The evolution of a gas in a radiation field from a kinetic point of view.

A. Nouri *

Abstract. An existence theorem is derived for a system of kinetic equations describing the evolution of a gas in a radiation field from a kinetic point of view. The geometrical setting is the slab and given indata. The photons ingoing distribution functions are Dirac measures.

Key words. Evolutionary kinetic equation, photoionisation.

Introduction. The evolution of a gas interacting with a radiation field is a subject of interest in astrophysics and the study of laboratory plasmas. The main models used so far are the radiative transfer equation for the photons distribution function, coupled with fluid equations describing the evolution of the gas ([3, 7, 11]). However, many astrophysical and laboratory plasmas show deviations from local thermodynamic equilibrium, which also requires a kinetic setting for the gas. Kinetic models have been derived in [10, 12]. In the frame of radiation gas dynamics, Burgers [4] provides a simplified kinetic model for the interaction of a gas with a radiation field. Gas molecules with only one excited energy level are considered, together with photons at a single frequency. This is certainly a simplifying assumption, but in many cases ([7], [9]) it is a good approximation. This model is improved in [13] by using the genuine Boltzmann collision operators instead of the BGK collision operators from [4] in the equations for the gas molecules. A remarkable feature of such a model is that Planck's law of radiation is recovered self-consistently, under thermodynamical equilibrium conditions. Moreover, a H-theorem is formally obtained. In this paper, a theorem of existence of a

*CMI, Université d'Aix-Marseille I, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France.

solution to this kinetic model is derived in the slab, when only elastic collisions within gas molecules are taken into account, for given indata on the boundary. Making use of the emission of the photons in beams perpendicular to the walls, a supplementary simplification is introduced in the model. On the mathematical side, what is new compared to the DiPerna and Lions result for the Boltzmann equation is the coupling between gas molecules and photons. The first ones are fermions, which allows L^1 renormalized solutions. The second ones are bosons, which only allows a measure setting for the photons distribution function. Theorem 1.1 states the existence of photons distribution functions that are measures in the velocity variable.

1 The model and the main result.

Let a gas of material particles of mass m endowed with only two internal energy levels E_1 and E_2, with $E_1 < E_2$. Denote by A_1 and A_2 particles A at the fundamental level 1 and the excited level 2 respectively, and by $f(t, x, v)$ and $g(t, x, v)$ their distribution functions. The time variable t belongs to a given interval $[0, T]$, the space variable x to the slab $[0, 1]$ and the velocity variable v to $\mathbb{R}^3$. A radiation field of photons p at a fixed frequency $\nu = \frac{\Delta E}{h}$, with $\Delta E = E_2 - E_1$ and h the Planck constant, interacts with the gas. The gas particles are assumed to interact elastically among themselves. The interactions between the gas molecules and the photons are, classically, of three types,

$$\begin{aligned} \text{Absorption}, &\quad A_1 + p \to A_2, \\ \text{Spontaneous emission}, &\quad A_2 \to A_1 + p, \\ \text{Stimulated emission}, &\quad A_2 + p \to A_1 + 2p. \end{aligned}$$

Let $c\Omega$, where $\Omega \in S^2$ and c is the speed of the light, be the photon velocities, θ their angle with the x-axis and $\tilde{I}(t, x, \Omega)$ the photons distribution function. Denote by $I(t, x, \Omega) = ch\nu\tilde{I}(t, x, \Omega)$ the specific intensity. Let β_{12}, α_{21} and β_{21} be the Einstein coefficients. Following [4, 12], the evolutionary equation for $I(t, x, \Omega)$ is given by

$$\frac{1}{c}I_t + cos\theta I_x = h\nu[(\alpha_{21} + \beta_{21}I)\int gdv - \beta_{12}I\int fdv]. \tag{1.1}$$

Since $\beta_{12} = \beta_{21}$, the subscripts of the Einstein coefficients can be dropped. Denote by ξ the first component of the velocity vector v. The Boltzmann equations for the two particle species A_1 and A_2 can be classically written as

$$f_t + \xi f_x = g \int (\alpha + \beta I) d\theta - \beta f \int I d\theta$$
$$+ \int_{I\!R^3 \times S^2} S(v, v_*, \omega)(f' f'_* - f f_*) dv_* d\omega$$
$$+ \int_{I\!R^3 \times S^2} S(v, v_*, \omega)(f' g'_* - f g_*) dv_* d\omega, \tag{1.2}$$

where S is a given collision kernel,

$$f' = f(t, x, v'), \;\; f'_* = f(t, x, v'_*), \;\; f_* = f(t, x, v_*),$$
$$v' = v - (v - v_*, \omega)\omega, \;\; v'_* = v + (v - v_*, \omega)\omega,$$

and

$$g_t + \xi g_x = -g \int (\alpha + \beta I) d\theta + \beta f \int I d\theta$$
$$+ \int S(v, v_*, \omega)(g' g'_* - g g_*) dv_* d\omega$$
$$+ \int S(v, v_*, \omega)(f'_* g' - f_* g) dv_* d\omega. \tag{1.3}$$

The physical conditions considered here are characterized by the following inequalities,

$$k_B T \ll mc^2, \quad \Delta E \ll c\sqrt{\frac{8mk_B T}{\pi}}, \tag{1.4}$$

where k_B is the Boltzmann constant and T the temperature of the gas. The first inequality implies that the relativistic effects can be neglected. Consequently, the velocities of the gas molecules do not exceed in modulus $\frac{\epsilon}{2}c$, for some positive number ϵ smaller than 1. Hence, the collision kernels S and S' are assumed to vanish for $|v|$ or $|v_*|$ or $|v'|$ or $|v'_*|$ bigger than $\frac{\epsilon}{2}c$. Moreover, S and S' are bounded and measurable positive functions having the usual symmetries of collision kernels. A further restriction on S is that $\frac{S(v,v_*,\omega)}{|\xi-\xi_*|}$, (resp. $\frac{S(v,v_*,\omega)}{|\xi'-\xi'_*|}$) is assumed to be bounded for small $\mid \xi - \xi_* \mid$, (resp. small $|\xi' - \xi'_*|$). The second inequality in (1.4) guarantees that the photon momentum is much smaller than the mean thermal momentum of the gas, so that any exchange of momentum between photons and molecules can be neglected. Initial conditions f_i, g_i and I_i are given for every distribution function,

$$f(0, x, v) = f_i(x, v), \quad g(0, x, v) = g_i(x, v), \quad I(0, x, \theta) = I_i(x, \theta). \tag{1.5}$$

The boundary conditions for the gas particles are given indata, i.e.

$$
\begin{aligned}
f(t,0,v) = f_0(t,v),\ \xi > 0, \quad f(t,1,v) = f_1(t,v),\ \xi < 0,\\
g(t,0,v) = g_0(t,v),\ \xi > 0, \quad g(t,1,v) = g_1(t,v),\ \xi < 0.
\end{aligned}
\tag{1.6}
$$

The photons are emitted at the boundaries in beams perpendicular to the walls, i.e.

$$
I(t,0,\theta) = I_0\delta_{\theta=0},\ cos\theta > 0 \quad I(t,1,\theta) = I_1\delta_{\theta=\pi},\ cos\theta < 0, \tag{1.7}
$$

where I_0 and I_1 are nonnegative constants. Because of this strong light source, directed along the x-axis, there is much higher intensity in this direction. Making use if this, it is assumed that the stimulated emission for $|cos\theta| < \epsilon$ is negligible compared to the stimulated emission in the other directions. Only stimulated emission and not also spontaneous emission or absorption are negligible for $|\ cos\theta\ | < \epsilon$, since a supplementary photon is emitted from a stimulated emission with the help of a primary photon hitting a molecule of gas. This is not required for spontaneous emission nor absorption. And so, instead of (1.1-3), the distribution functions (f, g, I) is assumed to satisfy

$$
\frac{1}{c}I_t + cos\theta I_x = h\nu[(\alpha+\beta I)\int g(v)dv - \beta I\int f(v)dv], \quad |cos\theta| > \epsilon, \tag{1.8}
$$

$$
\frac{1}{c}I_t + cos\theta I_x = h\nu[\alpha\int g(v)dv - \beta I\int f(v)dv], \quad |cos\theta| \leq \epsilon, \tag{1.9}
$$

$$
\begin{aligned}
f_t + \xi f_x = (\int \alpha d\theta + \int_{|cos\theta|>\epsilon} \beta I d\theta)g - \beta f\int I d\theta\\
+Q(f,f) + Q_1(f,g),
\end{aligned}
\tag{1.10}
$$

$$
\begin{aligned}
g_t + \xi g_x = -(\int \alpha d\theta + \int_{|cos\theta|>\epsilon} \beta I d\theta)g + \beta f\int I d\theta\\
+Q(g,g) + Q_2(f,g),
\end{aligned}
\tag{1.11}
$$

where

$$
\begin{aligned}
Q(f,f) &:= Q^+(f,f) - Q^-(f,f),\\
Q^+(f,f)(v) := \int_{I\!R^3\times S^2} Sf'f'_*dv_*d\omega,\ Q^-(f,f)(v) &:= f(v)\int_{I\!R^3\times S^2} Sf_*dv_*d\omega,\\
Q_1(f,g)(v) &:= \int_{I\!R^3\times S^2} S(f'g'_* - fg_*)dv_*d\omega,\\
Q_2(f,g)(v) &:= \int_{I\!R^3\times S^2} S(f'_*g' - f_*g)dv_*d\omega.
\end{aligned}
$$

The formal passage in (1.8-11) when $\epsilon \to 0$ results in model (1.1-3), since then (1.9) disappears and (1.8) becomes (1.1). For the sake of simplicity, the terms Q_1 and Q_2 will be skipped in the rest of the paper. Using that A_1 and A_2 are mechanically the same, the following existence theorem would also hold with them, with minor adaptations of the proof. Moreover, the constants $h\nu$, α and β do not play any role on the mathematical level. It is why they are taken as 1 in the rest of the paper.

