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STATIONARY SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR A MAXWELL–BOLTZMANN SYSTEM MODELLING DEGENERATE SEMICONDUCTORS *

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Abstract. One of the most accurate models for carrier transport in semiconductors is based on the Maxwell–Boltzmann system. Degeneracy effects are taken into account by the nonlinearity of the collisions operator. We use two recent techniques developed for the study of kinetic models, upper solutions and mean compactness results, to prove existence of stationary solutions with arbitrary large boundary data, in any kind of geometries.

Key words. boundary value problem, stationary solutions, Vlasov–Maxwell systems, Fermi–Dirac distribution, semiconductor

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Introduction. For bulk components, the drift diffusion equations give the basic model for the transport of carriers. However, transport phenomena that occur in submicron devices are due to hot and ballistic electrons. In these conditions, it is well known that the drift diffusion model is no longer valid. The physics description needs kinetic models. This paper is devoted to the analysis of one of the most accurate kinetic models, the Maxwell–Boltzmann system.

We use the upper solutions technique of [8] to construct solutions for stationary boundary value problems. In a previous paper [10] we analyzed the Maxwell–Boltzmann system for semiconductors but with a nondegeneracy assumption. Compared to this previous work, the new difficulty is to control the nonlinearity of the collision operator that takes into account the degeneracy effects. The main tools for that are the mean compactness results of [4] and a monotonicity property of the nonlinear collision the operator.

1. The kinetic model and the main result. In kinetic theory, the transport process of charged particles in a self-consistent electromagnetic field is modelled by the Vlasov–Maxwell equations. In semiconductor statistics [2], [3], the distribution function depends on the position x and the wavevector p, instead of the velocity, as in classical theory, in order to take into account some quantum phenomena. Then the velocity and the energy of an electron are given functions of the wavevector. They are related by the relation

$$v(p) = \frac{1}{\hbar}\nabla_p \mathcal{E}(p), \tag{1.1}$$

where $\mathcal{E}(p)$, the energy of the particles, belongs to $(C_b^2(\mathbb{R}^3))^3$, $v(p)$ is the velocity, and $\hbar$ is the reduced Planck constant. For instance, with the parabolic band approximation, we get

$$\mathcal{E}(p) = \hbar^2 \frac{|p|^2}{2m^*}, \qquad v(p) = \frac{\hbar}{m^*}p, \tag{1.2}$$

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where m^* is the effective mass of electrons. Then we find the classical identity

$$\mathcal{E} = \frac{1}{2} m^* |v|^2. \tag{1.3}$$

But another model, often used in semiconductor physics, is given implicitly by

$$\mathcal{E}(p) + \alpha \mathcal{E}(p)^2 = \frac{\hbar^2 |p|^2}{2m^*}, \tag{1.4}$$

where α is the coefficient of nonparabolicity. Hence, for the sake of generality, we will consider an arbitrary band diagram $\mathcal{E}(p)$. We only assume that there is a constant β such that for any unitary vector e of $\mathbb{R}^3$, and any positive reals R and γ,

$$\operatorname{meas}\{|p| \le R \text{ and } |v(p) \cdot e| \le \gamma\} \le c(R)\gamma^\beta. \tag{1.5}$$

This means that the velocity v cannot be concentrated along one direction. This assumption is needed for using a compactness property on averages on p of solutions of transport equations. Clearly, it is satisfied if $\mathcal{E}$ is given by (1.2) or (1.4).

Then the distribution $f = f(x,p)$ is determined as follows. Let Ω be an open bounded set of $\mathbb{R}^3$, modelling the device geometry. Let Σ_- denote the subset of the boundary where the velocities are inward:

$$\Sigma_- = \{(x,p) \in \partial\Omega \times \mathbb{R}^3 : v(p) \cdot \nu(x) < 0\}, \tag{1.6}$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega$.

The distribution f solves the following Vlasov–Maxwell equations:

$$v(p) \cdot \nabla_x f(x,p) + F(x,p) \cdot \nabla_p f(x,p) = C(f)(x,p), \quad x \in \Omega, \quad p \in \mathbb{R}^3, \tag{1.7}$$

$$-\Delta_x \phi(x) = \frac{q}{\varepsilon_r}[N(x) - \rho(x)], \quad x \in \Omega, \tag{1.8}$$

$$\nabla_x \wedge B(x) = \mu_r q j(x), \quad \nabla \cdot B(x) = 0, \quad x \in \Omega, \tag{1.9}$$

$$F(x,p) = \frac{q}{\hbar}[\nabla_x \phi(x) - v(p) \wedge B(x)], \quad x \in \Omega, \quad p \in \mathbb{R}^3. \tag{1.10}$$

The constants q, ε_r, and μ_r are, respectively, the charge of the electron, the permittivity, and the permeability of the semiconductor. The function N is the given doping profile. We assume N in $L^\infty(\Omega)$. The operator C is intended to model the collisions between the electrons, impurities, and phonons of the semiconductor [7]. It is defined by

(1.11)

$$C(f)(x,p) = \int_{\mathbb{R}^3} s(p,p')[m(p)f(x,p')(1 - f(x,p)) - m(p')f(x,p)(1 - f(x,p'))]\, dp'.$$

