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Asymptotic techniques for kinetic problems of Boltzmann type (**)

Contents

1 - Introduction

This section contains some background material on gases for the following presentation, including references to more complete introductions for each separate topic.

In physics, gases were in the beginning mainly treated from a gas dynamic perspective as continua in space, and later by Newtonian dynamics as interacting systems of particles. In the middle stands gas kinetics with its gases modelled in a

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probabilistic way with respect to velocity. Here the Boltzmann equation is the original paradigm with its evolution driven by pair-interactions. When on the other hand the kinetic evolution is dominated by collective influences on one particle from all the other particles, then the Vlasov equation takes central stage.

The Boltzmann equation can formally and in a few notable cases rigorously, be derived from particle mechanics via the so called BBGKY hierarchy ([L], [IP] and others). With an n-particle Hamiltonian for pair-interactions in a container Ω,

$$H_N^\Omega = \sum_1^N \frac{p_i^2}{2} + \sum_{i<j=1}^N \Phi(q_i - q_j) + \sum_1^N u^\Omega(q_i),\ u^\Omega = 0 \text{ in } \Omega,\ = \infty \text{ outside } \Omega,$$

the Hamiltonian system for the n-particle evolution becomes

$$\frac{\partial P_i}{\partial t} = -\frac{\partial H_N^\Omega(X)}{\partial Q_i},\quad \frac{\partial Q_i}{\partial t} = \frac{\partial H_N^\Omega(X)}{\partial P_i},\qquad t>0,\quad X=(Q,P),$$

$$P_i(0) = p_i,\quad Q_i(0) = q_i,\quad i = 1, ..., N,$$

or in Poisson brackets

$$\frac{d}{dt} f(X) = \{f(X), H_N^\Omega\},$$

where

$$\{f(X), H_N^\Omega\} = \Big(\sum_{i=1}^N (P_i, \frac{\partial f(X)}{\partial Q_i}) - \sum_{i\neq j=1}^N (\frac{\partial}{\partial Q_i}\Phi(Q_i - Q_j), \frac{\partial f(X)}{\partial P_i})$$
$$- \sum_{i=1}^N (\frac{\partial}{\partial Q_i} u^\Omega(qQ_i), \frac{\partial f(X)}{\partial P_i})\Big).$$

This is the Liouville equation for the evolution of the phase space density f_N. Integrating away all but s particles gives a hierarchy of equations, the BBGKY hierarchy, with the equation for the s-particle density

$$\frac{d}{dt} f_s = -\mathcal{H}_s f_s + ([\mathcal{H}, \int dx] f_{s+1}).$$

Here [,] denotes a certain commutator of operators. For finite N the hierarchy is equivalent to the Liouville equation, but letting N tend to infinity and s run from one to infinity, it can also be used for a coarse grained description of states of systems with infinitely many particles. In particular, $s = 1$ gives the Boltzmann equation under the hypothesis of molecular chaos (or factorization of f_s into one-particle products) in the so-called Boltzmann Grad limit with the radius of the molecules and the size of the vessel appropriately scaled when $N \to \infty$. (For a broad discussion of this topic, see [CGP], [CIP].)

The n-particle evolution is reversible, whereas the limiting coarse-grained Boltzmann equation has an inbuilt arrow of time given by its negative entropy dissipation rate.

Velocities in the pair collisions of the Boltzmann equation in $\mathbb{R}^n$ – (v, v_*) (before) $\to (v', v'_*)$ (after) – are connected by

$$\mathbf{v}' = \frac{\mathbf{v}+\mathbf{v}_*}{2} + \frac{|\mathbf{v}-\mathbf{v}_*|}{2}\sigma,$$
$$\mathbf{v}'_* = \frac{\mathbf{v}+\mathbf{v}_*}{2} - \frac{|\mathbf{v}-\mathbf{v}_*|}{2}\sigma,$$

where $\sigma \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$. The density of a kinetic gas is as usual modelled by nonnegative functions $f(x, v)$ with x the position and v the velocity. With respect to the velocities of the two particles before collision (v, v_*) and the ones after collision (v', v'_*), we shall write

$$f(v) = f, f(v_*) = f_*, f(v') = f', f(v'_*) = f'_*.$$

The x-domain Ω will in our main example be the position space between two coaxial cylinders with inner normal $n(x)$. On the ingoing boundary $\partial\Omega^+ = \{(x, v) \in \partial\Omega \times \mathbb{R}^n; v \cdot n(x) > 0\}$ indata f_b may be given, and a reflection operator $\mathcal{R}$ can be defined for diffuse reflection, e.g. the Maxwellian type

$$f(x, v) = cM(v) \int_{v' \cdot n(x) < 0} |v' \cdot n(x)| f(x, v') dv',$$

where the Maxwellian M is a Gaussian distribution. Combining these two boundary conditions leads to the so called mixed boundary conditions,

$$f = \Theta \mathcal{R} f + (1 - \Theta) f_b \qquad 0 \leq \Theta \leq 1. \tag{1.1}$$

The stationary Boltzmann equation in the domain Ω is

$$\begin{aligned} v \cdot \nabla_x f(x, v) &= Q(f, f)(x, v) = Q^+(x, v) - Q^-(x, v) = Q^+(x, v) - f\upsilon(f)(x, v) \\ &= \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \omega)[f'f'^* - ff^*] d\omega dv_*, \quad x \in \Omega, v \in \mathbb{R}^n, \end{aligned} \tag{1.2}$$

where $Q^+ - Q^-$ is the splitting into gain and loss parts of the collision operator Q, and υ is the collision frequency. In this equation $v \cdot \nabla_x f(x, v) dx dv$ is the transport term, i.e. represents the net variation per unit time due to the free flow in and out of the volume element $dx dv$ centered at (x, v) in phase space; $Q^-(x, v) dx dv$ represents the decrease per unit time of the number of particles in the same volume element by collisions with all other particles that are at the position x at the same time; and

$Q^+(x,v)dxdv$ represents the increase per unit time of the number of particles in the volume element as the result of collisions involving all particles at position x with velocities (v', v'_*). The kernel B describes the specific collision process under study. A discussion of how to compute B in particular cases can be found in [LaLi] Section 18. E.g. for interactions inversely proportional to some power of the distance, this function B has a non-integrable singularity in the angular variable at grazing collisions. To remove such singularities, the Grad cut-off assumption is usually added, replacing the divergent angular dependence by an integrable one, thereby guaranteeing separately convergent gain and loss terms.

Multiplying $Q(f,f)$ with a function $\psi(v)$, integrating with respect to velocity and changing variables, formally gives

$$\int_{\mathbb{R}^3} Q(f,f)(v)\psi(v)dv = \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} B(f'f'_* - ff_*)(\psi + \psi_* - \psi' - \psi'_*)dvdv_*dw.$$

In particular this integral vanishes for $\psi = 1, v, |v|^2$. In the cases of interest in these lectures, the formal calculations can be rigorously justified. Taking $\psi = \ln f$, we obtain the entropy dissipation rate

$$-e(f) = \int B(ff_* - f'f'_*)ln\frac{ff_*}{f'f'_*}dxdvdv_*dw.$$

The entropy dissipation rate is strictly negative except for Maxwell distributions $M_{\rho,u,\theta} = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}}exp(-\frac{(v-u)^2}{2\theta})$, i.e. the equilibria for which the entropy dissipation vanishes. For additional general introductory material on kinetic theory, you may consult [C] or [CIP] and their references.

Asymptotic studies of the Boltzmann equation like this work, require scalings for collision terms, for variables, and for boundary values. The variables are first rescaled to make the equation non-dimensional. Physically motivated additional scalings in some parameter like the mean free path, may then be introduced for particular situations to obtain formal comparison between the kinetic models at leading order and corresponding gas dynamic ones. To go from the kinetic microscopic to the macroscopic fluid dynamic descriptions, the conserved fields have to be slowly varying on the kinetic scale and have reasonable space variations over macroscopic distances. To expose these fluid fields, power series expansions in the scaling variable are inserted into the kinetic equations and coupled with formal truncations. A rest term is added to the truncated expansion for questions of rigorous kinetic existence, and likewise for convergence issues when the mean free path tends to zero. Let us consider some examples.

In the *incompressible* case, the expansions and the limit-takings may be carried out starting from a normalized global (i.e. space independent) Maxwellian equilibrium distribution function $M = (2\pi)^{-\frac{3}{2}}e^{-\frac{v^2}{2}}$, and with the scaling $F = MG_\varepsilon \geq 0$. A useful parameter is the Knudsen number, the ratio between the microscopic and macroscopic space units, such as the molecular mean free path (in ordinary air 10^{-5} cm) to a typical length scale for the flow, often based on the gradients occuring in the flows. With ε^j the Knudsen number or the mean free path, we get a Boltzmann equation in G_ε,

$$\varepsilon\partial_t G_\varepsilon + v \cdot \nabla_x G_\varepsilon = \frac{1}{\varepsilon^j} J(G_\varepsilon, G_\varepsilon).$$

Here J is the rescaled quadratic Boltzmann collision operator,

$$J(\Phi,\Psi)(v) := \frac{1}{2}\int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v')\Psi(v'_*) + \Phi(v'_*)\Psi(v')$$
$$- \Phi(v_*)\Psi(v) - \Phi(v)\Psi(v_*))dv_*d\omega.$$

Also its linearization around 1 is an important operator in kinetic theory;

$$(L\Phi)(v) := \int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v') + \Phi(v'_*) - \Phi(v_*)$$
$$- \Phi(v))dv_*d\omega = K(\Phi) - \nu\Phi.$$

With $G_\varepsilon = 1 + \varepsilon^m g_\varepsilon$, the term of order ε^m denoted by g_ε, determines the hydrodynamic fields (ρ, u, θ) representing the leading order density, velocity, and temperature fluctuations. The equation for the g_ε perturbation becomes

$$\varepsilon\partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon^j} L g_\varepsilon = \varepsilon^{m-j} J(g_\varepsilon, g_\varepsilon)$$

$\Longrightarrow$ (formally when $\varepsilon \to 0$)

$$g_\varepsilon \to \rho + u\cdot v + \theta(\frac{1}{2}v^2 - \frac{3}{2})$$

$\nabla_x \cdot u = 0$ (incompressibility), $\nabla_x(\rho + \theta) = 0$ (Boussinesq relation) together with

$j > 1, m = 1:$ $\partial_t u + u.\nabla_x u + \nabla_x p = 0,\quad \partial_t\theta + u\cdot\nabla_x\theta = 0$ E.E.

$j = 1, m > 1:$ $\partial_t u + \nabla_x p = \mu\Delta_x u,\quad \partial_t\theta = \kappa\Delta_x\theta$ (Stokes' eqn.)

$j = 1, m = 1:$ $\partial_t u + u.\nabla_x u + \nabla_x p = \mu\Delta_x u,\quad \partial_t\theta + u\cdot\nabla_x\theta = \kappa\Delta_x\theta$ N.S.E.

More generally we may start from a local Maxwellian

$$M_{\rho,u,\theta} = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}} exp(-\frac{(v-u)^2}{2\theta}),$$

and be interested in solutions f_ε to the Boltzmann equation

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f\varepsilon),$$

where f_ε is a perturbation of a Maxwellian $M_{\rho,u,\theta}$, which corresponds to the solution of some equations in *compressible* gas dynamics. Also f_ε is an approximate solution of order p if

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f\varepsilon) + O(\varepsilon^p).$$

Write f_ε as an asymptotic expansion plus a rest term,

$$f_\varepsilon = \sum_{j=0}^{j_1} \varepsilon^j f_j + \varepsilon^{j_0} R.$$

This may be inserted into the Boltzmann equation and followed by a formal identification as equations of one order at a time (the Hilbert expansion), either just ending at some suitable order j_1, or ending by rigorously solving the rest term problem. The procedure in its simplest form is

$$\begin{aligned} &\text{order} \quad -1: \quad Q(f_0, f_0) = 0 \implies f_0 = M_{\rho(x),u(x),\theta(x)}(v) \\ &\text{order} \quad 0: \quad \partial_t f_0 + v \cdot \nabla_x f_0 = Q(f_0, f_1) + Q(f_1, f_0). \end{aligned} \tag{1.3}$$

The expansion $\sum_{j=0}^{j_1} \varepsilon^j f_j$ is of course not by itself a density solution of the Boltzmann equation, since it satisfies the Boltzmann equation only up to some order, and may by its essentially polynomial character become negative, whereas a real density should be everywhere positive.

As basis for the kernel of L in $L^2_M(\mathbb{R}^3)$ (i.e. L^2 in velocity with Maxwellian weight function), we take $\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$. The right hand side in the zero order equation of (1.3) is orthogonal to the fluid dynamic $\psi_{..}$-moments, which span the kernel of L. A corresponding fluid dynamic projection gives the Euler equations of compressible gas dynamics

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ &\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x (\rho\theta) = 0, \\ &\partial_t (\rho(\frac{1}{2}u^2 + \frac{3}{2}\theta)) + \nabla_x \cdot (\rho u(\frac{1}{2}u^2 + \frac{5}{2}\theta)) = 0. \end{aligned}$$

Also mathematically and physically interesting, but not implied by the formal asymptotics, is in what sense the leading order gas dynamics equations are limits of the kinetic ones. The Euler equations obviously do not depend on any detailed information about the Boltzmann equation, not even the cross section of the collisions. They describe what happens at microscopic times of order ε^{-1}. Composite molecules on the other hand, require additional terms for unavoidable rotational and vibrational modes of interaction.

To reach instead the compressible Navier-Stokes equations, one could perform the Chapman-Enskog variant of the Hilbert expansion, adding a kind of equation expansion. Low orders are then of most interest for obtaining/improving/varying fluid dynamic models. Up to the Navier-Stokes level all is simple. We may start from

$$f_\varepsilon = M_{\rho_\varepsilon,u_\varepsilon,\theta_\varepsilon}(1 + \varepsilon f_{1\varepsilon} + \varepsilon^2 f_{2\varepsilon}), \tag{1.4}$$

and assume that $\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon$ solve the compressible Navier-Stokes system

$$\begin{aligned}
&\partial_t\rho_\varepsilon + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0,\\
&\rho_\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x)u_\varepsilon + \nabla_x(\rho_\varepsilon\theta_\varepsilon) = \varepsilon \nabla_x \cdot(\mu_\varepsilon(Du_\varepsilon)),\\
&\frac{3}{2}\rho_\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x)\theta_\varepsilon + \rho_\varepsilon\theta_\varepsilon \nabla_x . u_\varepsilon = \varepsilon \frac{1}{2}\mu_\varepsilon D(u_\varepsilon) : D(u_\varepsilon) + \varepsilon \nabla_x [\kappa_\varepsilon \nabla_x \theta_\varepsilon].
\end{aligned}$$

Inserting (1.4) into the Boltzmann equation, gives $f_{1\varepsilon}, f_{2\varepsilon}$, such that (1.4) becomes an approximative solution of the Boltzmann equation of order two (see [BGL]). The transport coefficients, μ_ε (viscosity) and κ_ε (thermal conductivity), are in physics usually taken as experimentally known. However, this is not always possible. From a kinetic perspective, on the other hand, they are always given by the equations, namely the collison operator dependent term $f_{1\varepsilon}$ which contains the main contribution to the momentum and heat flow dissipation. The first order microscopic term is thus the main responsible for the conversion of mechanical w ·k to heat and the transport of heat to the boundary. Adding a rest term, a true sol ·n can be obtained for the Boltzmann equation. Conversely a solution to the Bo' ıann equation may sometimes be used to derive rigorously a Navier-Stokes description from the Boltzmann one, that describes what happens for microscopic time s of order ε^{-2}. After mild changes in the set-up, extra terms may appear in the Navier-Stokes system (called ghost terms when their origin is not from leading order but comes from higher order terms). For a more extended introduction to such asymptotics, see Chapter 2 in [BGP] with references.

Proceeding beyond the Navier-Stokes level in the Chapman-Enskog procedure introduces undesired effects; well-posedness and the monotone entropy property may e.g. disappear. Among the many efforts to ameliorate this higher order situation, we mention two recent approaches, by M. Slemrod [S] using certain rational approximations, and A. Bobylev's operator calculus with projections [B], both delivering well-posed alternative equations.

This work will focus on stationary aspects. Stationary solutions are of importance in their own right, but also as time-asymptotics, and in rarefied gas dynamics. The latter deals with gas flows, where Navier Stokes type equations are not valid in some significant region of the flow field. The broad picture is one of normal regions where the gas flow follows the macroscopic fluid equations, plus thin shock layers, boundary layers, and initial layers, where matching conditions are sought between different fluid regions or between fluid regions and boundaries.

We shall here concentrate on the boundary layer case and present in detail some situations where the gas is contained between two concentric rotating cylinders, also considering the scaling limit for vanishing Knudsen number. The two-rolls set-up is classical on the fluid dynamics side with a surprisingly varied bifurcation behaviour, when the rotation rates of the cylinders change, which is well demonstrated in the experimental work of Andereck, Liu and Swinney [ALS]. An interesting question is how much of the bifurcations survive on the kinetic side. One may crudely expect that, as soon as there is a rigorous enough mathematical analysis of the fluid behavior, then those results should in a provable way carry over to the kinetic side. That is so in the the situations discussed in the present work. It is demonstrated how the leading order fluid terms dominate the higher order behaviour, when the solutions are close to equilibrium.

Systematic asymptotic studies close to equilibrium started already in the 1960-ies with Grad [G], Kogan [K], and Guiraud [Gu] among the pioneers, and with the main arguments based on fixed points and contraction mapping techniques. Two main approaches are presently in use, one based on energy methods in Sobolev spaces (i.e. involving L^p-estimates of derivatives). The other employs a setting of mixed weighted L-spaces, where precise spectral aspects are readily available. We shall here use the latter approach to study certain fully nonlinear stationary kinetic problems between rotating cylinders, including fluid limits when instabilities (bifurcations) arise. Part of the results were first published in [AN2] and [AN3].

The plan of the paper is as follows. In Section 2 weak type existence theory is first compared to the asymptotic approach, the latter being more accessible to numerical studies and also able to give detailed information on many questions of physical importance. The specific asymptotic set-up for the paper is introduced and further analysed in three specific situations with different scaling behaviour, including bifurcations.

The third section studies a priori estimates for the three cases, in the process introducing various technical approaches. Based on the previous asymptotic expansions and a priori estimates, Section 4 proves existence results for stationary solutions in the three cases and considers their fluid limits.

The results in Section 5 are new. There nonlinear stability of the solutions in the laminar case is proved in detail. The other two cases are briefly discussed.

Section 6 proves positivity of the earlier obtained stationary solutions, a result which is new for hard forces, extended from the case of Maxwellian molecules in [AN3].

2 - A kinetic gas between two coaxial cylinders

In this section asymptotic expansions are introduced and discussed for three archetypical two-rolls situations.

Consider the stationary Boltzmann equation in the space Ω between two coaxial cylinders with radii $r_A < r_B$. Denote by (r, θ, z) and (v_r, v_θ, v_z) respectively, the cylindrical spatial coordinates and the corresponding velocity coordinates. Let us start with parameter ranges where the system stays axially and rotationally uniform, the interesting solutions then being positive functions $f(r, v_r, v_\theta, v_z)$. In these coordinates the Boltzmann equation may be written

$$v_r \frac{\partial f}{\partial r} + \frac{1}{r} Nf = \frac{1}{\varepsilon^j} Q(f, f), \qquad r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in \mathbb{R}^3. \tag{2.1}$$

Here

$$Nf := v_\theta^2 \frac{\partial f}{\partial v_r} - v_\theta v_r \frac{\partial f}{\partial v_\theta}.$$

In the collision term Q the kernel $B = |v - v_*|^\beta b(\theta)$, where $b \in L^1_+(S^2)$, and $0 \le \beta \le 1$ in the hard force setting of these lectures.

The Knudsen number $k = \varepsilon^j$ will be considered for various j's. As boundary conditions, functions f_b are given on the ingoing cylinder boundary $\partial\Omega^+$, i.e. $\{(r_A, v); v_r > 0\}$ and $\{(r_B, v); v_r < 0\}$. For the axially homogeneous case we may assume that the solutions are even in the v_z-variable. The most general we are then able to say about the solvability of the problem is

Theorem 2.1 [AN1]. *Let β be the power of the relative velocity in the Boltzmann collision kernel. Given $m = \int_{r_A}^{r_B} \int_{\mathbb{R}^3} (1 + |v|)^\beta f dx dv$ and ingoing boundary values f_b with finite flow of mass, energy and entropy, then there exists a weak L^1-solution to the Boltzmann equation for hard forces in the two-rolls domain with β-moment m and the indata profile kf_b for some k depending on m.*

Thus for mere existence it is enough to require that the flows of mass, energy and entropy are finite for f_b. Also the mixed boundary conditions (1.1) can be handled. Results in this generality are based on weak L^1 compactness coming from the entropy dissipation control. It gives on the other hand no information about uniqueness, isolated solutions, fluid limits with extra terms, or possible ghost effects. Such results have instead to be based on the asymptotic methods inititated by Grad [G], Kogan [K] and Guiraud [Gu] a full generation ago. But still today many, if not most, important problems are open when it comes to rigorous mathematical analysis. The 1993 monograph by Maslova [M] is probably still the best introduction to the rigorous mathematics in the area. The present frontiers reached by rigorous mathematics unfortunately lag far behind what has been obtained in the approach by formal asymptotics and scientific computing. There two recent monographs by Sone, [S1] and [S2] give a good picture of the state of the art. In [S1] one also finds a thorough discussion about the asymptotic expansions for the two-rolls problems of this lecture series, including many aspects not covered here.

For the asymptotic problems in the domain between the two rotating cylinders, our main concern in this work will be with (multiple) isolated solutions, bifurcations and strict positivity, when the boundary indata are given as Maxwellians M_a with known boundary pressure P_a, temperature T_a, and rotation rate $v_{\theta a}$, where $a = A$ for the inner and B for the outer cylinder. Split the solution to the BE (2.1) as $f = M(1 + \varphi + \varepsilon^{j_0}R) = M(1 + \Phi)$ with φ an asymptotic expansion,

$$\varphi = \sum_{1}^{j_1} \varepsilon^j \Phi^j, \quad M = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{v^2}{2}\right), \tag{2.2}$$

and with R, the rest term, in turn split into

$$R = P_0 R + (I - P_0)R = R_{\|} + R_{\perp}.$$

The projection P_0 represents the fluid dynamic part. The asymptotic expansion (2.2) has boundary values equal the corresponding terms up to a suitable order in the ε-expansions of the boundary Maxwellians M_a. The remaining part of the boundary values are taken care of by the rest term.

As orthonormal basis for P_0 in $L^2_M(\mathrm{R}^3)$ (i.e. L^2 with weight function M), we take $\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$.

The new unknown $\Phi(r, z, v_r, v_\theta, v_z)$ should solve

$$v_r \frac{\partial \Phi}{\partial r} + v_z \frac{\partial \Phi}{\partial z} + \frac{1}{r} N\Phi = \frac{1}{\varepsilon}(L\Phi + J(\Phi, \Phi)).$$

Here J is the rescaled quadratic Boltzmann collision operator,

$$J(\Phi,\psi)(v) := \frac{1}{2}\int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v')\psi(v'_*)+\Phi(v'_*)\psi(v') - \Phi(v_*)\psi(v) - \Phi(v)\psi(v_*))dv_*d\omega,$$

and L is this operator linearized around 1,

$$(L\Phi)(v) := \int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)M(v_*)(\Phi(v')+\Phi(v'_*)-\Phi(v_*) - \Phi(v))dv_*d\omega = K(\Phi)-\nu\Phi.$$

By a change of variables

$$(\varphi, Lf) := \int MfL\varphi dv = \int MLf\varphi dv$$
$$= -\frac{1}{4}\int(\varphi'_*+\varphi'-\varphi_*-\varphi)(f'_*+f'-f_*-f)BMM_*dvdv_*d\omega.$$

In particular we notice that $\varphi = f$ gives $(f,Lf)\leq 0$. Taking φ as a fluid moment ψ_j, implies that $(\psi_j, Lf)=0$ for all f, hence that the fluid dynamic functions are in the kernel of L. There are no others since the only solutions to the equation of Cauchy type $f+f_*-f'-f'_*=0$ are the fluid moments, as first shown already by Boltzmann. Hence the kernel of L is spanned by the fluid moments. Moreover,

Lemma 2.2. *There exists a positive constant c such that*

$$-(f,Lf)\geq c\int(\nu^{\frac{1}{2}}(I-P_0)f)^2Mdv.$$

Proof of Lemma 2.2. We give the proof from [M]. Set

$$\bar{K}=\nu^{-\frac{1}{2}}K\nu^{-\frac{1}{2}},$$
$$\bar{\lambda}=\sup\{\lambda;\bar{K}f=\lambda f \quad \text{with} \quad P_0\nu^{-\frac{1}{2}}f=0, \int(\nu^{-\frac{1}{2}}f)^2Mdv=1\}.$$

The compactness of $\bar{K}$ (cf proof of Lemma 3.2 below) together with $(f,Lf)\leq 0$ imply that with $\bar{\lambda}<1$

$$((I-P_0)f, K(I-P_0)f)\leq\bar{\lambda}\int((I-P_0)f)^2\nu Mdv,$$

and so

$$(f,Lf)\leq(\bar{\lambda}-1)\int((I-P_0)f)^2\nu Mdv. \quad \square$$

Lengthy elementary computations show that $L(v_\theta v_r \bar{B}) = v_\theta v_r$, $L(v_r \bar{A}) = v_r(v^2-5)$ for some functions $\bar{B}(|v|)$ and $\bar{A}(|v|)$, with $v_\theta v_r \bar{B}(|v|)$ and $v_r \bar{A}(|v|)$ bounded in the L^2_M-norm (cf. [BGP] Lemma 2.2.3).

Our basic **Case 1** will be this two-rolls set-up with $j=1$ in (2.1) and $j_0=1$, $j_1=2$ in (2.2) with given Maxwellian ingoing, axially uniform boundary data, modelling for instance when the cylinder surfaces are of ice in the form of the solid phase of the gas between them. We assume that (no essential restriction) the inner cylinder is rotating with velocity $\varepsilon u_{\theta A}$, the outer cylinder is not rotating and the temperature and saturated pressure are the same at the two cylinders. Then

$$\begin{aligned} \gamma^+ f(r_A, z, v) &= \frac{1}{(2\pi)} e^{-\frac{1}{2}(v_r^2 + (v_\theta - \varepsilon u_{\theta A})^2 + v_z^2)}, \quad v_r > 0, \\ \gamma^+ f(r_B, z, v) &= \frac{1}{(2\pi)} e^{-\frac{1}{2}v^2}, \quad v_r < 0. \end{aligned} \tag{2.3}$$

We shall keep the same boundary values in the following Case 2-3. To simplify the exposition in these lectures, we shall take $u_{\theta A} = U_{\theta A}(r_B - r_A)$ with $U_{\theta A}$ fixed. This will allow for additional conditions on the size of $r_B - r_A$ when needed in the convergence studies. An alternative would be to have $r_B - r_A$ fixed (even large) and introduce more extended asymptotic expansions.

An axially homogeneous solution $M(1+\Phi)$ will be determined for (2.1), (2.3), with in Case 1 an approximate asymptotic expansion φ of order 2 with boundary values of first and second orders being $\Phi_{Ai}, \Phi_{Bi}(=0)$, $1 \le i \le 2$,

$$\begin{aligned} \Phi_{A1} &= \varepsilon u_{\theta A} v_\theta \\ \Phi_{A2} &= \frac{\varepsilon^2}{2} u_{\theta A}^2(-1 + v_\theta^2), \end{aligned}$$

plus a rest term εR,

$$\Phi(r,v) = \varphi(r,v) + \varepsilon R(r,v),$$

and

$$\varphi(r,v) = \varepsilon \Phi_{H1}(r,v) + \varepsilon^2(\Phi_{H2}(r,v) + \Phi_{K2A}(\frac{r-r_A}{\varepsilon}, v) + \Phi_{K2B}(\frac{r-r_B}{\varepsilon}, v)). \tag{2.4}$$

Here the Hilbert expansion term Φ_{H2} cannot by itself satisfy all boundary conditions. To remedy that, second order additional Knudsen boundary layer terms $\Phi_{K2.}$ are inserted.

