13).

The proof of Lemma 1.3 follows from [7, 8].

Theorem 1.4 *Let $T > 0$. Assume*

$(K1)$ $\quad \nu$ *is a convex function in* $W^1_\infty(I\!R_+)$,
$(K2)$ $\quad n_0 \in H^1_0(B) \cap L^\infty(B)$.

Then there exists $(\psi, n) \in (L^2(0,T; H^1(I\!R^3)))^2$ which is a weak solution of (0.12-13) in the sense that

$$\int_0^T \int_{I\!R^3} f(n) \bigtriangledown \psi \cdot \bigtriangledown \eta = -\int_0^T \int_{I\!R^3} f(n) E_\infty \cdot \bigtriangledown \eta,$$

and

$$\int_0^T \int_{I\!R^3} n\gamma_t - D(n) \bigtriangledown n \cdot \bigtriangledown \gamma + [\nu(\mid \bigtriangledown \psi \mid) - n] n\gamma$$
$$= \int_{I\!R^3} n_0(x) \gamma(0,x) dx,$$

for every $(\eta, \gamma) \in L^2(0,T; H^1_0(B)) \times H^1_0([0,T) \times B)$. Moreover (0.11) holds, whereas (0.10) is satisfied in a weaker sense, i.e. $\psi \in L^2(0,T; H^1_0(B))$.

Proof of Theorem 1.4.
Denote

$$K_j := \{n \in L^2(0,T; H^1_0(B_j)) \text{ s.t. } 0 \leq n(t,x) \leq \mid n_0 \mid_\infty e^{(1+|\nu|_\infty)t}\},$$

which is a closed and convex set of $L^2((0,T) \times B_j)$. Define $\mathcal{G}_|$ on K_j by $\mathcal{G}_|(\backslash) = \mathcal{N}$, where N is the solution defined in Lemma 1.3 for ψ which is the solution of Lemma 1.1 with n. A fixed point argument is now used. First $\mathcal{G}_| : \mathcal{L}^\in((\prime, \mathcal{T}) \times \mathcal{B}_|) \to \mathcal{L}^\in((\prime, \mathcal{T}); \mathcal{H}^\infty_\prime(\mathcal{B}_|))$ is continuous. Indeed, if $(n_i), n_i \in K_j$, tends to $\tilde{n}$ in $L^2((0,T) \times B_j)$, then

$$\int_{B_j} \mid \bigtriangledown \psi_i \mid^2 \leq \int_{B_j} f(n_i) \mid \bigtriangledown \psi_i \mid^2$$
$$\leq \int_{B_j} f(n_i) \mid E_\infty \cdot \bigtriangledown \psi_i \mid$$
$$\leq c(\int_{B_j} \mid \bigtriangledown \psi_i \mid^2)^{\frac{1}{2}},$$

so that $(\bigtriangledown \psi_i)$ is bounded in L^2 and weakly converges in L^2 to some $\bigtriangledown \tilde{\psi}$. Moreover $(\bigtriangledown \psi_i)$ strongly converges to $\bigtriangledown \tilde{\psi}$ in L^2. Indeed, $(f(n_i))$ converges in L^2 to $f(\tilde{n})$, since (n_i) converges in L^2 to $\tilde{n}$ and $n_i \in K_j$ (so (n_i) is bounded). Passing to the limit in

$$div(f(n_i) \bigtriangledown \psi_i) = -div(f(n_i) E_\infty)$$

implies

$$div(f(\tilde{n}) \bigtriangledown \tilde{\psi}) = -div(f(\tilde{n})E_\infty).$$

Hence

$$\int_{B_j} f(\tilde{n}) \mid \bigtriangledown \tilde{\psi} \mid^2 = -\int_{B_j} f(\tilde{n}) E_\infty \cdot \bigtriangledown \tilde{\psi}.$$