Definition 1.1 *(f, g, I) is called a solution to (1.5-11) in iterated integral form if*

$$(f,g,I) \in C_+([0,T],L^1((0,1)\times V)) \times C_+([0,T],L^1((0,1)\times V)) \times L^1((0,T)\times(0,1),M[0,2\pi]),$$

I satisfies (1.5), (1.7-9) in weak form with test functions in $C^1([0,T]\times[0,1]\times[0,2\pi])$*, compactly supported in* $[0,T[\times[0,1]\times[0,2\pi]$*,* $Q^{\pm}(f,f)(\cdot,x,v)$ *and* $Q^{\pm}(g,g)(\cdot,x,v)$ *belong to* $L^1(0,T)$ *for a.a.* $(x,v)\in(0,1)\times V$*, and for a.a.* $t\in(0,T)$*,*

$$\begin{aligned}
&\int_{(x,v);(x+t\xi,v)\in(0,1)\times V}(f^\sharp\varphi)(t,x,v)dxdv \\
&= \int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0}\Big((f^\sharp\varphi)(0\vee -\frac{x}{\xi},x,v) \\
&+\int_{\xi>0}\int_{0\vee-\frac{x}{\xi}}^t\Big(f^\sharp\frac{\partial\varphi}{\partial s}+(g(2\pi+\int_{|cos\theta|>\epsilon}Id\theta)-f\int Id\theta+Q(f,f))^\sharp\varphi\Big)(s,x,v)ds\Big)dxdv \\
&+\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0}\Big((f^\sharp\varphi)(0\vee\frac{1-x}{\xi},x,v) \\
&+\int_{\xi<0}\int_{0\vee\frac{1-x}{\xi}}^t\Big(f^\sharp\frac{\partial\varphi}{\partial s}+(g(2\pi+\int_{|cos\theta|>\epsilon}Id\theta)-f\int Id\theta+Q(f,f))^\sharp\varphi\Big)(s,x,v)ds\Big)dxdv,
\end{aligned}$$

$$\begin{aligned}
&\int_{(x,v);(x+t\xi,v)\in(0,1)\times V}(g^\sharp\psi)(t,x,v)dxdv \\
&= \int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0}\Big((g^\sharp\psi)(0\vee -\frac{x}{\xi},x,v) \\
&+\int_{\xi>0}\int_{0\vee-\frac{x}{\xi}}^t\Big(g^\sharp\frac{\partial\psi}{\partial s}+(-g(2\pi+\int_{|cos\theta|>\epsilon}Id\theta)+f\int Id\theta+Q(g,g))^\sharp\psi\Big)(s,x,v)ds\Big)dxdv \\
&+\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0}\Big((g^\sharp\psi)(0\vee\frac{1-x}{\xi},x,v) \\
&+\int_{\xi<0}\int_{0\vee\frac{1-x}{\xi}}^t\Big(g^\sharp\frac{\partial\psi}{\partial s}+(-g(2\pi+\int_{|cos\theta|>\epsilon}Id\theta)+f\int Id\theta+Q(g,g))^\sharp\psi\Big)(s,x,v)ds\Big)dxdv,
\end{aligned}$$

for any test functions φ and ψ such that $\varphi(\cdot,x,v)\in C^1([0,T])$, $\psi(\cdot,x,v)\in C^1([0,T])$ for a.a. $(x,v)\in(0,1)\times V$, and $(\varphi,\psi)\in(C^1([0,T];L^\infty((0,1)\times V)))^2$.

Here, $a\vee b$ (resp. $a\wedge b$) denotes the maximum (resp. the minimum) of a and b, $V:=\{v\in I\!R^3;|v|\leq\frac{\epsilon}{2}c\}$ and $M[0,2\pi]$ is the set of bounded measures in the θ variable belonging to $[0,2\pi]$. Moreover, $f^\sharp(t,x,v):=f(t,x+t\xi,v)$ denotes the value of f along the characteristics $(t,x+t\xi,v)$. Denote by $|f|_t(x,v):=supess_{s\leq t}f^\sharp(s,x,v)$. For any distribution function f, denote by the corresponding capital letter F its density function defined by $F(t,x):=\int f(t,x,v)dv$.
Remarks. For $\xi>0$, $(f^\sharp\varphi)(0\vee-\frac{x}{\xi},x,v)$ is either $f_i(x,v)\varphi(0,x,v)$ or $f_0(-\frac{x}{\xi},v)\varphi(-\frac{x}{\xi},x,v)$, and for $\xi<0$, $(f^\sharp\varphi)(0\vee\frac{1-x}{\xi},x,v)$ is either $f_i(x,v)\varphi(0,x,v)$ or $f_1(\frac{1-x}{\xi},v)\varphi(\frac{1-x}{\xi},x,v)$, which are known values.
As shown in [2], being a solution to (1.5-11) in iterated integral form is equivalent to be a solution to (1.5-11) in mild form.
The main result of this paper is the following.

Theorem 1.1 *Let $T>0$ be given. Assume that f_i, g_i, I_i, f_0, f_1, g_0 and g_1 are non negative functions satisfying*

$$\begin{aligned}&\int[f_i(1+lnf_i)+g_i(1+lng_i)]dxdv\\&+\int_0^T\int_{\xi>0}\xi(1+lnf_0)f_0(s,v)dsdv+\int_0^T\int_{\xi>0}\xi(1+lng_0)g_0(s,v)dsdv\\&+\int_0^T\int_{\xi<0}\mid\xi\mid(1+lnf_1)f_1(s,v)dsdv\\&+\int_0^T\int_{\xi<0}\mid\xi\mid(1+lng_1)g_1(s,v)dsdv+\int I_idxd\theta\\&+\sup_{(t,x)\in(0,T)\times(0,1)}\int_{|cos\theta|>\epsilon,0<x-tccos\theta<1}I_i(x-tccos\theta,\theta)d\theta<c_i.\end{aligned}\quad(1.12)$$

Then there is a solution (f,g,I) in iterated integral form of (1.5-11).

Here, and in the following, c_i denotes constants only depending on the initial data f_i, g_i, I_i, the given indata f_0, f_1, g_0 and g_1, and the given constants I_0 and I_1.
A priori bounds.
Solutions (f,g,I) to (1.5-11) satisfy the following a priori bounds, describing

the conservation of energy and the decrease of entropy with time. Multiplying (1.10) by v^2+E_1, (1.11) by v^2+E_2, integrating the sum on $(0,t)\times(0,1)\times V$, adding it to (1.8) integrated on $(0,t)\times(0,1)\times\{|cos\theta|>\epsilon\}$, then to (1.9) integrated on $(0,t)\times(0,1)\times\{|cos\theta|<\epsilon\}$, and using that $E_2-E_1=h\nu$, formally leads to the conservation of energy

$$\begin{aligned}
&\int[(v^2+E_1)f(t,x,v)+(v^2+E_2)g(t,x,v)]dxdv+\frac{1}{c}\int I(t,x,\theta)dxd\theta\\
&\qquad+\int_0^t\int_{\xi<0}\mid\xi\mid[(v^2+E_1)f+(v^2+E_2)g](s,0,v)dvds\\
&\qquad+\int_0^t\int_{\xi>0}\xi[(v^2+E_1)f+(v^2+E_2)g](s,1,v)dvds\\
&=\int[(v^2+E_1)f_i(x,v)+(v^2+E_2)g_i(x,v)]dxdv+\frac{1}{c}\int I_i(x,\theta)dxd\theta\\
&\qquad+\int_0^t\int_{\xi>0}\xi[(v^2+E_1)f_0+(v^2+E_2)g_0](s,v)dvds\\
&+\int_0^t\int_{\xi<0}\mid\xi\mid[(v^2+E_1)f_1+(v^2+E_2)g_1](s,v)dvds+(I_0+I_1)t.
\end{aligned}\tag{1.13}$$