In the asymptotic expansion the Hilbert terms Φ_{H1} and Φ_{H2} satisfy

$$L\Phi_{H1} = L\Phi_{H2} + J(\Phi_{H1}, \Phi_{H1}) - v \cdot \nabla_x \Phi_{H1} = 0.$$

Here $L\Phi_{H1} = 0$ implies that $\Phi_{H1} = a_1(r) + d_1(r)v^2 + b_1(r)v_\theta + c_1(r)v_r$. The v_z-term is zero due to the symmetry imposed. For compatibility reasons the hydrodynamic moments of the second equation are zero (cf. also (2.21), (2.41) below). In particular the 1-moment gives for Φ_{H1} that

$$c_1' + \frac{c_1}{r} = 0,$$

hence $c_1(r) = \frac{c}{r}$, where due to the boundary conditions $c = 0$.

Set $w_1 = \int v_r^2 v_\theta^2 \bar{B} M dv$, $w_2 = \int v_r^2 \bar{A} M dv, w_3 = \int v_r^2 v^2 \bar{A} M dv$. It is also consistent (and implied by the fluid dynamic projection equations) to take $a_1 = d_1 = 0$, giving

$$\Phi_{H1}(r,v) = b_1(r)v_\theta. \tag{2.5}$$

Then similarly

$$\Phi_{H2}(r,v) = a_2 + d_2 v^2 + b_2 v_\theta + c_2 v_r + \frac{1}{2} b_1^2 v_\theta^2 + (b_1' - \frac{1}{r} b_1) v_r v_\theta \bar{B}, \tag{2.6}$$

where by fluid dynamic projections and after some computations,

$$b_1(r) = \frac{u_{\theta A}}{r_B^2 - r_A^2}(\frac{r_B^2}{r} - r), \quad (a_2 + 5d_2)' + b_1 b_1' - \frac{1}{r} b_1^2 = 0, \quad c_2(r) = \frac{\gamma_2}{r}, \tag{2.7}$$

$$b_2'' + \frac{1}{r} b_2' - \frac{1}{r^2} b_2 = -\frac{1}{w_1}(b_1' + \frac{1}{r} b_1)c_2, \tag{2.8}$$

$$\begin{aligned}(w_3 - 5w_2)(d_2'' + \frac{1}{r} d_2') = (b_1(b_1' - \frac{1}{r} b_1))' \int M v_r (v^2 - 5)(\tilde{L}^{-1}(2\tilde{J}(v_\theta, v_r v_\theta \bar{B}) \\ - v_r(v_\theta^2 - 1))dv + (b_1 b_1' - \frac{1}{r} b_1^2) \int M(v^2 - 5) N(\tilde{L}^{-1}(2\tilde{J}(v_\theta, v_r v_\theta \bar{B}) - (v_r(v_\theta^2 - 1))dv,\end{aligned} \tag{2.9}$$

for some constant γ_2. With the term $(b_1' - \frac{1}{r} b_1) v_r v_\theta \bar{B}$, the function Φ_{H2} of (2.6) cannot satisfy the boundary conditions Φ_{A2} (resp. Φ_{B2}) at r_A (resp. r_B). That is instead handled by adding Knudsen boundary layers as will be discussed in next lemma. Inserting $1 + \Phi$ into the rescaled Boltzmann equation gives the pure φ-part

$$l = \frac{1}{\varepsilon}(L\varphi + J(\varphi, \varphi) - \varepsilon v \cdot \nabla_x \varphi), \tag{2.10}$$

which is of ε-order two, provided the Knudsen terms satisfy

$$L\Phi_{K2A} = v_r \frac{\partial \Phi_{K2A}}{\partial r}, \quad L\Phi_{K2B} = v_r \frac{\partial \Phi_{K2B}}{\partial r}.$$

Denote by $\eta = \frac{r - r_A}{\varepsilon}$ and $\mu = \frac{r - r_B}{\varepsilon}$.

Lemma 2.3. *There exist a second-order Hilbert term Φ_{H2} defined by (2.6) with a_2, d_2, b_2, c_2 satisfying (2.7-9), and Knudsen terms $\Phi_{K2A}(\eta, v)$, $\Phi_{K2B}(\mu, v)$ such that*

$$(2.11)\qquad \begin{aligned} &v_r \frac{\partial \Phi_{K2A}}{\partial \eta} = L\Phi_{K2A},\\ &\Phi_{K2A}(0, v) = \Phi_{A2}(v) - \Phi_{H2}(r_A, v), \quad v_r > 0,\\ &\lim_{\eta \to +\infty} \Phi_{K2A}(\eta, v) = 0, \end{aligned}$$

and

$$(2.12)\qquad \begin{aligned} &v_r \frac{\partial \Phi_{K2B}}{\partial \mu} = L\Phi_{K2B},\\ &\Phi_{K2B}(0, v) = \Phi_{B2}(v) - \Phi_{H2}(r_B, v), \quad v_r < 0,\\ &\lim_{\mu \to -\infty} \Phi_{K2B}(\mu, v) = 0. \end{aligned}$$

To prove this lemma, we need some properties of the Milne half-space problem:

$$(2.13)\qquad \begin{aligned} &v_r \frac{\partial \psi}{\partial \eta} = L\psi, \quad \eta > 0,\\ &\psi(0, v) = g, \quad v_r > 0,\\ &\int M v_r \psi(\eta, v) dv = m, \quad \eta > 0. \end{aligned}$$

Set $\mathrm{R}^3_+ = \mathrm{R}^3 \cap \{v_r > 0\}$ and take $b_\psi = (a(r), b(r), c(r), d(r))$ as the coefficients of the fluid dynamic moments of ψ (the v_z-moment in our present setting is identically zero by symmetry). The following results about the Milne problem were proved in [BCN] and [GP].

Theorem 2.4. *Let $m \in \mathbb{R}$ and $g \in L^2_{v_r M}(\mathbb{R}^3_+)$. There exists a unique solution ψ to (2.13), which belongs to $L^\infty(r > 0; L^2_{|v_r|M} \cap L^2_{vM}) \cap L^2_{vM}(r > 0)$ and has $b_\psi \in L^\infty(\mathbb{R}_+)$. If $M^{\frac{1}{2}}\varphi = O(|v|^{-n})$ for all $n > 0$, then $b_\infty = \lim_{r \to \infty} b_\psi$ exists, and $|b_\psi - b_\infty| = O(r^{-n})$ for any $n > 0$.*

Proof of Lemma 2.3. It follows from Theorem 2.4, that there are unique solutions ψ, ψ_{2A} and ψ_{2B} to

$$\begin{aligned} &v_r \frac{\partial \psi}{\partial \eta} = L\psi,\\ &\psi(0, v) = 0, \quad v_r > 0,\\ &\int M v_r \psi(\eta, v) dv = 1, \end{aligned}$$

$$v_r \frac{\partial \psi_{2A}}{\partial \eta} = L\psi_{2A},$$

$$\psi_{2A}(0,v) = -(b_1' - \frac{1}{r} b_1)(r_A) v_r v_\theta \bar{B}, \quad v_r > 0,$$

$$\int M v_r \psi_{2A}(\eta, v) dv = 0,$$

$$v_r \frac{\partial \psi_{2B}}{\partial \eta} = L\psi_{2B},$$

$$\psi_{2B}(0,v) = -(b_1' - \frac{1}{r} b_1)(r_B) v_r v_\theta \bar{B}, \quad v_r > 0,$$

$$\int M v_r \psi_{2B}(\eta, v) dv = 0.$$

Moreover,

$$\lim_{\eta \to +\infty} \psi(\eta, v) = a_\infty + d_\infty v^2 + b_\infty v_\theta + v_r,$$

$$\lim_{\eta \to +\infty} \psi_{2A}(\eta, v) = a_{2A,\infty} + d_{2A,\infty} v^2 + b_{2A,\infty} v_\theta,$$

$$\lim_{\eta \to +\infty} \psi_{2B}(\eta, v) = a_{2B,\infty} + d_{2B,\infty} v^2 + b_{2B,\infty} v_\theta,$$

for some constants a_∞, d_∞, b_∞, $a_{2A,\infty}$, $d_{2A,\infty}$, $b_{2A,\infty}$, $a_{2B,\infty}$, $d_{2B,\infty}$ and $b_{2B,\infty}$. Choose

$$a_2(r_A) = \gamma_2 a_\infty + a_{2A,\infty} - \frac{1}{2}(u_{\theta A})^2, \tag{2.14}$$

$$a_2(r_B) = -\frac{\gamma_2}{r_B} a_\infty + a_{2B,\infty}, \tag{2.15}$$

$$d_2(r_A) = \gamma_2 d_\infty + d_{2A,\infty}, \tag{2.16}$$

$$d_2(r_B) = -\frac{\gamma_2}{r_B} d_\infty + d_{2B,\infty}, \tag{2.17}$$

$$b_2(r_A) = \gamma_2 b_\infty + b_{2A,\infty}, \tag{2.18}$$

$$b_2(r_B) = \frac{\gamma_2}{r_B} b_\infty - b_{2B,\infty}. \tag{2.19}$$

Then

$$\Phi_{K2A} = \gamma_2(\psi - a_\infty - d_\infty v^2 - b_\infty v_\theta - v_r)$$
$$+ \psi_{2A} - a_{2A,\infty} - d_{2A,\infty} v^2 - b_{2A,\infty} v_\theta,$$

and

$$\Phi_{K2B}(\mu, v) = -\frac{\gamma_2}{r_B}(\psi(-\mu, -v) - a_\infty - d_\infty v^2 + b_\infty v_\theta + v_r)$$
$$+ \psi_{2B}(-\mu, -v) - a_{2B,\infty} - d_{2B,\infty} v^2 + b_{2B,\infty} v_\theta,$$

satisfy (2.11-12). The first equation in (2.7) defines $a_2 + 5d_2$ if and only if

$$(a_2 + 5d_2)(r_B) - (a_2 + 5d_2)(r_A) = \frac{1}{2}u_{\theta A}^2 + \int_{r_A}^{r_B} \frac{1}{s}\, b_1^2(s)ds,$$

i.e.

$$\gamma_2 = \frac{r_B}{(r_B + 1)(a_\infty + 5d_\infty)}\Big(a_{2B,\infty} - a_{2A,\infty} + 5d_{2B,\infty} - 5d_{2A,\infty} - \int_{r_A}^{r_B} \frac{1}{s} b_1^2(s)ds\Big). \tag{2.20}$$

This fixes γ_2, hence c_2 and $a_2 + 5d_2$. Finally the second-order differential equations (2.8-9) together with the boundary conditions (2.16-19) define b_2 and d_2. □

Case 2. If the Knudsen number is decreased by choosing $j = 4$ in (2.1), but still keeping the rotation velocity of the inner cylinder of order ε, then the boundary layer depth (of order ε) is no longer of the same order as the Knudsen number (i.e. ε^4). That gives rise to additional technical difficulties. In particular we now have to introduce an additional so-called suction boundary layer from first order in ε, and then from third order on also retain the previous Knudsen terms. For convenience we take $r_A = 1$ below in Case 2-3.

An asymptotic expansion φ of order 4 will thus be determined,

$$\begin{aligned}\varphi(r, v) = {} & \varepsilon\Big(\Phi_{H1}(r, v) + \Phi_{W1}(\frac{r - r_B}{\varepsilon}, v)\Big) + \varepsilon^2\Big(\Phi_{H2}(r, v) + \Phi_{W2}(\frac{r - r_B}{\varepsilon}, v)\Big) \\ & + \varepsilon^3\Big(\Phi_{H3}(r, v) + \Phi_{W3}(\frac{r - r_B}{\varepsilon}, v) + \Phi_{K3A}(\frac{r - 1}{\varepsilon^4}, v) + \Phi_{K3B}(\frac{r - r_B}{\varepsilon^4}, v)\Big) \\ & + \varepsilon^4\Big(\Phi_{H4}(r, v) + \Phi_{W4}(\frac{r - r_B}{\varepsilon}, v) + \Phi_{K4A}(\frac{r - 1}{\varepsilon^4}, v) + \Phi_{K4B}(\frac{r - r_B}{\varepsilon^4}, v)\Big).\end{aligned} \tag{2.21}$$

The successive asymptotic computations order by order, allow us to require by (hydrodynamic) orthogonality that

$$\begin{aligned}\int \Phi_{H1}(., v)(1, v_r, v^2)M(v)dv &= \int \Phi_{W1}(., v)(1, v_r, v^2)M(v)dv \\ &= \int \Phi_{H2}(., v)v_r M(v)dv = 0,\end{aligned} \tag{2.22}$$

$$\lim_{\frac{r-r_B}{\varepsilon} \to -\infty} \Phi_{Wi}(\frac{r - r_B}{\varepsilon}, v) = 0, \quad 1 \leq i \leq 4, \tag{2.23}$$

$$\lim_{\frac{r-1}{\varepsilon^4} \to +\infty} \Phi_{KiA}(\frac{r - 1}{\varepsilon^4}, v) = 0, \quad \lim_{\frac{r-r_B}{\varepsilon^4} \to -\infty} \Phi_{KiB}(\frac{r - r_B}{\varepsilon^4}, v) = 0, \quad 3 \leq i \leq 4. \tag{2.24}$$

Here $(\varepsilon\Phi_{H1}+\varepsilon^2\Phi_{H2}+\varepsilon^3\Phi_{H3}+\varepsilon^4\Phi_{H4})(r,v)$ denotes the Hilbert terms up to fourth order. The sum $(\varepsilon\Phi_{W1}+\varepsilon^2\Phi_{W2})(\frac{r-r_B}{\varepsilon},v)$ consists of correction terms allowing the boundary conditions to be satisfied to first and second order. They correspond to suction boundary layer terms at r_B. At third and fourth orders, supplementary boundary layers of Knudsen type described by

$$\varepsilon^3(\Phi_{K3A}(\frac{r-1}{\varepsilon^4},v)+\Phi_{K3B}(\frac{r-r_B}{\varepsilon^4},v))+\varepsilon^4(\Phi_{K4A}(\frac{r-1}{\varepsilon^4},v)+\Phi_{K4B}(\frac{r-r_B}{\varepsilon^4},v)),$$

are also required in order to get all the boundary conditions satisfied.

Let $\psi(\eta,v)$ be the solution to the half-space problem

$$\begin{aligned}&v_r\frac{\partial\psi}{\partial\eta}=L\psi,\quad \eta>0,\quad v\in\mathbb{R}^3,\\ &\psi(0,v)=0,\quad v_r>0,\\ &\int\psi(\eta,v)v_rM(v)dv=1,\quad \eta>0.\end{aligned}\tag{2.25}$$

From Theorem 2.4 about the Milne problem, it follows that there are constants A, D, and E, such that

$$\lim_{\eta\to+\infty}\psi(\eta,v)=A+Dv^2+Ev_\theta+v_r.\tag{2.26}$$

Let the nondimensional density, perturbed temperature and saturated pressure at r_B be

$$\omega_B=\frac{\varepsilon^2}{1+\tau_B}(P_{SB2}-\tau_{B2}),\quad \tau_B=\varepsilon^2\tau_{B2},\quad P_{SB}=\varepsilon^2P_{SB2}.$$

We may here in Case 2 couple the angular velocity to the Knudsen number through

$$P_{SB2}-\frac{r_{B2}^2-1}{r_B^2}u_{\theta A1}^2=\Delta\varepsilon.$$

The boundary condition at r_B in (2.3) is replaced by

$$\gamma^+f(r_B,z,v)=(2\pi)^{-\frac{3}{2}}\frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}}e^{-\frac{v^2}{2(1+\tau_B)}},\quad v_r<0.$$

For the third order asymptotic term that will lead to a bifurcation of the radial velocity – see (2.40) below – if $A+5D<0$.

Lemma 2.5. *Assume that*

$$(A+5D)<0,$$

and set

$$\Delta_{bif} := -\Big(2w_1 \frac{r_B+1}{r_B^3}(A+5D)(3u_{\theta A1}^2)\Big)^{\frac{1}{2}}.$$

For $\Delta > \Delta_{bif}$, *there is no solution* φ *to the family defined in* (2.21-24).

For $\Delta = \Delta_{bif}$, *there is a unique solution* φ *to the family defined in* (2.21-24).

For $\Delta < \Delta_{bif}$, *there are two solutions* φ *to the family defined in* (2.21-24).

Proof of Lemma 2.5. Define $Y := \frac{r - r_B}{\varepsilon}$, and let the expansions $\sum_{k\geq 1} \varepsilon^k \Phi_{Hk}(r,v)$ and $\sum_{k\geq 1} \varepsilon^k(\Phi_{Hk}(r_B+\varepsilon Y, v) + \Phi_{Wk}(Y,v))$ formally satisfy the Boltzmann equation order by order. Then,

$$\begin{aligned}(2.27)\qquad L\Phi_{H1} &= L\Phi_{H2} + J(\Phi_{H1},\Phi_{H1}) = L\Phi_{H3} + 2J(\Phi_{H1},\Phi_{H2})\\ &= L\Phi_{H4} + 2J(\Phi_{H1},\Phi_{H3}) + J(\Phi_{H2},\Phi_{H2}) = 0,\end{aligned}$$

$$(2.28)\qquad v_r\frac{\partial \Phi_{Hk-4}}{\partial r} + \frac{1}{r}N\Phi_{Hk-4} = L\Phi_{Hk} + \sum_{j=1}^{k-1} J(\Phi_{Hj},\Phi_{Hk-j}), \quad k\geq 5,$$

and

$$\begin{aligned}(2.29)\quad L\Phi_{W1} &= L\Phi_{W2} + J(\Phi_{W1}, 2\Phi_{H1}(r_B,.) + \Phi_{W1})\\ &= L\Phi_{W3} + 2J(\Phi_{H1}(r_B,.) + \Phi_{W1},\Phi_{W2}) + 2J(\Phi_{W1},\Phi_{H2}(r_B,.) + Y\Phi'_{H1}(r_B,.))\\ &= L\Phi_{W4} + 2J(\Phi_{W3},\Phi_{H1}(r_B,.) + \Phi_{W1}) + J(\Phi_{W2},\Phi_{W2} + 2\Phi_{H2}(r_B,.)\\ &+ 2Y\Phi'_{H1}(r_B,.)) + 2J(\Phi_{W1},\Phi_{H3}(r_B,.) + Y\Phi'_{H2}(r_B,.)\\ &+ \frac{Y^2}{2}\Phi''_{H1}(r_B,.)) - v_r\frac{\partial \Phi_{W1}}{\partial Y} = 0,\end{aligned}$$

$$\begin{aligned}(2.30)\quad v_r\frac{\partial \Phi_{Wk-3}}{\partial Y} &+ \frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) + \Phi_{Wk-4-i})\\ &= L\Phi_{Wk} + \sum_{j=1}^{k-1} J(2\Phi_{Hj}(r_B,.) + \Phi_{Wj},\Phi_{Wk-j}), \quad k\geq 5.\end{aligned}$$

Similarly to (2.5), by (2.27) $\Phi_{H1}(r,v) = b_1(r)v_\theta$ for some function b_1, and $\Phi_{Hi}, i\geq 2$ split into a fluid dynamical part $a_i(r) + d_i(r)v^2 + b_i(r)v_\theta + c_i(r)v_r$ and a non-fluid-dynamic part involving Hilbert terms of lower order than i. In particular for

$1 \le j \le 4$ we get

$$\Phi_{H1}(r,v) = b_1(r)v_\theta,$$

$$\Phi_{H2} = a_2 + d_2v^2 + b_2v_\theta + \frac{1}{2}b_1^2v_\theta^2,$$

$$\Phi_{H3} = a_3 + d_3v^2 + b_3v_\theta + c_3v_r + b_1d_2v_\theta v^2 + b_1b_2v_\theta^2 + \frac{1}{6}b_1^3v_\theta^3,$$

$$\Phi_{H4} = a_4 + d_4v^2 + b_4v_\theta + c_4v_r + (b_1d_3 + b_2d_2)v_\theta v^2 + (b_1b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2a_2)v_\theta^2$$
$$+ b_1c_3v_rv_\theta + \frac{1}{2}b_1^2b_2v_\theta^3 + \frac{1}{2}d_2^2v^4 + \frac{1}{24}b_1^4v_\theta^4 + \frac{1}{2}b_1^2d_2v_\theta^2v^2.$$

Equations (2.28) have solutions if and only if the following compatibility conditions hold,

$$\int \Big(v_r\frac{\partial \Phi_{Hi}}{\partial r} + \frac{1}{r}N\Phi_{Hi}\Big)(1, v^2-5, v_\theta, v_r)M(v)dv = 0, \quad i \ge 1.$$

They provide first-order differential equations for the functions $a_i(r)$, $b_i(r)$, $c_i(r)$ and $d_i(r)$, $i \ge 1$. In particular,

(2.31) $$(rb_1)' = 0, \quad (10d_2 + b_1^2)' = 0,$$

(2.32) $$(r^2c_3b_2)' = w_1r^2(b_1' - \frac{1}{r}b_1)' + (2w_1 - w_2)r(b_1' - \frac{1}{r}b_1),$$
$$(a_2 + 5d_2 + \frac{1}{2}b_1^2)' = \frac{1}{r}b_1^2,$$

(2.33) $$(a_3 + 5d_3 + b_1b_2)' = \frac{2}{r}b_1b_2,$$

(2.34) $$(rc_3)' = 0,$$

(2.35) $$(a_4 + 5d_4 + b_1b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2a_2 + \frac{35}{2}d_2^2 + \frac{7}{2}b_1^2d_2)'$$
$$= \frac{2}{r}(b_1b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2a_2) + \frac{1}{2r}b_1^4 + \frac{7}{r}b_1^2d_2,$$
$$(rc_4)' = 0.$$

Together with the boundary condition at r_A of first and second orders, this fixes

$$\Phi_{H1}(r,v) = \frac{u_{\theta A}}{r}v_\theta, \quad \Phi_{H2} = -\frac{u_{\theta A}^2}{2r^2} + \frac{u_{\theta A}^2}{10}(1 - \frac{1}{r^2})v^2 + \frac{u_{\theta A}^2}{2r^2}v_\theta^2,$$

and $c_3(r) = \frac{u_3}{r}$, for some constant $u_3 \neq 0$. Moreover, (2.22) and (2.29-30) give that $\Phi_{W1}(Y,v) = z_1(Y)v_\theta$, for some function z_1, and that $\Phi_{Wi}, i \ge 2$ split into a fluid dynamical part $x_i(Y) + y_i(Y)v^2 + z_i(Y)v_\theta + t_i(Y)v_r$ and a non-fluid-dynamic part involving Hilbert terms of lower order than i. Notice that Φ_{W4} is the sum of $z_1'v_\theta v_r\bar{B}$ and a polynomial in the v-variable with bounded coefficients in the r-variable. More precisely,

$$\Phi_{W2} = x_2 + y_2 v^2 + z_2 v_\theta + (b_1(r_B)z_1 + \frac{1}{2}z_1^2)v_\theta^2,$$

$$\begin{aligned}\Phi_{W3} &= x_3 + y_3 v^2 + z_3 v_\theta + t_3 v_r + (b_1(r_B)y_2 + z_1 y_2 + z_1 d_2(r_B))v_\theta v^2 \\ &+ (b_1(r_B)z_2 + z_1 z_2 + z_1 b_2(r_B) + Yb_1'(r_B)z_1)v_\theta^2 \\ &+ (\frac{1}{2}b_1^2(r_B)z_1 + \frac{1}{2}b_1(r_B)z_1^2 + \frac{1}{6}z_1^3)v_\theta^3,\end{aligned}$$

$$\Phi_{W4} = x_4 + y_4 v^2 + z_4 v_\theta + t_4 v_r + z_1' v_r v_\theta \bar{B}(v) + \dots .$$

Equations (2.29) have solutions if and only if the following compatibility (orthogonality) conditions hold,

$$\int \Big(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) + \Phi_{Wk-4-i}\Big)(v^2-5, v_\theta)M(v)dv = 0, \quad k \geq 5, \tag{2.36}$$

and

$$\int \Big(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) + \Phi_{Wk-4-i}\Big)(1, v_r)M(v)dv = 0, \quad k \geq 5. \tag{2.37}$$

Equations (2.36) (resp. (2.37)) provide second-order (resp. first-order) differential equations for y_i and z_i (resp. $x_i + 5y_i$ and t_i). In particular,

$$\begin{aligned} & w_1 z_1'' - \frac{u_3}{r_B} z_1' = 0, \\ & (x_2 + 5y_2 + b_1(r_B)z_1 + \frac{1}{2}z_1^2)' = 0, \\ & w_2 y_2'' + \frac{10}{r_B} y_2' + A_1 = 0, \quad w_1 z_2'' - \frac{u_3}{r_B} z_2' + A_1 = 0, \\ & t_3' = 0, \end{aligned}$$

$$(x_3 + 5y_3 + b_1(r_B)z_2 + z_1 z_2 + z_1 b_2(r_B) + Yb_1'(r_B)z_1)' = \frac{1}{r_B}(2b_1(r_B)z_1 + z_1^2), \tag{2.38}$$

$$\begin{aligned} & w_2 y_3'' + \frac{10}{r_B} y_3' + A_2 = 0, \\ & w_1 z_3'' - \frac{u_3}{r_B} z_3' + \Big((b_1(r_B) + z_1)(c_5(r_B) + t_5)\Big)' + A_2 = 0, \\ & t_4' + \frac{1}{r_B}(t_3 + c_3(r_B)) + c_3'(r_B) = 0, \end{aligned}$$

$$(x_4 + 5y_4)' + A_3 = 0. \tag{2.39}$$

Here $A_i, 1 \leq i \leq 3$, denote terms involving Hilbert and suction coefficients up to order i. Together with the boundary conditions at first and second orders, and the conditions (2.23), this fixes

$$\Phi_{W1}(Y, v) = -\frac{u_{\theta A1}}{r_B} e^{\frac{x_3 Y}{w_1 r_B}} \cdot v_\theta,$$

as well as Φ_{W2} in terms of u_3, and implies that $t_3 = t_4 = 0$. Then, giving the value 0 to any coefficient of order bigger than 5 in the second-order differential equations satisfied by y_i and $z_i, 3 \leq i \leq 4$ and taking into account (2.21-24) fixes the functions y_i and $z_i, 3 \leq i \leq 4$ in terms of u_i. A Knudsen analysis at third and fourth orders makes the first-order differential equations satisfied by $x_3 + 5y_3$ and $x_4 + 5y_4$ compatible with (2.23) at third and fourth orders. Finally u_3 must solve the equation

$$u_3^2(A + 5D)\frac{r_B + 1}{r_B} - \Delta u_3 + \frac{w_1}{2r_B^2}(-3u_{\theta A1}^2) = 0. \tag{2.40}$$

A study of the positive roots u_3 to (2.40) leads to the three cases described in the theorem for Δ with respect to Δ_{bif} . That proof requires a non-degeneracy in the Milne asymptotics (2.26),

$$A + 5D < 0. \tag{2.41}$$

The condition is expected to hold on physical grounds and has been verified numerically for hard spheres and Maxwellian molecules. A mathematical proof of (2.41) related to the numerical approach seems feasible, but has not been undertaken. □

Case 3. The techniques developed for the previous particular two-rolls situations, also hold the key to resolving other and sometimes more famous problems. This third example is such a generalization. The density f will now be allowed to depend on the axial variable z, assuming periodicity in the axial direction. The previous transport term in (2.1) is then extended to include also a z-derivative term $v_z \frac{\partial f}{\partial z}$. We consider the Knudsen number ε^j for $j = 1$ in (2.1), and keep the earlier ingoing Maxwellian data. For small enough parameters, there is an axially uniform solution as in the Case 1. This axially homogeneous cylindrical Couette flow of Case 1 will bifurcate into axially periodic ones – Taylor rolls – when the rotation of the inner cylinder is started from rest and then is being sufficiently increased. The equations for the successive terms in the asymptotic expansions are now no longer ordinary but partial differential equations, which here may be solved by elementary and explicit Fourier methods. We shall only allow bifurcations to a fixed axial period, for convenience taken as $c(r_B - r_A)$, and carry out the computations when $c = 1$. Denote the first (lowest) bifurcation velocity value by $u_{\theta Ab}$, and require that all functions have the symmetry $f(r,z,v_r,v_\theta,v_z)=f(r,-z,v_r,v_\theta,-v_z)$. The first order a symptotic expansion term Φ_{H1}, should satisfy $L\Phi_{H1}=0$, i.e. belong to the kernel of L, hence

$$\Phi_{H1}(r,v)=a_1(r,z)+d_1(r,z)v^2+b_1(r,z)v_\theta+c_1(r,z)v_r+e_1(r,z)v_z. \tag{2.42}$$

The fluid dynamic orthogonality arguments leading to (2.5) in Case 1, here imply that in a one-sided neighbourhood of $u_{\theta Ab}$, the first order coefficients may satisfy a steady (secondary) Taylor Couette fluid flow problem ((4.9) below) with corresponding boundary values. This fluid bifurcation problem was first rigorously studied in [V] using topological Leray Schauder degree, to be followed over the years by a number of alternative treatments and expansions - see [CI] for properties, references and an overview. It follows from that theory that the coefficients in (2.42) are smooth functions with uniform bounds in a neighbourhood of $u_{\theta Ab}$.