The function m is a Maxwellian:

$$m(p) = \exp(-\mathcal{E}(p)/\theta), \tag{1.12}$$

where θ is a physical constant related to the fixed temperature of the semiconductor. The terms $(1 - f)$ in (1.11) come from Pauli's exclusion principle. They express the fact that there is at most one particle for a given quantum state (x, p). Thus, the physically admissible distribution f will satisfy

$$0 \leq f \leq 1. \tag{1.13}$$

The function s is given and satisfies

$$s > 0, \qquad s(p,p') = s(p',p), \tag{1.14}$$

$$m(p)s(p,p') \in L^2(\mathbb{R}^6), \tag{1.15}$$

and the collision frequency is assumed to be bounded:

$$\sigma(p) = \int_{\mathbb{R}^3} s(p,p')m(p')\,dp' \in L^\infty(\mathbb{R}^3). \tag{1.16}$$

We recall the null-space of the operator C, which gives the thermal equilibrium distributions (see [9]): it consists of the Fermi–Dirac distributions

$$N(C) = \left\{ n(p) = \left[1 + \exp\left(\frac{\mathcal{E}(p) - \kappa}{\theta}\right)\right]^{-1} ; \kappa \in [-\infty, +\infty] \right\}. \tag{1.17}$$

The concentration ρ and the flux j depend on the particle distribution f through the relations

$$\rho(x) = \int_\Omega f(x,p)\,dp, \quad j(x) = \int_\Omega v(p)f(x,p)\,dp, \quad x \in \Omega. \tag{1.18}$$

The system (1.6)–(1.9) is completed with the boundary conditions

$$f(x,p) = g_0(x,p), \qquad (x,p) \in \Sigma_-, \tag{1.19}$$

$$\phi(x) = \phi_0(x), \qquad x \in \partial\Omega, \tag{1.20}$$

$$B(x) \cdot \nu(x) = b(x), \qquad x \in \partial\Omega. \tag{1.21}$$

In order to allow the extension of the boundary data, we assume the following.

(H1) Ω is a smooth bounded set of $\mathbb{R}^3$. Its boundary $\partial\Omega$ is compact and connected.

(H2) $\phi_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$.

(H3) $b \in H^{-1/2}(\partial\Omega)$; $\langle b, 1\rangle_{H^{-1/2},H^{1/2}} = 0$.

Then there are two functions Φ_0 and B_0 such that:

$$\Phi_0 \in H^1(\Omega) \cap L^\infty(\Omega), \quad -\Delta_x \Phi_0 = \frac{q}{\varepsilon_r} N, \quad \Phi_{0/\partial\Omega} = \phi_0,$$

$$B_0 \in H(\text{div, curl}, \Omega), \quad \nabla_x \cdot B_0 = 0, \quad \nabla_x \wedge B_0 = 0, \quad B_0 \cdot \frac{\nu}{\partial\Omega} = b.$$

Finally, we assume that the boundary distribution g_0 is nonnegative and bounded by a Fermi–Dirac distribution:

$$0 \leq g_0 \leq \left[1 + \exp\left(\frac{\mathcal{E}(p) - \kappa}{\theta}\right)\right]^{-1}. \tag{H4}$$

We now state the main result of this paper.

THEOREM 1.1. *Under the assumptions* (H1)–(H4), *the stationary Vlasov-Maxwell system* (1.7)–(1.10), (1.19)–(1.21) *has at least one solution* (f, ϕ, B) *belonging to* $L^2(\Omega \times \mathbb{R}^3) \times H^1(\Omega) \times H(\text{div, curl}, \Omega)$ *and verifying*

$$(1.22) \qquad 0 \leq f \leq \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\Phi_0(x) - \nu}{\theta}\right)\right]^{-1}, \quad \textit{where } \nu = \kappa - \frac{q}{\hbar}\|\phi_0\|_\infty.$$

Remark. There is no uniqueness of the solution of the system (1.7)–(1.10), (1.19)–(1.21). We give the following counterexample, based on the idea of trapping particles with a potential created by a background charge density. Let n_0 be an arbitrary positive real number. Define ψ by

$$(1.23) \qquad -\Delta_x \psi = \frac{q}{\varepsilon_r} n_0, \qquad \psi/\partial\Omega = 0.$$

We let the background charge density N be equal to

$$(1.24) \qquad \int_{\mathbb{R}^3} \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\psi(x)}{\theta}\right)\right]^{-1} dp + n_0,$$

and we define Φ_1 by

$$(1.25) \qquad -\Delta_x \Phi_1 = \frac{q}{\varepsilon_r} N, \qquad \Phi_{1/\partial\Omega} = 0.$$

Then $f_1 = 0$, associated with $\Phi = \Phi_1$, and

$$f_2 = \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\psi(x)}{\theta}\right)\right]^{-1},$$

associated with $\Phi_2 = \psi$, are two solutions of (1.7)–(1.10), (1.19)–(1.21).

The paper is organized as follows: in §2, a modified Vlasov–Poisson problem is solved and a maximum principle property is stated, in order to obtain uniform bounds on the concentrations and the fluxes of the modified problem. Then §3 is devoted to the proof of the full stationary Vlasov–Maxwell problem. Finally, §4 details some compactness results used throughout the paper.

2. A modified Vlasov problem. In this section, the electrostatic potential ϕ and the magnetic field B are assumed to be known, such that

$$(2.1) \qquad \phi \in C_b^2(\Omega), \qquad B \in (C_b^1(\Omega))^3.$$

Because zero lies in the spectrum of Vlasov operators, uniqueness fails for boundary value problems. Therefore [1], we add an absorption term and solve the following system:

$$\alpha f(x,p) + v(p) \cdot \nabla_x f(x,p) + F(x,p) \cdot \nabla_p f(x,p) = C(f)(x,p), \quad x \in \Omega, \quad p \in \mathbb{R}^3;$$

$$(2.2) \qquad f(x,p) = g_0(x,p), \qquad (x,p) \in \Sigma_-,$$

where F is given by (1.10).