For any set Y, denote by χ_Y the characteristic function of Y. Multiplying (1.10), (1.11), (1.8) and (1.9) by lnf, lng, $\chi_{|cos\theta|>\epsilon}ln\frac{I}{1+I}$ and $\chi_{|cos\theta|<\epsilon}lnI$ respectively, integrating the two first resulting equations on $(0,t)\times(0,1)\times V$, the two last ones on $(0,t)\times(0,1)\times(0,2\pi)$ and adding them, formally leads to the entropy inequality

$$\begin{aligned}
&\int(flnf+glng)(t,x,v)dxdv+\frac{1}{c}\int_{|cos\theta|<\epsilon}(IlnI-I)(t,x,\theta)dxd\theta\\
&+\int_0^t\int_{\xi>0}\xi(flnf+glng)(s,1,v)dsdv+\int_0^t\int_{\xi<0}|\xi|(flnf+glng)(s,0,v)dsdv\\
&\qquad+\frac{1}{c}\int_0^t\int_{cos\theta>\epsilon}cos\theta(IlnI-(1+I)ln(1+I))(s,1,\theta)dxd\theta\\
&\qquad+\frac{1}{c}\int_0^t\int_{cos\theta<-\epsilon}|cos\theta|(IlnI-(1+I)ln(1+I))(s,0,\theta)dxd\theta\\
&\qquad+\frac{1}{c}\int_0^t\int_{cos\theta\in(0,\epsilon)}cos\theta(IlnI-I)(s,1,\theta)d\theta ds\\
&\qquad+\frac{1}{c}\int_0^t\int_{cos\theta\in(-\epsilon,0)}|cos\theta|(IlnI-I)(s,0,\theta)d\theta ds\\
&\qquad+e(f,f)+e(g,g)+D_1(f,g,I)+D_2(f,g,I)
\end{aligned}$$

$$\leq \frac{1}{c}\int_{|cos\theta|>\epsilon}[(1+I)ln(1+I)-IlnI](t,x,\theta)dxd\theta$$
$$+\int(f_i lnf_i+g_i lng_i)(x,v)dxdv$$
$$+\frac{1}{c}\int_{|cos\theta|>\epsilon}(I_i lnI_i-(1+I_i)ln(1+I_i))(x,\theta)dxd\theta$$
$$+\frac{1}{c}\int_{|cos\theta|<\epsilon}(I_i lnI_i-I_i)(x,\theta)dxd\theta$$
$$+\int_0^t(\int_{\xi>0}\xi(f_0 lnf_0+g_0 lng_0)(s,v)dv$$
$$+\int_{\xi<0}\mid\xi\mid(f_1 lnf_1+g_1 lng_1)(s,v)dv)ds.$$

Here, $\int(flnf+glng)(t,x,v)dxdv$ and $\frac{1}{c}\int[(1+I)ln(1+I)-IlnI](t,x,\theta)dxd\theta$ are the entropies at time t of the gas molecules and the light respectively. Moreover, $e(f,f)$, $e(g,g)$, $D_1(f,g,I)$ and $D_2(f,g,I)$ are the nonnegative entropy production terms defined by

$$e(f,f):=\int S(f'f'_*-ff_*)ln\frac{f'f'_*}{ff_*}dsdxdvdv_*d\omega,$$
$$D_1(f,g,I):=\int_{|cos\theta|>\epsilon}((1+I)g-If)ln\frac{(1+I)g}{If}dsdxdvd\theta,$$
$$D_2(f,g,I):=\int_{|cos\theta|<\epsilon}(g-If)ln\frac{g}{If}dsdxdvd\theta.$$

Then,

$$\int_0^t(\int_{\xi>0,f\leq 1}\xi f|lnf|(s,1,v)dv+\int_{\xi>0,g\leq 1}\xi g|lng|(s,1,v)dv)ds$$
$$+\int_0^t(\int_{\xi<0,f\leq 1}|\xi|f|lnf|(s,0,v)dv+\int_{\xi<0,g\leq 1}|\xi|g|lng|(s,0,v)dv)ds$$
$$\leq ce^{-1},$$

since the volume of integration is bounded. Moreover,

$$\frac{1}{c}\int_0^t\int_{cos\theta\in(0,\epsilon),I\leq e}cos\theta|IlnI-I|(s,1,\theta)dsd\theta$$
$$+\frac{1}{c}\int_0^t\int_{cos\theta\in(-\epsilon,0),I\leq e}|cos\theta||IlnI-I|(s,0,\theta)dsd\theta\leq c.$$

Then, for some $c > 0$ and $I_c > 0$,

$$(1 + I)ln(1 + I) - IlnI - cI \leq 0, \quad I \geq I_c.$$

Hence,

$$\int_{|cos\theta|>\epsilon} [(1 + I)ln(1 + I) - IlnI](t, x, \theta)dxd\theta$$
$$\leq c_1 \int_{I \geq I_c} I(t, x, \theta)dxd\theta + c_2[(1 + I_c)ln(1 + I_c) - I_c lnI_c] \leq c_i,$$

by (1.13). Hence

$$\int (flnf + glng)(t, x, v)dxdv + \int_{|cos\theta|<\epsilon} IlnI(t, x, \theta) \leq c_i. \tag{1.14}$$

From (1.13-14), it classically holds that

$$\int [f(1+ \mid lnf \mid) + g(1+ \mid lng \mid)](t, x, v)dxdv$$
$$+ \int I(t, x, \theta)dxd\theta + \int_{|cos\theta|<\epsilon} I|lnI|(t, x, \theta) < c_i. \tag{1.15}$$

Sketch of the proof of Theorem 1.1.
Approximations that are bounded in the time and space variables are first derived in Section 2. Mass, energy and entropy bounds of the type of the previous a priori bounds are stated for these approximations. They provide L^1 weak compactness for the gas distribution functions, together with compactness in the weak * topology of measures for the photons distribution function. This is not sufficient neither to pass to the limit in the nonlinear terms, nor to be in the frame of application of the averaging lemma for the gas distribution functions. Some supplementary work is also required in order to take into account the difference of characteristics for the gas i.e. $(t, x + t\xi, v)$ from characteristics for photons i.e. $(t, x + ctcos\theta, \theta)$. This is done in Lemma 2.2 and is crucial for the whole proof. It provides weak $L^1((0, T) \times (0, 1) \times \{| \, cos\theta \, |< \epsilon\})$ compactness of a subsequence of photons distribution approximations, as well as a frame of application of the averaging lemma for the gas distribution approximations. The passage to the limit is finally performed in Section 3, in the frame of the iterated integral formulation of solutions to the problem.
Remark. When explicitly expressing I from (1.7-9), the Dirac measure $\delta_{\theta=0}$ is applied to the function

$$e^{\int_{t-\frac{x}{ccos\theta}}^{t} (G-F)(s,x+(s-t)ccos\theta)ds},$$

which is continuous at $\theta = 0$. (The term $ccos\theta$ means the product of the speed of light c by $cos\theta$). Indeed, for $|cos\theta| > \epsilon$ and $v \in V$, $|ccos\theta - \xi| > \frac{\epsilon}{2}c$ and

$$\int_{t-\frac{x}{ccos\theta}}^{t}(G-F)(s,x+(s-t)ccos\theta)ds =$$

$$\int_V\int_{D_{\theta,v}}(g-f)^\sharp(\frac{y-x+tccos\theta}{ccos\theta-\xi},y,v)dy\frac{dv}{|ccos\theta-\xi|},$$

where

$$D_{\theta,v} := \{y \in (0,1); \frac{y-x+tccos\theta}{ccos\theta-\xi} \in (t-\frac{x}{ccos\theta},t)\}.$$

This function is continuous at $\theta = 0$, by the continuity in time of $f^\sharp$ and $g^\sharp$ proven in Lemma 2.4 and the bounds

$$\int supess_{t<T}f^\sharp(t,x,v)dxdv < c_i \text{ and } \int supess_{t<T}g^\sharp(t,x,v)dxdv < c_i,$$

proven in Lemma 2.2. Analogously, the Dirac measure $\delta_{\theta=\pi}$ can be applied to the function

$$e^{\int_{t+\frac{1-x}{ccos\theta}}^{t}(G-F)(s,x+(s-t)ccos\theta)ds},$$

which is continuous at $\theta = \pi$.

2 Approximations.

Let n be a fixed integer bigger than 2. In this section, a solution

$$(f,g,I) \in C_+([0,T],L^\infty((0,1)\times V))\times C_+([0,T],L^\infty((0,1)\times V))$$
$$\times L^\infty((0,T)\times(0,1),M[0,2\pi])$$

to the following system will be determined,

$$\begin{aligned} f_t+\xi f_x &= \frac{g}{1+\frac{g}{n}}(2\pi+\int_{|cos\theta|>\epsilon}Id\theta)-\frac{f}{1+\frac{f}{n}}\int Id\theta \\ &+\int S(\frac{f'}{1+\frac{f'}{n}}\frac{f'_*}{1+\frac{f'_*}{n}}-f\frac{f_*}{1+\frac{f_*}{n}})dv_*d\omega, \quad (2.1) \\ g_t+\xi g_x &= -\frac{g}{1+\frac{g}{n}}(2\pi+\int_{|cos\theta|>\epsilon}Id\theta)+\frac{f}{1+\frac{f}{n}}\int Id\theta \end{aligned}$$

$$+\int S(\frac{g'}{1+\frac{g'}{n}}\frac{g'_*}{1+\frac{g'_*}{n}}-g\frac{g_*}{1+\frac{g_*}{n}})dv_*d\omega, \quad (2.2)$$

$$\frac{1}{c}I_t+cos\theta I_x=(1+I)\int\frac{g}{1+\frac{g}{n}}dv-I\int\frac{f}{1+\frac{f}{n}}dv, \quad |cos\theta|>\epsilon, \quad (2.3)$$

$$\frac{1}{c}I_t+cos\theta I_x=\int\frac{g}{1+\frac{g}{n}}dv-I\int\frac{f}{1+\frac{f}{n}}dv, \quad |cos\theta|<\epsilon, \quad (2.4)$$

together with the initial conditions

$$f(0,x,v)=f_i\ \wedge n, \quad g(0,x,v)=g_i\ \wedge n, \quad (2.5)$$

$$I(0,x,\theta)=I_i\ \wedge n, \quad (2.6)$$

and the boundary conditions

$$f(t,0,v)=f_0(t,v)\wedge n,\ \xi>0, \quad f(t,1,v)=f_1(t,v)\wedge n,\ \xi<0,$$

$$g(t,0,v)=g_0(t,v)\wedge n,\ \xi>0, \quad g(t,1,v)=g_1(t,v)\wedge n,\ \xi<0, \quad (2.7)$$

$$I(t,0,\theta)=I_0\delta_{\theta=0},\ cos\theta>0, \quad I(t,1,\theta)=I_1\delta_{\theta=\pi},\ cos\theta<0. \quad (2.8)$$