Then, since $f \geq 1$,

$$\begin{aligned}
&\textstyle\int_{B_j} \mid \bigtriangledown \psi_i - \bigtriangledown \tilde{\psi} \mid^2 \leq \int_{B_j} f(n_i) \mid \bigtriangledown \psi_i - \bigtriangledown \tilde{\psi} \mid^2 \\
&\textstyle= \int_{B_j} f(n_i) \mid \bigtriangledown \psi_i \mid^2 - 2\int_{B_j} f(n_i) \bigtriangledown \psi_i \cdot \bigtriangledown \tilde{\psi} + \int_{B_j} f(n_i) \mid \bigtriangledown \tilde{\psi} \mid^2 \\
&\textstyle= -\int_{B_j} f(n_i) \bigtriangledown \psi_i \cdot E_\infty - 2\int_{B_j} f(n_i) \bigtriangledown \psi_i \cdot \bigtriangledown \tilde{\psi} + \int_{B_j} f(n_i) \mid \bigtriangledown \tilde{\psi} \mid^2,
\end{aligned}$$

and the right-hand side tends to

$$-\int_{B_j} f(\tilde{n}) \bigtriangledown \tilde{\psi} \cdot E_\infty - \int_{B_j} f(\tilde{n}) \mid \bigtriangledown \tilde{\psi} \mid^2 = 0.$$

It follows from the convergence of $(\bigtriangledown \psi_i)$ to $\bigtriangledown \tilde{\psi}$ in L^2 and from $\nu \in W^1_\infty(I\!R_+)$ that $(\nu(\mid \bigtriangledown \psi_i \mid))$ converges to $\nu(\mid \bigtriangledown \tilde{\psi} \mid)$ in $L^2((0,T) \times B_j)$. On the other hand, $(N_i) := (\mathcal{G}_|(\backslash_\rangle))$ weakly converges in $L^2(0,T; H^1_0(B_j))$. Indeed,

$$\begin{aligned}
&\frac{1}{2}\int_{B_j} N_i^2(t,x)dx + \int_0^t \int_{B_j} \mid \bigtriangledown N_i \mid^2 \\
\leq &\mid \nu \mid_\infty \int_0^t \int_{B_j} N_i^2(\tau,x) d\tau dx + \frac{1}{2}\int_{B_j} n_0^2(x)dx,
\end{aligned}$$

so that (N_i) weakly converges in $L^2(0,T; H^1_0(B_j))$ to some $\tilde{N}$. Indeed $(\bigtriangledown N_i)$ is bounded in $L^2((0,T) \times B_j)$. Moreover, under the hypothesis $n_0 \in H^1_0(B)$, it follows by the theory of parabolic equations that $((N_i)_t)$ is bounded in $L^2((0,T) \times B_j)$. Hence, up to a subsequence the convergence of (N_i) to $\tilde{N}$ is strong in $L^2((0,T) \times B_j)$. Passing to the limit in

$$(N_i)_t - div(D(N_i) \bigtriangledown N_i) - \nu(\mid \bigtriangledown \psi_i \mid) N_i + N_i^2 = 0$$

leads to

$$\tilde{N}_t - div(D(\tilde{N}) \bigtriangledown \tilde{N}) - \nu(\mid \bigtriangledown \tilde{\psi} \mid)\tilde{N} + \tilde{N}^2 = 0,$$

so that $\tilde{N} = \mathcal{G}_|(\heartsuit)$. That ends the proof of the continuity of $\mathcal{G}_|$.
Let us prove the compactness of $\mathcal{G}_|$. Let (n_i) be a sequence in K_j, therefore bounded in $L^2((0,T) \times B_j)$. Then (N_i) is bounded in $L^2(0,T; H^1_0(B_j))$,

whereas by the previous parabolic argument $((N_i)_t)$ is bounded in $L^2((0,T)\times B_j)$. It follows that (N_i) is precompact in $L^2((0,T)\times B_j)$.
Finally the Schauder fixed point theorem applied to every $\mathcal{G}_|$ proves the existence of a sequence $(\psi_j, n_j) \in (L^2(0,T;H_0^1(B_j)))^2$ that satisfies

$$\int_0^T \int_{B_j} f(n_j) \bigtriangledown \psi_j \cdot \bigtriangledown \eta = -\int_0^T \int_{B_j} f(n_j) E_\infty \cdot \bigtriangledown \eta, \qquad \eta \in L^2(0,T;H_0^1(B_j))$$

and

$$\int_0^T \int_{B_j} n_j \gamma_t - D(n_j) \bigtriangledown n_j \cdot \bigtriangledown \gamma + [\nu(\mid \bigtriangledown \psi_j \mid) - n_j] n_j \gamma = \int_{B_j} n_0(x)\gamma(0,x)dx,$$
$$\gamma \in H^1((0,T)\times B_j), \text{compactly supported.}$$