Denote by the index b when an axially homogeneous term Φ_{Hj} is evaluated at the first bifurcation velocity $u_{\theta A}=u_{0Ab}$, and let δ^2 denote the deviation from this bifurcation value. With $\Phi_{Hj}=\Phi_{Hjb}+\delta\Phi_j^1$, and Φ_1^1 given by the smooth perturbation to the fluid Taylor Couette problem, we can successively construct higher order terms in the asymptotic expansion. E.g. for $j=2,3$, the perturbations $\Phi_2^1(x,v,\delta)$ and $\Phi_3^1(x,v,\delta)$ should satisfy

$$\tilde{L}\Phi_2^1+g_{1\perp}-v_r\frac{\partial\Phi_1^1}{\partial r}-v_z\frac{\partial\Phi_1^1}{\partial z}-Nh_1=0, \tag{2.43}$$

$$\tilde{L}\Phi_3^1+g_{2\perp}-v_r\frac{\partial\Phi_2^1}{\partial r}-v_z\frac{\partial\Phi_2^1}{\partial z}-Nh_2=0, \tag{2.44}$$

with

$$g_{1\perp}=2\tilde{J}(\Phi_{H1b},\Phi_1^1)+\delta\tilde{J}(\Phi_1^1,\Phi_1^1),\quad h_1=\frac{1}{r}\Phi_1^1,$$
$$g_{2\perp}=2\tilde{J}(\Phi_{H1},\Phi_2^1)+2\tilde{J}(\Phi_{H2b},\Phi_1^1),\quad h_2=\frac{1}{r}\Phi_2^1.$$

The locally uniform smoothness of Φ_{H1} (for small δ), implies by (2.43) spacewise smoothness for $\Phi_{2,\perp}^1$ uniformly for small δ. We may also prove by Fourier techniques, that the fluid dynamic moments of Φ_2^1 and its derivatives are uniformly bounded in L^∞ in a δ^2-neighbourhood of the bifurcation point u_{0Ab} for small enough ε. The procedure may be repeated for the Φ_3^1-term.

To provide the correct boundary values for the problem, we add boundary layer corrections to Φ_2^1 and Φ_3^1 of Knudsen type. Our previous boundary layer analysis based on [GP] applies, when the equations are taken in Fourier space for the periodic z-variable. This is so since at the crucial steps in the decay study for the Milne problem in [GP], the relevant squared L^2-integrals in velocity space of the Fourier coefficients can be added to give (by Parseval's identity) analogous estimates for the corresponding squared L^2-norms with respect to z of Φ_2^1 and Φ_3^1. This also holds for their z-derivatives, which in turn via Sobolev embedding leads to uniform bounds with respect to z for the Knudsen layer terms.

For the interested reader we end this section with a proof of the appearance of this Taylor bifurcation in the present context. Extend the asymptotic expansion of Case 1 by third and fourth order terms $\Phi^3(r,v)$ and $\Phi^4(r,v)$, and denote it by

$$\begin{aligned}&\varepsilon b_1 v_\theta + \varepsilon^2(\varphi_{2u} + \Phi_{K2A}(\eta, v) + \Phi_{K2B}(\mu, v))\\ &\quad + \varepsilon^3(\varphi_{3u} + \Phi_{K3A}(\eta, v) + \Phi_{K3B}(\mu, v)) + \varepsilon^4(\varphi_{4u} + \Phi_{K4A}(\eta, v) + \Phi_{K4B}(\mu, v)),\end{aligned}$$

where $\varphi_{2u} = \Phi_{H2}$ of Case 1. This expansion is uniform with respect to the variable z, and $\eta = \dfrac{r-1}{\varepsilon}$, $\mu = \dfrac{r - r_B}{\varepsilon}$. Consider the following z-periodic perturbation $\varphi(r,z,v)$ of the z-homogeneous expansion,

$$\begin{aligned}\varphi(r,z,v) &= \varepsilon\big(b_1 v_\theta + \delta \cos az(Uv_\theta + Vv_r) + \delta(\sin az)Wv_z + \delta^2 U_{20} v_\theta\big)\\ &+ \varepsilon^2\big(\varphi_{2u} + \Phi_{K2A} + \Phi_{K2B} + \delta(\cos az)(\varphi_{11}^2 + \Phi_{K21A}(\eta, v) + \Phi_{K21B}(\mu, v))\\ &+ \delta(\sin az)(\psi_{11}^2 + \psi_{K21A} + \psi_{K21B}) + \delta^2(\varphi_{20}^2 + \Phi_{K20A} + \Phi_{K20B})\\ &+ \delta^2(\cos 2az)(\varphi_{22}^2 + \Phi_{K22A} + \Phi_{K22B})\\ &+ \delta^2(\sin 2az)(\psi_{22}^2 + \psi_{K22A} + \psi_{K22B})\big)\\ &+ \varepsilon^3\big(\varphi_{3u} + \Phi_{K3A} + \Phi_{K3B} + \delta(\cos az)(\varphi_{11}^3 + \Phi_{K31A} + \Phi_{K31B})\\ &+ \delta(\sin az)(\psi_{11}^3 + \psi_{K31A} + \psi_{K31B})\big)\\ &+ \varepsilon^4(\phi_{4u} + \Phi_{K4A} + \Phi_{K4B}).\end{aligned}$$

Here all coefficient functions are taken with respect to space as functions of r only. Look for boundary conditions where the rotational velocity of first order in ε, $b_1 + \delta(cosaz)U + \delta^2 U_{20}$, at $r_A = 1$ deviates from b_1 by a δ^2-order term $\Delta u_{\theta A}$. All the unknowns $U, V, W, \ldots$ should then vanish at r_A and r_B, except U_{20}, for which

$$U_{20}(r_A) = \Delta u_{\theta A}, \quad U_{20}(r_B) = 0.$$

Lemma 2.6. *Let*

$$l = \frac{1}{\varepsilon}(L\varphi + J(\varphi, \varphi) - \varepsilon v \cdot \nabla_x \varphi).$$

If $\delta \leq \varepsilon$ and if (U,V) are solutions to

$$\begin{aligned}&L_\theta(U) - q_\theta V = 0, \quad L_r(V) + q_r U = 0,\\ &U(r) = V(r) = V'(r) = 0 \text{ at } r = r_A, r = r_B,\end{aligned} \tag{2.45}$$

where

$$L_\theta(U) = U'' + \frac{1}{r}U' - (\frac{1}{r^2} + a^2)U, \quad L_r(V) = V^{(4)} + \frac{2}{r}V^{(3)} - (\frac{3}{r^2} + 2a^2)V''$$
$$+ (\frac{3}{r^3} - \frac{2a^2}{r})V' + (- \frac{3}{r^4} + \frac{2a^2}{r^2} + a^4)V,$$
$$q_\theta = \frac{2u_{\theta A}}{w_1(r_B^2 - 1)}, \quad q_r = -\frac{2a^2 u_{\theta A}}{w_1(r_B^2 - 1)}(\frac{r_B^2}{r^2} - 1),$$

then the function φ can be taken z-dependent, and so that $l = l_\perp$ is of order ε^4 in $\tilde{L}^\infty$.

The function φ is the asymptotic expansion for an axially periodic solution bifurcating from the axially homogeneous one at $u_{\theta A} = u_{\theta Ab}$.

Proof of Lemma 2.6. Replacing in l, φ by its expansion implies that

$$l = \varepsilon\delta \cos az\Big(L(\varphi_{11}^2 - b_1 U v_\theta^2 - b_1 V v_r v_\theta) - (U' - \frac{1}{r}U)v_r v_\theta$$
$$-(V' v_r^2 + \frac{1}{r}V v_\theta^2 + aW v_z^2) + L\Phi_{K21A} - v_r\frac{\partial \Phi_{K21A}}{\partial \eta} + L\Phi_{K21B} - v_r\frac{\partial \Phi_{K21B}}{\partial \mu}\Big)$$
$$+\varepsilon\delta \sin az\Big(L(\psi_{11}^2 - b_1 W v_\theta v_z) + aU v_\theta v_z + (aV - W')v_r v_z$$
$$+L\psi_{K21A} - v_r\frac{\partial \psi_{K21A}}{\partial \eta} + L\psi_{K21B} - v_r\frac{\partial \psi_{K21B}}{\partial \mu}\Big)$$
$$+\varepsilon\delta^2\Big(L(\varphi_{20}^2 - \frac{1}{4}U^2 v_\theta^2 - \frac{1}{4}V^2 v_r^2 - \frac{1}{2}UV v_r v_\theta - \frac{1}{4}W^2 v_z^2 - b_1 U_{20} v_\theta^2)$$
$$-(U_{20}' - \frac{1}{r}U_{20})v_r v_\theta + L\Phi_{K20A} - v_r\frac{\partial \Phi_{K20A}}{\partial \eta} + L\Phi_{K20B} - v_r\frac{\partial \Phi_{K20B}}{\partial \mu}\Big)$$
$$+\varepsilon\delta^2 \cos 2az\Big(L(\varphi_{22}^2 - \frac{1}{4}U^2 v_\theta^2 - \frac{1}{4}V^2 v_r^2 - \frac{1}{2}UV v_r v_\theta + \frac{1}{4}W^2 v_z^2)$$
$$+L\Phi_{K22A} - v_r\frac{\partial \Phi_{K22A}}{\partial \eta} + L\Phi_{K22B} - v_r\frac{\partial \Phi_{K22B}}{\partial \mu}\Big)$$
$$+\varepsilon\delta^2 \sin 2az\Big(L(\psi_{22}^2 - UW v_\theta v_z - VW v_r v_z) + L\psi_{K22A} - v_r\frac{\partial \psi_{K22A}}{\partial \eta} + L\psi_{K22B} - v_r\frac{\partial \psi_{K22B}}{\partial \mu}\Big)$$
$$+\varepsilon^2\delta \cos az\Big(L\varphi_{11}^3 + 2J(b_1 v_\theta, \varphi_{11}^2) + 2J(\varphi_{2u}, U v_\theta + V v_r) - (v_r\frac{\partial \varphi_{11}^2}{\partial r} + \frac{1}{r}N\varphi_{11}^2 + a\psi_{11}^2 v_z)$$
$$+L\Phi_{K31A} + 2J(b_1 v_\theta, \Phi_{K21A}) + 2J(U v_\theta + V v_r, \Phi_{K2A}) - N\Phi_{K21A} - v_r\frac{\partial \Phi_{K31A}}{\partial \eta}$$
$$+L\Phi_{K31B} + 2J(b_1 v_\theta, \Phi_{K21B}) + 2J(U v_\theta + V v_r, \Phi_{K2B}) - \frac{1}{r_B}N\Phi_{K21B} - v_r\frac{\partial \Phi_{K31B}}{\partial \mu}\Big)$$

$$+\varepsilon^2\delta \sin az\Big(L\psi^3_{11} + 2J(b_1v_\theta, \psi^2_{11}) + 2J(\varphi_{2u}, Wv_z) - (v_r\frac{\partial \psi^2_{11}}{\partial r} + \frac{1}{r}N\psi^2_{11} - a\varphi^2_{11}v_z)$$

$$+L\psi_{K31A} + 2J(b_1v_\theta, \psi_{K21A}) + 2J(Wv_z, \Phi_{K2A}) - N\psi_{K21A} - v_r\frac{\partial \psi_{K31A}}{\partial \eta}$$

$$+L\psi_{K31B} + 2J(b_1v_\theta, \psi_{K21B}) + 2J(Wv_z, \Phi_{K2B}) - \frac{1}{r_B}N\psi_{K21B} - v_r\frac{\partial \psi_{K31B}}{\partial \mu}\Big) + O(\varepsilon^4).$$

The compatiblity conditions in the $\varepsilon\delta\cos az$ term write

$$aW = -V' - \frac{1}{r}V. \tag{2.46}$$

And so φ^2_{11} can be taken as

$$\varphi^2_{11} = a^2_{11} + d^2_{11}v^2 + b^2_{11}v_\theta + c^2_{11}v_r + e^2_{11}v_z + b_1Uv^2_\theta + b_1Vv_rv_\theta$$
$$+ (U' - \frac{1}{r}U)v_rv_\theta\bar{B} + \frac{1}{r}V(v^2_\theta - v^2_r)\bar{B} + aW(v^2_z - v^2_r)\bar{B},$$

for some functions a^2_{11}, d^2_{11}, b^2_{11}, c^2_{11} and e^2_{11}. Moreover,

$$\psi^2_{11} = \alpha^2_{11} + \delta^2_{11}v^2 + \beta^2_{11}v_\theta + \gamma^2_{11}v_r + \eta^2_{11}v_z + b_1Wv_\theta v_z - aUv_\theta v_z\bar{B}$$
$$- aVv_rv_z\bar{B} + W'v_rv_z\bar{B},$$

for some functions α^2_{11}, δ^2_{11}, β^2_{11}, γ^2_{11} and η^2_{11}. Then, the compatibility conditions of the $\varepsilon^2\delta\cos az$-term of l are

$$(c^2_{11})' + \frac{1}{r}(c^2_{11}) + a\eta^2_{11} = 0, \tag{2.47}$$

$$\frac{1}{w_1}(a^2_{11} + 5d^2_{11} + b_1U)' = aW' + \frac{2a}{r}W + \frac{2}{r}V' + (\frac{2}{r^2} + a^2)V + \frac{2}{w_1r}b_1U, \tag{2.48}$$

$$(b_1V)' + \frac{2}{r}b_1V + w_1(U' - \frac{1}{r}U)' + \frac{2w_1}{r}(U' - \frac{1}{r}U) + ab_1W - a^2w_1U = 0, \tag{2.49}$$

$$\alpha^2_{11} + 5\delta^2_{11} = 0. \tag{2.50}$$

Taking (2.46) into account in (2.49) implies that

$$L_\theta U + q_\theta V = 0.$$

The compatibility conditions of the $\varepsilon^2\delta\sin az$-term of l are

$$(\gamma^2_{11})' + \frac{1}{r}(\gamma^2_{11}) - ae^2_{11} = 0, \tag{2.51}$$

$$(\alpha^2_{11} + 5\delta^2_{11})' = 0, \tag{2.52}$$

$$\frac{1}{w_1}(a^2_{11} + 5d^2_{11} + b_1U) = W'' + \frac{1}{r}W' - 2a^2W - a(V' + \frac{1}{r}V). \tag{2.53}$$

Differentiating (2.53) with respect to the variable r and taking (2.48) and (2.46) into account, implies that

$$L_r V + q_r U = 0.$$

It follows that the coefficients $\varphi^2_{20}, \varphi^2_{22}, \psi^2_{22}, \varphi^3_{11}, \psi^3_{11}$, as well as the Knudsen terms can be defined so that l be of order 4 provided (2.45) holds. □

Lemma 2.7. *Let $a > 0$ be given. There are nonnegative functions u_1 and v_1, and $u_{\theta A} = u_{\theta Ab} > 0$, such that for $r_B - r_A$ small enough, the problem* (2.45) *has the solutions* $\{(U, V) = x(u_1, v_1); x \in \mathbb{R}\}$.

Proof of Lemma 2.7. The equation $L_\theta U = 0$ is disconjugate on $[1, r_B]$ for any $r_B > 1$ since

$$\int_1^{r_B} (ry'^2 + (\frac{1}{r} + a^2)y^2)dr$$

is nonneggative ([Co]). Hence there is a continuous Green function G such that for any continuous founction f, the problem

$$L_\theta U = f, \quad U(1) = U(r_B) = 0,$$

has the unique solution

$$U(r) = \int_1^{r_B} G(r, s)f(s)ds.$$

Moreover,

$$G(r, s)(r - 1)(r - r_B) \geq 0, \quad (r, s) \in [1, r_B]^2,$$

so that G is non positive. It also satisfies

$$rG(r, s) = sG(s, r), \quad (r, s) \in [1, r_B]^2,$$

since

$$\int_1^{r_B} rL_\theta(U)Xdr = \int_1^{r_B} rL_\theta(X)Udr.$$

By [Co] the equation

$$L_r(V) = 0, \quad V(1) = V(r_B) = V'(1) = V'(r_B) = 0,$$

is disconjugate on $[1, r_B]$ for $r_B - 1$ small enough. Hence there is a C^2 Green function H such that for any continuous function f, the problem

$$L_r V = f, \quad V(1) = V(r_B) = V'(1) = V'(r_B) = 0,$$

has the unique solution

$$V(r) = \int_1^{r_B} H(r,s) f(s) ds.$$

Moreover,

$$H(r,s)(r-1)^2(r-r_B)^2 \geq 0, \quad (r,s) \in [1, r_B]^2,$$

so that H is nonnegative. It also satisfies

$$rH(r,s) = sH(s,r), \quad (r,s) \in [1, r_B]^2,$$

since

$$\int_1^{r_B} rL_r(V)Ydr = \int_1^{r_B} rL_r(Y)Vdr.$$

And so, solving (2.45) comes back to finding $u_{\theta Ab} := U_{\theta Ab}(r_B - 1)$ and $V \geq 0$ such that

$$KV = \left(\frac{w_1(r_B^2 - 1)}{4au_{\theta Ab}}\right)^2 V, \tag{2.54}$$

where K is the operator defined by

$$KV(r) = -\int_1^{r_B}\int_1^{r_B} H(r,s)\left(\frac{r_B^2}{s^2} - 1\right)G(s,t)V(t)dtds.$$

K is compact in $L^2(1, r_B)$. It maps the cone of the nonneggative functions of L^2 into its interior, since G is nonpositive, H is nonnegative, and neither G nor H are identically zero. And so the Krein-Rutman theorem applies. There is an eigenvector $v_1 \geq 0$ corresponding to a positive eigenvalue of K, $\left(\frac{w_1(r_B^2-1)}{2au_{\theta Ab}}\right)^2 = \left(\frac{w_1(r_B+1)}{2aU_{\theta Ab}}\right)^2$ with algebraic and geometric multiplicity equal to one. Denote by

$$u_1(r) = -q_\theta \int_1^{r_B} G(r,s)v_1(s)ds, \quad r \in [1, r_B].$$

Then any (xu_1, xv_1), $x \in \mathbb{R}_+$ is solution to (2.45). □

3 - Fluid dynamic and non-fluid-dynamic estimates

This section discusses a priori estimates for the two-rolls cases introduced in Section 2.

We recall that the orthonormal basis $\psi_0 = 1$, $\psi_\theta = v_\theta$, $\psi_r = v_r$, $\psi_z = v_z$, $\psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$ for the kernel of L in $L^2_M(\mathbb{R}^3)$ was introduced in Section 2 together with an orthogonal splitting of functions $f \in L^2_M([r_A, r_B] \times \mathbb{R}^3)$ into $f = f_\parallel + f_\perp = P_0 f + (I - P_0)f$, where for the fluid dynamic part

$$f_\parallel(r,v) = f_0(r) - \frac{\sqrt{6}}{2} f_4(r) + f_\theta(r)v_\theta + f_r(r)v_r + f_z(r)v_z + \frac{\sqrt{6}}{6} f_4(r)v^2,$$

$$\int M(v)(1, v, v^2) f_\perp(r,z,v)dv = 0,$$

$$\int M\psi_0 f(r,v)dv = f_0(r), \quad \int M\psi_4 f(r,v)dv = f_4(r),$$

$$\int M\psi_\theta f(r,v)dv = f_\theta(r), \quad \int M\psi_r f(r,v)dv = f_r(r)$$

$$\int M\psi_z f(r,v)dv = f_z(r).$$

Set $Df := v_r \frac{\partial f}{\partial r}(+ v_z \frac{\partial f}{\partial z}) + \frac{1}{r} Nf$ with N defined in (2.1). In Case 1 due to the symmetries, the position space may be changed from the two-cylinder domain $\Omega \subset \mathbb{R}^3$ with measure dx, to $[r_A, r_B] \subset \mathbb{R}^+$ with measure rdr. All functions considered are even in v_z giving in particular $f_z = 0$. The relevant ingoing boundary space becomes

$$L^+ := \{f; |f|_\sim := \Big(\int_{v_r > 0} v_r M(v)\, |f(r_A, v)|^2 \, dv\Big)^{\frac{1}{2}} + \Big(\int_{v_r<0} |v_r| M(v) |f(r_B, v)|^2 dv\Big)^{\frac{1}{2}} < +\infty\}.$$

Set

$$\tilde{L}^q := \{f; |f|_q := \Big(\int M(v) (\int |f(x,v)|^q dx)^{\frac{2}{q}} dv\Big)^{\frac{1}{2}} < +\infty\},$$

$$\mathcal{W}^{q-}([r_A, r_B] \times \mathbb{R}^3) = \mathcal{W}^{q-} := \{f; v^{\frac{1}{2}} f \in \tilde{L}^q, v^{-\frac{1}{2}} Df \in \tilde{L}^q, \gamma^+ f \in L^+\}.$$

Lemma 3.1. *Let $v^{-\frac{1}{2}} g \in \tilde{L}^q$, $F_b \in L^+$, $2 \le q < \infty$, be given. There exists a unique solution $F \in \mathcal{W}^{q-}$ to*

$$DF = \frac{1}{\varepsilon}(LF + 2\sum_{j=1}^{j_1} \varepsilon^j J(F, \Phi^j) + g), \quad F_{/\partial\Omega^+} = F_b, \tag{3.1}$$

where the terms Φ^j of the axially homogeneous asymptotic expansion were introduced in (2.2), and the boundary data F_b are given on the ingoing boundary $\partial\Omega^+$.

Notice first that the a priori estimates (3.2), (3.4) below imply uniqueness in L^2. Then use the solution formula $F = WF_b + Ug + UKF$ from the proof of Lemma 3.2 below in the case φ of (2.2) equals zero. Here UK is compact in L^2 (e.g by first proving the compactness of UE for $EF := \int MFdv$ and then using the splitting $K = K' + K''$ below), so the L^2 case follows from Fredholm's alternative. The L^∞ case then follows from (3.3), and the intermediate cases hold similarly. Finally the addition of the small perturbation $J(F, \varphi)$ does not change the result.

To obtain uniform control of the final non-linear Boltzmann equation all the way to the fluid dynamic limit, we shall use this section to secure sufficiently strong a priori estimates in $\tilde{L}^q$ for the linear problem (3.1). With regard to the shortest, the most transparent or the most elegant method of proof, various approaches are the best suited depending on the situation. We shall varyingly be using straight forward direct computations, dual estimates, ODE methods or Fourier techniques.

For the non-fluid-dynamic part $F_\perp$ of the solution and for the comparison of the solution in different $\tilde{L}^q$-spaces, in the simplest Case 1 we may use quite explicit computations. Define a specular reflection operator $\mathcal{S}$ at $r = r_A,\ r_B$ as $\mathcal{S}f(r, v) = f(r, -v_r, v_\theta, v_z)$.

Lemma 3.2. *Let $q = 2, \infty$, and let F be a solution in $\mathcal{W}^{q-}$ to (3.1) for $g = g_\perp$. The following estimates hold for small enough $\varepsilon > 0$;*

$$\varepsilon^{\frac{1}{2}} \mid \mathcal{S}F \mid_\sim + \mid \nu^{\frac{1}{2}} F_\perp \mid_2 \leq c(\mid \nu^{-\frac{1}{2}} g \mid_2 + \varepsilon^{\frac{1}{2}} \mid F_b \mid_\sim + \varepsilon(\|F_r\|_2 + \|F_\theta\|_2 + \|F_0\|_2 + \|F_4\|_2)), \tag{3.2}$$

$$\mid \nu^{\frac{1}{2}} F \mid_\infty \leq c(\mid \nu^{-\frac{1}{2}} g \mid_\infty + \varepsilon^{-\frac{2}{q}} \mid \nu^{\frac{1}{2}} F \mid_q + \mid F_b \mid_\sim). \tag{3.3}$$

The estimate (3.3) also holds in this form, when g has a non-vanishing fluid dynamic component $g_\|$.

Proof of Lemma 3.2. We first turn to the estimate (3.3). To prove it, we shall need some estimates which are suitably discussed in the original coordinates of (1.2). Consider the exponential form of (3.1) with φ of (2.2) equal zero;

$$\frac{d}{ds}(F(x + sv, v)e^{s\frac{\nu}{\varepsilon}}) = \frac{e^{s\frac{\nu}{\varepsilon}}(KF + g)}{\varepsilon}(x + sv, v),$$

or integrated

$$F(x,v) = e^{-s_0\frac{\nu}{\varepsilon}}F_b(x - s_0v, v) + \int_{-s_0}^{0} e^{s\frac{\nu}{\varepsilon}}\,\frac{(KF+g)}{\varepsilon}\,(x+sv,v)ds$$

$$=: WF_b + UKF + Ug.$$

Here s_0 denotes the time to reach the ingoing boundary along the characteristic.

Split the kernel k of K into $k_n = \text{sign}k \min(|k|, n)$ and the remaining part $k - k_n$, and denote the corresponding operators by K' and K''. The operator norm of $K - K' = K''$ tends to zero, and K is compact in L^2_M. It immediately follows that F can be written as

$$F = (UK')^2F + (UK''UK + UK'UK'')F + (UKU + U)g + (UKW + W)F_b$$

$$=: (UK')^2F + Z_1F + Z_2g + Z_3F_b.$$

The K''-factor makes the operator norm of Z_1 in $\tilde{L}^\infty$ tend to zero (uniformly in ε) when the cut-off $n \to \infty$. Also by straight forward computations

$$|\, \nu^{\frac{1}{2}} Z_2 g \,|_\infty \leq c \,|\, \nu^{-\frac{1}{2}} g \,|_\infty, \quad |\, \nu^{\frac{1}{2}} Z_3 F_b \,|_\infty \leq c \,|\, F_b \,|_\sim .$$

It remains the term $UK'UK'$. The first U is (uniformly in ε) bounded in $\tilde{L}^\infty$, so it is enough to consider $K'UK'$. Setting $EF(x) = \int F(x,v)M(v)dv$, we can estimate $K'UK'$ by a cut-off dependent multiple of EUE in the operator norm. For fixed ε the operator EUE is bounded from L^p into L^q for $p > d$, $q = \infty$, $d \geq 1$, as well as for $1 < p \leq d$, $\quad q < dp(d-p)^{-1}$, $d > 1$. Here d is the dimension of the x-space. For the proof of this estimate of EUE we follow [M] Chapter 6. Let us first consider the case $\varepsilon = 1$, $\quad d = 2$, our main concern being the domain Ω equal an open annulus between the radii r_A and r_B.