THEOREM 2.1. *Under the assumption* (H4) *the problem* (2.2) *has, for every* $\alpha > 0$, *a unique solution in the set of the square integrable functions on* $\Omega \times \mathbb{R}^3$ *that satisfies*

$$0 \leq f \leq G_{\phi,\nu}.$$

Let us introduce the Maxwell–Boltzmann distribution

$$G_{\phi,\nu}(x,p) = \left[1 + \exp\left(\frac{\mathcal{E}(p) + \frac{q}{\hbar}\phi(x) - \nu}{\theta}\right)\right]^{-1}, \qquad \nu \in [-\infty, +\infty]. \tag{2.3}$$

This distribution solves the Vlasov equation and will be used as an upper solution in the proof of the existence of a solution of (2.2). It will provide a priori estimates on the density ρ and the flux j that will be useful in the following. In order to obtain some maximum principle, we assumed (H4); it is then possible to bound g_0 by $G_{\phi,\nu}$, where

$$\nu = \kappa - \frac{q}{\hbar}\|\phi_0\|_\infty. \tag{2.4}$$

Next we prove the uniqueness of the solution of (2.2), using a monotonicity property of the collision operator. A similar strategy of proof has been used in [8].

Let us prove the existence of a solution of (2.2). We need the following compactness result, which is an easy variant of the mean compactness results of [4], [5], and [6].

PROPOSITION 2.2. *Let* $(f_n), (g_n)$, *and* (h_n) *be bounded sequences of* $L^2(\Omega \times \mathbb{R}^3)$ *that satisfy:*

$$v(p) \cdot \nabla_x f_n = \mathrm{div}_p g_n + h_n \tag{2.5}$$

in the sense of distributions.

Then, for any Hilbert–Schmidt operator K defined on $L^2(\mathbb{R}^6)$, the sequence $(K(f_n(x,\cdot)))$ is relatively compact in $L^2(\Omega \times \mathbb{R}^3)$.

The proof of Proposition 2.2 is given in §4.

Proof of Theorem 2.1. The system (2.2) to be solved gives

$$[\alpha + \sigma(p) + \lambda(f)]f(x,p) + v(p)\cdot\nabla_x f(x,p) + F(x,p)\cdot\nabla_p f(x,p) = \mu(f)(x,p) \quad \text{on } \Omega \times \mathbb{R}^3,$$

$$f/\Sigma_- = g_0, \tag{2.6}$$

with σ defined as in (1.16) and

$$\lambda(f)(x,p) = \int_{\mathbb{R}^3} s(p,p')[m(p) - m(p')]f(x,p')\,dp', \tag{2.7}$$

$$\mu(f)(x,p) = \int_{\mathbb{R}^3} s(p,p')m(p)f(x,p')\,dp'. \tag{2.8}$$

Let us denote

$$\gamma(f)(x,p) = (\lambda - \mu)(f)(x,p) + \sigma(p) = \int_{\mathbb{R}^3} s(p,p')m(p')[1 - f(x,p')]\,dp'. \tag{2.9}$$

Solving (2.6) results in determining a fixed point of the following operator T. Let us denote

$$X = \{(L, M) \in L^2(\Omega\times\mathbb{R}^3)\times L^2(\Omega\times\mathbb{R}^3) : 0 \le M \le \mu(G_{\phi,\nu}) \text{ and } L-M+\sigma \ge \gamma(G_{\phi,\nu})\}.$$

Clearly, X is a closed convex set of $L^2(\Omega \times \mathbb{R}^3) \times L^2(\Omega \times \mathbb{R}^3)$. For every (L, M) in X, let f be the unique solution in $L^2(\Omega \times \mathbb{R}^3)$ of

$$[\alpha+\sigma(p)+L(x,p)]f(x,p)+v(p)\cdot\nabla_x f(x,p)+F(x,p)\cdot\nabla_p f(x,p) = M(x,p) \quad \text{on } \Omega\times\mathbb{R}^3,$$

$$f/\Sigma_- = g_0. \tag{2.10}$$

f is well defined since $\alpha + \sigma + L$ is positive and v and F belong to $C^1(\mathbb{R}^3)^3$ and $C^1(\Omega \times \mathbb{R}^3)^3$, respectively. Moreover, since M and g_0 are nonnegative, then $f \ge 0$.

We define the operator T on X by

$$T(L, M) = (\lambda(f), \mu(f)).$$

Let us now show that T maps X in X. $G_{\phi,\nu} - f$ is a solution of

$$(\alpha+\sigma+L)(G_{\phi,\nu}-f)+v\cdot\nabla_x(G_{\phi,\nu}-f)+F\cdot\nabla_p(G_{\phi,\nu}-f) = \alpha G_{\phi,\nu}+LG_{\phi,\nu}-M. \tag{2.11}$$

Since, thanks to (1.17),

$$\mu(G_{\phi,\nu}) - (\sigma + \lambda(G_{\phi,\nu}))G_{\phi,\nu} = C(G_{\phi,\nu}) = 0, \tag{2.12}$$

we obtain

$$\alpha G_{\phi,\nu}+LG_{\phi,\nu}-M = \alpha G_{\phi,\nu}+[\sigma+L-M-\gamma(G_{\phi,\nu})]G_{\phi,\nu}+(\mu(G_{\phi,\nu})-M)(1-G_{\phi,\nu}) \ge 0.$$