In the rest of the proof, c will denote constants that do not depend on j. First by Gronwall's lemma,

$$\int_0^T \int_{B_j} n_j^2 + \mid \bigtriangledown n_j \mid^2 \leq c.$$

Moreover,

$$\begin{aligned}\int_0^T \int_{B_j} \mid \bigtriangledown \psi_j \mid^2 &\leq \int_0^T \int_{B_j} f(n_j) \mid \bigtriangledown \psi_j \mid^2 \\ &= \int_0^T \int_{B_j} [div(f(n_j)E_\infty)]\psi_j \\ &= \int_0^T \int_{B_j} \frac{n_j}{\sqrt{1+n_j^2}} \psi_j E_\infty \cdot \bigtriangledown n_j \\ &\leq c \parallel \psi_j \parallel_{L^6} \\ &\leq c[\parallel \psi_j \parallel_{L^2} + \parallel \bigtriangledown \psi_j \parallel_{L^2}]\end{aligned}$$

due to the continuous imbedding of $H_0^1(B_j)$ into $L^6(B_j)$. This in turn is bounded by $c \parallel \bigtriangledown \psi_j \parallel$ from the Poincaré inequality, here applicable since B_j is bounded in the y, z-directions. Hence $(\bigtriangledown \psi_j)$ is uniformly in j bounded in $L^2((0,T)\times B_j)$. Extend ψ_j and n_j by 0 outside of B_j. It follows from a diagonal extraction that there exists n and a subsequence still denoted (n_j) of (n_j), which converges to n in $L^2(0,T;H^1_{loc}(I\!R^3))$ weak and in $L^2(0,T;L^2_{loc}(I\!R^3))$

strongly. Analogously there exist ψ and a subsequence, still denoted (ψ_j) of (ψ_j), with $(\bigtriangledown\psi_j)$ converging to $\bigtriangledown\psi$ in $L^2(0,T;L^2_{loc}(I\!R^3))$ weak. Moreover, the same way as in the proof of the continuity of $\mathcal{G}_|$, it can be proved that $(\nu(\mid \bigtriangledown\psi_j \mid))$ strongly converges to $L^2((0,T)\times I\!R^3)$. It is then possible to pass to the limit in the weak formulations respectively satisfied by $\bigtriangledown\psi_j$ and n_j, which proves that (ψ, n) is a weak solution of (0.12-13) in the sense of Theorem 1.4.

References

1. Boeuf, J. P., Belenguer, Ph., 'Collisional absorption during microwave propagation through the atmosphere', Proceedings of the International symposium on electromagnetic environments and consequences, ed. D. J. Serafin, Bordeaux, 1994.

2. Gajewski, H., 'On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors', Z. Angew. Math. Mech. 65, 1985 2, 101-108.

3. Gildenburg, V. B., Gushchin, I. S., Dvinin S. A. & Kim A. V., Sov. Phys. JETP 70 (4), 645, 1990.

4. Krall, N., Trivelpiece, A. W., Principles of plasma physics, McGraw-Hill, 1973.

5. Lions, J. L., Quelques méthodes de résolution de problèmes aux limites non lineaires, Paris, Dunod; Gauthier-Villars, 1969.

6. Mock, M. S., 'An initial value problem from semiconductor device theory', SIAM J. Math. Anal. , 1974, Vol. 5, No. 4, 597-612.

7. Raizer, Y. P., Gas discharge physics, Springer-Verlag, 1991.

8. Seidman, T. I., Troianiello G. M., 'Time-dependent solutions of a nonlinear system arising in semiconductor theory', Nonlinear Analysis, Theory, Methods & Applications, 1985, Vol. 9, No. 11, 1137-1157.

9. Seidman, T. I., 'Time-dependent solutions of a nonlinear system arising in semiconductor theory-II. Boundedness and periodicity', Nonlinear analysis, Theory, Methods & Applications, 1986, Vol. 10, No. 5, 491-502.