Let $p\geq n$ be given. Let $(f^j,g^j,I^j)_{j\in I\!N}$ be defined by $f^0=g^0=0$, $I^0=0$ and

$$f^{j+1}_t+\xi f^{j+1}_x=\frac{g^j}{1+\frac{g^j}{n}}(2\pi+\int_{|cos\theta|>\epsilon}I^jd\theta)-\frac{f^{j+1}}{1+\frac{f^j}{n}}\int I^jd\theta$$

$$+\int S(\frac{f^{j'}}{1+\frac{f^{j'}}{n}}\frac{f^{j'}_*}{1+\frac{f^{j'}_*}{n}}-f^{j+1}\frac{f^j_*}{1+\frac{f^j_*}{n}})dv_*d\omega, \quad (2.9)$$

$$g^{j+1}_t+\xi g^{j+1}_x=-\frac{g^{j+1}}{1+\frac{g^j}{n}}(2\pi+\int_{|cos\theta|>\epsilon}I^jd\theta)+\frac{f^j}{1+\frac{f^j}{n}}\int I^jd\theta$$

$$+\int S(\frac{g^{j'}}{1+\frac{g^{j'}}{n}}\frac{g^{j'}_*}{1+\frac{g^{j'}_*}{n}}-g^{j+1}\frac{g^j_*}{1+\frac{g^j_*}{n}})dv_*d\omega, \quad (2.10)$$

$$\frac{1}{c}I^{j+1}_t+cos\theta I^{j+1}_x=(1+I^{j+1})\int\frac{g^j}{1+\frac{g^j}{n}}dv-I^{j+1}\int\frac{f^j}{1+\frac{f^j}{n}}dv, \quad |cos\theta|>\epsilon, \quad (2.11)$$

$$\frac{1}{c}I^{j+1}_t+cos\theta I^{j+1}_x=\int\frac{g^j}{1+\frac{g^j}{n}}dv-I^{j+1}\int\frac{f^j}{1+\frac{f^j}{n}}dv, \quad |cos\theta|<\epsilon, \quad (2.12)$$

together with the initial conditions (2.5-6) and the boundary conditions (2.7-8). The sequence $(f^j,g^j,\int I^jd\theta)$ can be explicitly defined, is non negative as well as I^j, and bounded by $(n^3e^{n^2},n^3e^{n^2},n^2e^{n^2})$ for n large enough. Hence,

$\left(\frac{f^j}{1+\frac{f^j}{n}}\right)$ and $\Big((\partial_t+\xi\partial_x)\frac{f^j}{1+\frac{f^j}{n}}\Big)$ are weakly compact in L^1. And so, by the averaging lemma [6], $(\int \frac{f^j}{1+\frac{f^j}{n}}dv)$, and analogously $(\int \frac{g^j}{1+\frac{g^j}{n}}dv)$, are strongly compact in L^1. Consequently, the passage to the limit in (2.9-12) is possible, and $(f^n, g^n, I^n) := \lim_{j\to+\infty}(f^j, g^j, I^j)$ satisfies the system (2.1-8).

Lemma 2.1 *The solution (f^n, g^n, I^n) to (2.1-8) satisfies*

$$\begin{aligned}
&\int [f^n(\mid lnf^n \mid +1) + g^n(\mid lng^n \mid +1)](t,x,v)dxdv\\
&+\int_0^t\int_{\xi>0}\xi(f^n(|lnf^n|+1)+g^n(|lng^n|+1))(s,1,v)dsdv\\
&+\int_0^t\int_{\xi<0}|\xi|(f^n(|lnf^n|+1)+g^n(|lng^n|+1))(s,0,v)dsdv\\
&+\int I^n(t,x,\theta)dxd\theta+\int_{|cos\theta|<\epsilon}I^n|lnI^n|(t,x,\theta)dxd\theta\\
&+\tilde{e}(f^n,f^n)+\tilde{e}(g^n,g^n)+\tilde{D}_1(f^n,g^n,I^n)+\tilde{D}_2(f^n,g^n,I^n)<c,\quad t\in(0,T),
\end{aligned}\tag{2.13}$$

where

$$\tilde{e}(f,f) := \int S\Big(\frac{f'}{1+\frac{f'}{n}}\frac{f'_*}{1+\frac{f'_*}{n}}-\frac{f}{1+\frac{f}{n}}\frac{f_*}{1+\frac{f_*}{n}}\Big)ln\frac{\frac{f'}{1+\frac{f'}{n}}\frac{f'_*}{1+\frac{f'_*}{n}}}{\frac{f}{1+\frac{f}{n}}\frac{f_*}{1+\frac{f_*}{n}}}dsdxdvdv_*d\omega,$$

$$\tilde{D}_1(f,g,I) := \int_{|cos\theta|>\epsilon}[(1+I)\frac{g}{1+\frac{g}{n}}-I\frac{f}{1+\frac{f}{n}}]ln\frac{(1+I)\frac{g}{1+\frac{g}{n}}}{I\frac{f}{1+\frac{f}{n}}}dsdxdvd\theta,$$

$$\tilde{D}_2(f,g,I) := \int_{|cos\theta|<\epsilon}[\frac{g}{1+\frac{g}{n}}-I\frac{f}{1+\frac{f}{n}}]ln\frac{\frac{g}{1+\frac{g}{n}}}{I\frac{f}{1+\frac{f}{n}}}dsdxdvd\theta.$$

Proof of Lemma 2.1 Multiplying (2.1) by E_1, (2.2) by E_2, integrating the sum over $(0,t)\times(0,1)\times V$, adding it to (2.3) integrated over $(0,t)\times(0,1)\times\{|cos\theta|>\epsilon\}$, then to (2.4) integrated over $(0,t)\times(0,1)\times\{|cos\theta|<\epsilon\}$, and using that $E_2-E_1=h\nu=1$ leads to

$$\begin{aligned}
&\int(f^n+g^n)(t,x,v)dxdv+\frac{1}{c}\int I^n(t,x,\theta)dxd\theta\\
&+\int_0^t\int_{\xi>0}\xi(f^n+g^n)(s,1,v)dsdv\\
&+\int_0^t\int_{\xi<0}|\xi|(f^n+g^n)(s,0,v)dsdv
\end{aligned}$$

$$+\int_0^t\int_{cos\theta>0} cos\theta I^n(s,1,\theta)d\theta ds+\int_0^t\int_{cos\theta<0}|cos\theta|I^n(s,0,\theta)d\theta ds$$
$$\leq c_i. \quad (2.14)$$

Moreover, multiplying (2.1) by $ln\frac{f^n}{1+\frac{f^n}{n}}$, (2.2) by $ln\frac{g^n}{1+\frac{g^n}{n}}$, (2.3) by $ln\frac{I^n}{1+I^n}$, (2.4) by $ln(I^n)$, integrating the two first over $(0,t)\times(0,1)\times V$, the third on $(0,t)\times(0,1)\times\{|cos\theta|>\epsilon\}$, the last on $(0,t)\times(0,1)\times\{|cos\theta|<\epsilon\}$, adding them and using (2.18) and (1.13), leads to

$$\int[f^nlnf^n-n(1+\frac{f^n}{n})ln(1+\frac{f^n}{n})+g^nlng^n-n(1+\frac{g^n}{n})ln(1+\frac{g^n}{n})](t,x,v)dxdv$$
$$+\int_0^t\int_{\xi>0}\xi[f^nlnf^n-n(1+\frac{f^n}{n})ln(1+\frac{f^n}{n})+g^nlng^n-n(1+\frac{g^n}{n})ln(1+\frac{g^n}{n})](s,1,v)dsdv$$
$$+\int_0^t\int_{\xi<0}|\xi|[f^nlnf^n-n(1+\frac{f^n}{n})ln(1+\frac{f^n}{n})$$
$$+g^nlng^n-n(1+\frac{g^n}{n})ln(1+\frac{g^n}{n})](s,0,v)dsdv+\frac{1}{c}\int_{|cos\theta|<\epsilon}I^nlnI^n(t,x,\theta)dxd\theta$$
$$+\tilde{e}(f^n,f^n)+\tilde{e}(g^n,g^n)+\tilde{D}_1(f^n,g^n,I^n)+\tilde{D}_2(f^n,g^n,I^n)$$
$$\leq c_i+\int_{\frac{f^n}{1+\frac{f^n}{n}}\leq 1}S\frac{(f^n)^2}{n(1+\frac{f^n}{n})}f^n_*|ln\frac{f^n}{1+\frac{f^n}{n}}|+\int_{\frac{g^n}{1+\frac{g^n}{n}}\leq 1}S\frac{(g^n)^2}{n(1+\frac{g^n}{n})}g^n_*|ln\frac{g^n}{1+\frac{g^n}{n}}|.$$

Then,

$$\int_{\frac{f^n}{1+\frac{f^n}{n}}<1}S\frac{(f^n)^2}{n(1+\frac{f^n}{n})(1+\frac{f^n_*}{n})}|ln\frac{f^n}{1+\frac{f^n}{n}}|\leq c_i\int Sf^n\leq c_j.$$