Let $v' = (v_x, v_y)$ for $v = (v_x, v_y, v_z)$ and let g be a function from $L^p(\Omega)$ where we let $g(x - sv', v)$ for $x \in \Omega$ take the value zero after $x - sv$ has for the first time left Ω. This gives

$$Ug(x,v) = \int_0^\infty g(x - sv', v)e^{(-\nu(v)s)}ds.$$

Set $G(x) = Eg(x,v)$. Then

$$EUG(x) = \int_{\mathrm{R}^3\times(0,\infty)} e^{-\nu(v)s}G(x - sv')M(v)dvds$$

$$\leq \int_{\mathrm{R}^3\times(0,\infty)} e^{-s\nu_0}G(x - sv')M(v)dvds,$$

where $\upsilon_0 = \inf \upsilon(v) > 0$. It follows that the v_z-integral can be added after concluding the estimate of the $dv'ds$-integral. We continue the discussion for $v' \in \mathbb{R}^2$ using the notation $v' = v$. A change of variables $(s, v) \to (r, y)$ with $r = |v|, \quad y = x - sv$ gives $EUG \leq G * \varphi$ with

$$\varphi(y) = c_1 |y|^{-1} \int_0^\infty k(r, y) dr, \quad c_1 > 0,$$

$$k(r, y) = M(r) e^{-r^{-1}|y|\upsilon_0}.$$

Since $M(v) \leq c_2 e^{-c_3|v|}$, we get

$$k(r, y) \leq c_4 e^{-\frac{c_3 r}{2}} e^{-\frac{c_3 r}{2} - \frac{\upsilon_0 |y|}{r}}$$

$$\leq c_4 r^{-1} e^{\frac{-c_3 r}{2}} e^{-c_5 |y|^{\frac{1}{2}}}.$$

It follows that $\varphi \in L^\rho$ if $\rho < 2$. If $1 < p \leq 2$ the result now follows from Young's inequality (i.e. from $*\varphi : L^p \to L^q$ for $q^{-1} = p^{-1} - \rho'^{-1}$). By Hölder's inequality $EUG \in L^\infty$ if $p > 2$. The proof for $\Omega \in \mathbb{R}^3$ is analogous whereas the case $d = 1$ requires a slightly different estimate of k.

For the desired estimate of the solution in L^∞ by L^2-terms for $d = 2$ we have to apply the estimate of $UK'UK'$ twice (also the solution formula). Including the ε-dependence in the above estimate of EUE gives the factor $\varepsilon^{-\frac{2}{q}}$.

With this estimate of EUE and choosing the cut-off n large enough, (3.3) follows when $\varphi = 0$. Recalling that φ is of order ε, and taking ε small enough, it follows that the addition of $J(F, \varphi)$ to g does not change the result in this part of the proof, neither does the addition of a fluid component to g.

Consider next the mapping from $\upsilon^{-\frac{1}{2}}\tilde{L}^q \times L^+$ into $\mathcal{W}^{q-}$ given by $(g, F_b) \to F$, with F a solution to (3.1) for $\varphi = 0$. Green's formula and the spectral inequality of Lemma 2.2 for the linearized collision operator L, i.e.

$$-\int M f L f dv \geq c \int M \upsilon f_\perp^2 dv,$$

give

$$\varepsilon \mid SF \mid_\sim^2 + \mid \upsilon^{\frac{1}{2}} F_\perp \mid_2^2 \leq \frac{c}{\delta} \mid \upsilon^{-\frac{1}{2}} g_\perp \mid_2^2 + \delta \mid \upsilon^{\frac{1}{2}} F_\perp \mid_2^2 + \varepsilon \mid F_b \mid_\sim^2 .$$

This completes the estimate (3.2) when $\varphi = 0$. The inclusion of $J(F, \varphi)$ to g, adds $c\varepsilon \mid \upsilon^{\frac{1}{2}} F_\perp \mid_2^2$, which is incorporated in the left hand side, and a term

$$c\varepsilon(\|F_r\|_2 + \|F_\theta\|_2 + \|F_0\|_2 + \|F_4\|_2). \qquad \square$$

The control of the fluid part $F_\parallel$ of the solution, i.e. the kernel of L, is less efficient.

In particular Case 2 requires a careful analysis. For this we have chosen a direct computation of each moment in order to obtain sharp estimates. The method is here illustrated in some detail in the following lemmas for the simpler Case 1.

Lemma 3.3. *Let $g = g_{\|} + g_{\perp}$ (i.e. with a possible fluid dynamic part $g_{\|}$ in g), and let F be a solution in $\mathcal{W}^{2-}$ to (3.1). For $\varepsilon > 0$ and small enough,*

$$\|F_r\|_2 + \|F_\theta\|_2 + \|F_0\|_2 + \|F_4\|_2 \le c(\mid F_\perp \mid_2 + \frac{1}{\varepsilon} \mid v^{-\frac{1}{2}} g_\perp \mid_2 + \frac{1}{\varepsilon^2} \mid g_{\|} \mid_2 + \mid F_b \mid_\sim). \tag{3.4}$$

Proof of Lemma 3.3. Define

$$f_{\theta^i r^j}(r) := \int M v_\theta^i v_r^j f_\perp(r, v) dv, \quad i + j \ge 2,$$

and $f_{\theta^i r^j 2}(r)$ correspondingly, when there is an extra factor $\mid v \mid^2$ in the integrand.

A multiplication of (3.1) with $v_\theta M$ (resp. $v^2 M$) and integration over $\mathbb{R}^3_v$ leads to

$$F_{\theta r}(r) = \frac{F_{\theta r}(1)}{r^2} + \frac{1}{r^2} \int_1^r s^2 \frac{g_0}{\varepsilon} ds,$$

$$F_{r2}(r) = \frac{c_{r2}}{r} + \frac{1}{r\varepsilon} \int_1^r s(\sqrt{6} g_4 - 2 g_0) ds.$$

Multiply equation (3.1) with $\bar{A}(|v|) v_r M$ and integrate over $\mathbb{R}^3_v$,

$$\begin{aligned}\Big(\int v_r^2 \bar{A} M F dv\Big)' &= \Big(k_4 F_4 + F_{r^2 \bar{A}}\Big)' = \frac{1}{r}\Big(F_{\theta^2 \bar{A}} - F_{r^2 \bar{A}}\Big) \\ &+ \frac{1}{\varepsilon}\Big(\frac{c_{r2}}{r} + \frac{1}{r\varepsilon} \int_1^r s(\sqrt{6} g_4 - 2 g_0) ds + \int v_r \bar{A} J(F_\perp, \varepsilon \Phi^1) M dv\Big) \\ &+ \sum_{j=2}^{j_1} \varepsilon^{j-1} \int v_r \bar{A} J(F, \Phi^j) M dv + \frac{1}{\varepsilon} \int g v_r \bar{A} M dv.\end{aligned} \tag{3.5}$$

Using the spectral inequality of Lemma 2.2, we notice that

$$\begin{aligned}k_4 &:= \int v_r^2 \psi_4 \bar{A} M dv = \frac{1}{\sqrt{6}} \int v_r v^2 v_r \bar{A} M dv \\ &= \frac{1}{\sqrt{6}} \int v_r (v^2 - 5) v_r \bar{A} M dv = \frac{1}{\sqrt{6}} \int L(v_r \bar{A}) v_r \bar{A} M dv < -c \int |v_r \bar{A}|^2 M dv < 0.\end{aligned}$$

Set $\tilde{F}_4 = k_4 F_4 + F_{r^2\bar{A}}$ and regroup the terms in (3.5) as

$$\tilde{F}_4' = \frac{c_{r2}}{r\varepsilon} + \Big\{ \frac{1}{r}(F_{\theta^2\bar{A}} - F_{r^2\bar{A}})$$
$$+ \frac{1}{\varepsilon}\Big(\frac{1}{r\varepsilon}\int_1^r s(\sqrt{6}\, g_4 - 2g_0)ds + \int v_r \bar{A} J(F_\perp, \varepsilon\Phi^1) M dv \Big)$$
$$+ \sum_{j=2}^{j_1} \varepsilon^{j-1} \int v_r \bar{A} J(F, \Phi^j) M dv + \frac{1}{\varepsilon}\int g v_r \bar{A} M dv \Big\}.$$

Denoting the expression within $\{...\}$ by G_4 gives

$$(\tilde{F}_4)' = \frac{c_{r2}}{r\varepsilon} + G_4,$$

which integrates to give

$$\tilde{F}_4(r_B) - \tilde{F}_4(r_A) = \frac{c_{r2}}{\varepsilon}(\ln r_B - \ln r_A) + \int_{r_A}^{r_B} G_4(s)ds,$$

$$\tilde{F}_4(r) = \tilde{F}_4(r_B) + \frac{c_{r2}}{\varepsilon}(\ln r - \ln r_B) + \int_{r_B}^{r} G_4(s)ds.$$

Eliminating c_{r2}, it follows that

$$\tilde{F}_4(r) = \tilde{F}_4(r_B) + \frac{\ln r - \ln r_B}{\ln r_B - \ln r_A}\Big(\tilde{F}_4(r_B) - \tilde{F}_4(r_A) + \int_{r_A}^{r_B} G_4(s)ds \Big) - \int_r^{r_B} G_4(s)ds. \tag{3.6}$$

With $w_1 = (v_r^2 v_\theta^2 \bar{B}, 1)$, an analogous solution formula for $\frac{\tilde{F}_\theta}{r} := \frac{w_1 F_\theta}{r} + \frac{F_{\theta r^2 \bar{B}}}{r}$ can be obtained in the same way. Namely, multiply the equation (3.1) with $M v_r v_\theta \bar{B}(|v|)$ and integrate over $\mathbb{R}^3_v$. It follows that

$$(\frac{\tilde{F}_\theta}{r})' = \frac{F_{\theta^3\bar{B}} - 3F_{\theta r^2\bar{B}}}{r^2}$$
$$+ \frac{1}{r\varepsilon}\Big(\frac{c_{\theta r}}{r^2} + \frac{1}{r^2}\int_1^r s^2 \frac{g_0}{\varepsilon} + 2\int v_r v_\theta \bar{B} J(v_r F_r + F_\perp, \varepsilon\Phi^1) M dv \Big)$$
$$+ 2\sum_{j=2}^{j_1} \varepsilon^{j-1} \int v_r v_\theta \bar{B} J(F, \Phi^j) M dv$$
$$+ \frac{1}{\varepsilon}\int v_r v_\theta \bar{B} M g dv = \frac{c_{\theta r}}{r^3 \varepsilon} + G_\theta.$$

And so

$$\frac{\tilde{F}_\theta(r_B)}{r_B} - \frac{\tilde{F}_\theta(r_A)}{r_A} = \frac{2c_{\theta r}}{\varepsilon}\left(\frac{1}{r_A^2} - \frac{1}{r_B^2}\right) + \int_{r_A}^{r_B} G_\theta(s)ds,$$

$$\frac{\tilde{F}_\theta(r)}{r} - \frac{\tilde{F}_\theta(r_B)}{r_B} = \frac{2c_{\theta r}}{\varepsilon}\left(\frac{1}{r_B^2} - \frac{1}{r^2}\right) + \int_{r_B}^{r} G_\theta(s)ds.$$

Eliminating $c_{\theta r}$ gives

$$\frac{\tilde{F}_\theta(r)}{r} = \frac{\tilde{F}_\theta(r_B)}{r_B} + \frac{(r^2 - r_B^2)r_A^2}{(r_B^2 - r_A^2)r^2}\Big(\frac{\tilde{F}_\theta(r_B)}{r_B} - \frac{\tilde{F}_\theta(r_A)}{r_A} - \int_{r_A}^{r_B} G_\theta(s)ds\Big) + \int_{r_B}^{r} G_\theta(s)ds. \tag{3.7}$$

Multiplying the equation (3.1) with M and integrating over R_v^3, leads to $(rF_r)' = r\,\frac{g_0}{\varepsilon}$, i.e.

$$F_r(r) = \frac{F_r(1)}{r} + \frac{1}{r}\int_1^r s\frac{g_0}{\varepsilon}ds. \tag{3.8}$$

By definition of $F_r(1)$,

$$| F_r(1) | = \Big| \int v_r F(1,v)Mdv \Big| \leq c\Big(\int | v_r | F^2(1,v)Mdv\Big)^{\frac{1}{2}} \leq c(| SF |_\sim + | F_b |_\sim).$$

And so by (3.8)

$$\|F_r\|_2 \leq c(\frac{1}{\varepsilon} | g_\| |_2 + | SF |_2 + | F_b |_\sim). \tag{3.9}$$

Multiply the equation (3.1) with $v_r M$ and integrate with respect to v. It follows that

$$\Big(\int v_r^2 F(r,v)Mdv\Big)' = \Big(F_0 + \sqrt{\frac{2}{3}}F_4 + F_{r^2}\Big)' = \frac{F_{\theta^2} - F_{r^2}}{r} + \frac{g_r}{\varepsilon}.$$

Multiply this with $2\Big(F_0 + \sqrt{\frac{2}{3}}F_4 + F_{r^2}\Big)$ and integrate with respect to r on (r, r_B), then on (r_A, r_B), to obtain

$$\|F_0 + \sqrt{\frac{2}{3}}F_4\|_2 \leq c\Big(| F_\perp |_2 + \frac{1}{\varepsilon}\|g_r\|_2 + \Big| \int v_r^2 F(r_B, v)Mdv \Big| \Big).$$

But

$$\Big| \int v_r^2 F(r_B, v)Mdv \Big| \leq c\Big(\int M | v_r | F^2(r_B, v)dv\Big)^{\frac{1}{2}} \leq c(| SF |_\sim + | F_b |_\sim).$$

Hence

$$\|F_0+\sqrt{\frac{2}{3}}F_4\|_2 \leq c\Big(\mid F_\perp \mid_2 + \frac{1}{\varepsilon}\|g_r\|_2 + \mid SF \mid_\sim + \mid F_b \mid_\sim\Big). \tag{3.10}$$

It follows from (3.6) and (3.7) that

$$\|F_4\|_2 + \|F_\theta\|_2 \leq c\Big(\mid F_\perp \mid_2 + \frac{1}{\varepsilon^2}\mid g_\parallel \mid_2 + \frac{1}{\varepsilon}\mid g_\perp \mid_2 + \mid SF \mid_\sim + \mid F_b \mid_\sim + \varepsilon \mid F_\parallel \mid_2\Big).$$

This together with (3.9-10) gives (3.4). □

Analogous estimates hold in the axially homogeneous **Case 2.** Care is here needed to remove terms of low ε-order in the proof of the fluid dynamic estimates. This complication has its origin in the fact that the boundary scalings (of order ε) here are larger than the Knudsen number (ε^4). For upcoming negative order terms in $F_\perp$ the example $a = \int MdvJ(F_\perp, v_\theta)v_r\bar{A}$ will suffice to clarify the technique. That moment can obviously be written as $\int MdvF_\perp\chi$ for some non-fluid-dynamic function χ. Projecting the whole equation along $L^{-1}\chi$, increases the epsilon order of the term a by one. This can be repeated until all appearing moments of $F_\perp$ are of non-negative order. A corresponding raising of order for the fluid dynamic estimates is more involved (see [AN2]). The resulting a priori estimates are

Lemma 3.4. *If $0 < \delta'$ is small enough and $g = g_\perp$, then for small enough $\varepsilon > 0$ the following estimates hold for a solution of* (3.1) *in* $\mathcal{W}^2$,

$$\mid v^{\frac{1}{2}}F_\perp \mid_2 \leq c\Big(\varepsilon^{-3}\mid v^{-\frac{1}{2}}g_\perp \mid_2 + \varepsilon^2 \mid F_b \mid_\sim\Big), \tag{3.11}$$

$$\|F_0\|_2 + \|F_r\|_2 + \|F_4\|_2 + \|F_\theta\|_2 \leq c\Big(\varepsilon^{-5}\mid v^{-\frac{1}{2}}g_\perp \mid_2 + \mid F_b \mid_\sim\Big). \tag{3.12}$$

If F is a solution of (3.1) *in* $\mathcal{W}^{\infty -}$, *then the following estimate holds for small enough $\varepsilon > 0$;*

$$\mid v^{\frac{1}{2}}F \mid_\infty \leq c(\mid v^{-\frac{1}{2}}g \mid_\infty + \varepsilon^{-\frac{8}{q}}\mid v^{\frac{1}{2}}F \mid_q + \mid F_b \mid_\sim), \quad q \leq \infty. \tag{3.13}$$

A fluid dynamic component in g does not change the results in (3.14-15).

In **Case 3** the partial differential nature of the problem requires more work than the ordinary differential equations appearing in Cases 1 and 2. But the two-roll domain is bounded and has a simple geometry that allows the use of a direct approach involving orthogonal (Fourier) expansions. For more complicated geometries in other bounded domains one may first by similar Fourier based methods study the dual problem in say a box containing the domain in question, and then via dual estimates and trace theorems obtain corresponding results for more arbitrary bounded domains (cf. [M]).

With the change of variables from $(r,z)\in(1,r_B)\times(-\frac{r_B-1}{2},\frac{r_B-1}{2})$ to $(s,Z)\in(-\pi,\pi)^2$ and with $\eta=\frac{r_B-1}{2\pi}$, we will be interested in the case when the new unknown $\tilde{F}(s,Z,v):=F(\eta s+\frac{r_B+1}{2},\eta Z,v)$ solves

$$v_r\frac{\partial\tilde{F}}{\partial s}+v_z\frac{\partial\tilde{F}}{\partial Z}+\eta\mu(s)N\tilde{F}=\frac{\eta}{\varepsilon}(L\tilde{F}+\tilde{g}), \tag{3.14}$$

where $\mu(s)=\frac{2}{2\eta s+r_B+1}$. The control of the fluid dynamic moments will be obtained by Fourier series expansions. Write (in the new variables) the Fourier expanded density function $\tilde{F}$ as

$$\tilde{F}(s,Z,v)=\sum_{(n,j)\in Z^2}a^{nj}(v)e^{i(ns+jZ)}.$$

The fluid dynamic moments $\tilde{F}_0$, $\tilde{F}_4$, $\tilde{F}_r$, $\tilde{F}_\theta$, and $\tilde{F}_z$ become

$$\tilde{F}_0(s,Z)=\sum_{(n,j)}m_0^{nj}e^{i(ns+jZ)},\quad \tilde{F}_4(s,Z)=\sum_{(n,j)}m_4^{nj}e^{i(ns+jZ)},$$
$$\tilde{F}_r(s,Z)=\sum_{(n,j)}u_r^{nj}e^{i(ns+jZ)},\quad \tilde{F}_\theta(s,Z)=\sum_{(n,j)}u_\theta^{nj}e^{i(ns+jZ)},\quad \tilde{F}_z(s,Z)=\sum_{(n,j)}u_z^{nj}e^{i(ns+jZ)},$$

where

$$m_0^{nj}:=(a^{nj},1),\quad m_4^{nj}:=(a^{nj},\psi_4),$$
$$u_r^{nj}:=(a^{nj},\psi_r),\quad u_\theta^{nj}:=(a^{nj},\psi_\theta),\quad u_z^{nj}:=(a^{nj},\psi_z).$$

Recall that (α,β) denotes the scalar product $\int\alpha(v)\beta(v)M(v)dv$, and notice that $u_z^{n0}=0$ due to the symmetry $\tilde{F}(s,Z,v_r,v_\theta,v_z)=\tilde{F}(s,-Z,v_r,v_\theta,-v_z)$. Notice that the Fourier coefficients of the first r-derivative contain a multiple of the boundary value difference,

$$a^{nj}(\frac{\partial F}{\partial r})=ina^{nj}(F)+\frac{(-1)^n}{2\pi}d^j,\quad (n,j)\in Z^2,$$

whereas for the first z-derivative no such term is present. Set $d=(\tilde{F}(\pi-0)-\tilde{F}(-\pi+0))\frac{1}{2\pi}$ with d^j its j'th Fourier coefficient in the Z-direction. Denote by $\lambda:=(v_r^2\bar{A},\psi_4)$, $w_1=(v_r^2v_\theta^2\bar{B},1)$, and by $Q=I-P_0$, and write

$$e(Z,v):=\frac{1}{2\pi}\left((\mu\tilde{F})(\pi-0,Z,v)-(\mu\tilde{F})(-\pi+0,Z,v)\right)=\sum_{j\in Z}e^j(v)e^{ijZ}.$$

Set

$$
\begin{aligned}
\Lambda_r^{nj} :=& -\frac{3i}{\varepsilon}(g^{nj}, v_r) - 3j(g^{nj}, v_r v_z \bar{B}) + n(g^{nj}, (2v_r^2 - v_\theta^2 - v_z^2)\bar{B}) \\
& - in\varepsilon(-1)^n d^j_{v_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}} + 3i(-1)^n d^j_{r^2} - 3ij\varepsilon(-1)^n d^j_{v_r^2 v_z \bar{B}} \\
& - i\varepsilon n^2 (Qv_r(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, Qa^{nj}) - i\varepsilon nj(Qv_z(2v_r^2 - v_\theta^2 - v_z^2)\bar{B}, Qa^{nj}) \\
& - 3i\varepsilon nj(Qv_r^2 v_z \bar{B}, Qa^{nj}) - i\varepsilon j^2 (Qv_r v_z^2 \bar{B}, Qa^{nj}) \\
& - \varepsilon\eta n(\mu\tilde{F})^{nj}_{v_r(2v_r^2 - 7v_\theta^2 - v_z^2)\bar{B}} - 3\varepsilon\eta j(\mu\tilde{F})^{nj}_{(v_r^2 - v_\theta^2)\bar{B}} + 3i\eta(\mu\tilde{F})^{nj}_{v_r^2 - v_\theta^2},
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_z^{nj} :=& -\frac{3i}{\varepsilon}(g^{nj}, v_z) + j(g^{nj}, (2v_z^2 - v_r^2 - v_\theta^2)\bar{B}) + 3n(g^{nj}, v_r v_z \bar{B}) \\
& - 3in\varepsilon(-1)^n d^j_{v_r^2 v_z \bar{B}} - ij\varepsilon(-1)^n d^j_{v_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}} + 3i(-1)^n d^j_{rz} \\
& - 3i\varepsilon n^2 (Qv_r^2 v_z \bar{B}, Qa^{nj}) - 3i\varepsilon nj(Qv_r v_z^2 \bar{B}, Qa^{nj}) \\
& - i\varepsilon nj(Qv_r(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, Qa^{nj}) - i\varepsilon j^2(Qv_z(2v_z^2 - v_r^2 - v_\theta^2)\bar{B}, Qa^{nj}) \\
& - 3\varepsilon\eta n(\mu\tilde{F})^{nj}_{(v_r^2 - v_\theta^2)\bar{B}} - \varepsilon\eta j(\mu\tilde{F})^{nj}_{v_r(-v_r^2 - v_\theta^2 + 2v_z^2)\bar{B}} + i\eta(\mu\tilde{F})^{nj}_{v_r v_z}.
\end{aligned}
$$

Lemma 3.5. *Let $\tilde{F}$ be a solution to (3.14). Denote by $\varepsilon_1 = \frac{\varepsilon}{\eta}$. For $(n, j) \neq (0, 0)$,*

$$
\begin{aligned}
m_0^{nj} =& -\frac{4}{3}w_1(g^{nj}, 1) + \frac{4}{3}w_1(-1)^n d^j_r + \frac{n\Lambda_r + j\Lambda_z}{3(n^2 + j^2)} + \frac{4}{3}w_1(\mu\tilde{F})^{nj}_{v_r} \\
& + \sqrt{\frac{2}{3}}\frac{1}{\lambda(n^2 + j^2)}\Big(\frac{1}{\varepsilon_1^2}(g^{nj}, v^2 - 5) + i\frac{n}{\varepsilon_1}g^{nj}_{v_r\bar{A}} + \frac{ij}{\varepsilon_1}(g^{nj}, v_z\bar{A}) - \frac{\eta}{\varepsilon_1}(\mu g)^{nj}_{v_r\bar{A}} \\
& - \frac{(-1)^n}{\varepsilon_1}d^j_{v_r(v^2-5)} - i(-1)^n n d^j_{v_r^2\bar{A}} - i(-1)^n j d^j_{v_r v_z\bar{A}} + \eta(-1)^n e^j_{v_r^2\bar{A}} \\
& + n^2(Qv_r^2\bar{A}, Qa^{nj}) + j^2(Qv_z^2\bar{A}, Qa^{nj}) + 2nj(v_r v_z\bar{A}, Qa^{nj}) \\
& - i\eta n(\mu\tilde{F})^{nj}_{v_\theta^2 - v_r^2} - i\eta j(\mu\tilde{F})^{nj}_{v_r v_z} + i\eta j(\mu\tilde{F})^{nj}_{v_r v_z\bar{A}} \\
& - \eta^2(\mu^2\tilde{F})^{nj}_{v_r^2 - v_\theta^2} + i\eta n(\mu\tilde{F})^{nj}_{v_r^2\bar{A}} - \eta(\mu'\tilde{F})^{nj}_{v_r^2\bar{A}}\Big),
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
m_4^{nj} =& \frac{1}{\lambda(n^2 + j^2)}\Big(-\frac{1}{\varepsilon_1^2}(g^{nj}, v^2 - 5) - i\frac{n}{\varepsilon_1}g^{nj}_{v_r\bar{A}} - \frac{ij}{\varepsilon_1}(g^{nj}, v_z\bar{A}) + \frac{\eta}{\varepsilon_1}(\mu g)^{nj}_{v_r\bar{A}} \\
& + \frac{(-1)^n}{\varepsilon_1}d^j_{v_r(v^2-5)} + i(-1)^n n d^j_{v_r^2\bar{A}} + ij(-1)^n d^j_{v_r v_z\bar{A}} - \eta(-1)^n e^j_{v_r^2\bar{A}} \\
& - n^2(Qv_r^2\bar{A}, Qa^{nj}) - j^2(Qv_z^2\bar{A}, Qa^{nj}) - 2nj(v_r v_z\bar{A}, Qa^{nj}) \\
& + i\eta n(\mu\tilde{F})^{nj}_{v_\theta^2 - v_r^2} + i\eta j(\mu\tilde{F})^{nj}_{v_r v_z} - i\eta j(\mu\tilde{F})^{nj}_{v_r v_z\bar{A}} + \eta^2(\mu\tilde{F})^{nj}_{v_r^2 - v_\theta^2} \\
& - i\eta n(\mu\tilde{F})^{nj}_{v_r^2\bar{A}} + \eta(\mu'\tilde{F})^{nj}_{v_r^2\bar{A}}\Big),
\end{aligned}
\tag{3.16}
$$

$$(3.17)\quad u_\theta^{nj} = \frac{1}{w_1(n^2+j^2)}\Big(-\frac{1}{\varepsilon_1^2}(g^{nj}, v_\theta) - \frac{in}{\varepsilon_1}(g^{nj}, v_r v_\theta \bar{B}) - \frac{ij}{\varepsilon_1}(g^{nj}, v_\theta v_z \bar{B}) - 2\frac{\eta}{\varepsilon_1}(\mu g)^{nj}_{v_r v_\theta \bar{B}}$$
$$+ \frac{(-1)^n}{\varepsilon_1} d^j_{r\theta} + in(-1)^n d^j_{v_r^2 v_\theta \bar{B}} + ij(-1)^n d^j_{v_r v_\theta v_z \bar{B}} + 2\eta(-1)^n e^j_{v_r^2 v_\theta \bar{B}}$$
$$- n^2(Qv_r^2 v_\theta \bar{B}, Qa^{nj}) - j^2(Qv_\theta v_z^2 \bar{B}, Qa^{nj}) - 2nj(v_r v_\theta v_z \bar{B}, Qa^{nj})$$
$$+ i\eta n(\mu\tilde{F})^{nj}_{(v_\theta^3 - 2v_r^2 v_\theta)\bar{B}} + 4i\eta j(\mu\tilde{F})^{nj}_{v_r v_\theta v_z \bar{B}} + 2i\eta n(\mu\tilde{F})^{nj}_{v_r^2 v_\theta \bar{B}}$$
$$- 2\eta(\mu'\tilde{F})^{nj}_{v_r^2 v_{\theta B}} + 2\eta^2(\mu^2\tilde{F})^{nj}_{v_r v_\theta \bar{B}}\Big),$$

$$(3.18)\quad u_r^{nj} = \frac{i}{n^2+j^2}\Big(-\frac{n}{\varepsilon_1}(g^{nj},1) + \frac{-j^2 A_r^{nj} + nj A_z^{nj}}{3\varepsilon_1 w_1(n^2+j^2)} + n(-1)^n d_r^j + \eta n(\mu\tilde{F})^{nj}_{v_r}\Big),$$

$$(3.19)\quad u_z^{nj} = \frac{i}{n^2+j^2}\Big(-\frac{j}{\varepsilon_1}g^{nj},1) + \frac{nj A_r^{nj} - n^2 A_z^{nj}}{3\varepsilon_1 w_1(n^2+j^2)} + j(-1)^n d_r^j + \eta j(\mu\tilde{F})^{nj}_{v_r}\Big).$$