Moreover, the boundary condition on $G_{\phi,\nu} - f$ gives us

$$(G_{\phi,\nu} - f)/\Sigma_- = G_{\phi,\nu} - g_0, \tag{2.13}$$

which, from hypothesis (H4), is nonnegative. Hence $G_{\phi,\nu} - f \ge 0$, so we have

$$0 \le f \le G_{\phi,\nu} \quad \text{on } \Omega \times \mathbb{R}^3. \tag{2.14}$$

Since μ and γ are, respectively, increasing and decreasing,

$$0 \le \mu(f) \le \mu(G_{\phi,\nu}) \tag{2.15}$$

and

$$\gamma(f) = \sigma + \lambda(f) - \mu(f) \ge \gamma(G_{\phi,\nu}). \tag{2.16}$$

Finally, $\mu(f)$ belongs to $L^2(\Omega \times \mathbb{R}^3)$ because

$$\|\mu(f)\|_2 \le \|s(p,p')m(p)\|_{L^2(\mathbb{R}^6)}\|f\|_2 \le c\|f\|_2. \tag{2.17}$$

A similar proof establishes that $\lambda(f)$ belongs to $L^2(\Omega \times \mathbb{R}^3)$.

Let us prove the continuity of the operator T. Let $(L_n), (M_n)$ be convergent sequences in $L^2(\Omega \times \mathbb{R}^3)$ towards L and M, respectively. Knowing that g_0 belongs to $L^2(\Sigma_-; -v \cdot \nu(x) dv d\sigma(x))$, the associated solutions f_n of the system (2.10) form a bounded sequence of $L^2(\Omega \times \mathbb{R}^3)$; therefore, there is a subsequence (f_{n_k}) of (f_n) which converges to some f in $L^2(\Omega \times \mathbb{R}^3)$ weak. Passing to the limit in (2.10) as n_k tends to infinity gives

$$[\alpha+\sigma(p)+L(x,p)]f(x,p)+v(p)\cdot\nabla_x f(x,p)+F(x,p)\cdot\nabla_p f(x,p) = M(x,p) \quad \text{on } \Omega\times\mathbb{R}^3,$$

$$f/\Sigma_- = g_0, \tag{2.18}$$

thus f is unique and the complete sequence (f_n) converges weakly to f in $L^2(\Omega \times \mathbb{R}^3)$.

Moreover, λ and μ being linear functionals, $(\lambda(f_n))$ and $(\mu(f_n))$, respectively, converge weakly to $\lambda(f)$ and $\mu(f)$ in $L^2(\Omega \times \mathbb{R}^3)$.

In view of (2.14), f_n satisfies

$$0 \le f_n \le G_{\phi,\nu}. \tag{2.19}$$

Thus (f_n) is bounded in $L^\infty(\Omega \times \mathbb{R}^3)$. On the other hand,

$$v(p) \cdot \nabla_x f_n(x,p) = -\nabla_p[F(x,p) f_n(x,p)] + g_n, \tag{2.20}$$

with $(f_n), (F f_n)$, and $(g_n) = (-(\alpha + \sigma - L_n) f_n + M_n)$ bounded in $L^2(\Omega \times \mathbb{R}^3)$.

Then, according to Proposition 2.2, $(\lambda(f_n))$ and $(\mu(f_n))$ belong to a compact set of $L^2(\Omega \times \mathbb{R}^3)$. So, in view of the weak convergence of $(\lambda(f_n))$ and $(\mu(f_n))$ in $L^2(\Omega \times \mathbb{R}^3)$, the complete sequence $(\lambda(f_n))$ and $(\mu(f_n))$, respectively, converges to $\lambda(f)$ and $\mu(f)$ in $L^2(\Omega \times \mathbb{R}^3)$. This proves the continuity of T.

Let us show the compactness of T. If (L_n) and (M_n) are bounded in $L^2(\Omega \times \mathbb{R}^3)$, the associated sequence (f_n) is bounded in $L^2(\Omega \times \mathbb{R}^3)$, so Proposition 2.2 implies that $T(L_n, M_n)$ belongs to a compact set of $L^2(\Omega \times \mathbb{R}^3)$. Therefore, the Schauder fixed point theorem gives the existence of a solution of (2.2).

We now prove the uniqueness of the solution of (2.2). Let f and g be two solutions of (2.2). A small computation leads to

(2.21)

$$C(f) - C(g) = -\int_{\mathbb{R}^3} s(p,p')\{(f-g)[m'(1-f')+mg'] - (f'-g')[m(1-f)+m'g]\}\, dp',$$

where f' denotes $f(x,p')$.

For any function h and with the help of the symmetry of s, we get

$$\int_{\mathbb{R}^3} [C(f) - C(g)] h \, dp = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} s(h-h')(f-g)[m'(1-f')+mg']\, dp\, dp'. \tag{2.22}$$

For any small and positive δ, we define the odd function sg^δ and the function abs^δ by

$$sg^\delta(x) = \begin{cases} \frac{4}{\delta}\left(x - \frac{1}{\delta}x^2\right) & \text{if } x \in \left[0, \frac{\delta}{2}\right], \\ 1 & \text{if } x \ge \frac{\delta}{2}, \end{cases} \tag{2.23}$$

and $abs^\delta(x) = x sg^\delta(x)$.