Analogously,

$$\int_{\frac{g^n}{1+\frac{g^n}{n}}<1}S\frac{(g^n)^2}{n(1+\frac{g^n}{n})(1+\frac{g^n_*}{n})}|ln\frac{g^n}{1+\frac{g^n}{n}}|\leq c_j.$$

Then,

$$\frac{9}{10}tlnt-n(1+\frac{t}{n})ln(1+\frac{t}{n})\geq 0,\quad n>100,\quad t\in(100,n^3).$$

Adding (2.1) and (2.2) leads to $f^n+g^n\in[0,n^3]$, hence $f^n\in[0,n^3]$, $g^n\in[0,n^3]$, for n large enough. Hence,

$$\frac{1}{10}\int(f^nlnf^n+g^nlng^n)(t,x,v)dxdv$$

$$+\frac{1}{10}\int_0^t\int_{\xi>0}\xi(f^n lnf^n+g^n lng^n)(s,1,v)dsdv$$
$$+\frac{1}{10}\int_0^t\int_{\xi<0}|\xi|(f^n lnf^n+g^n lng^n)(s,0,v)dsdv$$
$$+\frac{1}{c}\int_{|cos\theta|<\epsilon}I^n lnI^n(t,x,\theta)dxd\theta$$
$$+\tilde{e}(f^n,f^n)+\tilde{e}(g^n,g^n)+\tilde{D}_1(f^n,g^n,I^n)+\tilde{D}_2(f^n,g^n,I^n)\leq c_i.$$

Lemma 2.2

$$\int supess_{t<T}(f^n+g^n)^\sharp(t,x,v)dxdv<c_i. \tag{2.15}$$

Proof of Lemma 2.2. The proof follows the lines of [1]. By Lemma 2.1, for $\delta>0$,

$$\sup_{|M|<\delta}\int_M f^n(t,x,v)dxdv,\quad \sup_{|M|<\delta}\int_M g^n(t,x,v)dxdv,$$
$$\int_0^\delta\Big(\int_{\xi>0}\xi f^n(\sigma,1,v)dv+\int_{\xi<0}|\xi|f^n(\sigma,0,v)dv\Big)d\sigma$$
$$\text{and}\quad\int_0^\delta\Big(\int_{\xi>0}\xi g^n(\sigma,1,v)dv+\int_{\xi<0}|\xi|g^n(\sigma,0,v)dv\Big)d\sigma,$$

are $o(1)$ in δ, uniformly with respect to n. Choose $\delta_0>0$ such that

$$\sup_{|M|<\delta_0}\int_M(f^n+g^n)(t,x,v)dxdv<\frac{2}{25c_0},\ \text{and}$$
$$\int_0^{\delta_0}\Big(\int_{\xi>0}\xi(f^n+g^n)(\sigma,1,v)dv+\int_{\xi<0}|\xi|(f^n+g^n)(\sigma,0,v)dv\leq\frac{2}{25c_0}, \tag{2.16}$$

where $c_0=2\pi^2\sup_{(v,v_*,\omega)}\frac{S(v,v_*,\omega)}{|\xi-\xi_*|}$. Notice that δ_0 only depends on the initial data and the given indata. Let $T'=min\{T,\delta_0,\frac{6\delta_0}{\pi\epsilon^4c^4}\}$. Let us prove that

$$\int supess_{t<T'}(f^n+g^n)^\sharp(t,x,v)dxdv\leq c_i. \tag{2.17}$$

Since T' only depends on T, the initial data and the given indata, it will then be possible to extend the result to $[T',2T']$ ..., $[(k-1)T',kT']$, $[kT',T]$, where $k=E(\frac{T}{T'})$, i.e. to $[0,T]$. Adding equations (2.1) and (2.2) leads to

$$(f^n+g^n)^{\sharp'}=\int S(\frac{f^{n'}}{1+\frac{f^{n'}}{n}}\frac{f_*^{n'}}{1+\frac{f_*^{n'}}{n}}-f^n\frac{f_*^n}{1+\frac{f_*^n}{n}})dv_*d\omega$$

$$+\int S(\frac{g^{n'}}{1+\frac{g^{n'}}{n}}\frac{g^{n'}_*}{1+\frac{g^{n'}_*}{n}}-g^n\frac{g^n_*}{1+\frac{g^n_*}{n}})dv_*d\omega.$$

Since f^n and g^n are bounded on $(0,T)\times(0,1)\times V$ by n^3 for n large enough, let T_n be the largest time smaller than T' such that

$$\sup_{x\in(0,1)}\{\sup_{\xi>0}\int_{-\frac{x}{\xi}\vee 0}^{T_n\wedge\frac{1-x}{\xi}}\int(f^n+g^n)^\sharp(s,x+s(\xi-\xi_*),v_*)|\xi-\xi_*|dv_*d\omega ds,$$
$$\sup_{\xi<0}\int_{\frac{1-x}{\xi}\vee 0}^{T_n\wedge-\frac{x}{\xi}}\int(f^n+g^n)^\sharp(s,x+s(\xi-\xi_*),v_*)|\xi-\xi_*|dv_*d\omega ds\}\leq\frac{1}{2c_0}.\quad(2.18)$$

Then, for $\xi>0$ and $t\in(-\frac{x}{\xi}\vee 0,T_n\wedge\frac{1-x}{\xi})$

$$(f^n+g^n)^\sharp(t,x,v)\leq(f^n+g^n)(-\frac{x}{\xi}\vee 0,x,v)$$
$$+\int_{-\frac{x}{\xi}\vee 0}^{T_n\wedge\frac{1-x}{\xi}}\int S(\frac{f^{n'}}{1+\frac{f^{n'}}{n}}\frac{f^{n'}_*}{1+\frac{f^{n'}_*}{n}}+\frac{g^{n'}}{1+\frac{g^{n'}}{n}}\frac{g^{n'}_*}{1+\frac{g^{n'}_*}{n}})^\sharp$$
$$=(f^n+g^n)^\sharp(T_n\wedge\frac{1-x}{\xi},x,v)+\int_{-\frac{x}{\xi}\vee 0}^{T_n\wedge\frac{1-x}{\xi}}\int S(f^n\frac{f^n_*}{1+\frac{f^n_*}{n}}+g^n\frac{g^n_*}{1+\frac{g^n_*}{n}})^\sharp dv_*d\omega$$
$$\leq(f^n+g^n)^\sharp(T_n\wedge\frac{1-x}{\xi},x,v)+\int_{-\frac{x}{\xi}\vee 0}^{T_n\wedge\frac{1-x}{\xi}}\int S[f^{n\sharp}(s,x,v)f^{n\sharp}(s,x+s(\xi-\xi_*),v_*)$$
$$+g^{n\sharp}(s,x,v)g^{n\sharp}(s,x+s(\xi-\xi_*),v_*)]dv_*d\omega.$$

Hence, for $\xi>0$,

$$supess_{t\in(-\frac{x}{\xi}\vee 0,T_n\wedge\frac{1-x}{\xi})}(f^n+g^n)^\sharp(t,x,v)\leq(f^n+g^n)^\sharp(T_n\wedge\frac{1-x}{\xi},x,v)$$
$$+supess_{t\in(-\frac{x}{\xi}\vee 0,T_n\wedge\frac{1-x}{\xi})}(f^n+g^n)^\sharp(t,x,v)$$
$$\int_{-\frac{x}{\xi}\vee 0}^{T_n\wedge\frac{1-x}{\xi}}\int S(f^n+g^n)^\sharp(s,x+s(\xi-\xi_*),v_*)dv_*d\omega ds.$$

By the definition of T_n, it holds that, for $\xi>0$,

$$supess_{t\in(-\frac{x}{\xi}\vee 0,T_n\wedge\frac{1-x}{\xi})}(f^n+g^n)^\sharp(t,x,v)\leq 2(f^n+g^n)^\sharp(T_n\wedge\frac{1-x}{\xi},x,v).\quad(2.19)$$

Analogously, for $\xi<0$,

$$supess_{t\in(\frac{1-x}{\xi}\vee 0,T_n\wedge-\frac{x}{\xi})}(f^n+g^n)^\sharp(t,x,v)\leq 2(f^n+g^n)^\sharp(T_n\wedge-\frac{x}{\xi},x,v).\quad(2.20)$$