Proof of Lemma 3.5. This is proved by moment projections and direct computations from the Fourier expanded (3.14), see [AN3]. □

Lemma 3.6. *Let $\tilde{F}$ be a solution to* (3.14). *Then for η small enough,*

$$| m_0^{00} | + | m_4^{00} | + | u_\theta^{00} | + | u_r^{00} | + | u_z^{00} |$$
$$\leq c\Big(\frac{| g_{\|} |_2}{\varepsilon_1^2} + \frac{| \upsilon^{-\frac{1}{2}} g_\perp |_2}{\varepsilon_1} + | S\tilde{F} |_\sim + \frac{| \tilde{F}_b |_\sim}{\sqrt{\varepsilon_1}} + \eta\|\tilde{F}\|_2\Big).$$

Proof of Lemma 3.6. For $(n,j) = (0,0)$, it holds that

$$(3.20)\quad a^{00} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dZ[\frac{1}{2}(\tilde{F}(\pi-0) + \tilde{F}(-\pi+0)) - \sum_{n\neq 0} a^{n0}e^{in\pi}] = \Delta - \sum_{n\neq 0} a^{n0}e^{in\pi},$$

where $\Delta = \frac{1}{4\pi}\int_{-\pi}^{\pi} dZ(\tilde{F}(\pi-0) + \tilde{F}(-\pi+0))$. First,

$$a^{00}_{v_r^2\bar{A}} = \frac{1}{\sqrt{6}} a_4^{00} \int v_r^2 v^2 \bar{A} M dv + a^{00}_{\perp v_r^2 \bar{A}}.$$

A multiplication of (3.20) with $Mv_r^2\bar{A}$ and v-integration gives

$$a^{00}_{v_r^2\bar{A}} = \Delta_{v_r^2\bar{A}} - \sum_{n\neq 0} a^{n0}_{v_r^2\bar{A}}(-1)^n.$$

To proceed, take the scalar product of (3.14) with $v_r\bar{A}$ and identify the Fourier coefficients,

$$(3.21)\qquad \begin{aligned}&(-1)^n d^j_{v_r^2\bar{A}} + in(v_r^2\bar{A}, a^{nj}) + ij(v_r v_z\bar{A}, a^{nj}) + \eta(\mu\tilde{F}_{v_\theta^2-v_r^2})^{nj}\\ &=\frac{1}{\varepsilon_1}\Big((v_r(v^2-5), a^{nj}) + (g^{nj}, v_r\bar{A})\Big).\end{aligned}$$

Also take the scalar product of (3.16) with v^2-5, and identify the Fourier coefficients,

$$(3.22)\qquad \begin{aligned}&-i(-1)^n d^j_{v_r(v^2-5)} + n(v_r(v^2-5), a^{nj}) + j(v_z(v^2-5), a^{nj})\\ &=-\frac{i}{\varepsilon_1}\Big((g^{nj}, v^2-5) + \varepsilon_1\eta(\mu\tilde{F})^{nj}_{v_r(v^2-5)}\Big).\end{aligned}$$

Moreover, (3.14) writes

$$v_r\frac{\partial}{\partial s}(\mu\tilde{F}) + v_z\frac{\partial}{\partial Z}(\mu\tilde{F}) - v_r\mu'\tilde{F} + \eta\mu^2 N\tilde{F} = \frac{1}{\varepsilon_1}(L(\mu\tilde{F}) + \mu g),$$

so that

$$\begin{aligned}&i(nv_r + jv_z)(\mu\tilde{F})^{nj} + (-1)^n v_r e^j(v) - v_r(\mu'\tilde{F})^{nj}\\ &+\eta(\mu^2 N\tilde{F})^{nj} = \frac{1}{\varepsilon_1}(L(\mu\tilde{F})^{nj} + (\mu g)^{nj}),\end{aligned}$$

where

$$e(Z,v) := \frac{1}{2\pi}\Big((\mu\tilde{F})(\pi-0, Z, v) - (\mu\tilde{F})(-\pi+0, Z, v)\Big) = \sum_{j\in Z} e^j(v)e^{ijZ}.$$

Taking the scalar product with $v_r\bar{A}$ leads to

$$(3.23)\qquad \begin{aligned}&(-1)^n e^j_{v_r^2\bar{A}} + in(v_r^2\bar{A}, (\mu\tilde{F})^{nj}) + ij(v_r v_z\bar{A}, (\mu\tilde{F})^{nj} - (v_r^2\bar{A}, (\mu'\tilde{F})^{nj})\\ &+\eta(\mu^2\tilde{F}_{v_r^2-v_\theta^2})^{nj} = \frac{1}{\varepsilon_1}\big((v_r(v^2-5), (\mu\tilde{F})^{nj}) + (v_r\bar{A}, (\mu g)^{nj})\big).\end{aligned}$$

By (3.21-23) for $n\neq 0$,

$$\begin{aligned}a^{n0}_{v_r^2\bar{A}} &= -\frac{1}{\varepsilon_1^2 n^2}g^{n0}_{v^2-5} - \frac{i}{\varepsilon_1}g^{n0}_{v_r\bar{A}} + \frac{(-1)^n}{\varepsilon_1 n^2}d^0_{v_r(v^2-5)}\\ &+i\frac{(-1)^n}{n}d^0_{v_r^2\bar{A}} - \frac{\eta}{\varepsilon_1 n^2}(\mu\tilde{F})^{n0}_{v_r(v^2-5)} + i\frac{\eta}{n}(\mu\tilde{F})^{n0}_{v_\theta^2-v_r^2}\\ &= -\frac{1}{\varepsilon_1^2 n^2}g^{n0}_{v^2-5} - \frac{i}{\varepsilon_1}g^{n0}_{v_r\bar{A}} + \frac{\eta}{n^2}(\mu g)^{n0}_{v_r\bar{A}}\end{aligned}$$

$$+\frac{(-1)^n}{\varepsilon_1 n^2} d^0_{v_r(v^2-5)} + i\frac{(-1)^n}{n} d^0_{v_r^2\bar{A}} - \eta\frac{(-1)^n}{n^2} e^0_{v_r^2\bar{A}}$$

$$+ i\frac{\eta}{n}(\mu\tilde{F})^{n0}_{v_\theta^2-v_r^2} - i\frac{\eta}{n}(\mu\tilde{F})^{n0}_{v_r^2\bar{A}}$$

$$+\frac{\eta}{n^2}(\mu'\tilde{F})^{n0}_{v_r^2\bar{A}} - \frac{\eta^2}{n^2}(\mu^2\tilde{F})^{n0}_{v_r^2-v_\theta^2}.$$

From here, using

$$d^0_{v_r(v^2-5)} + \eta(\mu\tilde{F})^{00}_{v_r(v^2-5)} = \frac{1}{\varepsilon_1} g^{00}_{(v^2-5)},$$

it follows that

$$|\, m_4^{00}\,|_2 \le c\Big(\frac{|\, g_0\,|_2 + |\, g_4\,|_2}{\varepsilon_1^2} + \frac{\eta\,|\, g_r\,|_2 + |\, \nu^{-\frac{1}{2}} g_\perp\,|_2}{\varepsilon_1} + |\,\tilde{F}_\perp|_2 + |\, S\tilde{F}|_\sim + |\,\tilde{F}_b\,|_\sim + \eta\,|\tilde{F}_\parallel|\Big).$$

Since

$$m_0^{00} = a^{00}_{r^2} - \frac{\sqrt{6}}{3} m_4^{00} - a^{00}_{\perp r^2}, \quad u_\theta^{00} = \frac{1}{w_1}(a^{00}_{v_r^2 v_\theta \bar{B}} - a^{00}_{\perp v_r^2 v\theta\bar{B}}),$$

$$u_r^{00} = \Delta_r - \sum_{n\neq 0}(-1)^n u_r^{n0}, \quad u_z^{00} = \frac{1}{w_1}(a^{00}_{v_r^2 v_z\bar{B}} - a^{00}_{\perp v_r^2 v_z\bar{B}}),$$

similar inequalities can be obtained for m_4^{00}, u_θ^{00}, u_r^{00}, and u_z^{00} and the lemma follows. □

Lemma 3.7. *Let $\nu^{\frac{1}{2}}\beta \in \tilde{L}^\infty$ be given. Then there is $\eta_0 > 0$ such that for $\eta < \eta_0$, a solution $\tilde{F}$ in $\mathcal{W}^{2-}$ to*

$$v_r\frac{\partial\tilde{F}}{\partial s} + v_z\frac{\partial\tilde{F}}{\partial Z} + \eta\mu N\tilde{F} = \frac{\eta}{\varepsilon}\Big(L\tilde{F} + \varepsilon J(\tilde{F},\beta) + g\Big), \quad \tilde{F}_{/\partial\Omega^+} = \tilde{F}_b, \tag{3.24}$$

satisfies

$$|\,\nu^{\frac{1}{2}}\tilde{F}\,|_2 \le c\Big(\frac{\eta^2}{\varepsilon^2}\,|\, g_\parallel\,|_2 + \frac{\eta}{\varepsilon}\,|\,\nu^{-\frac{1}{2}} g_\perp\,|_2 + \sqrt{\frac{\eta}{\varepsilon}}\,|\,\tilde{F}_b\,|_\sim\Big). \tag{3.25}$$

Proof of Lemma 3.7 Consider first the case where $\beta = 0$. As in the axially homogeneous situation, Green's formula and Lemma 2.2 imply that

$$\varepsilon_1\,|\, S\tilde{F}\,|_\sim^2 + |\,\nu^{\frac{1}{2}}\tilde{F}_\perp\,|_2^2 \le c(\,|\,\nu^{-\frac{1}{2}} g_\perp\,|_2^2 + \int(g_\parallel, \tilde{F}_\parallel) + \varepsilon_1\,|\,\tilde{F}_b\,|_\sim^2\,). \tag{3.26}$$

Then Parseval's identity, Lemma 3.6 for $(n,j) = (0,0)$, and an estimate of the

Fourier coefficients $(n, j) \neq (0,0)$ as given in Lemma 3.5, imply that

$$|\tilde{F}_{\|}|_2 \leq c\Big(\frac{|g_{\|}|_2}{\varepsilon_1^2} + \frac{\|\nu^{-\frac{1}{2}} g_{\perp}\|_2}{\varepsilon_1} + \frac{|\tilde{F}_b|_{\sim}}{\sqrt{\varepsilon_1}} + |\nu^{\frac{1}{2}}\tilde{F}_{\perp}|_2 + \eta |\tilde{F}_{\|}|_2\Big).$$

And so (3.25) holds in the $\beta = 0$ case, since $|F_{\|}|_2 \simeq |\nu^{\frac{1}{2}} F_{\|}|_2$,. The case $\beta \neq 0$ can be handled as the case $\beta = 0$ with g in the right hand side, by taking instead $g + \varepsilon \tilde{J}(\tilde{F}, \beta)$ in the right hand side. This gives

$$|\tilde{F}_{\|}|_2 \leq c\Big(\frac{|g_{\|}|_2}{\varepsilon_1^2} + \frac{\|\nu^{-\frac{1}{2}}(g_{\perp} + \varepsilon\tilde{J}(\tilde{F}, \beta))\|_2}{\varepsilon_1} + \frac{1}{\sqrt{\varepsilon_1}} |\tilde{F}_b|_{\sim}\Big)$$

$$\leq c\Big(\frac{|g_{\|}|_2}{\varepsilon_1^2} + \frac{|\nu^{-\frac{1}{2}} g_{\perp}|_2}{\varepsilon_1} + \frac{1}{\sqrt{\varepsilon_1}} |\tilde{F}_b|_{\sim} + \eta |\nu^{\frac{1}{2}}\tilde{F}|_2 |\nu^{\frac{1}{2}}\beta|_{\infty}\Big).$$

Thus the lemma holds for η small enough. □

Remark. If we had access to the estimates in this section of the non-hydrodynamic part with respect to $\tilde{L}^q$ for (large) $q > 2$, then the actual asymptotic expansions required in the existence proofs of the following Section 4 would be considerably shortened in the Cases 2 and 3.

4 - Existence theorems and fluid dynamic limits

Based on the discussions about asymptotic expansions and a priori estimates in Sections 2-3, this section studies existence results and fluid dynamic limits for our three choices of archetypical two-rolls behaviour.

Given the asymptotic expansion φ of (2.4), the aim for Case 1 is to prove the existence of a rest term R, so that

$$f = M(1 + \varphi + \varepsilon R) \tag{4.1}$$

is a solution to (2.1), (2.3) in Case 1 with $M^{-1}f \in \tilde{L}^{\infty}$. This corresponds to the function R being a solution to

$$DR = \frac{1}{\varepsilon}\big(LR + 2J(R, \varphi) + \varepsilon J(R, R) + l\big),$$

where l was defined in (2.10). Recall that the asymptotic expansion φ is of order two in ε with correct boundary values up to order two and that l of (2.10) - the pure φ- part of the equation - is of ε-order two and η-order one in $\tilde{L}^q$, where $\eta = r_B - r_A$. Notice that Φ^j, $j = 1, 2$, may be constructed so that "practically" $D\Phi^j = (I - P_0)D\Phi^j$, hence $l = l_{\perp}$. This holds modulo a possible higher order fluid dynamic component, neglected in this section, that does not change the line of reasoning or its results.

Let the sequences $(R^n)_{n\in N}$ be defined by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\varepsilon}\Big(LR^{n+1} + 2\sum_{j=1}^{2}\varepsilon^j J(R^{n+1}, \Phi^j) + g^n\Big), \tag{4.2}$$

$$R^{n+1}(1,v) = R_A(v),\ v_r > 0, \quad R^{n+1}(r_B, v) = R_B(v),\ v_r < 0. \tag{4.3}$$

In (4.2-3)

$$g^n := \varepsilon J(R^n, R^n) + l,$$
$$\varepsilon R_A(v) := e^{\varepsilon u_{\theta A1} v_\theta - \frac{\varepsilon^2}{2} u_{\theta A1}^2} - 1 - \sum_{j=1}^{2}\varepsilon^j \Phi^j(r_A, v), \quad v_r > 0,$$
$$\varepsilon R_B(v) := -\sum_{j=1}^{2}\varepsilon^j \Phi^j(r_B, v), \quad v_r < 0,$$

with R_A, R_B of ε-order two.

For the rest term iteration scheme (4.2-3) the following holds.

Lemma 4.1. *For $0 < \varepsilon$, $0 < r_B - r_A$ small enough, there is a unique sequence (R^n) of solutions to (4.2-3) in the set $X := \{R; |\, \tilde{\nu}^{\frac{1}{2}} R\,|_q \le C\}$ for some constant C. The sequence converges in $\tilde{L}^q$ for $2 \le q \le \infty$, to an isolated solution of*

$$DR = \frac{1}{\varepsilon}\Big(LR + \varepsilon J(R,R) + 2J(R,\varphi) + l\Big), \tag{4.4}$$

$$R(1,v) = R_A(v),\ v_r > 0, \quad R(r_B, v) = R_B(v),\ v_r < 0. \tag{4.5}$$

Proof of Lemma 4.1 Denote by $\eta = r_B - r_A$. The existence result of Lemma 3.1 holds for the boundary value problem

$$DF = \frac{1}{\varepsilon}\big(LF + 2J(F,\varphi) + g\big),$$
$$F(1,v) = R_A(v),\ v_r > 0, \quad F(r_B, v) = R_B(v),\ v_r < 0.$$

Here $g = g_\perp$ and by Lemma 3.2-3

$$\begin{aligned} |\, \nu^{\frac{1}{2}} F\,|_2 &\le c_1\Big(\frac{1}{\varepsilon}\,|\, \nu^{-\frac{1}{2}} g_\perp\,|_2 + |\, R_b\,|_\sim\Big),\\ |\, \nu^{\frac{1}{2}} F\,|_\infty &\le c_1\Big(\,|\, \nu^{-\frac{1}{2}} g\,|_\infty + \frac{1}{\varepsilon}\,|\, \nu^{\frac{1}{2}} F\,|_2 + |\, \nu^{\frac{1}{2}} R_b\,|_\sim\Big). \end{aligned} \tag{4.6}$$

We note the obvious L^2-norm equivalence $|\, F_\parallel\,|_2 \simeq |\, \nu^{\frac{1}{2}} F_\parallel\,|_2$, and the Grad type inequality

$$|\, \nu^{-\frac{1}{2}} J(g,h)\,|_q \le C\,|\, \nu^{\frac{1}{2}} g\,|_\infty |\, \nu^{\frac{1}{2}} h\,|_q, \tag{4.7}$$

which follows by an easy, direct computation. This will next be used to show by induction that

$$(4.8) \qquad | v^{\frac{1}{2}}(R^{n+1}-R^n) |_2 \leq c\eta \, | v^{\frac{1}{2}}(R^n-R^{n-1}) |_2, \quad | v^{\frac{1}{2}}R^n |_\infty \leq c\eta, n\in \mathbb{N}, n>0.$$

For $n=0$, R^1 is the solution to

$$DR^1 = \frac{1}{\varepsilon}(LR^1 + 2J(\varphi, R^1) + l),$$
$$R^1(1,v) = R_A(v), v_r > 0, \quad R^1(r_B, v) = R_B(v), v_r < 0,$$

so that by (4.6-7) $| v^{\frac{1}{2}}R^1 |_2 \leq c\eta\varepsilon$, $| v^{\frac{1}{2}}R^1 |_\infty \leq c\eta$, where $\eta = r_B - r_A$. Also, $R^{n+2}-R^{n+1}$ is a solution to

$$D(R^{n+2}-R^{n+1}) = \frac{1}{\varepsilon}(L(R^{n+2}-R^{n+1}) + 2J(\varphi, R^{n+2}-R^{n+1})$$
$$+ \varepsilon J(R^{n+1}+R^n, R^{n+1}-R^n)), \quad R^{n+2}-R^{n+1} = 0, \quad \partial\Omega^+,$$

which by (4.6-7) and the induction hypothesis (4.8) leads to

$$| v^{\frac{1}{2}}(R^{n+2}-R^{n+1}) |_2 \leq c \, | v^{-\frac{1}{2}}J(R^{n+1}+R^n, R^{n+1}-R^n) |_2$$
$$\leq c(| v^{\frac{1}{2}}R^{n+1} |_\infty + | v^{\frac{1}{2}}R^n |_\infty) \, | v^{\frac{1}{2}}(R^{n+1}-R^n) |_2$$
$$\leq 2c^2\eta \, | v^{\frac{1}{2}}(R^{n+1}-R^n) |_2 .$$

Moreover,

$$| v^{\frac{1}{2}}R^{n+2} |_\infty \leq | v^{\frac{1}{2}}(R^{n+2}-R^{n+1}) |_\infty + \ldots + | v^{\frac{1}{2}}(R^2-R^1) |_\infty + | v^{\frac{1}{2}}R^1 |_\infty \leq c\eta,$$

for sufficiently small $\eta > 0$. And so (R^n) converges to some R, solution to (4.4-5) in $\tilde{L}^q$ for $q \leq \infty$. The contraction mapping construction guarantees that the solution is isolated. □

The existence of isolated solutions to (2.1), (2.3) is an immediate consequence of Lemma 4.1. It also follows that, when ε tends to zero the fluid dynamic moments converge to the (Hilbert type) corresponding leading (first) order limiting fluid solution given by (2.5). This is obvious in L^2 from the estimate of R^1 in Lemma 4.1, and holds in L^∞ for the following reason. If the asymptotic expansion were carried out to third order, then R^1 would be of order ε also in L^∞. Grouping it together with the new third order term from the asymptotic expansion, shows that the R^1 of our present Lemma 4.1 also is of order ε. We have thus proved

Theorem 4.2. *For $0 < \varepsilon$, $0 < r_B - r_A$ small enough and $j = 1$, there is an isolated axially homogeneous solution of* (2.1), (2.3). *When ε tends to zero, the corresponding fluid dynamic moments of ϕ converge to solutions of the limiting fluid equations at the leading order ε.*

In Case 1 the (incompressible) fluid dynamics behaviour is given by the limiting first order (angular) velocity $\frac{u_{\theta A}}{r}\frac{r_B^2 - r^2}{r_B^2 - r_A^2} = \frac{U_{\theta A}}{r}\frac{r_B^2 - r^2}{r_B + r_A}$.

Using similar arguments but more extended asymptotic expansions, the same type of results can be proved in the other cases. In Case 2 our present estimates give (see [AN2])

Theorem 4.3. *Assume that $r_B - r_A$ is small enough and that $(A + 5D) < 0$. There is a negative value Δ_{bif} of the parameter Δ, such that for the quantity $\Delta_{bif} - \Delta$ positive and small enough, there are for ε positive and small enough, two isolated, non-negative L^1-solutions f_ε^j, $j = 1, 2$ of* (2.1), (2.3) *coexisting with $M^{-1} f_\varepsilon^j \in \tilde{L}^\infty$,*

$$\int M^{-1} supess_{r \in (r_A, r_B)} \mid f_\varepsilon^j(r, v) \mid^2 dv < +\infty.$$

The two solutions have different outward radial bulk velocities of order ε^3. For fixed ε, they converge to the same solution, when Δ increases to Δ_{bif}. Their fluid dynamic moments converge to solutions of the corresponding limiting fluid equations at leading order, when $\varepsilon \to 0$.

Here the leading order (in ε) fluid dynamics behaviour is given by the first order angular velocity $\frac{u_{\theta A1}}{r} - \frac{u_{\theta A1}}{r_B} e^{\frac{u_3 Y}{w_1 r_B}}$ and the two possible third order radial velocities $\frac{u_3}{r}$, where u_3 solves (2.40).

Finally in Case 3 one obtains

Theorem 4.4. *For $j = 1$, $0 < \varepsilon$, $0 < r_B - r_A$ small enough, there is a smallest bifurcation value $u_{\theta Ab} > 0$, such that the axially homogeneous solution to the problem* (2.1), (2.3) *bifurcates at $u_{\theta Ab}$ with a steady secondary solution appearing locally for $u_{\theta Ab} < u_{\theta A}$, which is axially symmetric and axially $(r_B - r_A)$-periodic. When ε tends to zero, the corresponding fluid dynamic moments converge to solutions of the limiting fluid equations at the leading order ε (bifurcated solution of Taylor-Couette type).*

In this case the limiting fluid Taylor-Couette equations of incompressible Navier-Stokes type are

$$
\begin{aligned}
&u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}=-\frac{1}{2}\frac{\partial P_1}{\partial r}+\mu\Big(\Delta u_r-\frac{u_r}{r^2}\Big),\\
&\frac{u_r}{r}\frac{\partial(ru_\theta)}{\partial u_\theta}+u_z\frac{\partial u_\theta}{\partial z}=\mu\Big(\Delta u_\theta-\frac{u_\theta}{r^2}\Big),\\
&u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{2}\frac{\partial P_1}{\partial z}+\mu\Delta u_z,\\
&\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z}=0,
\end{aligned}
\tag{4.9}
$$

where μ depends on the molecular model, and P_1 is the next order term in ε of the perturbed relative pressure.

Proof of Theorem 4.4 Given the asymptotic expansion (4.1) in Case 3 and its bifurcation point, the aim is to prove the existence of a rest term R, so that for the parameters near the bifurcation point, there is an axially periodic solution

$$f=M(1+\varphi+\varepsilon R)$$

to (2.1) with an added $\dfrac{\partial}{\partial z}$-term and boundary values (2.3) with $M^{-1}f\in\tilde{L}^\infty$. This corresponds to the rest term R being a solution of the same type to

$$DR=\frac{1}{\varepsilon}\big(LR+2\tilde{J}(R,\varphi)+\varepsilon J(R,R)+l\big).$$

In Section 2 a third order asymptotic expansion in ε was constructed in a δ^2-neighbourhood of the bifurcation velocity $u_{\theta Ab}$ with correct boundary values up to ε-order three, and so that l - the φ-part of the equation - is smooth in r, z and of order ε^3 in $\tilde{L}^q$. Notice that Φ^j can be constructed so that $D\Phi^j=(I-P_0)D\Phi^j$, hence that $l=l_\perp$.

Let the sequences $(R^n)_{n\in\mathbb{N}}$ be defined as in the earlier Couette case by $R^0=0$, and

$$DR^{n+1}=\frac{1}{\varepsilon}\Big(LR^{n+1}+2\sum_{j=1}^{3}\varepsilon^j J(R^{n+1},\Phi^j)+g^n\Big),\tag{4.10}$$

$$R^{n+1}(1,v)=R_A(v),\ v_r>0,\quad R^{n+1}(r_B,v)=R_B(v),\ v_r<0.\tag{4.11}$$

In (4.10-11)

$$
\begin{aligned}
&g^n:=\varepsilon^2J(R^n,R^n)+l,\\
&\varepsilon R_A(v):=e^{\varepsilon u_{\theta A1}v_\theta-\frac{2}{2}u_{\theta A1}^2}-1-\sum_{j=1}^{3}\varepsilon^j\Phi^j(r_A,v),\quad v_r>0,\\
&\varepsilon R_B(v):=0,\quad v_r<0,
\end{aligned}
$$

with $R=(R_A,R_B)$ of ε-order three.

For the rest term iteration scheme (4.10-11) the following proposition holds and with it the proof of Theorem 4.4 is complete.