We choose $h = sg^\delta(f-g)$. In view of (2.14), we obtain

$$(h-h')(f-g) = abs^\delta(f-g) - sg^\delta(f'-g')(f-g) \geq -2\delta. \tag{2.24}$$

Then (2.22) implies

$$\int_{\mathbb{R}^3} [C(f) - C(g)] sg^\delta(f-g)\, dp \leq 2c\delta, \tag{2.25}$$

where c is the constant given by

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} s[m'(1-f') + mg']\, dp\, dp' \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} s[m'+m]\, dp\, dp' = c. \tag{2.26}$$

Since f and g are solutions of (2.2),

$$\alpha(f-g) + v(p)\cdot\nabla_x(f-g) + F\cdot\nabla_p(f-g) = C(f) - C(g).$$

Multiplying this equation by $sg^\delta(f-g)$ and integrating it on $\Omega\times\mathbb{R}^3$ leads to

$$\alpha\int_{\Omega\times\mathbb{R}^3}(f-g)sg^\delta(f-g) + \int_\Sigma v(p)\cdot\nu\, abs^\delta(f-g) = \int_{\Omega\times\mathbb{R}^3}[C(f)-C(g)]sg^\delta(f-g). \tag{2.27}$$

But

$$f-g=0 \quad \text{on } \Sigma_-.$$

Then

$$\alpha\int_{\Omega\times\mathbb{R}^3}(f-g)sg^\delta(f-g) \leq c\delta. \tag{2.28}$$

As δ tends to zero, we get

$$\alpha\int_{\Omega\times\mathbb{R}^3}|f-g| = 0, \tag{2.29}$$

so $f=g$.

3. The Vlasov–Maxwell problem. This section is devoted to the proof of Theorem 1.1. First let us give a sketch of this proof. We regularize the force field and add an absorption term in order to be in the frame of §2. We solve the regularized problem by means of the Schauder fixed point theorem and obtain a solution $(f_\alpha, \phi_\alpha, B_\alpha)$. Seeing that the potentials ϕ_α and Φ_0 satisfy

$$-\Delta_x\phi_\alpha = \frac{q}{\varepsilon_r}(N-\rho_\alpha), \quad \Delta_x\Phi_0 = \frac{q}{\varepsilon_r}N, \quad \Phi_{0/\partial\Omega} = \phi_0 = \phi_{\alpha/\partial\Omega},$$

the following inequality holds:

$$q\phi_\alpha \leq q\Phi_0.$$

It follows that the maximum principle of §2 applies to (f_α, ϕ_α): uniform bounds on the flux j_α and the concentration ρ_α are obtained, which gives compactness properties for F in $L^2(\Omega\times\mathbb{R}^3)$.

Finally, we pass to the limit in the modified system: we overcome the problem of passing to the limit in the nonlinear collision operator with the help of the compactness result given in Proposition 2.2.

We define a regularized force field in the following way. For any $\alpha > 0$,

$$F_\alpha = F_\alpha(\phi, B) = \frac{q}{\hbar}[\nabla_x \psi_\alpha(\phi) - v(p) \wedge H_\alpha(B)], \tag{3.1}$$

where the modified magnetic field H_α is obtained by regularizing in the classical way:

$$H_\alpha = \bar{B} * \zeta_\alpha.$$

$\bar{B}$ is the prolongation of B by zero outside Ω, and ζ_α is a regularizing sequence:

$$\zeta_\alpha(x) = \frac{1}{\alpha^3}\, \zeta\Big(\frac{x}{\alpha}\Big), \quad \int_{\mathbb{R}^3} \zeta(x)\, dx = 1, \quad \zeta \in C_0^\infty(\mathbb{R}^3).$$

To get a regularized potential ψ_α of ϕ such that ψ_α belongs to $C_b^2(\Omega)$, and $\psi_{\alpha/\partial\Omega} = \phi_0$ and $q\phi_\alpha \leq q\Phi_0$ as soon as $\phi/\partial\Omega = \phi_0$ and $q\phi \leq q\Phi_0$, we choose

$$\psi_\alpha = \Phi_0 + (I - \alpha\Delta)^{-2}(\phi_0 - \Phi_0),$$

where the operator Δ is considered an unbounded operator on $L^2(\Omega)$ whose domain is $H^2(\Omega) \cap H_0^1(\Omega)$. We remark that $(I - \alpha\Delta)^{-2}(\phi_0 - \Phi_0)$ belongs to $H^4(\Omega)$, then to $C_b^2(\Omega)$. Thus, in order that ψ_α belongs to $C_b^2(\Omega)$, we assume that

(H5) $\Phi_0 \in C_b^2(\Omega)$.

Then the properties of the above regularization are summarized in the following lemma.

LEMMA 3.1. *The map $F_\alpha = F_\alpha(\phi, B)$ is continuous from $H^1(\Omega) \times L^2(\Omega)$ into $C_b^1(\Omega \times \mathbb{R}^3)$. For any potential ϕ such that*

$$\phi \in H^1(\Omega), \quad \phi/\partial\Omega = \phi_0, \quad \text{and} \quad q\phi \leq q\Phi_0,$$

the modified potential $\psi_\alpha = \psi_\alpha(\phi)$ satisfies

$$\psi_\alpha \in C_b^2(\Omega), \quad \psi_{\alpha/\partial\Omega} = \phi_0, \quad q\phi_\alpha \leq q\Phi_0.$$

Furthermore, for any sequence (α_n, ϕ_n, B_n) such that

$$\begin{aligned} &\alpha_n \to 0, \\ &(\phi_n) \text{ is uniformly bounded in } H^2(\Omega), \quad \phi_{n/\partial\Omega} = \phi_0, \quad \phi_n \to \phi \quad \text{in } H^1(\Omega), \\ &B_n \to B \quad \text{in } L^2(\Omega), \end{aligned}$$

the regularized force field $F_{\alpha_n}(\phi_n, B_n)$ converges towards $F = \frac{q}{\hbar}(\nabla_x\phi - v(p) \wedge B)$ in $L^2_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3)$.