Consider (x,v) such that $\xi > 0$. By the change of variables $s \to y = x + s(\xi - \xi_*)$,

$$\int_{-\frac{x}{\xi}\vee 0}^{T_n\wedge\frac{1-x}{\xi}} \int (f^n+g^n)^\sharp(s, x+s(\xi-\xi_*), v_*)|\xi-\xi_*|dv_* d\omega ds$$
$$\leq \int_{(y,v_*)\in\Delta} supess_{\sigma\in(-\frac{x}{\xi}\vee 0, T_n\wedge\frac{1-x}{\xi})}(f^n+g^n)^\sharp(\sigma,y,v_*)dydv_*,$$

where Δ is a subset of $(0,1)\times V$ of measure smaller than $\frac{\pi T_n\epsilon^4c^4}{6} \leq \delta_0$. Splitting Δ into

$$\{(y,v_*)\in\Delta; \xi_*>0, T_n\leq\frac{1-y}{\xi_*}\}\cup\{(y,v_*)\in\Delta; \xi_*>0, T_n>\frac{1-y}{\xi_*}\}$$
$$\cup\{(y,v_*)\in\Delta; \xi_*<0, T_n\leq-\frac{y}{\xi_*}\}\cup\{(y,v_*)\in\Delta; \xi_*<0, T_n>-\frac{y}{\xi_*}\},$$

and using (2.19-20) implies that

$$\begin{aligned}
&\int_{(y,v_*)\in\Delta} supess_{\sigma\in(-\frac{x}{\xi}\vee 0, T_n\wedge\frac{1-x}{\xi})}(f^n+g^n)^\sharp(\sigma,y,v_*)dydv_* \\
&\leq 2\int_\Delta (f^n+g^n)^\sharp(T_n,y,v_*)dydv_* \\
&+2\int_{\Delta;\xi_*>0,T_n>\frac{1-y}{\xi_*}}(f^n+g^n)^\sharp(\frac{1-y}{\xi_*},y,v_*)dydv_* \\
&+2\int_{\Delta;\xi_*<0,T_n>-\frac{y}{\xi_*}}(f^n+g^n)^\sharp(-\frac{y}{\xi_*},y,v_*)dydv_* \\
&\leq 2\int_\Delta (f^n+g^n)^\sharp(T_n,y,v_*)dydv_* + 2\int_{\xi_*>0,T_n>\frac{1-y}{\xi_*}}(f^n+g^n)(\frac{1-y}{\xi_*},1,v_*)dydv_* \\
&+2\int_{\Delta;\xi_*<0,T_n>-\frac{y}{\xi_*}}(f^n+g^n)(-\frac{y}{\xi_*},0,v_*)dydv_* \\
&= 2\int_\Delta (f^n+g^n)^\sharp(T_n,y,v_*)dydv_* + 2\int_0^{T_n}\int_{\xi_*>0}\xi_*(f^n+g^n)(\tau,1,v_*)dv_*d\tau \\
&+2\int_0^{T_n}\int_{\xi_*<0}|\xi_*|(f^n+g^n)(\tau,0,v_*)dv_*d\tau \leq \frac{12}{25c_0},
\end{aligned}$$

by (2.16). And so, the right-hand side in (2.18) can be improved by replacing $\frac{1}{2}$ by $\frac{12}{25}$. This implies that $T_n = T'$. Consequently, the inequalities (2.19-20) hold for $T_n = T'$. This ends the proof of Lemma 2.2.

Lemma 2.3 *The sequence (I^n) is weakly compact in $L^1((0,T)\times(0,1))\times\{|cos\theta|<\epsilon\}$. The sequences (F^n), (G^n), and $(\int_{|cos\theta|>\epsilon} I^n d\theta)$ are strongly compact in $L^1((0,T)\times(0,1))$. Moreover, $(\int_{|cos\theta|>\epsilon} I^n(t,x,\theta)d\theta)$ is bounded in $L^\infty((0,T)\times(0,1))$.*

By Lemma 2.1, (f^n), (g^n) and (I^n) are weakly compact in $L^1((0,T)\times(0,1)\times V)$, $L^1((0,T)\times(0,1)\times V)$, and $L^1((0,T)\times(0,1)\times\{|cos\theta|<\epsilon\})$ respectively. Moreover,

$$\int_{|cos\theta|>\epsilon} I^n(t,x,\theta)d\theta = c\int_{(Y\times V)\times\{|cos\theta|>\epsilon\}} \frac{g^n}{1+\frac{g^n}{n}}^{\sharp}(s,x-tccos\theta+s(ccos\theta-\xi),v)$$
$$e^{c\int_s^t\int(\frac{g^n}{1+\frac{g^n}{n}}-\frac{f^n}{1+\frac{f^n}{n}})^\sharp(\sigma,x-tccos\theta+\sigma(ccos\theta-\xi_*),v_*)dv_*d\sigma} dsd\theta dv$$
$$+\int_{|cos\theta|>\epsilon,x-tccos\theta\in(0,1)} I_i(x-tccos\theta,\theta)e^{c\int_0^t\int(\frac{g^n}{1+\frac{g^n}{n}}-\frac{f^n}{1+\frac{f^n}{n}})(\sigma,x+(\sigma-t)ccos\theta,v)d\sigma dv}d\theta$$
$$+I_0H(ct-x)e^{c\int_{t-x}^t\int(\frac{g^n}{1+\frac{g^n}{n}}-\frac{f^n}{1+\frac{f^n}{n}})(\sigma,x+(\sigma-t)c,v)d\sigma dv}$$
$$+I_1H(x+ct-1)e^{c\int_{x+t-1}^t\int(\frac{g^n}{1+\frac{g^n}{n}}-\frac{f^n}{1+\frac{f^n}{n}})(\sigma,x-(\sigma-t)c,v)d\sigma dv}, \quad (2.21)$$

where

$$Y:=\{(s,\theta);\ (0<s<t,x-tccos\theta\in(0,1))$$
$$\cup(t-\frac{x}{ccos\theta}<s<t,x-tccos\theta<0)\cup(t+\frac{1-x}{ccos\theta}<s<t,x-tccos\theta>1)\},$$

and H is the Heavyside function. Performing the change of variables $s\to y:=x-tccos\theta+s(ccos\theta-\xi)$ and $\sigma\to z:=x-tccos\theta+\sigma(ccos\theta-\xi_*)$ in the first term of the right-hand side, and noticing that, on $(Y\times V)\cap\{|cos\theta|>\epsilon\}$, $|ccos\theta-\xi|\geq c\frac{\epsilon}{2}$, leads to

$$\int_{(Y\times V)\cap\{|cos\theta|>\epsilon\}} g^{n\sharp}(s,x-tccos\theta+s(ccos\theta-\xi),v)$$
$$e^{c\int_s^t\int(g^n-f^n)^\sharp(\sigma,x-tccos\theta+\sigma(ccos\theta-\xi_*),v_*)dv_*d\sigma}dsd\theta dv$$
$$\leq\frac{2}{\epsilon c}\int_{(0,1)\times V}|g^n|_T(y,v)e^{\frac{2}{\epsilon c}\int_{(0,1)\times V}|g^n|_T(z,v_*)dv_*dz}dydv\leq c_i,$$

by Lemma 2.2. The other exponential terms in $\int_{|cos\theta|>\epsilon} I^n(t,x,\theta)d\theta$ can be treated analogously. Hence,

$$\int_{|cos\theta|>\epsilon} I^n(t,x,\theta)d\theta\leq c_i+c_j\int_{|cos\theta|>\epsilon,x-tccos\theta\in(0,1)} I_i(x-tccos\theta,\theta)d\theta.$$

It follows from the last assumption on I_i made in (1.12) that $(\int_{|cos\theta|>\epsilon} I^n(t,x,\theta)d\theta)$ is bounded from above, hence weakly compact in $L^1((0,T)\times(0,1))$. Moreover, for $\delta>0$,

$$
\begin{aligned}
&(\partial_t+\xi\partial_x)\frac{1}{\delta}ln(1+\delta f^n)\\
=&\frac{g^n}{(1+\delta f^n)(1+\frac{g^n}{n})}(2\pi+\int_{|cos\theta|>\epsilon} I^n d\theta)-\frac{f^n}{(1+\delta f^n)(1+\frac{f^n}{n})}\int I^n d\theta\\
&+\frac{1}{1+\delta f^n}\int S\frac{f^{n'}}{1+\frac{f^{n'}}{n}}\frac{f^{n'}_*}{1+\frac{f^{n'}_*}{n}}dv_*d\omega-\frac{f^n}{1+\delta f^n}\int S\frac{f^n_*}{1+\frac{f^n_*}{n}}dv_*d\omega.
\end{aligned}
$$

First,

$$
\begin{aligned}
&\frac{f^n}{(1+\delta f^n)(1+\frac{f^n}{n})}\int I^n d\theta+\frac{f^n}{1+\delta f^n}\int S\frac{f^n_*}{1+\frac{f^n_*}{n}}dv_*d\omega\\
&\qquad\leq\frac{1}{\delta}(\int I^n d\theta+\int S f^n_* dv_*d\omega)
\end{aligned}
$$

is weakly compact in L^1. It follows from the weak L^1 compactness of (f^n) and (g^n) and the boundedness of $\int_{|cos\theta|>\epsilon} I^n d\theta$ that

$$
(\frac{g^n}{1+\delta f^n}(2\pi+\int_{|cos\theta|>\epsilon} I^n d\theta))
$$

is weakly compact in L^1. Then, it classically follows from the weak compactness of (f^n) and the boundedness of $\tilde{e}(f^n,f^n)$, that

$$
(\frac{1}{1+\delta f^n}\int S\frac{f^{n'}}{1+\frac{f^{n'}}{n}}\frac{f^{n'}_*}{1+\frac{f^{n'}_*}{n}}dv_*d\omega)
$$

is weakly compact in L^1. Hence the sequence (F^n) is strongly compact in $L^1((0,T)\times(0,1))$. The same argument holds for proving that (G^n) is strongly compact in $L^1((0,T)\times(0,1))$. Then $(\int_{|cos\theta|>\epsilon} I^n(t,x,\theta)d\theta)$, as written in (2.21), is strongly compact in L^1, since $(\int\frac{g^n}{1+\frac{g^n}{n}}dv)$ is strongly compact in L^1 and $(t,\theta)\to\int_0^t(F^n+G^n)(s,x+(s-t)ccos\theta)ds$ is uniformly bounded with respect to n, t, x and θ such that $|cos\theta|>\epsilon$.