Proposition 4.5. *For $\varepsilon > 0$ and small enough together with $\eta = r_B - r_A$, there is a unique sequence (R^n) of solutions to (4.10-11) in the set $X := \{R; | \nu^{\frac{1}{2}} R |_q \leq K\varepsilon\}$ for some constant K. The sequence converges in $\tilde{L}^q$ for $2 \leq q \leq \infty$, to an isolated solution of*

$$DR = \frac{1}{\varepsilon}\left(LR + \varepsilon J(R,R) + 2J(R,\varphi) + l\right), \tag{4.12}$$

$$R(1,v) = R_A(v),\ v_r > 0, \quad R(r_B,v) = R_B(v),\ v_r < 0. \tag{4.13}$$

Proof of Proposition 4.5. The existence result of Lemma 3.1 holds for the boundary value problem

$$Df = \frac{1}{\varepsilon}\Big(Lf + 2\sum_{j=1}^{3}\varepsilon^j J(f,\Phi^j) + g\Big),$$

$$f(1,v) = R_A(v),\ v_r > 0, \quad f(r_B,v) = R_B(v),\ v_r < 0.$$

Rescale in space to $(-\pi,\pi)^2$ and consider the approximation (4.10-11) in the case $n = 0$ with $g^0 = l$. As discussed before (4.10), this $g^0 = g^0_\perp$ is of order ε^3 in $\tilde{L}^\infty$, and

$$| \nu^{-\frac{1}{2}} l |_\infty + | R_b |_\sim \leq c_1\varepsilon^3,$$

for some constant c_1. By (3.25) and (3.3) it holds that for some constant c_2

$$| \nu^{\frac{1}{2}} R^1 |_2 \leq c_1c_2\eta\varepsilon^2, \quad | \nu^{\frac{1}{2}} R^1 |_\infty \leq 2c_1c_2\eta\varepsilon, \tag{4.14}$$

for η and ε small enough. Let us prove by induction that

$$\begin{aligned} &| \nu^{\frac{1}{2}} R^n |_\infty \leq 4c_1c_2\varepsilon, \\ &| \nu^{\frac{1}{2}}(R^{n+1} - R^n) |_2 \leq 2c_1c_2\varepsilon\, | \nu^{\frac{1}{2}}(R^n - R^{n-1}) |_2, \quad n \geq 1. \end{aligned} \tag{4.15}$$

For $n = 1$, $R^2 - R^1$ satisfies

$$D(R^2 - R^1) = \frac{\eta}{\varepsilon}\Big(L(R^2 - R^1) + 2\sum_{j=1}^{3}\varepsilon^j J(R^2 - R^1,\Phi^j) + \varepsilon J(R^1,R^1)\Big),$$

$$(R^2 - R^1)(r_A,z,v) = 0, v_r > 0, \quad (R^2 - R^1)(r_B,z,v) = 0, v_r < 0,$$

so that, by (3.25),

$$| \nu^{\frac{1}{2}}(R^2 - R^1) |_2 \leq c_2\eta\, | \nu^{-\frac{1}{2}} J(R^1,R^1) |_2 .$$

Recall that for any $g \in \tilde{L}^\infty$ (resp. $h \in \tilde{L}^q$,

$$| \nu^{-\frac{1}{2}} J(g,h) |_q \leq c_3\, | \nu^{\frac{1}{2}} g |_\infty | \nu^{\frac{1}{2}} h |_q . \tag{4.16}$$

Hence

$$| \nu^{\frac{1}{2}}(R^2 - R^1) |_2 \leq c_1 \eta^2 \varepsilon \, | \nu^{\frac{1}{2}}(R^1 - R^0) |_2,$$

for η small enough. If (4.15) holds until n, then

$$\begin{aligned} | \nu^{\frac{1}{2}} R^{n+1} |_\infty &\leq | \nu^{\frac{1}{2}}(R^{n+1} - R^n) |_\infty + \ldots + | \nu^{\frac{1}{2}}(R^1 - R^0) |_\infty \\ &\leq \frac{c_4}{\varepsilon}(| \nu^{\frac{1}{2}}(R^{n+1} - R^n) |_2 + \ldots + | \nu^{\frac{1}{2}}(R^1 - R^0) |_2) \\ &\leq 4 c_1 c_2 \varepsilon, \end{aligned}$$

for η small enough. Then $R^{n+2} - R^{n+1}$ satisfies

$$\begin{aligned} D(R^{n+2} - R^{n+1}) &= \frac{1}{\varepsilon}\Big(L(R^{n+2} - R^{n+1}) + 2\sum_{j=1}^{3} \varepsilon^j J(R^{n+2} - R^{n+1}, \Phi^j) \\ &+ \varepsilon J(R^{n+1} + R^n, R^{n+1} - R^n) \Big) \\ &(R^{n+2} - R^{n+1})(r_A, z, v) = 0, \ \ v_r > 0, \quad (R^{n+2} - R^{n+1})(r_B, z, v) = 0, \ \ v_r < 0, \end{aligned}$$

so that by (3.25) and the bound on $| \nu^{\frac{1}{2}} R^n |_\infty$ and $| \nu^{\frac{1}{2}} R^{n+1} |_\infty$,

$$\begin{aligned} | \nu^{\frac{1}{2}}(R^{n+2} - R^{n+1}) |_2 &\leq c_3 \eta (| \nu^{\frac{1}{2}} R^{n+1} |_\infty + | \nu^{\frac{1}{2}} R^n |_\infty) \, | \nu^{\frac{1}{2}}(R^{n+1} - R^n) |_2 \\ &\leq 2 c_1 c_2 \varepsilon \, | \nu^{\frac{1}{2}}(R^{n+1} - R^n) |_2, \end{aligned}$$

for ε and η small enough.

And so (R^n) converges for sufficiently small $\eta > 0$ to some R, solution to (4.12-13) in $\tilde{L}^q$ for $q \leq \infty$. The contraction mapping construction guarantees that this solution is isolated.

5 - Stability

We next come to the question of stability for the solutions obtained in the previous sections. Only Case 1 will be discussed. It turns out that the well known *fluid stability* of the leading order term is the prime mover behind the *kinetic stability*, which in a certain way is uniform down to the fluid level. More precisely we shall devote this section to prove the following new result.

Theorem 5.1. *The steady Couette problem for the Boltzmann equation in the two rolls problem is stable. The stability is uniform in the following sense for small enough mean free path ε. When the gap between the cylinders is small and the angular, axial and energy moments are perturbed of order ε or ε^2, then uniformly* *in ε the perturbation vanishes asymptoticaly in time. Also an initial perturbation of order ε^3, with small but otherwise arbitrary fluid dynamic as well as non fluid dynamic part, vanishes asymptotically in time.*

This type of results is expected to carry over to the cases 2-3, where also the fluid stability is well understood.

Among the few earlier rigorous non-linear kinetic stability results outside the situation with global Maxwellian limits, are the studies in [UYY] dealing with stability of half-space Milne problems, and [UYZ] dealing with the Boltzmann equation in full space with an external force.

With $\tilde{\Phi}_s = 1 + \Phi_s$ the rescaled stationary solution, the stability problem consists in proving that the distribution function $\tilde{\Phi}$ tends to $\tilde{\Phi}_s$ when $t \to \infty$, where $\tilde{\Phi}$ solves the evolutionary problem

$$\begin{aligned}
&\frac{\partial \tilde{\Phi}}{\partial t} + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{\Phi} = \frac{1}{\varepsilon^2}(L\tilde{\Phi} + J(\tilde{\Phi}, \tilde{\Phi})),\\
&\tilde{\Phi}(0, r, v) = \tilde{\Phi}_s(r, v) + P(r, v), \quad r \in (r_A, r_B), \ v \in \mathbb{R}^3,\\
&\tilde{\Phi}(t, r_A, v) = \tilde{\Phi}_s(r_A, v), \ t > 0, \ v_r > 0,\\
&\tilde{\Phi}(t, r_B, v) = \tilde{\Phi}_s(r_B, v), \ t > 0, \ v_r < 0,
\end{aligned}$$

and P is a small perturbation of $\tilde{\Phi}_s$.

Denote by $\tilde{\psi} = \tilde{\Phi} - \tilde{\Phi}_s$. It should then be a solution to

$$\frac{\partial \tilde{\psi}}{\partial t} + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{\psi} = \frac{1}{\varepsilon^2}\left(L\tilde{\psi} + J(\tilde{\psi}, \tilde{\psi}) + 2J(\tilde{\psi}, \tilde{\Phi}_s)\right), \tag{5.1}$$

$$\tilde{\psi}(0, r, v) = P(r, v), \quad r \in (r_A, r_B), \ v \in \mathbb{R}^3, \tag{5.2}$$

$$\tilde{\psi}(t, r_A, v) = 0, \ t > 0, \ v_r > 0, \quad \tilde{\psi}(t, r_B, v) = 0, \ t > 0, \ v_r < 0, \tag{5.3}$$

and tend to zero when $t \to \infty$.

Here the following perturbations P are considered,

$$P(r, v) = \varepsilon(a_1(v^2 - 5) + \beta_1 v_\theta + \gamma_1 v_z) + \varepsilon^2(a_2(v^2 - 5) + \beta_2 v_\theta + \gamma_2 v_z) + \varepsilon^3 p_3(x, v, \varepsilon),$$

where a_i, β_i, γ_i, $1 \le i \le 2$ are L^∞-functions of the space variable, and the function $p_3(x, v, \varepsilon)$ is measurable with $\|p_3\|_{\infty,2} < c$ uniformly in ε, where

$$\|p_3\|_{\infty,2} = \Big(\int_{\mathbb{R}^3} \sup_{x \in \Omega} p_3^2(x, v, \varepsilon) M(v) dv\Big)^{\frac{1}{2}}.$$

As in Section 4, the stationary solution $\tilde{\Phi}_s$ is here determined by an approximate asymptotic expansion Φ_s of terms of up to third order in ε with boundary values being those of the same order of $e^{\frac{1}{2}(v_\theta^2 - (v_\theta - \varepsilon u_{\theta A})^2)}$ at $\{(r_A, v), \ v_r > 0\}$ (resp. 0 at

$\{(r_B, v),\ v_r < 0\}$, plus a rest term εS,

$$\tilde{\Phi}_s(r,v) = 1 + \Phi_s(r,v) + \varepsilon S(r,v),$$

where $\|S\|_{\infty,2} \le c|r_B - r_A|\varepsilon$, $\|S\|_{2,2} \le c|r_B - r_A|\varepsilon^2$, and

$$\Phi_s(r,v) = \varepsilon\Phi_{H1}(r,v) + \varepsilon^2\Phi_2 + \varepsilon^3\Phi_3,$$
$$\Phi_i = \Phi_{Hi}(r,v) + \Phi_{KiA}(\frac{r-1}{\varepsilon}, v) + \Phi_{KiB}(\frac{r-r_B}{\varepsilon}, v), \quad 2 \le i \le 3.$$

The Hilbert terms Φ_{Hi}, $1 \le i \le 3$ satisfy

$$L\Phi_{H1} = L\Phi_{H2} + J(\Phi_{H1}, \Phi_{H1}) - v\cdot\nabla_x\Phi_{H1} = L\Phi_{H3} + 2J(\Phi_{H1}, \Phi_{H2}) - v\cdot\nabla_x\Phi_{H2} = 0.$$

They are given by

$$\Phi_{H1}(r,v) = b_1(r)v_\theta,$$
$$\Phi_{H2}(r,v) = a_2 + d_2v^2 + b_2v_\theta + c_2v_r + \frac{1}{2}b_1^2v_\theta^2 + (b_1' - \frac{1}{r}b_1)v_rv_\theta\bar{B},$$
$$\Phi_{H3}(r,v) = a_3 + d_3v^2 + b_3v_\theta + c_3v_r + d_2'v_r\bar{A} + b_1d_2v_\theta v^2 + b_1b_2v_\theta^2$$
$$+ b_1c_2v_rv_\theta + \frac{1}{6}b_1^3v_\theta^3 - b_1(b_1' - \frac{1}{r}b_1)L^{-1}(J(v_\theta, v_rv_\theta\bar{B}))$$
$$+ (b_1b_1' - \frac{1}{r}b_1^2)L^{-1}(v_r(v_\theta^2 - 1)) + \frac{1}{r}c_2L^{-1}((v_\theta^2 - v_r^2))$$
$$+ \frac{1}{r}(b_1' - \frac{1}{r}b_1)L^{-1}((v_\theta^3 - 3v_r^2v_\theta)\bar{B}) + b_2'v_rv_\theta\bar{B}.$$

We take $r_A = 1$ (implying $r_B > 1$). For compatibility reasons

$$b_1(r) = \frac{u_{\theta A}}{r_B^2 - 1}(\frac{r_B^2}{r} - r), \tag{5.4}$$

$a_i + 5d_i$ (resp. c_i), $2 \le i \le 3$, satisfy first-order differential equations, whereas b_i (resp. d_i), $2 \le i \le 3$, satisfy second-order differential equations. Knudsen terms Φ_{KAi} (resp. Φ_{KBi}), $2 \le i \le 3$ are added in order to satisfy the given zero ingoing boundary conditions up to third order.

The solution $\tilde{\psi}$ to the evolutionary problem (5.1-3) is determined as the sum of an asymptotic expansion ψ and a rest term εR,

$$\tilde{\psi} = \psi + \varepsilon R,$$

where

$$\psi(t,r,v) = \varepsilon\psi_{H1}(t,r,v) + \varepsilon^2\psi_2 + \varepsilon^3\psi_3,$$
$$\psi_i = \psi_{Hi}(t,r,v) + \psi_{KiA}(t, \frac{r-1}{\varepsilon}, v) + \psi_{KiB}(t, \frac{r-r_B}{\varepsilon}, v), \quad 2 \le i \le 3.$$

The initial values of ψ_3 is taken as zero, those of ψ_{H1}, ψ_2 are the corresponding orders of P and finally $R_0 := \varepsilon^2 p_3$ is taken as initial value for R. For (5.1) to be satisfied up to zeroth order in ε included, it is required that

$$\begin{aligned}
0 &= L\psi_{H1} = L\psi_{H2} + J(\psi_{H1}, \psi_{H1} + 2\Phi_{H1}) - v\cdot\nabla_x\psi_{H1}\\
&= L\psi_{H3} + 2J(\psi_{H1}, \psi_{H2} + \Phi_{H2}) + 2J(\psi_{H2}, \Phi_{H1}) - \frac{\partial\psi_{H1}}{\partial t} - v\cdot\nabla_x\psi_{H2}\\
&= L\psi_{K2A} - v_r\frac{\partial\psi_{K2A}}{\partial\eta} = L\psi_{K2B} - v_r\frac{\partial\psi_{K2B}}{\partial\mu}\\
&= L\psi_{K3A} + 2J(\psi_{H1}(r_A), \psi_{K2A} + \Phi_{K2A}) + 2J(\psi_{K2A}, \Phi_{H1}(r_A)) - \frac{1}{r}N\psi_{K2A} - v_r\frac{\partial\psi_{K3A}}{\partial\eta}\\
&= L\psi_{K3B} + 2J(\psi_{H1}(r_B), \psi_{K2B} + \Phi_{K2B}) + 2J(\psi_{K2B}, \Phi_{H1}(r_B)) - \frac{1}{r}N\psi_{K2B} - v_r\frac{\partial\psi_{K3B}}{\partial\mu}.
\end{aligned}$$

The rest term R should then be a solution to

$$\frac{\partial R}{\partial t} + \frac{1}{\varepsilon}v\cdot\nabla_x R = \frac{1}{\varepsilon^2}LR + \frac{1}{\varepsilon}J(R,R) + \frac{2}{\varepsilon}H(R) + a,$$

where

$$H(R) = \frac{1}{\varepsilon}J(\psi + \Phi_s, R) + J(S,R),$$

and

$$\begin{aligned}
a = 2\varepsilon\Big(&-\frac{\partial\psi_2}{\partial t} - v\cdot\nabla_x\psi_{H3} - \frac{1}{r}(N\psi_{K3A} + N\psi_{K3B}) + J(\psi_2,\psi_2)\\
&+ 2J(\psi_{H1}, \psi_3 + \Phi_3) + 2J(\psi_2, \Phi_2) + 2J(\psi_3, \Phi_{H1})\Big)\\
&+ \varepsilon^2\Big(2J(\psi_2, \psi_3 + \Phi_3) + 2J(\psi_3, \Phi_2) - \frac{\partial\psi_3}{\partial t}\Big)\\
&+ \varepsilon^3\big(J(\psi_3,\psi_3) + 2J(\psi_3, \Phi_3)\big) + \frac{2}{\varepsilon^2}J(\psi, S).
\end{aligned}$$

The equations involving $L\psi_{Hi}$, $1 \le i \le 3$ give the v-dependence of ψ_{Hi},

$$\begin{aligned}
\psi_{H1}(t,r,v) &= A_1 + D_1v^2 + B_1v_\theta + C_1v_r + E_1v_z,\\
\psi_{H2}(t,r,v) &= A_2 + D_2v^2 + B_2v_\theta + C_2v_r + E_2v_z + g_2,\\
\psi_{H3}(t,r,v) &= A_3 + D_3v^2 + B_3v_\theta + C_3v_r + E_3v_z + g_3.
\end{aligned}$$

Here A_i, B_i, C_i, D_i and E_i, $1 \le i \le 3$ denote functions in the (t,r) variables. By the compatibility conditions

$$\int v\cdot\nabla_x\psi_{H1}(1, v_r)M\,dv = 0,$$

and the initial and boundary conditions at first order, it holds that

$$A_1 + 5D_1 = C_1 = 0.$$

$\bar{A}(|v|)$ was introduced after Lemma 2.2 from the nonhydrodynamic solution to $L(v_r\bar{A}) = v_r(v^2-5)$ together with $\bar{B}(|v|)$ from the corresponding solution to $L(v_r v_\theta \bar{B}) = v_r v_\theta$. Further,

$$\begin{aligned} g_2 &= \frac{1}{2}D_1^2 v^4 + (\frac{1}{2}B_1^2 + B_1 b_1)v_\theta^2 + \frac{1}{2}E_1^2 v_z^2 \\ &+ (B_1+b_1)D_1 v_\theta v^2 + D_1 E_1 v_z v^2 + (B_1+b_1)E_1 v_\theta v_z \\ &+ \frac{\partial D_1}{\partial r} v_r \bar{A} + (\frac{\partial B_1}{\partial r} - \frac{1}{r}B_1)v_r v_\theta \bar{B} + \frac{\partial E_1}{\partial r} v_r v_z \bar{B}, \end{aligned}$$

and g_3 is a similar expression depending on ψ_{H1}, ψ_{H2}, Φ_{H1} and Φ_{H2}. By the compatibility conditions

$$\int (\frac{\partial \psi_{H1}}{\partial t} + v\cdot\nabla_x \psi_{H2})(v_\theta, v^2-5, v_z)Mdv = 0,$$

the functions B_1, D_1 and E_1 are solutions to the parabolic equations

$$\begin{aligned} &\frac{\partial B_1}{\partial t} + w_1(\frac{\partial^2 B_1}{\partial r^2} + \frac{1}{r}\frac{\partial B_1}{\partial r} - \frac{1}{r^2}B_1) = 0, \\ &B_1(0,r) = \beta_1(r), \\ &B_1(t,r_A) = B_1(t,r_B) = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\partial D_1}{\partial t} + \frac{w_3 - 5w_2}{10}(\frac{\partial^2 D_1}{\partial r^2} + \frac{1}{r}\frac{\partial D_1}{\partial r}) = 0, \\ &D_1(0,r) = a_1(r), \\ &D_1(t,r_A) = D_1(t,r_B) = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\partial E_1}{\partial t} + w_1(\frac{\partial^2 E_1}{\partial r^2} + \frac{1}{r}\frac{\partial E_1}{\partial r}) = 0, \\ &E_1(0,r) = \gamma_1(r), \\ &E_1(t,r_A) = E_1(t,r_B) = 0. \end{aligned}$$

Here,

$$w_1 = \int v_r^2 v_\theta^2 \bar{B}Mdv, \quad w_2 = \int v_r^2 \bar{A}Mdv, \quad w_3 = \int v_r^2 v^2 \bar{A}Mdv.$$

The convergence to zero when $t \to \infty$ of ψ_{H1} is well known from the fluid dynamics context (see e.g. [V]). Here the convergence follows from classical asymptotic properties of the solutions to the above linear parabolic equations [LSU].

The compatibility conditions

$$\int \Big(\frac{\partial \psi_{H1}}{\partial t} + v\cdot\nabla_x \psi_{H2}\Big)(1, v_r)Mdv = 0,$$

write

$$
\frac{\partial}{\partial r}\Big(r(C_2 + \frac{w_2}{5}\frac{\partial D_1}{\partial r})\Big) = 0,
$$

(5.5)

$$
\frac{\partial}{\partial r}\Big(A_2 + 5D_2 + \frac{35}{2}D_1^2 + \frac{1}{2}B_1^2 + B_1 b_1 + \frac{1}{2}E_1^2\Big) = \frac{B_1(B_1 + 2b_1)}{r}.
$$

Let $\lambda(\eta, v)$, $\rho_{A2}(t, \eta, v)$ and $\rho_{B2}(t, \eta, v)$ be the solutions of Theorem 2.4 to the half-space problems

$$
\begin{aligned}
&v_r \frac{\partial \lambda}{\partial \eta} = L\lambda,\\
&\lambda(0, v) = 0, \quad v_r > 0,\\
&\int v_r \lambda(\eta, v) dv = 1,
\end{aligned}
$$

$$
\begin{aligned}
&v_r \frac{\partial \rho_{A2}}{\partial \eta} = L\rho_{A2},\\
&\rho_{A2}(t, 0, v) = -\tilde{\psi}_{H2}(t, r_A, v), \quad v_r > 0,\\
&\int v_r \rho_{A2}(\eta, v) dv = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&v_r \frac{\partial \rho_{B2}}{\partial \mu} = L\rho_{B2},\\
&\rho_{B2}(t, 0, v) = -\tilde{\psi}_{H2}(t, r_B, v), \quad v_r < 0,\\
&\int v_r \rho_{B2}(\mu, v) dv = 0.
\end{aligned}
$$

As η (resp. μ) tends to $+\infty$ (resp. $-\infty$), λ and ρ_{A2} (resp. ρ_{B2}) tend to some $a_\infty + \delta_\infty v^2 + \beta_\infty v_\theta + v_r + \gamma_\infty v_z$ and $a_{\infty A} + \delta_{\infty A} v^2 + \beta_{\infty A} v_\theta + \gamma_{\infty A} v_z$ (resp. $a_{\infty B} + \delta_{\infty B} v^2 + \beta_{\infty B} v_\theta + \gamma_{\infty B} v_z$). Give as boundary conditions to A_2, B_2, D_2 and E_2,

$$
\begin{aligned}
A_2(t, r_A) &= a_\infty C_2(t, r_A) + a_{\infty A}(t), \quad & A_2(t, r_B) &= a_\infty C_2(t, r_B) + a_{\infty B}(t),\\
B_2(t, r_A) &= \beta_\infty C_2(t, r_A) + \beta_{\infty A}(t), \quad & B_2(t, r_B) &= \beta_\infty C_2(t, r_B) + \beta_{\infty B}(t),\\
D_2(t, r_A) &= \delta_\infty C_2(t, r_A) + \delta_{\infty A}(t), \quad & D_2(t, r_B) &= \delta_\infty C_2(t, r_B) + \delta_{\infty B}(t),\\
E_2(t, r_A) &= \gamma_\infty C_2(t, r_A) + \gamma_{\infty A}(t), \quad & E_2(t, r_B) &= \gamma_\infty C_2(t, r_B) + \gamma_{\infty B}(t),
\end{aligned}
$$

with

$$
C_2(t, r_A) = r_B C_2(t, r_B) + \frac{w_2}{5}\Big(r_B \frac{\partial D_1}{\partial r}(t, r_B) - \frac{\partial D_1}{\partial r}(t, r_A)\Big).
$$

Then there is a solution $A_2 + 5D_2$ to (5.5) if and only if

$$
(A_2 + 5D_2)(t, r_B) - (A_2 + 5D_2)(t, r_A) = \int_{r_A}^{r_B} B_1(B_1 + 2b_1)(t, r)\frac{dr}{r},
$$

which fixes $C_2(t, r_B)$. Finally, the linear parabolic problems for B_2, D_2, and for E_2 provided by the compatibility conditions

$$\int \Big(\frac{\partial \psi_{H2}}{\partial t} + v \cdot \nabla_x \psi_{H3} \Big)(v_\theta, v^2 - 5, v_z) M dv = 0,$$

have unique solutions;

$$\begin{aligned}
&\frac{\partial B_2}{\partial t} + w_1 \Big(\frac{\partial^2 B_2}{\partial r^2} + \frac{1}{r}\frac{\partial B_2}{\partial r} - \frac{1}{r^2} B_2 \Big) = f_1, \\
&B_2(0, r) = \beta_2(r), \\
&B_2(t, r_A) = \beta_\infty C_2(t, r_A) + \beta_{\infty A}(t), \\
&B_2(t, r_B) = \beta_\infty C_2(t, r_A) + \beta_{\infty B}(t),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial D_2}{\partial t} + \frac{w_3 - 5 w_2}{10} \Big(\frac{\partial^2 D_2}{\partial r^2} + \frac{1}{r}\frac{\partial D_2}{\partial r} \Big) = \bar{f}_1, \\
&D_2(0, r) = a_2(r), \\
&D_2(t, r_A) = \delta_\infty C_2(t, r_A) + \delta_{\infty A}(t), \\
&D_2(t, r_B) = \delta_\infty C_2(t, r_A) + \delta_{\infty B}(t),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial E_2}{\partial t} + w_1 \Big(\frac{\partial^2 E_2}{\partial r^2} + \frac{1}{r}\frac{\partial E_2}{\partial r} \Big) = \tilde{f}_1, \\
&E_1(0, r) = \gamma_2(r), \\
&E_2(t, r_A) = \gamma_\infty C_2(t, r_A) + \gamma_{\infty A}(t), \\
&E_2(t, r_B) = \gamma_\infty C_2(t, r_A) + \gamma_{\infty B}(t).
\end{aligned}$$

Here, $f_1, \bar{f}_1$ and $\tilde{f}_1$ are given functions depending on ψ_{H1}, b_1 and c_2.
Let ψ_{K2A} and ψ_{K2B} be defined by

$$\begin{aligned}
&\psi_{K2A} = C_2(t, r_A)(\lambda - a_\infty - \delta_\infty v^2 - \beta_\infty v_\theta - v_r - \gamma_\infty v_z) \\
&+ \rho_{2A} - a_{\infty A}(t) - \delta_{\infty A}(t) v^2 - \beta_{\infty A}(t) v_\theta - \gamma_{\infty A}(t) v_z,
\end{aligned}$$

$$\begin{aligned}
&\psi_{K2B}(t, \mu, v) = C_2(t, r_B)(\lambda(-\mu, -v) - a_\infty - \delta_\infty v^2 + \beta_\infty v_\theta + v_r + \gamma_\infty v_z) \\
&+ \rho_{2B}(t, -\mu, -v) - a_{\infty B}(t) - \delta_{\infty B}(t) v^2 + \beta_{\infty B}(t) v_\theta + \gamma_{\infty B}(t) v_z.
\end{aligned}$$

They satisfy

$$\begin{aligned}
&v_r \frac{\partial \psi_{K2A}}{\partial \eta} = L \psi_{K2A}, \\
&\psi_{H2}(t, r_A, v) + \psi_{K2A}(t, 0, v) = 0, \quad t > 0, \ v_r > 0, \\
&\lim_{\eta \to +\infty} \psi_{K2A}(t, \eta, v) = 0,
\end{aligned}$$

and

$$v_r \frac{\partial \psi_{K2B}}{\partial \mu} = L\psi_{K2B},$$

$$\psi_{H2}(t, r_B, v) + \psi_{K2A}(t, 0, v) = 0, \quad t > 0, \quad v_r < 0,$$

$$\lim_{\mu \to -\infty} \psi_{K2B}(t, \mu, v) = 0.$$

The convergence to zero of $\psi_{H2} + \psi_{K2A} + \psi_{K2B}$ when $t \to \infty$, follows from the convergence of ψ_{H1}, and from the properties of the parabolic problems and of the Knudsen terms.

The Knudsen terms ψ_{K3A} and ψ_{K3B} are defined analogously, so that the boundary conditions at third order be satisfied by ψ. The third order terms are constructed similarly to the second order ones, and analogously converge to zero when $t \to \infty$.