For a proof of this lemma, we refer the reader to [8].

The modified Vlasov–Maxwell system. The regularized problem is

$$\begin{gathered} \alpha f_\alpha + v(p) \cdot \nabla_x f_\alpha + F_\alpha \cdot \nabla_p f_\alpha = C(f_\alpha) \quad \text{on } \Omega \times \mathbb{R}^3, \\ f_{\alpha/\Sigma_-} = g_0. \end{gathered} \tag{3.2}$$

The regularized force field is given by (3.1). The potential solves

$$(3.3)\qquad \begin{aligned} -\Delta_x \phi_\alpha &= \frac{q}{\varepsilon_r}(N - \rho_\alpha), \\ \rho_\alpha(x) &= \int_{\mathbb{R}^3} f_\alpha(x,p)\,dp, \\ \phi_\alpha(x) &= \phi_0(x) \quad \text{on } \partial\Omega. \end{aligned}$$

The magnetostatic problem has to be modified because the flux of a solution of (3.2) is no longer divergence free. Instead we obtain

$$\alpha\rho_\alpha + \nabla_x \cdot j_\alpha = 0.$$

But (3.3) shows that

$$\nabla_x \cdot \left[j_\alpha + \alpha \frac{\varepsilon_r}{q} \nabla_x(\phi_\alpha - \Phi_0) \right] = 0.$$

Therefore, we define the new magnetostatic problem by

$$(3.4)\qquad \begin{aligned} &\nabla_x \wedge B_\alpha = \mu_r q \left[j_\alpha + \alpha \frac{\varepsilon_r}{q} \nabla_x(\phi_\alpha - \Phi_0) \right], \\ &\nabla_x \cdot B_\alpha = 0, \\ &j_\alpha(x) = \int_{\mathbb{R}^3} v(p) f_\alpha(x,p)\,dp, \\ &B_\alpha(x) \cdot \nu(x) = b(x) \quad \text{on } \partial\Omega. \end{aligned}$$

PROPOSITION 3.2 (*existence for the modified problem*). *Let $\alpha > 0$. Under the hypotheses* (H1)–(H5), *the modified Vlasov–Maxwell system* (3.2)–(3.4) *has at least one solution $(f_\alpha, \phi_\alpha, B_\alpha) \in L^2(\Omega \times \mathbb{R}^3) \times H^2(\Omega) \times H^1(\Omega)$, which satisfies uniformly with respect to α:*

$$(3.5)\qquad 0 \le f_\alpha \le \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\Phi_0(x) - \nu}{\theta} \right) \right]^{-1}.$$

ϕ_α is uniformly bounded in $H^2(\Omega)$; $B_\alpha - B_0$ is uniformly bounded in $H^1(\Omega)$.

Proof of Proposition 3.2. Let Ξ be the following nonempty convex closed set of $H^1(\Omega) \times L^2(\Omega)$:

$$\Xi = \{(\phi, B) \in H^1(\Omega) \times L^2(\Omega) : \phi/\partial\Omega = \phi_0 \text{ and } q\phi \le q\Phi_0\}.$$

We define a map on Ξ in the following way. For every (ϕ, B) in Ξ, let $f_{\phi,B}$ be the unique solution of the modified Vlasov–Maxwell problem (3.2). Then we define

$$\begin{aligned} \rho_{\phi,B}(x) &= \int_{\mathbb{R}^3} f_{\phi,B}(x,p)\,dp, \\ j_{\phi,B}(x) &= \int_{\mathbb{R}^3} v(p) f_{\phi,B}(x,p)\,dp. \end{aligned}$$

Let (ϕ_1, B_1) be the solution of (3.3), (3.4) with the corresponding concentration and flux. The map Γ is defined by $\Gamma(\phi, B) = (\phi_1, B_1)$. The following property and the

Schauder fixed point theorem establish the existence of a solution $(f_\alpha, \phi_\alpha, B_\alpha)$ of (3.2)–(3.4).

Property 3.3. $\Gamma : \Xi \to \Xi$ is continuous and compact for the topology of $H^1(\Omega \times \mathbb{R}^3) \times L^2(\Omega \times \mathbb{R}^3)$.

Let us first show the compactness of Γ. From (H2) and the maximum principle established in Theorem 2.1, we know that

$$
\begin{aligned}
0 \le f_{\phi,B} &\le \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\phi(x) - \nu}{\theta}\right)\right]^{-1} \\
&\le \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\Phi_0(x) - \nu}{\theta}\right)\right]^{-1}.
\end{aligned} \tag{3.6}
$$

Therefore, $\rho_{\phi,B}$ and $j_{\phi,B}$ are uniformly bounded by a constant c depending only on $\|\Phi_0\|_\infty$ and ν:

$$0 \le \rho_{\phi,B} \le c \quad \text{and} \quad |j_{\phi,B}| \le c.$$

Then the solution η in $H^1_0(\Omega)$ of

$$-\Delta_x \eta = -\frac{q}{\varepsilon_r}\rho_{\phi,B}$$

is uniformly bounded in $H^2(\Omega)$ and satisfies $q\eta \le 0$. Hence the function $\phi_1 = \Phi_0 + \eta$ lies in a bounded set of $H^2(\Omega)$, which is a compact set of $H^1(\Omega)$ and satisfies

$$\phi_{1/\partial\Omega} = \phi_0, \qquad q\phi_1 \le q\Phi_0.$$

The function

$$j_{\phi,B} + \alpha\frac{\varepsilon_r}{q}\nabla_x(\phi_1 - \Phi_0)$$

belongs to a bounded set of $L^2(\Omega)$ and satisfies

$$\nabla_x \cdot \left[j_{\phi,B} + \alpha\frac{\varepsilon_r}{q}\nabla_x(\phi_1 - \Phi_0)\right] = 0.$$