Lemma 2.4 *The sequences (f^n) and (g^n) belong to $C([0,T],L^1((0,1)\times V))$.*

Proof of Lemma 2.4. The temporal regularity of (f^n) and (g^n) is proven as in [8], by using the weak L^1 compactness of (f^n), (g^n),

$$(\frac{1}{1+\delta f^n}((\int d\theta + \int_{|cos\theta|>\epsilon} I^n d\theta)g^n - f^n \int I^n d\theta + Q(f^n, f^n))),$$

and

$$(\frac{1}{1+\delta g^n}(-(\int d\theta + \int_{|cos\theta|>\epsilon} I^n d\theta)g^n + f^n \int I^n d\theta + Q(g^n, g^n))).$$

3 The passage to the limit.

Lemma 3.1 *Let f, g and I be the weak limits in L^1 of subsequences of (f^n), (g^n) and (I^n) respectively. Then I satisfies the equations (1.5), (1.7-9) in weak form.*

Proof of Lemma 3.1. Let a test function μ be given. It is straightforward that I satisfies (1.5), (1.7-8) in weak form, since up to a subsequence, $(\int_{|cos\theta|>\epsilon} \mu I^n(t,x,\theta)d\theta)$ converges to $\int_{|cos\theta|>\epsilon} \mu I(t,x,\theta)d\theta$ in L^∞ weak $*$, and $(\int \frac{f^n}{1+\frac{f^n}{n}}dv)$ and $(\int \frac{g^n}{1+\frac{g^n}{n}}dv)$ converge strongly in L^1 to F and G. Let us prove that I satisfies (1.5), (1.7) and (1.9) in weak form. Let $\alpha > 0$ and a test function μ be given. From

$$\frac{f^n}{1+\frac{f^n}{n}} I^n \leq K \frac{g^n}{1+\frac{g^n}{n}} + \frac{1}{lnK}(\frac{g^n}{1+\frac{g^n}{n}} - I^n \frac{f^n}{1+\frac{f^n}{n}}) ln \frac{\frac{g^n}{1+\frac{g^n}{n}}}{\frac{I^n f^n}{1+\frac{f^n}{n}}}, \ K \geq 2,$$

the weak L^1 compactness of (g^n) and the bound on $\tilde{D}_2(f^n, g^n, I^n)$ derived in Lemma 2.1, it follows that $(\frac{f^n}{1+\frac{f^n}{n}} I^n)$ is weakly compact in L^1, so that there is $\beta_1 > 0$ such that for any subset Z of $(0,T)\times(0,1)$ such that$|Z| < \beta_1$,

$$\int_{Z\times V\times\{|cos\theta|<\epsilon\}} \frac{f^n}{1+\frac{f^n}{n}} I^n < \frac{\alpha}{|\mu|_\infty}.$$

By the averaging lemma and Egoroff's theorem, for any $\gamma > 0$ there is a set $Y_\gamma \subset (0,T)\times(0,1)$ such that $|Y_\gamma| < \gamma$, F is bounded on Y^c_γ and $(\int \frac{f^n}{1+\frac{f^n}{n}}dv)$ converges to F when $n \to +\infty$, uniformly with respect to $(t,x) \in Y^c_\gamma$. Hence, $\int_{Y^c_\gamma \times\{|cos\theta|<\epsilon\}} FI \leq c$. Using a decreasing sequence (Y_{γ_n}) with $\lim_{n\to+\infty} \gamma_n =$

0, implies that $\int FI \leq c$. And so, there is $\beta_2 \in (0, \beta_1)$, such that for any subset Z of $(0,T) \times (0,1)$ with $|Z| < \beta_2$,

$$\int_{Z\times\{|cos\theta|<\epsilon\}} FI \leq \frac{\alpha}{|\mu|_\infty}.$$

Then,

$$\begin{aligned} &\left| \int \left(\frac{f^n}{1+\frac{f^n}{n}} I^n - fI\right) \mu dt dx dv d\theta \right| \\ &\leq \int_{Y^c_{\beta_2}\times V\times\{|cos\theta|<\epsilon\}} \left| \int \left(\frac{f^n}{1+\frac{f^n}{n}} - f\right) dv \right| I^n \mu dt dx dv d\theta \\ &\quad + \int_{Y^c_{\beta_2}\times\{|cos\theta|<\epsilon\}} F\mu(I^n - I) dt dx d\theta \\ &+ \int_{Y_{\beta_2}\times V\times\{|cos\theta|<\epsilon\}} \frac{f^n}{1+\frac{f^n}{n}} I^n \mu dt dx dv d\theta + \int_{Y_{\beta_2}\times\{|cos\theta|<\epsilon\}} FI\mu dt dx d\theta. \end{aligned}$$

The first term in the right-hand side tends to 0 when $n \to +\infty$, by the uniform convergence with respect to $(t,x) \in Y^c_{\beta_2}$ of $\int(\frac{f^n}{1+\frac{f^n}{n}} - f)dv$ to 0 when $n \to +\infty$. The second term tends to 0 when $n \to +\infty$, since F is bounded on $Y^c_{\beta_2}$ and $\int(I^n - I)\mu d\theta$ tends to 0 weakly in L^1. The third and fourth terms are smaller than α, uniformly with respect to n. And so, the passage to the limit when $n \to +\infty$ in (2.4) can be performed. This ends the proof of Lemma 3.1.

Lemma 3.2 *Let $\rho > 0$ be given. There are a constant $c_\rho > 0$ and a subset $A_\rho \subset (0,1) \times V$ such that $|A^c_\rho| < \rho$ and*

$$(f+g)^\sharp(t,x,v) \leq c_\rho, \quad a.a.\ t \in (0,T), \quad (x,v) \in A_\rho.$$

Moreover, for any $\delta > 0$, there is a sequence $(Y_{n,\delta})$ of subsets of $(0,1) \times V$ such that

$$meas\{(t, x+t\xi, v), t \in (0,T), (x,v) \in Y^c_{n,\delta}\} < \delta,$$

and

$$(f^n + g^n)^\sharp(t,x,v) < \frac{c}{\delta}, \quad a.a.\ t \in (0,T), \quad (x,v) \in Y_{n,\delta}.$$

Proof of Lemma 3.2. Adding (2.1) and (2.2) implies that

$$(f^n+g^n)_t+\xi(f^n+g^n)_x+(f^n+g^n)\int S(\frac{f^n_*}{1+\frac{f^n_*}{n}}+\frac{g^n_*}{1+\frac{g^n_*}{n}})dv_*d\omega$$
$$\geq (f^n+g^n)_t+\xi(f^n+g^n)_x$$
$$+f^n\int S\frac{f^n_*}{1+\frac{f^n_*}{n}}dv_*d\omega+g^n\int S\frac{g^n_*}{1+\frac{g^n_*}{n}}dv_*d\omega \geq 0.$$

Hence, for (t,x,v) such that $(x,x+t\xi,x+T\xi)\in(0,1)^3$,

$$(f^n+g^n)^\sharp(t,x,v)\leq(f^n+g^n)^\sharp(T,x,v)e^{\int_0^T\int S(f^n+g^n)(s,x+s\xi,v_*)dv_*d\omega ds}.$$

By the averaging lemma,

$$(x,v)\to\int_0^T\int S(f^n+g^n)(s,x+s\xi,v_*)dv_*d\omega ds$$

is strongly compact in $L^1((0,1)\times V)$. Hence, it is bounded in L^∞, uniformly with respect to n, on the complementary A^1_ρ of some subset of $(0,1)\times V$ with measure smaller than $\frac{\rho}{2}$. Consequently,

$$\int_0^T\int S(f^n+g^n)(s,x+s\xi,v_*)dv_*d\omega ds\leq c_1,\quad \text{a.a. } (x,v)\in A^1_\rho, \quad (3.1)$$

and

$$(f^n+g^n)^\sharp(t,x,v)\leq e^{c_1}(f^n+g^n)^\sharp(T,x,v),\quad \text{a.a. } t\in(0,T),\quad (x,v)\in A^1_\rho. \quad (3.2)$$

And so,

$$(f+g)^\sharp(t,x,v)\leq e^{c_1}(f+g)^\sharp(T,x,v),\quad \text{a.a. } t\in(0,T),\quad (x,v)\in A^1_\rho.$$

For some $A^2_\rho\subset(0,1)\times V$ with complementary of measure smaller than $\frac{\rho}{2}$,

$$(f+g)^\sharp(T,x,v)\leq c_2,\quad (x,v)\in A^2_\rho.$$

Hence, for $A_\rho=A^1_\rho\cap A^2_\rho$, such that $|A^c_\rho|\leq\rho$,

$$(f+g)^\sharp(t,x,v)\leq c,\quad \text{a.a. t}\in(0,T),\quad (x,v)\in A_\rho.$$

The cases where $x<0$, $(x+t\xi,x+T\xi)\in(0,1)^2$ and $x>1$, $(x+t\xi,x+T\xi)\in(0,1)^2$ can be treated analogously. The cases where $x+t\xi\in(0,1)$, $x+T\xi>1$ (resp. $x+t\xi\in(0,1)$, $x+T\xi<0$) can be treated with a comparison with the outgoing values of $f^\sharp$ at the boundary point 1 (resp. 0). The existence of the sequence of sets $(Y_{n,\delta})$ stated in Lemma 3.2 follows from (3.3) and the inequality

$$\int (f^n + g^n)^\sharp (T,x,v)dxdv \leq c, \quad a.a.(x,v) \text{ s.t. } x+t\xi \in (0,1).$$