For the a priori estimates of the rest term the following norms will be used,

$$\|R\|_{2t,2,2} = \Big(\int_0^t \int_{\Omega \times \mathbf{R}^3} R^2(s, x, v) M(v) ds dx dv \Big)^{\frac{1}{2}},$$

$$\|R\|_{\infty,2,2} = \sup_{t>0} \Big(\int_{\Omega \times \mathbf{R}^3} R^2(t, x, v) M(v) dx dv \Big)^{\frac{1}{2}},$$

$$\|R\|_{\infty,\infty,2} = \sup_{t>0} \Big(\int_{\mathbf{R}^3} \sup_{x \in \Omega} R^2(t, x, v) M(v) dv \Big)^{\frac{1}{2}},$$

$$\|f\|_{2t,2,\sim} = \Big(\int_0^t \int_{v_r > 0} v_r M(v) \mid f(s, r_A, v) \mid^2 dv ds \Big)^{\frac{1}{2}}$$

$$+ \Big(\int_0^t \int_{v_r < 0} \mid v_r \mid M(v) \mid f(s, r_B, v) \mid^2 dv ds \Big)^{\frac{1}{2}} < +\infty.$$

The rest term R can be split into $R = R_1 + R_2$, where

$$v \cdot \nabla_x R_1 = \frac{1}{\varepsilon} L R_1 + 2H(R_1), \tag{5.6}$$

$$\begin{aligned} R_1(t, r_A, v) &= -\frac{1}{\varepsilon} \psi(t, r_A, v), \quad t > 0, \quad v_r > 0, \\ R_1(t, r_B, v) &= -\frac{1}{\varepsilon} \psi(t, r_B, v), \quad t > 0, \quad v_r < 0, \end{aligned} \tag{5.7}$$

and

$$\frac{\partial R_2}{\partial t} + \frac{1}{\varepsilon} v \cdot \nabla_x R_2 = \frac{1}{\varepsilon^2} L R_2 + \frac{1}{\varepsilon} J(R_1 + R_2, R_1 + R_2) + \frac{2}{\varepsilon} H(R_2) + \bar{a},$$
$$R_2(0, r, v) = R_0(r, v),$$
$$R_2(t, r_A, v) = 0, \quad t > 0, \ v_r > 0,$$
$$R_2(t, r_B, v) = 0, \quad t > 0, \ v_r < 0,$$

where $\bar{a} = a - \frac{\partial R_1}{\partial t}$. Notice that a can be taken non-hydrodynamic modulo higher order terms in ε, which converge uniformly to zero when time tends to infinity. Hence only $\frac{\partial R_1}{\partial t}$ contributes to the hydro-dynamics in $\bar{a}$. A priori bounds on R_1 are first derived, and also hold for $\frac{\partial R_1}{\partial t}$. The ingoing boundary values as given by (5.7) are subexponentially decreasing in ε, and tend to zero when time tends to infinity.

Lemma 5.1. *With R_1^{in} (R_1^{out}) the ingoing (outgoing) boundary values of R_1, any solution to* (5.6-7) *satisfies*

$$\sqrt{\varepsilon} \| R_1^{out} \|_{2t,2,\sim} + \| \nu^{\frac{1}{2}} (I - P_0) R_1 \|_{2t,2,2} \leq c \sqrt{\varepsilon} \| R_1^{in} \|_{2t,2,\sim},$$
$$\| P_0 R_1 \|_{2t,2,2} \leq c \| R_1^{in} \|_{2t,2,\sim},$$
$$\| \nu^{\frac{1}{2}} R_1 \|_{\infty t, \infty, 2} \leq \frac{c}{\varepsilon} \| R_1^{in} \|_{\infty t, 2, \sim}.$$

Proof of Lemma 5.1. Denote by

$$| R_1 |_2 = \Big(\int_{\Omega \times \mathbf{R}^3} M (R_1)^2 (t, x, v) dx dv \Big)^{\frac{1}{2}}$$

with t acting as a parameter.

By (3.2-4)

$$\sqrt{\varepsilon} | R_1^{out} |_\sim + | \nu^{\frac{1}{2}} (I - P_0) R_1 |_2 \leq c \Big(\sqrt{\varepsilon} | R_1^{in} |_\sim + \varepsilon | \nu^{-\frac{1}{2}} H(R_1) |_2 \Big),$$
$$| P_0 R_1 |_2 \leq c \Big(| \nu^{-\frac{1}{2}} H(R_1) |_2 + | R_1^{in} |_\sim \Big),$$
$$| \nu^{\frac{1}{2}} R_1 |_\infty \leq c \Big(\varepsilon | \nu^{-\frac{1}{2}} H(R_1) |_\infty + \frac{1}{\varepsilon} | \nu^{\frac{1}{2}} R_1 |_2 + | R_1^{in} |_\sim \Big).$$

Then,

$$| \nu^{-\frac{1}{2}} H(R_1) |_2 \leq c \Big(| \nu^{\frac{1}{2}} (\psi_1 + \varepsilon \psi_2 + \varepsilon^2 \psi_3) |_\infty$$
$$+ | \nu^{\frac{1}{2}} (\Phi_1 + \varepsilon \Phi_2 + \varepsilon^2 \Phi_3) |_\infty + | \nu^{\frac{1}{2}} S |_\infty \Big) | \nu^{\frac{1}{2}} R_1 |_2 \leq c \eta | \nu^{\frac{1}{2}} R_1 |_2,$$

and

$$| v^{-\frac{1}{2}}H(R_1) |_\infty \leq c\eta \, | v^{\frac{1}{2}}R_1 |_\infty,$$

where $\eta = r_B - r_A$.

Including the estimates in t, this ends the proof of the lemma. □

The a priori bounds on R_2 are obtained by an approach adapted from [M], and involve dual, space-periodic solutions discussed in the following two lemmas.

Lemma 5.2. *Let $\pi a > r_B$. Let g be such that*

$$\int_{[0,2\pi a]^2} g(\bar{\tau}, x, v)dx = 0, \quad a.a.\ \bar{\tau} \in [0,\infty),\ \ v \in \mathbb{R}^3. \tag{5.8}$$

Let $\varphi(\bar{\tau}, x, v)$ be periodic of period $(2\pi a)^2$ in the space variable, solution to

$$\begin{aligned} &\frac{\partial \varphi}{\partial \bar{\tau}} + v \cdot \nabla_x \varphi = \frac{1}{\varepsilon} L\varphi + g, \\ &\varphi(0, x, v) = 0. \end{aligned} \tag{5.9}$$

Then,

$$\|\varphi\|_{\infty,2,2} \leq c\Big(\sqrt{\varepsilon}\|v^{-\frac{1}{2}}(I - P_0)g\|_{2,2,2} + \frac{1}{\sqrt{\varepsilon}}\|P_0 g\|_{2,2,2}\Big),$$

$$\|v^{\frac{1}{2}}(I - P_0)\varphi\|_{2,2,2} \leq c\Big(\varepsilon\|v^{-\frac{1}{2}}(I - P_0)g\|_{2,2,2} + \|P_0 g\|_{2,2,2}\Big),$$

$$\|P_0\varphi\|_{2,2,2} \leq c\Big(\|v^{-\frac{1}{2}}(I - P_0)g\|_{2,2,2} + \frac{1}{\varepsilon}\|P_0 g\|_{2,2,2}\Big).$$

Proof of Lemma 5.2. First, multiplying (5.9) by φ and integrating the resulting equation on $[0, \bar{T}] \times [0, 2\pi a]^2 \times \mathbb{R}^3$ leads to

$$\begin{aligned} &\|\varphi\|^2_{\infty\bar{T},2,2} + \frac{1}{\varepsilon}\|v^{\frac{1}{2}}(I - P_0)\varphi\|^2_{2\bar{T},2,2} \\ &\leq c(\varepsilon\|v^{-\frac{1}{2}}(I - P_0)g\|^2_{2\bar{T},2,2} + \eta_1\|P_0\varphi\|^2_{2\bar{T},2,2} + \frac{1}{\eta_1}\|P_0 g\|^2_{2\bar{T},2,2}). \end{aligned} \tag{5.10}$$

By (5.8) it holds that

$$\frac{\partial}{\partial \bar{\tau}} \int_{[0,2\pi a]^2} P_0\varphi(\bar{\tau}, x, v)dx = 0, \quad \bar{\tau} \geq 0,\ \ v \in \mathbb{R}^3,$$

so that

$$\int_{[0,2\pi a]^2} P_0\varphi(\bar{\tau},x,v)dx = 0, \quad \bar{\tau} \geq 0, \ v \in \mathbb{R}^3. \tag{5.11}$$

Denote by $\bar{\varphi}(\bar{\tau},\xi,v)$, $\xi \in Z^2$ the Fourier series of φ with respect to space, and define $\bar{g}$ analogously. Then for $\xi \neq (0,0)$,

$$\frac{\partial\bar{\varphi}}{\partial\bar{\tau}} = \Big(\frac{1}{\varepsilon}L + i\xi\cdot v\Big)\bar{\varphi} + \bar{g}.$$

Let β be a truncation function belonging to $C^1(\mathrm{R})$ with support $[0,\infty]$, and such that $\beta(\bar{\tau}) = 1$ for $\bar{\tau} > \delta$ for some $\delta > 0$. Let $\tilde{\varphi} = \bar{\varphi}\beta$. Then

$$\frac{\partial\tilde{\varphi}}{\partial\bar{\tau}} = (\frac{1}{\varepsilon}L + i\xi\cdot v)\tilde{\varphi} + \bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}} + \bar{g}\beta, \quad \xi \in Z^2.$$

Let $\mathcal{F}$ be the Fourier transform in $\bar{\tau}$ with Fourier variable σ. Denote by

$$\Phi = \mathcal{F}\tilde{\varphi}, \tilde{Z} = \mathcal{F}(\varepsilon^{-1}L\tilde{\varphi} + \bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}} + \bar{g}\beta), Z = \mathcal{F}(\varepsilon^{-1}L\tilde{\varphi} + \bar{g}\beta), \hat{U} = (i\sigma + i\xi\cdot v)^{-1}.$$

Let χ be the indicatrix function of the set

$$\{v;\ |\sigma + \xi\cdot v| < a\},$$

for some positive a to be chosen later. Let $\psi_s(v) = (1+|v|)^s$. First,

$$\begin{aligned}
\|P_0(\chi\Phi)\|_H &\leq c\Big(\Big|\int\chi\Phi(\sigma,\xi,v)Mdv\Big|\,\|1\|_H + \Big|\int\chi\Phi(\sigma,\xi,v)v^2Mdv\Big|\,\|v^2\|_H \\
&+ \Big|\int\chi\Phi(\sigma,\xi,v)v_rMdv\Big|\,\|v_r\|_H + \Big|\int\chi\Phi(\sigma,\xi,v)v_\theta Mdv\Big|\,\|v_\theta\|_H\Big) \\
&\leq c\|\psi_{-s}\Phi\|_H\Big(\|\chi\psi_s\|_H + \|\chi\psi_{s+2}\|_H\Big) \\
&\leq c\sqrt{\frac{a}{|\xi|}}\|\psi_{-s}\Phi\|_H.
\end{aligned}$$

Now $\Phi = -\hat{U}\tilde{Z}$, and so

$$\begin{aligned}
&\|P_0(1-\chi)\Phi\|_H^2 \leq c\Big(\|\psi_s(1-\chi)\hat{U}\|_H^2 + \|\psi_{s+2}(1-\chi)\hat{U}\|_H^2\Big)\|\psi_{-s}Z\|_H^2 \\
&-\sum_0^4\int\psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int\psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^* \\
&\leq \frac{c}{|\xi||a|}\|\psi_{-s}Z\|_H^2 - \sum_0^4\int\psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int\psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.
\end{aligned}$$

Choosing $a = \|\psi_{-s}\Phi\|_H^{-1}\|\psi_{-s}Z\|_H$ leads to

$$| \xi | \, \|P_0\Phi\|_H^2 \le c\|\psi_{-s}\Phi\|_H\|\psi_{-s}Z\|_H$$
$$- | \xi | \sum_0^4 \int \psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.$$

Hence,

$$| \xi | \, \|P_0\Phi\|_H^2 \le c\Big(\|P_0\Phi\|_H + \|\psi_{-s}(I-P_0)\Phi\|_H\Big)\|\psi_{-s}Z\|_H$$
$$- | \xi | \sum_0^4 \int \psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.$$

Consequently,

$$| \xi | \, \|P_0\Phi\|_H^2 \le c\Big(| \xi |^{-1} \|\psi_{-s}Z\|_H^2 + \|\psi_{-s}(I-P_0)\Phi\|_H\|\psi_{-s}Z\|_H\Big)$$
$$- | \xi | \sum_0^4 \int \psi_j(1-\chi)\hat{U}(\mathcal{F}\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.$$

And so,

$$\frac{\xi^2}{1+ | \xi |}\|P_0\Phi\|_H^2 \le c\Big(\|\psi_{-s}Z\|_H^2 + \|\psi_{-s}(I-P_0)\Phi\|_H^2\Big)$$
$$- \frac{|\xi|^2}{1+|\xi|}\sum_0^4 \int \psi_j(1-\chi)\hat{U}\mathcal{F}(\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.$$

Therefore, for $s \ge \frac{\beta}{2}$,

$$\frac{\xi^2}{1+ | \xi |}\int (P_0\Phi)^2(\sigma,\xi,v)Mdvd\sigma$$
$$\le c\Big(\frac{1}{\varepsilon^2}\int \|\psi_{-s}(v)L((I-P_0)\Phi)(\sigma,\xi,\cdot)\|_H^2 d\sigma + \int \|\psi_{-s}(v)(I-P_0)\Phi(\sigma,\xi,\cdot)\|_H^2 d\sigma$$
$$+ \int \|\psi_{-s}\bar{g}\beta(\bar{\tau},\xi,\cdot)\|_H^2 d\bar{\tau}\Big)$$
$$- \frac{|\xi|^2}{1+|\xi|}\sum_0^4 \int d\sigma \int \psi_j(1-\chi)\hat{U}(\mathcal{F}\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*$$
$$\le c\Big(\frac{1}{\varepsilon^2}\int \|v^{\frac{1}{2}}(I-P_0)\Phi(\sigma,\xi,\cdot)\|_H^2 d\sigma + \int \|\psi_{-s}\bar{g}\beta(\bar{\tau},\xi,\cdot)\|_H^2 d\bar{\tau}\Big)$$
$$- \frac{|\xi|^2}{1+|\xi|}\sum_0^4 \int d\sigma \int \psi_j(1-\chi)\hat{U}(\mathcal{F}\bar{\varphi}\frac{\partial\beta}{\partial\bar{\tau}})Mdv(\int \psi_j(1-\chi)(\mathcal{F}(\bar{\varphi}\beta) - \hat{U}Z)Mdv)^*.$$

Making δ tend to zero implies that

$$\int_0^\infty \int (P_0\bar{\varphi})^2(\bar{\tau},\xi,v)Mdvd\bar{\tau}$$
$$\leq c\Big(\frac{1}{\varepsilon^2}\int_0^\infty \int \nu((I-P_0)\bar{\varphi})^2(\bar{\tau},\xi,v)Mdvd\bar{\tau} + \int_0^\infty \nu_{-s}\bar{g}^2(\bar{\tau},\xi,v)Mdvd\bar{\tau}\Big).$$

Summing the former inequalities over all $\xi \in Z^2$ with $\xi \neq (0,0)$ and taking (5.11) into account, implies by Parseval that

$$\int_0^\infty \int (P_0\varphi)^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}$$
$$\leq c\Big(\frac{1}{\varepsilon^2}\int_0^\infty \int \nu((I-P_0)\varphi)^2(\bar{\tau},x,v)Mdvdxd\bar{\tau} + \int_0^\infty \int \nu^{-1}g^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}\Big).$$

Together with (5.10) this ends the proof of the lemma. □

Lemma 5.3. *Let $\pi a > r_B$. Let g be such that*

$$\int_{[0,2\pi a]^2} g(\bar{\tau},x,v)dx = 0, \quad a.a.\ \bar{\tau}\in[0,\infty),\ \ v\in \mathbb{R}^3. \tag{5.12}$$

Let $\varphi(\bar{\tau},x,v)$ be periodic of period $(2\pi a)^2$ in the space variable x and solution to

$$\frac{\partial\varphi}{\partial\bar{\tau}} + v\cdot\nabla_x\varphi = \frac{1}{\varepsilon}L\varphi + g, \qquad \varphi(0,x,v) = 0. \tag{5.13}$$

Then,

$$\int_0^\infty \int_{|x|=r_B}\int_{v_r>0} v_r\varphi^2(\bar{\tau},x,v)Mdvd\sigma(x)d\bar{\tau} + \int_0^\infty \int_{|x|=r_A}\int_{v_r<0} |\, v_r\,|\,\varphi^2(\bar{\tau},x,v)Mdvd\sigma(x)d\bar{\tau}$$
$$\leq \frac{c}{\varepsilon^2}\int_0^\infty \int g^2(\bar{\tau},x,v)Mdvdxd\bar{\tau}).$$

(Here $d\sigma(x)$ is the surface measure of the circles.)

Proof of Lemma 5.3. Let $C_{(0,1)}$ be the set in the (x,y)-plane consisting of the half with $y\geq 0$ of the circle with radius r_B and center at the origin together with the rectangle given by $|x| \leq r_B, -\eta < y < 0$, where $\eta > 0$ taken small enough that any rotation of the set $C_{(0,1)}$ around the origin stays within the square $\{|x|, |y| < \pi a\}$. Let $C_{(v_x,v_y)}$ be the set $C_{(0,1)}$ rotated from the $(0,1)$-direction to the (v_x, v_y)-direction. Let $\chi_{(0,1)}$ be defined and continuous in $C_{(0,1)}$, monotone and continuously differentiable in the y-direction, equal zero at $y = -\frac{\eta}{2}$ and equal one at $y \geq 0$. Define $\chi_{(v_x,v_y)}(x,y)$ correspondingly by rotation. Then

$$\frac{\partial}{\partial \bar{\tau}}(\chi^2_{(v_x,v_y)}\varphi^2) + v \cdot \nabla_x(\chi^2_{(v_x,v_y)}\varphi^2)$$
$$= \frac{2}{\varepsilon}\chi^2_{(v_x,v_y)}\varphi L\varphi + 2\chi^2_{(v_x,v_y)}\varphi g + 2(v \cdot \nabla_x \chi_{(v_x,v_y)})\chi_{(v_x,v_y)}\varphi^2.$$

Hence,

$$\int_0^{\bar{T}} \int_{|x|=r_B} v_r \chi^2_{(v_x,v_y)}\varphi^2(\bar{\tau}, x, v) M d\sigma(x) d\bar{\tau} \leq A_v + B_v + C_v,$$

where by Lemma 5.2

$$\int_{v_r>0} A_v dv := \frac{2}{\varepsilon}\int_0^{\bar{T}}\int \chi^2_{(v_x,v_y)}\varphi L\varphi M dv dx d\bar{\tau}$$

$$= \frac{2}{\varepsilon}\int_0^{\bar{T}}\int \chi^2_{(v_x,v_y)}\varphi L((I-P_0)\varphi) M dv dx d\bar{\tau} \leq \frac{c}{\varepsilon^2}\|v^{-\frac{1}{2}} g\|^2_{2,2,2},$$

$$\int B_v dv := 2\int_0^{\bar{T}}\int \chi^2_{(v_x,v_y)}\varphi g M dv dx d\bar{\tau} \leq \frac{c}{\varepsilon}\|v^{-\frac{1}{2}} g\|^2_{2,2,2},$$

$$\int C_v dv := 2\int_0^{\bar{T}}\int (v \cdot \nabla_x \chi_{(v_x,v_y)})\chi_{(v_x,v_y)}\varphi^2 M dv dx d\bar{\tau}$$

$$\leq c\|v^{\frac{1}{2}}\varphi\|^2_{2\bar{T},2,2} \leq \frac{c}{\varepsilon^2}\|v^{-\frac{1}{2}} g\|^2_{2,2,2}.$$

Here the C_v-estimate was carried out for hard spheres, but holds also for hard forces for the particular g appearing in the applications below. The r_A-part is treated similarly. □

For the iteration procedure to obtain R_2 we shall be using systems of the type

$$\frac{\partial R_2}{\partial t} + \frac{1}{\varepsilon} v \cdot \nabla_x R_2 = \frac{1}{\varepsilon^2} L R_2 + \frac{1}{\varepsilon} G, \tag{5.14}$$

$$R_2(0,r,v) = R_0(r,v), \tag{5.15}$$

$$\begin{aligned} &R_2(t,r_A,v) = 0, \quad t>0,\ v_r>0,\\ &R_2(t,r_B,v) = 0, \quad t>0,\ v_r<0. \end{aligned} \tag{5.16}$$

Multiply (5.14) with R_2M, integrate over $[0,t] \times \Omega \times \mathbb{R}^3$ and use the spectral inequality, so that

$$\begin{aligned} &\|R_2(t)\|_{2,2}^2 + \frac{1}{\varepsilon}\|R_2^{out}\|_{2t,2,2}^2 + \frac{1}{\varepsilon^2}\|\nu^{\frac{1}{2}}(I-P_0)R_2\|_{2t,2,2}^2 \\ &\le c\Big(\|R_0\|_{2,2}^2 + \|\nu^{-\frac{1}{2}}(I-P_0)G\|_{2t,2,2}^2 + \frac{\eta_1}{2\varepsilon}\|P_0R_2\|_{2t,2,2}^2 + \frac{1}{2\eta_1\varepsilon}\|P_0G\|_{2t,2,2}^2\Big), \end{aligned}$$

for every $\eta_1 > 0$.

The a priori bounds on P_0R_2 are discussed in the following two lemmas. They are based on dual techniques using the space periodic solutions introduced above. Denote by

$$h(t,x,v) := P_0R_2 - \langle P_0R_2\rangle, \quad \langle f(t,v)\rangle := \int_\Omega f(t,x,v)dx.$$

Lemma 5.4. *For any $0<\eta<1$ there is ε_η such that, for $0<\varepsilon<\varepsilon_\eta$,*

$$\|h\|_{2,2,2}^2 \le \frac{c}{\varepsilon}\Big(\|R_0\|_{2,2}^2 + \|\nu^{-\frac{1}{2}}(I-P_0)G\|_{2,2,2}^2 + \frac{1}{\varepsilon^3}\|P_0G\|_{2,2,2}^2\Big) + \eta\| < P_0R_2 > \|_{2,2,2}^2.$$

Proof of Lemma 5.4. In the variables $(\bar{t},x,v) := (\frac{t}{\varepsilon},x,v)$, the function R_2 is solution to

$$\begin{aligned} &\frac{\partial R_2}{\partial \bar{t}} + v\cdot\nabla_x R_2 = \frac{1}{\varepsilon}LR_2 + G, \quad R_2(0,r,v) = R_0(r,v),\\ &R_2(\bar{t},r_A,v) = 0, \quad \bar{t}>0,\ v_r>0,\\ &R_2(\bar{t},r_B,v) = 0, \quad \bar{t}>0,\ v_r<0. \end{aligned}$$

Let φ be the $(2\pi a)^2$-periodic φ function solution to

$$\frac{\partial\varphi}{\partial\bar{t}} + v\cdot\nabla_x\varphi = \frac{1}{\varepsilon}L\varphi + h, \quad \varphi(0,x,v) = 0,$$

where h is taken as zero outside the gap between the cylinders and periodically continued. Denote by

$$(f,g)_H = \int f(v)g(v)M(v)dv.$$

Then,

$$\frac{\partial}{\partial \bar{\tau}}(R_2,\varphi)_H + \int div_x(vR_2\varphi)Mdv = \frac{2}{\varepsilon}(LR_2,(I-P_0)\varphi)_H + (G,\varphi)_H + (h,P_0R_2)_H.$$

Integrating with respect to $\bar{\tau}$ and x gives

$$\begin{aligned}
\|h\|^2_{2\bar{\tau},2,2} &\le \frac{K_1}{2}\|R_2(\bar{\tau},\cdot,\cdot)\|^2_{2,2} + \frac{1}{2K_1}\|\varphi(\bar{\tau},\cdot,\cdot)\|^2_{2,2}\\
&+\frac{K_2}{2}\|R_2^{out}\|^2_{2\bar{\tau},2,\sim} + \frac{1}{2K_2}\|\varphi^{out}\|^2_{2\bar{\tau},2,\sim}\\
&+\frac{K_3}{2\varepsilon}\|v^{\frac12}(I-P_0)R_2\|^2_{2\bar{\tau},2,2} + \frac{1}{2K_3\varepsilon}\|v^{\frac12}(I-P_0)\varphi\|^2_{2\bar{\tau},2,2}\\
&+\frac{K_4}{2}\|v^{-\frac12}(I-P_0)G\|^2_{2\bar{\tau},2,2} + \frac{1}{2K_4}\|v^{\frac12}(I-P_0)\varphi\|^2_{2\bar{\tau},2,2}\\
&+\frac{K_5}{2}\|P_0G\|^2_{2\bar{\tau},2,2} + \frac{1}{2K_5}\|P_0\varphi\|^2_{2\bar{\tau},2,2},
\end{aligned}$$

for any positive constants $K_j, j = 1, ..., 5$.