Then the solution D of

$$
\begin{aligned}
&\nabla_x \wedge D = \mu_r q\left[j_{\phi,B} + \alpha\frac{\varepsilon_r}{q}\nabla_x(\phi_1 - \Phi_0)\right], \qquad \nabla_x \cdot D = 0, \\
&D \cdot \nu = 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

belongs to a bounded set of $H^1(\Omega)$. Thus $B_1 = B_0 + D$ belongs to a compact set of $L^2(\Omega)$ and we have proved that (ϕ_1, B_1) lies in a compact subset of Ξ.

Let us show the continuity of Γ. Let (ϕ_n, B_n) in Ξ be such that ϕ_n converges to ϕ in $H^1(\Omega)$ and B_n converges to B in $L^2(\Omega)$. Then $F_\alpha(\phi_n, B_n)$ converges towards $F_\alpha(\phi, B)$ in $C^1_b(\Omega \times \mathbb{R}^3)$. (3.6) implies that there is a subsequence $f_n = f_{\phi_n,B_n}$ which converges weakly in $L^2(\Omega \times \mathbb{R}^3)$ towards some f. Then, with the help of Proposition 2.2, $\lambda(f_n)$ and $\mu(f_n)$ converge towards $\lambda(f)$ and $\mu(f)$, respectively, in $L^2(\Omega \times \mathbb{R}^3)$, so $C(f_n)$ converges to $C(f)$ in the distributional sense. It follows that f solves the Poisson equation associated with (ϕ, B) and hence is equal to $f_{\phi,B}$. Then ρ_n and j_n, respectively, converge weakly to $\rho_{\phi,B}$ and $j_{\phi,B}$ in $L^2(\Omega)$. Since the sequence $\Gamma(\phi_n, B_n) = (\phi_{n,1}, B_{n,1})$

belongs to a compact set of $H^1(\Omega)\times L^2(\Omega)$, the last convergences show that $(\phi_{n,1}, B_{n,1})$ converges to (ϕ_1, B_1) in $H^1(\Omega) \times L^2(\Omega)$.

Therefore, the Schauder fixed point theorem applies, which shows the existence of a solution $(f_\alpha, \phi_\alpha, B_\alpha)$ of (3.2)–(3.4). Moreover, f_α satisfies

$$0 \le f_\alpha \le \left[1 + \exp\left(\frac{\mathcal{E}(p) - \frac{q}{\hbar}\Phi_0(x) - \nu}{\theta}\right)\right]^{-1},$$

as $f_{\phi,B}$ does.

We then deduce from the proof of the compactness of Γ that ϕ_α belongs to a bounded set of $H^2(\Omega)$ and that $B_\alpha - B_0$ belongs to a bounded set of $H^1(\Omega)$.

We now prove the existence of a solution of the complete Vlasov–Maxwell system. We first assume that Φ_0 satisfies (H5). Let $(f_\alpha, \phi_\alpha, B_\alpha)$ be the solution of the modified problem for any $\alpha > 0$. In view of the uniform estimates (3.5), there is a subsequence α_n converging to 0, such that $(f_{\alpha_n}, \phi_{\alpha_n}, B_{\alpha_n})$, denoted by (f_n, ϕ_n, B_n), satisfies

$$\begin{aligned}
&f_n \to f \text{ weakly} \quad \text{in } L^2(\Omega \times \mathbb{R}^3),\\
&(\phi_n) \text{ is uniformly bounded in } H^2(\Omega), \quad \phi_{n/\partial\Omega} = \phi_0, \quad \phi_n \to \phi \quad \text{in } H^1(\Omega),\\
&B_n \to B \quad \text{in } L^2(\Omega).
\end{aligned}$$

Then

$$F_{\alpha_n}(\phi_n, B_n) \to F = \frac{q}{\hbar}(\nabla_x \phi - v \wedge B) \quad \text{in } L^2_{\text{loc}}(\bar{\Omega} \times \mathbb{R}^3).$$

As in the proof of the continuity of Γ, $C(f_n)$ converges to $C(f)$ in the distributional sense. Hence f is a solution of (1.7)–(1.10). Moreover, in view of (3.5) and the choice of the constant ν,

$$\rho_n(x) = \int_{\mathbb{R}^3} f_n(x,p)\,dp \quad \text{and} \quad j_n(x) = \int_{\mathbb{R}^3} v(p) f_n(x,p)\,dp$$

are uniformly bounded, so that

$$\rho_n(x) \to \rho(x) = \int_{\mathbb{R}^3} f(x,p)\,dp \quad \text{in } L^\infty(\Omega) \text{ weak star},$$

and

$$j_n(x) \to j(x) = \int_{\mathbb{R}^3} v(p) f(x,p)\,dp \quad \text{in } L^\infty(\Omega) \text{ weak star}.$$

Then it is straightforward to pass to the limit in (3.3), (3.4) and obtain a solution of the Vlasov–Maxwell system. To get rid of the restriction (H5), we introduce a sequence $\Phi_{0,n}$ such that

$$\Phi_{0,n} \in C^2_b(\Omega), \quad \Phi_{0,n} \to \Phi_0 \quad \text{in } H^1(\Omega), \quad \|\Phi_0\|_\infty \le c_1$$

and pass to the limit of the corresponding solutions (f_n, ϕ_n, B_n).

Appendix: A compactness result. This section is devoted to the proof of Proposition 2.2, which we now restate.

PROPOSITION 2.2. *Let $(f_n), (g_n)$, and (h_n) be bounded sequences of $L^2(\Omega \times \mathbb{R}^3)$ that satisfy*

$$v(p) \cdot \nabla_x f_n = \operatorname{div}_p g_n + h_n \tag{4.1}$$

in the sense of distributions. Then for any Hilbert–Schmidt operator K defined on $L^2(\mathbb{R}^3)$, the sequence $(K(f_n(x,\cdot))$ is relatively compact in $L^2(\Omega \times \mathbb{R}^3)$.

Proof of Proposition 2.2. Since K is a Hilbert–Schmidt operator, there is a kernel ϕ in $L^2(\mathbb{R}^6)$ such that

$$K : f \to K(f)(x,p) = \int_{\mathbb{R}^3} \phi(p,q) f(x,q)\, dq. \tag{4.2}$$

Let ϕ^N be a sequence converging to ϕ in $L^2(\mathbb{R}^6)$ and verifying

$$\phi^N(p,p') = \sum_{i=1}^{i=M} \psi_i^N(p)\varphi_i^N(p'), \tag{4.3}$$

where ψ_i^N and φ_i^N are compactly supported and indefinitely differentiable. The Hilbert–Schmidt operator K is the uniform limit of the integral operators K^N of kernel ϕ^N, since the norm of $K^N - K$ is the norm of $\phi^N - \phi$ in $L^2(\mathbb{R}^6)$. From classical compactness results (see [4], [5], and [6]), the sequences

$$(\psi_i^N(p) \int_{\mathbb{R}^3} f_k(x,p')\psi_i^N(p')\, dp')_{k\geq 1}$$

belong to a compact set of $L^2(\mathbb{R}^6)$, so finite sums of such sequences also belong to a compact set of $L^2(\mathbb{R}^6)$. By the diagonal process we construct a subsequence (f_{k_p}) such that the sequence $(K^N(f_{k_p}))_{p\geq 1}$ converges.

Let us show that $(K(f_{k_p}))_{p\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^6)$:

$$\begin{aligned} \|K(f_{k_p}) - K(f_{k_q})\| &\leq \|(K^N - K)(f_{k_p} - f_{k_q})\| + \|K^N(f_{k_p}) - K^N(f_{k_q})\| \\ &\leq 2\|K^N - K\|M + \|K^N(f_{k_p}) - K^N(f_{k_q})\|, \end{aligned} \tag{4.4}$$

where M is a bound of $\|f_k\|_{L^2(\mathbb{R}^6)}$. $\varepsilon > 0$ being given, there is an integer N_0 such that

$$2\|K^{N_0} - K\|M < \frac{\varepsilon}{2}. \tag{4.5}$$

$(K^{N_0}(f_{k_p}))$ being a Cauchy sequence, there is an integer P such that for every $p \geq P$ and $q \geq P$,

$$\|K^{N_0}(f_{k_p}) - K^{N_0}(f_{k_q})\|_{L^2(\mathbb{R}^6)} \leq \frac{\varepsilon}{2}. \tag{4.6}$$

Using (4.4)–(4.6) leads to

$$\|K(f_{k_p}) - K(f_{k_q})\|_{L^2(\mathbb{R}^6)} \leq \varepsilon, \tag{4.7}$$

which proves that $(K(f_{k_p}))_{p\geq 1}$ is a Cauchy sequence in $L^2(\mathbb{R}^6)$, and ends the proof.

REFERENCES

[1] C. BARDOS, *Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d'approximations; application à l'équation de transport*, Ann. Sci. Ecole Norm. Sup., 4e série, 3 (1970), pp. 185–233.

[2] J. S. BLAKEMORE, *Semiconductor Statistics*, Pergamon Press, Oxford, 1962.

[3] P. Degond, F. Poupaud, B. Niclot, and F. Guyot, *Semiconductor modelling via the Boltzmann equation*, Lectures in Appl. Math., 25 (1990), pp. 51–73.

[4] R. J. Di Perna and P. L. Lions, *Global weak solutions of Vlasov–Maxwell systems*, Comm. Pure Appl. Math., 62 (1989), pp. 729–757.

[5] F. Golse, P. L. Lions, B. Perthame, and R. Sentis, *Regularity of the moments of the solution of a transport equation*, J. Funct. Anal., 76 (1988), pp. 110–125.

[6] F. Golse and F. Poupaud, *Limite fluide des équations de Boltzmann des semiconducteurs pour une statistique de Fermi-Dirac*, Asymptotic Anal., 6 (1992), pp. 135–160.

[7] F. Poupaud, *A half-space problem for a non-linear Boltzmann equation arising in semiconductor statistics*, Math. Models Methods Appl. Sci., 14 (1991), pp. 121–137.

[8] ———, *Boundary value problems for the stationary Vlasov–Maxwell system*, Forum Math., 4 (1992), pp. 499–527.

[9] ———, *On a system of non linear Boltzmann equations of semiconductor physics*, SIAM J. Appl. Math., 50 (1990), pp. 1593–1607.

[10] ———, *Boundary value problems in semiconductors for the stationary Vlasov–Maxwell–Boltzmann equations*, IMA Vol. Math. Appl. 59, Springer-Verlag, Berlin, 1994.