It then follows from the preceeding estimates that

$$\|h\|^2_{2,2,2} \le c\Big(\frac{1}{\varepsilon^2}\|R_0\|^2_{2,2} + \frac{1}{\varepsilon}\|v^{-\frac12}(I-P_0)G\|^2_{2,2,2} + \frac{1}{\varepsilon^4}\|P_0G\|^2_{2,2,2}\Big) + \eta\|\langle P_0R_2\rangle\|^2_{2,2,2}.$$

This ends the proof of Lemma 5.4 when coming back to the t-variable. □

Lemma 5.5.

$$\| < P_0R_2 > \|^2_{2,2,2} \le c(\frac{1}{\varepsilon}\|R_0\|^2_{2,2} + \frac{1}{\varepsilon}\|v^{-\frac12}(I-P_0)G\|^2_{2,2,2}) + \frac{1}{\varepsilon^4}\|P_0G\|^2_{2,2,2}).$$

Proof of Lemma 5.5. For $t > 0$ let $\varphi(t,x,v)$ be the solution to the (stationary) problem

$$\begin{aligned}
&v\cdot\nabla_x\varphi = \frac{1}{\varepsilon}L\varphi - \varepsilon\langle P_0R_2\rangle,\\
&\varphi(t,r_A,v) = 0, \quad t>0,\ v_r>0,\\
&\varphi(t,r_B,v) = 0, \quad t>0,\ v_r<0.
\end{aligned}$$

By (3.2-4),

$$\begin{aligned}
&\|v^{\frac12}(I-P_0)\varphi\|_{2,2} \le \varepsilon\| < P_0R_2 > \|_{2,2},\\
(5.17)\qquad &\|P_0\varphi\|_{2,2} \le \| < P_0R_2 > \|_{2,2},\\
&\|\varphi^{out}\|_\sim \le \sqrt{\varepsilon}\| < P_0R_2 > \|_{2,2}.
\end{aligned}$$

Then

$$\varepsilon\frac{\partial}{\partial t}(R_2,\varphi)-\varepsilon(R_2,\frac{\partial\varphi}{\partial t})+\int div_x(vR_2\varphi)Mdv$$
$$=\frac{2}{\varepsilon}(LR_2,(I-P_0)\varphi)_H+(G,\varphi)_H-\varepsilon(<P_0R_2>,R_2)_H.$$

Hence for η_1 of order ε

$$\|\langle P_0R_2\rangle\|^2_{2t,2,2}\leq c\Big(\|R_0\|^2_{2,2}+\|v^{-\frac{1}{2}}(I-P_0)G\|^2_{2t,2,2}+\frac{1}{\eta_1\varepsilon}\|P_0G\|^2_{2t,2,2}$$
$$+\frac{\eta_1}{\varepsilon}\|P_0R_2\|^2_{2t,2,2}+\int_0^t\int R_2\frac{\partial\varphi}{\partial t}(s,x,v)Mdvdxds\Big).$$

And so, by Lemma 5.4, and for η_1 of order ε and small enough,

$$\|\langle P_0R_2\rangle\|^2_{2t,2,2}\leq\frac{c}{\varepsilon}\Big(\|R_0\|^2_{2,2}+\|v^{-\frac{1}{2}}(I-P_0)G\|^2_{2,2,2}+\frac{1}{\varepsilon^3}\|P_0G\|^2_{2,2,2}\Big)$$
$$+\int_0^t\int R_2\frac{\partial\varphi}{\partial t}(s,x,v)Mdvdxds.$$

It remains to bound the term $\int_0^t R_2\frac{\partial\varphi}{\partial t}(s,x,v)Mdvdxds$ from above. Differentiate the equation satisfied by φ with respect to t. Similarly to (5.17),

$$\|P_0\frac{\partial\varphi}{\partial t}\|_{2t,2,2}\leq\|\langle P_0\frac{\partial R_2}{\partial t}\rangle\|_{2t,2,2}.$$

Taking the hydrodynamic part of the equation (5.14) leads to

$$P_0\frac{\partial R_2}{\partial t}+\frac{1}{\varepsilon}P_0(v\cdot\nabla_xR_2)=\frac{1}{\varepsilon}P_0G.$$

Moreover,

$$\langle P_0(v\cdot\nabla_xR_2\rangle=c\Big(r_BP_0(v_rR_2(t,r_B,v))-r_AP_0(v_rR_2(t,r_A,v))\Big).$$

Hence,

$$\|P_0\frac{\partial\varphi}{\partial t}\|^2_{2t,2,2}\leq c\|\langle P_0\frac{\partial R_2}{\partial t}\rangle\|^2_{2t,2,2}\leq c\Big(\frac{1}{\varepsilon^2}\|R_2^{out}\|^2_{2t,2}+\frac{1}{\varepsilon^2}\|P_0G\|^2_{2t,2,2}\Big).$$

And so, Lemma 5.5 follows. □

Lemma 5.6. *Any solution R_2 to the system*

$$\begin{aligned}
&\frac{\partial R_2}{\partial t}+\frac{1}{\varepsilon}v\cdot\nabla_x R_2=\frac{1}{\varepsilon^2}LR_2+\frac{2}{\varepsilon}H(R_2)+\frac{1}{\varepsilon}G,\\
&R_2(0,r,v)=R_0(r,v),\\
&R_2(t,r_A,v)=0,\quad t>0,\quad v_r>0,\\
&R_2(t,r_B,v)=0,\quad t>0,\quad v_r<0,
\end{aligned}$$

satisfies

$$\begin{aligned}
&\|R_2\|_{2,2,2}\le c\Big(\frac{1}{\sqrt{\varepsilon}}\|R_0\|_{2,2}+\frac{1}{\sqrt{\varepsilon}}\|\nu^{-\frac12}(I-P_0)G\|_{2,2,2}+\frac{1}{\varepsilon^2}\|P_0G\|_{2,2,2}\Big),\\
&\|R_2\|_{\infty,2,2}\le \frac{c\eta}{\sqrt{\varepsilon}}\Big(\|R_0\|_{2,2}+\|\nu^{-\frac12}(I-P_0)G\|_{2,2,2}+\frac{1}{\varepsilon\sqrt{\varepsilon}}\|P_0G\|_{2,2,2}\Big),\\
&\|R_2\|_{\infty,\infty,2}\le c\Big(\frac{1}{\varepsilon}\|R_0\|_{2,2}+\|R_0\|_{\infty,2}+\frac{1}{\varepsilon}\|\nu^{-\frac12}(I-P_0)G\|_{2,2,2}\\
&+\frac{1}{\varepsilon^2\sqrt{\varepsilon}}\|P_0G\|_{2,2,2}+\varepsilon\|G\|_{\infty,\infty,2}\Big).
\end{aligned}$$

Proof of Lemma 5.6. Consider first for $H=0$ the solution R_2 to

$$\begin{aligned}
&\frac{\partial R_2}{\partial t}+\frac{1}{\varepsilon}v\cdot\nabla_x R_2=\frac{1}{\varepsilon^2}LR_2+\frac{1}{\varepsilon}G,\\
&R_2(0,r,v)=R_0(r,v),\\
&R_2(t,r_A,v)=0,\quad t>0,\quad v_r>0,\\
&R_2(t,r_B,v)=0,\quad t>0,\quad v_r<0.
\end{aligned}$$

It satisfies

$$\begin{aligned}
&\sup_{t\ge 0}\|R_2(t)\|_{2,2}+\frac{1}{\varepsilon}\|\nu^{-\frac12}(I-P_0)R_2\|_{2,2,2}\\
&\le c\Big(\|R_0\|_{2,2}+\|(I-P_0)G\|_{2,2,2}+\frac{\eta}{\sqrt{\varepsilon}}\|P_0R_2\|_{2,2,2}+\frac{1}{\eta\sqrt{\varepsilon}}\|P_0G\|_{2,2,2}\Big),
\end{aligned}$$

for any $\eta>0$. Moreover, it follows from Lemmas 5.4-5 that

$$\|P_0R_2\|_{2,2,2}\le c\Big(\frac{1}{\sqrt{\varepsilon}}\|R_0\|_{2,2}+\frac{1}{\sqrt{\varepsilon}}\|\nu^{-\frac12}(I-P_0)G\|_{2,2,2}+\frac{1}{\varepsilon^2}\|P_0G\|_{2,2,2}\Big).$$

Choosing $\eta=\sqrt{\varepsilon}$ leads to the first inequality of Lemma 5.6, and choosing $\eta=\varepsilon$ leads to the second one with a $\sqrt{\varepsilon}$ improvement of the order in the lemma. Then, by some additional computations similar to what we have done in previous sections,

$$\|R_2\|_{\infty,\infty,2}\le c\Big(\frac{1}{\sqrt{\varepsilon}}\|R_2\|_{\infty,2,2}+\|R_0\|_{\infty,2}+\varepsilon\|G\|_{\infty,\infty,2}\Big),$$

which leads to the last inequality of Lemma 5.6 again with some $\sqrt{\varepsilon}$ improvement. A more careful computation shows that adding the small perturbation $\frac{1}{\varepsilon}H(R_2)$ only changes the previous orders to those of the lemma. □

Proof of Theorem 5.1. The convergence to zero when $t \to \infty$ of the asymptotic expansion ψ for the difference $\tilde{\Phi} - \tilde{\Phi}_s$ was discussed at the beginning of this section. The corresponding rest term εR was split into $\varepsilon R_1 + \varepsilon R_2$, where by Lemma 5.1 and by the boundary conditions being satisfied by ψ up to third order in ε,

$$\|v^{\frac{1}{2}}R_1\|_{2,2,2} \leq c \mid R_1^{in} \mid_{2,\sim}, \quad \|v^{\frac{1}{2}}R_1\|_{\infty,\infty,2} \leq \frac{c}{\varepsilon}|R_1^{in}|_{\infty,\sim},$$

i.e. subexponential decrease in ε and convergence to zero when time tends to infinity.

So it only remains to show the existence of R_2 and its convergence to zero when $t \to +\infty$. We shall prove that R_2 can be obtained as the limit of an approximating sequence and that

$$\int_0^{+\infty}\int_\Omega\int_{\mathbb{R}^3} (R_2)^2(t,x,v)M(v)dtdxdv < c\varepsilon^2. \tag{5.18}$$

This in turn implies the L^2-convergence to zero of R_2 when time tends to infinity, i.e. $\lim_{t\to\infty} \int R_2(t,x,v)^2 M dxdv = 0$.

Let the approximating sequence (R_2^n) be defined by $R_2^0 = 0$, and

$$\begin{aligned}
&\frac{\partial R_2^{n+1}}{\partial t} + \frac{1}{\varepsilon}v\cdot\nabla_x R_2^{n+1} = \frac{1}{\varepsilon^2}LR_2^{n+1} + \frac{2}{\varepsilon}H(R_2^{n+1}) + \frac{1}{\varepsilon}J(R_1+R_2^n, R_1+R_2^n) + \bar{a},\\
&R_2^{n+1}(0,r,v) = R_0(r,v),\\
&R_2^{n+1}(t,r_A,v) = 0, \quad t>0, \quad v_r>0,\\
&R_2^{n+1}(t,r_B,v) = 0, \quad t>0, \quad v_r<0,
\end{aligned}$$

where R_0 is of ε-order two and

$$\bar{a} = a - \frac{\partial R_1}{\partial t}.$$

The function R_2^1 is solution to

$$\begin{aligned}
&\frac{\partial R_2^1}{\partial t} + \frac{1}{\varepsilon}v\cdot\nabla_x R_2^1 = \frac{1}{\varepsilon^2}LR_2^1 + \frac{2}{\varepsilon}H(R_2^1) + \frac{1}{\varepsilon}J(R_1,R_1) + \bar{a},\\
&R_2^1(0,r,v) = R_0(r,v),\\
&R_2^1(t,r_A,v) = 0, \quad t>0, \quad v_r>0,\\
&R_2^1(t,r_B,v) = 0, \quad t>0, \quad v_r<0,
\end{aligned}$$

so that by Lemma 5.6 and the subexponential decrease of R^{in} together with the orders 2 of R_0 and 1 of $a_\perp$ and 2 of $a_\|$,

$$\|R_2^1\|_{\infty,\infty,2} \leq c_1\varepsilon^{\frac{1}{2}}, \quad \|R_2^1\|_{2,2,2} \leq c_1\varepsilon,$$

for some constant c_1. A closer inspection shows that $c_1 = O(r_B - r_A)$ when the coefficients in the perturbation P are $O(r_B - r_A)$.

By induction, for $r_B - r_A$ small enough

$$\|R_2^j\|_{\infty,\infty,2} \leq 2c_2|r_B - r_A|\varepsilon^{\frac{1}{2}}, \quad j \leq n,$$
$$\|R_2^{n+1} - R_2^n\|_{2,2,2} \leq c_3\sqrt{r_B - r_A}\|R_2^n - R_2^{n-1}\|_{2,2,2}, \quad n \geq 1,$$

for some constants c_2, c_3. Namely, if this holds up to n^{th} order, then

$$\begin{aligned}
&\frac{\partial}{\partial t}(R_2^{n+2} - R_2^{n+1}) + \frac{1}{\varepsilon}v \cdot \nabla_x(R_2^{n+2} - R_2^{n+1}) \\
&= \frac{1}{\varepsilon^2}L(R_2^{n+2} - R_2^{n+1}) + \frac{2}{\varepsilon}H(R_2^{n+2} - R_2^{n+1}) + \frac{1}{\varepsilon}G^{n+1}, \\
&(R_2^{n+2} - R_2^{n+1})(0, r, v) = 0, \\
&(R_2^{n+2} - R_2^{n+1})(t, r_A, v) = 0, \quad t > 0, \quad v_r > 0, \\
&(R_2^{n+2} - R_2^{n+1})(t, r_B, v) = 0, \quad t > 0, \quad v_r < 0,
\end{aligned}$$

with

$$G^{n+1} = (I - P_0)G^{n+1} = 2J(R_1, R_2^{n+1} - R_2^n) + J(R_2^{n+1} + R_2^n, R_2^{n+1} - R_2^n),$$

and where by Lemma 5.6

$$\begin{aligned}
&\|R_2^{n+2} - R_2^{n+1}\|_{2,2,2} \leq \frac{c}{\sqrt{\varepsilon}}\|G^{n+1}\|_{2,2,2} \\
&\leq \frac{c}{\sqrt{\varepsilon}}\Big(\|R_1\|_{\infty,\infty,2} + \|R_2^{n+1}\|_{\infty,\infty,2} + \|R_2^n\|_{\infty,\infty,2}\Big)\|R_2^{n+1} - R_2^n\|_{2,2,2} \\
&\leq c_2\sqrt{r_B - r_A}\| R_2^{n+1} - R_2^n\|_{2,2,2}.
\end{aligned}$$

This ends the first induction step, and also implies that

$$\|R_2^{n+2}\|_{2,2,2} \leq \|R_2^{n+2} - R_2^{n+1}\|_{2,2,2} + \ldots + \|R_2^2 - R_2^1\|_{2,2,2} + \|R_2^1\|_{2,2,2} \leq 2c_1\varepsilon,$$

for $r_B - r_A$ small enough. Similarly $\|R_2^{n+2}\|_{\infty,\infty,2} \leq 2c_2|r_B - r_A|\varepsilon^{\frac{1}{2}}$. In particular (R_2^n) is a Cauchy sequence in $L^2([0, +\infty[\times \Omega \times \mathbb{R}^3_M)$. The existence of R_2 follows, and the estimate (5.18) holds. This completes the study of the R_2-term and Theorem 5.1 follows. □

The dependence on a small enough $r_B - r_A$ was introduced to be able to use a short ε-expansion. With an ε-expansion of higher order the same proof shows that the existence of R_2 and the stability result of Theorem 5.1 hold for an arbitrary fixed $r_B - r_A$, when ε is small enough.

6 - Positivity

We shall in this final section discuss the positivity of the earlier solutions.

In the time-dependent small data case, positivity of sufficiently regular solutions can be proved by Gronwall based ideas, see [LZ]. But in the stationary small data case the question whether certain solutions are positive remains an interesting open problem. For general time-dependent problems, positivity is usually introduced at the beginning of the approximation procedure and then kept throughout, so the solutions *constructed* are positive, but not necessarily other solutions. When there is uniqueness around, time-dependent positivity may alternatively be obtained by comparison with some other equation already known to have only positive solutions (see [A]). That turns out to be a possible approach also here for our stationary solutions using a new type of comparison equation.

The proof starts by considering a variant of the stationary Boltzmann equation with a particular extra term depending only on the negative part of the solution. This new equation is then proved only to have positive solutions, the extra term disappears and the solutions solve the BE. The proof goes on to construct a solution to the new equation of the type we already discussed for the original problem, and to show that this new solution coincides with the original solution. There is the following technical problem. In one step of the proof, growth estimates are needed for terms like $v_r v_\theta \bar{B} = L^{-1} v_r v_\theta$. For Maxwellian molecules such estimates are provided in [C]. For the strictly hard force case, suitable types of growth estimates - also of interest in other contexts -have been studied by C. Mouhot [Mo].

Write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Suppose f satisfies the related problem (6.1-2) below. Then $f^- = 0$ by Theorem 6.1 below, and $f = f^+$ is a non-negative solution also to (2.1), (2.3). If the contraction mapping approach used above can be extended to the construction of suitable solutions for the problem (6.1-2), then as a consequence, any solution from the previous sections would coincide with such a non-negative solution.

Theorem 6.1. *Let Ω be a bounded set in $\mathbb{R}^n$ with smooth boundary, and f_b a nonnegative function defined on $\partial\Omega^+$. If $M^{-1}f \in \tilde{L}^\infty(\Omega \times \mathbb{R}^3)$ and f solves the* *boundary value problem*

$$v \cdot \nabla_x f = Q(f^+, f^+) - ML(M^{-1}f^-), \quad (x,v) \in \Omega \times \mathbb{R}^3, \tag{6.1}$$

$$f = f_b, \quad \partial\Omega^+, \tag{6.2}$$

then $f^- = 0$, *and* $f = f^+$ *solves the corresponding boundary value problem for the Boltzmann equation,*

$$v \cdot \nabla_x f = Q(f,f), \quad \Omega \times \mathbb{R}^3,$$
$$f = f_b, \quad \partial\Omega^+.$$

Proof of Theorem 6.1 The function $F = M^{-1}f$ satisfies

$$v \cdot \nabla_x F = J(F^+, F^+) - L(F^-), \quad F = M^{-1}f_b, \quad \partial\Omega^+.$$

Define J^+ and J^- by $J(\varphi,\varphi) = J^+(\varphi,\varphi) - J^-(\varphi,\varphi)$, where

$$J^+(\varphi,\varphi)(v) := \int | v - v_* |^\beta \, b(\theta) M_* \varphi' \varphi'_* dv_* d\omega,$$

$$J^-(\varphi,\varphi)(v) := \varphi(v) \int | v - v_* |^\beta \, b(\theta) M_* \varphi_* dv_* d\omega.$$

Also, F^- satisfies

$$- v \cdot \nabla_x F^- = \chi_{F^- \neq 0}(J^+(F^+, F^+) - L(F^-)), \quad F^- = 0, \quad \partial\Omega^+. \tag{6.3}$$

Multiplying (6.3) with $-MF^-$, integrating on $\Omega \times \mathbb{R}^3$ and using that

$$- \int MF^- \chi_{F^- \neq 0} L(F^-) dv = - \int MF^- L(F^-) dv \geq c \int M\nu \, | (I - P_0)F^- |^2 \, dv,$$

implies that

$$\frac{1}{2} \int_{\partial\Omega^-} | v \cdot n | \, M(F^-)^2 + c \int_{\Omega \times \mathbb{R}^3} M\nu \, | (I - P_0)F^- |^2 \leq - \int MF^- \chi_{F^- \neq 0} J^+(F^+, F^+) \leq 0.$$

It follows that

$$F^- = 0 \text{ on } \partial\Omega^-, \quad L(F^-) = 0.$$

And so, F^- satisfies

$$F^- = 0, \; \partial\Omega^- \cup \partial\Omega^+, \quad v \cdot \nabla_x F^- \leq 0.$$

This implies that F^- is identically zero. □

Corollary 6.2. *If there is a solution f to (6.1-2) in a ball of contraction from the proofs of Theorem 4.2-4, then* $f^- = 0$ *and* $f = f^+$ *is the unique and* *strictly positive solution in that ball of the corresponding boundary value problem* (2.1), (2.3).

Theorem 6.3. *The solutions obtained in Theorem 4.2-4 are strictly positive.*

Proof of Theorem 6.3. For the case of Maxwellian molecules there is indeed in all three cases a solution to (6.1-2), i.e. the hypothesis of the corollary holds. We start with the axially homogeneous situation of Case 1. Set $\bar{\chi} = \chi_{|v|<\varepsilon^{-\frac{1}{n}}}$ and denote again by φ the previous asymptotic expansion of order two,

$$\varphi(r,v) = \sum_{i=1}^{2} \varepsilon^i \Phi^i.$$

If the terms in Φ^i, $1 \leq i \leq 2$ are polynomially bounded in the v-variable, with bounded coefficients in the r-variable, then for ε and $\frac{1}{n}$ small enough and positive, it would hold that

$$1 + \bar{\chi}\varphi = 1 + \bar{\chi}\Big(\sum_{i=1}^{2} \varepsilon^i \Phi^i\Big) \geq 0. \tag{6.4}$$

The required bounds follow from the previous discussion of the terms in φ except the $\bar{B}$-term in Φ^2 (and also some $\bar{A}$-terms in Case 2-3). But it is well known that also such $\bar{A}$ and $\bar{B}$ terms are polynomially bounded in the Maxwellian case (cf [C]). Notice that the $\tilde{L}^q$-norm of $(1-\bar{\chi})\Phi$ for any q is of arbitrarily high order in ε because of the factor M in the v-integrand.

Using the approach of Section 4, the positivity under the cut-off $\bar{\chi}$ in (6.4), and the corresponding splitting

$$f = M(1 + \bar{\chi}\varphi + \varepsilon R),$$

lead to a nonnegative solution of (6.1-2) with $M^{-1}f \in \tilde{L}^\infty$ as follows. Namely, the rest term R should be a solution to

$$DR = \frac{1}{\varepsilon}\Big(LR + 2J(\bar{R}, \bar{\chi}\varphi) + \varepsilon J(\bar{R}, \bar{R}) + \bar{l}\Big), \tag{6.5}$$

where

$$\bar{l} = \frac{1}{\varepsilon}\Big(L(\bar{\chi}\varphi) + J(\bar{\chi}\varphi, \bar{\chi}\varphi) - \varepsilon D(\bar{\chi}\varphi)\Big),$$

and

$$\bar{R}(r,v) = R(r,v) \ \text{ when } \ \varepsilon R(r,v) \geq -\Big(1 + \bar{\chi}\sum_{i=1}^{2} \varepsilon^i \Phi^i(r,v)\Big),$$

$$\bar{R}(r,v) = -\frac{1}{\varepsilon}\Big(1 + \bar{\chi}\sum_{i=1}^{2} \varepsilon^i \Phi^i(r,v)\Big) \ \text{ otherwise.}$$

Here $\bar{l}$ can be decomposed as $\bar{l}_\perp$ as in Section 4, and $\bar{l}_\parallel$ which in $\tilde{L}^q$ is of arbitrarily high order in ε. The approximating sequences $(R^n)_{n\in\mathbb{N}}$ and $(\bar{R}^n)_{n\in\mathbb{N}}$ are defined by $R^0 = \bar{R}^0 = 0$, and

$$DR^{n+1} = \frac{1}{\varepsilon}\Big(LR^{n+1} + 2\sum_{j=1}^{2}\varepsilon^j J(\bar{R}^{n+1}, \bar{\chi}\Phi^j) + g^n\Big), \tag{6.6}$$

$$R^{n+1}(1,v) = R_A(v),\ v_r > 0, \quad R^{n+1}(r_B, v) = R_B(v),\ v_r < 0, \tag{6.7}$$

with

$$g^n := \varepsilon J(\bar{R}^n, \bar{R}^n) + \bar{l},$$

$$\varepsilon R_A(v) := e^{\varepsilon u_{\theta A1} v_\theta - \frac{\varepsilon^2}{2} u_{\theta A1}^2 v_\theta^2} - 1 - \bar{\chi}\Phi(r_A, v), \quad v_r > 0,$$

$$\varepsilon R_B(v) := -\bar{\chi}\Phi(r_B, v), \quad v_r < 0,$$

and

$$\bar{R}^n(r,v) = R^n(r,v) \text{ when } \varepsilon R^n(r,v) \geq -\Big(1 + \bar{\chi}\sum_{i=1}^{2}\varepsilon^i \Phi^i(r,v)\Big),$$

$$\bar{R}^n(r,v) = -\frac{1}{\varepsilon}\Big(1 + \bar{\chi}\sum_{i=1}^{2}\varepsilon^i \Phi^i(r,v)\Big) \text{ otherwise.}$$

From here the only difference with respect to the contraction mapping analysis of Section 4, is related to the appearance of factors $\bar{R}^n$ instead of the previous R^n in J. The existence result in Lemma 3.1 is not changed by the replacements $\bar{R}$. Arguing similarly to the previous cases, the contribution to the a priori non fluid dynamic estimate (3.2) due to $g_\parallel$ gives rise to an extra term $|\, g_\parallel\, |_2\, \varepsilon^{-1}$, hence

$$\varepsilon^{\frac{1}{2}} \mid SF \mid_\sim + \mid \tilde{v}^{\frac{1}{2}} F_\perp \mid_2 \leq c(\mid \tilde{v}^{-\frac{1}{2}} g_\perp \mid_2 + \varepsilon^{-1} \mid \tilde{v}^{-\frac{1}{2}} g_\parallel \mid_2 + \varepsilon \mid F_\parallel \mid_2 + \varepsilon^{\frac{1}{2}} \mid F_b \mid_\sim).$$

The proof of the fluid dynamic Lemma 3.3 is essentially unchanged in the present situation (with the $\bar{R}$-terms included in $g_\perp$), and its estimate (3.4) follows.

We turn to the existence proof for (6.5), (6.7). In the new situation the contraction mapping arguments from the proof of Theorem 4.2 still hold. That leads to an isolated solution for (6.5), (6.7) which defines the positive solution of Corollary 6.2. The solution lies in the same ball of contraction as the solution constructed in Section 4, so they coincide and the solution of Section 4 is positive. That completes the proof of Theorem 6.3 in the axially homogeneous case. The other cases for Maxwellian molecules are similarly proved.

Extending the above approach to hard forces, requires suitable growth estimates for some terms in the asymptotic expansion φ, like the terms $v_r\bar{A}$ and $v_\theta v_r\bar{B}$. The following growth estimates are due to C.Mouhot (personal communication). Case 1 follows from

Lemma 6.4. *The preimage for L of the polynomials in $(I - P_0)L^2_M$ is contained in $L^2_{M^s}$ for $0 < s \leq \frac{1}{2}$, with in particular $| v_\theta v_r \bar{B} | \leq C_s M^{-s}(v)$.*

Proof of Lemma 6.4. The proof is based on an extension of well known Grad estimates [G] for the linearized collision operator L. Set

$$M_s = M^{\frac{s}{2}},$$
$$g := v_\theta v_r \bar{B} = -\frac{v_\theta v_r}{\upsilon} + \frac{Kg}{\upsilon}.$$

Write $K = -K_1 + K_2$ where

$$K_1 g(v) = \int g(v_*) M(v_*) B(v_* - v, \theta) d\omega dv_*.$$

For $0 < s \leq 1$ the pointwise estimate

$$| M_s K_1 g(v) | < C_{1s} \sqrt{\int (M_s g(v_*))^2}$$

holds. Grad's arguments also give for K_2 that pointwise

$$| M_s K_2 g(v) | < C_{2s} \sqrt{\int (M_s g(v_*))^2}.$$

It follows that

$$| v_\theta v_r \bar{B} | \leq | \frac{v_\theta v_r}{\upsilon} | + \frac{C_s M_s^{-1}}{\upsilon} \sqrt{\int (M_s v_\theta v_r \bar{B})^2},$$

where $C_s = C_{1s} + C_{2s}$. The square root integral term is finite. Namely, the mapping $\frac{L}{\upsilon}$ is continuous and injective on the non-hydrodynamic space $(I - P_0)L^2_M$ intersected with $L^2_{M_{2s}}$ in which $\frac{K}{\upsilon}$ is compact. So by the Fredholm theory $\frac{L}{\upsilon}$ is also surjective there, in particular the preimage for L of the polynomials in $(I - P_0)L^2_M$ is contained in $L^2_{M^s}$. □

Using this lemma we can complete the proof of positivity in Case 1 for hard forces. By Section 4 it is enough to make the previous χ-splitting so that $\bar{l}$ of (6.5) is of ε-order two, i.e. so that the exterior part under the splitting of φ is of ε-order three.

Write in the second order asymptotic term, $v_\theta v_r \bar{B} = -\frac{v_\theta v_r}{v} + \frac{Kg}{v}$, and treat all other first and second order terms as well as $\frac{v_\theta v_r}{v}$ as in the previous Maxwellian case. There remains $\frac{Kg}{v}$. Choose $s < \frac{1}{4}$ and v_ε so that $| v_\theta v_r \bar{B} | \leq C_s M^{-s}(v_\varepsilon) := \varepsilon^{-1}$ for $| v | \leq v_\varepsilon$. Then $\varepsilon^2 v_\theta v_r \bar{B} << 1$ for $|v| \leq v_\varepsilon$, and

$$\int_{|v|>v_\varepsilon} | v_\theta v_r \bar{B} |^2 M < \varepsilon^2 \int | M^s v_\theta v_r \bar{B} |^2 .$$

Hence a cutoff χ_ε for $v_\theta v_r \bar{B}$ at v_ε gives an exterior term $\varepsilon^2 \chi_\varepsilon v_\theta v_r \bar{B}$ of ε-order three. Case 2 and 3 for hard forces are treated analogously. □

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Abstract

The problems discussed in this work concern asymptotic techniques and detailed quantitative properties close to global equilibrium in classical kinetic theory. The discussion is mainly centered on a particular two-rolls model problem for the Boltzmann equation and hard forces, with the understanding that such a program can be applied in many other contexts for single and multi-component gases. The topics include asymptotic expansions, a priori estimates, existence and positivity results, fluid dynamic limits, bifurcations and stability questions.

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