This ends the proof of Lemma 3.2. Denote by $\chi_{n,\delta}$ the characteristic function of the set of characteristics

$$\{(t, x+t\xi, v); t \in (0,T), (x,v) \in Y_{n,\delta}\}.$$

Let $\rho > 0$ and $\delta > 0$ be fixed. The equations (2.1-2) satisfied by (f^n, g^n) are first written in weak form for test functions $\chi_{n,\delta}\varphi$, i.e.

$$\begin{aligned}
&\int_{(x,v);(x+t\xi,v)\in(0,1)\times V} (f^{n\sharp}\chi^\sharp_{n,\delta}\varphi)(t,x,v)dxdv \\
&= \int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0} (f^{n\sharp}\chi^\sharp_{n,\delta}\varphi)(0\vee -\frac{x}{\xi},x,v)dxdv + \int_{\xi>0}\int_{0\vee -\frac{x}{\xi}}^t f^{n\sharp}\chi^\sharp_{n,\delta}\frac{\partial\varphi}{\partial s} \\
&+ \int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0} (f^{n\sharp}\chi^\sharp_{n,\delta}\varphi)(0\vee \frac{1-x}{\xi},x,v)dxdv + \int_{\xi<0}\int_{0\vee \frac{1-x}{\xi}}^t f^{n\sharp}\chi^\sharp_{n,\delta}\frac{\partial\varphi}{\partial s} \\
&+ \int_{\xi>0}\int_{0\vee -\frac{x}{\xi}}^t (\frac{g^n}{1+\frac{g^n}{n}}(2\pi + \int_{|cos\theta|>\epsilon} I^n d\theta) - \frac{f^n}{1+\frac{f^n}{n}}\int I^n d\theta + Q_n(f^n,f^n))^\sharp \chi^\sharp_{n,\delta}\varphi \\
&+ \int_{\xi<0}\int_{0\vee \frac{1-x}{\xi}}^t (\frac{g^n}{1+\frac{g^n}{n}}(2\pi + \int_{|cos\theta|>\epsilon} I^n d\theta) - \frac{f^n}{1+\frac{f^n}{n}}\int I^n d\theta + Q_n(f^n,f^n))^\sharp \chi^\sharp_{n,\delta}\varphi,
\end{aligned}$$

and an analogous equation for g^n. Here, Q_n denotes the approximation of the Boltzmann collision operator used in Section 2. Since $\lim_{\delta\to 0} w \lim_{n\to+\infty} f^n\chi_{n,\delta} = f$, the limit of the first three lines when $n \to +\infty$, then $\delta \to 0$ is

$$\int_{(x,v);(x+t\xi,v)\in V} (f^\sharp\varphi)(t,x,v)dxdv,$$

$$\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0} f^\sharp\varphi(0\vee -\frac{x}{\xi},x,v) + \int_{\xi>0}\int_{0\vee -\frac{x}{\xi}}^t f^\sharp\frac{\partial\varphi}{\partial s},$$

$$\text{and } \int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0} f^\sharp\varphi(0\vee \frac{1-x}{\xi},x,v) + \int_{\xi<0}\int_{0\vee \frac{1-x}{\xi}}^t f^\sharp\frac{\partial\varphi}{\partial s}.$$

The passage to the limit in

$$\int_{\{\xi>0, 0\vee -\frac{x}{\xi}<s<t\}\cup\{\xi<0, 0\vee\frac{1-x}{\xi}<s<t\}} (\frac{g^n}{1+\frac{g^n}{n}}(2\pi + \int I^n d\theta) - \frac{f^n}{1+\frac{f^n}{n}}\int I^n d\theta)^\sharp \chi^\sharp_{n,\delta}\varphi$$

and in

$$\int_{\{\xi>0,0\vee-\frac{x}{\xi}<s<t\}\cup\{\xi<0,0\vee\frac{1-x}{\xi}<s<t\}} Q_n(f^n,f)^\sharp\chi^\sharp_{n,\delta}\varphi$$

are performed in Lemma 3.3, and in Lemmas 3.4-6 respectively.

Lemma 3.3 *For any bounded function φ, the sequence* $(\int_{|cos\theta|<\epsilon}\frac{f^n}{1+\frac{f^n}{n}}^\sharp\chi^\sharp_{n,\delta}\varphi(t,x,v)I^n(t,x+t\xi,\theta)d\theta dtdxdv)$, *(resp.* $(\int_{|cos\theta|<\epsilon}\frac{g^n}{1+\frac{g^n}{n}}^\sharp\chi^\sharp_{n,\delta}\varphi(t,x,v)I^n(t,x+t\xi,\theta)d\theta dtdxdv)$, $(\int_{|cos\theta|>\epsilon}\frac{f^n}{1+\frac{f^n}{n}}^\sharp\chi^\sharp_{n,\delta}\varphi(t,x,v)I^n(t,x+t\xi,\theta)d\theta dtdxdv)$, $(\int_{|cos\theta|>\epsilon}\frac{g^n}{1+\frac{g^n}{n}}^\sharp\chi^\sharp_{n,\delta}\varphi(t,x,v)I^n(t,x+t\xi,\theta)d\theta dtdxdv)$) *tends to* $\int_{|cos\theta|<\epsilon}f^\sharp\varphi(t,x,v)I(t,x+t\xi,\theta)d\theta dtdxdv$, *(resp.* $\int_{|cos\theta|<\epsilon}g^\sharp\varphi(t,x,v)I(t,x+t\xi,\theta)d\theta dtdxdv$, $\int_{|cos\theta|>\epsilon}f^\sharp\varphi(t,x,v)I(t,x+t\xi,\theta)d\theta dtdxdv$, $\int_{|cos\theta|>\epsilon}g^\sharp\varphi(t,x,v)I(t,x+t\xi,\theta)d\theta dtdxdv$) *when n tends to infinity.*

<u>Proof of Lemma 3.3.</u> The proof of the first statement of the lemma is similar to the last part of the proof of Lemma 3.1, after noticing that for $\delta>0$ fixed, the averaging lemma applies to $\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta}$. Indeed, $(\frac{f^n}{1+\frac{f^n}{n}})$, and

$$(\partial_t+\xi\partial_x)\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta}=\frac{\chi_{n,\delta}}{(1+\frac{f^n}{n})^2}(\frac{g^n}{1+\frac{g^n}{n}}(2\pi+\int_{|cos\theta|>\epsilon}I^nd\theta)$$
$$-\frac{f^n}{1+\frac{f^n}{n}}\int I^nd\theta+Q_n(f^n,f^n))$$

are weakly compact in L^1. The proof of

$$\lim_{\delta\to0}\lim_{n\to+\infty}\int_{|cos\theta|>\epsilon}(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta})^\sharp\varphi(t,x,v)I^n(t,x+t\xi,\theta)dtdxdvd\theta$$
$$=\int_{|cos\theta|>\epsilon}f^\sharp\varphi(t,x,v)I(t,x+t\xi,\theta)dtdxdvd\theta$$

then follows from the weak $*$ convergence in L^∞ of $\int_{|cos\theta|>\epsilon}I^n(t,x+t\xi)d\theta$ to $\int_{|cos\theta|>\epsilon}I(t,x+t\xi)d\theta$ when $n\to+\infty$.
The passage to the limit when $n\to+\infty$, then $\delta\to0$ will now be performed in the Boltzmann collision operators. Let $\rho>0$ be given. Test functions with support outside of $(0,T)\times A_\rho$ will first be considered, following the lines of the proof of the passage to the limit in [2]. Then this restriction on the support of the test function will be removed.

Lemma 3.4 *Let $\rho > 0$ be given. For any test function φ defined in Definition 1.1 and such that φ vanishes on $(0,T) \times A^c_\rho$,*

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\int S(f^n\chi_{n,\delta})^\sharp\varphi(t,x,v)\frac{f^n}{1+\frac{f^n}{n}}(t,x+t\xi,v_*)dv_*d\omega dtdxdv$$
$$= \int S(f^\sharp\varphi)(t,x,v)f(t,x+t\xi,v_*)dv_*d\omega dtdxdv.$$

Proof of Lemma 3.4. Denote by $g_\delta = w\lim_{n\to+\infty} f^n\chi_{n,\delta}$ in L^1. For $\delta > 0$ fixed, let us first prove that

$$\lim_{n\to+\infty}\int S(f^n\chi_{n,\delta})^\sharp\varphi(t,x,v)\frac{f^n}{1+\frac{f^n}{n}}(t,x+t\xi,v_*)dtdxdvdv_*$$
$$= \int Sg^\sharp_\delta\varphi(t,x,v)f(t,x+t\xi,v_*)dtdxdvdv_*.$$

Then

$$\lim_{\delta\to 0}\int S(g^\sharp_\delta\varphi)(t,x,v)f(t,x+t\xi,v_*)dtdxdvdv_*$$
$$= \int S(f^\sharp\varphi)(t,x,v)f(t,x+t\xi,v_*)dtdxdvdv_*,$$

since the family (g_δ) increasingly converges to f in L^1, and

$$(g^\sharp_\delta\varphi)(t,x,v)f(t,x+t\xi,v_*) \leq (f^\sharp\varphi)(t,x,v)f(t,x+t\xi,v_*),$$

which is integrable, since $f^\sharp\varphi$ is bounded.

$$|\int S\varphi(f^n\chi_{n,\delta})^\sharp\frac{f^n}{1+\frac{f^n}{n}}(t,x+t\xi,v_*) - g^\sharp_\delta f(t,x+t\xi,v_*))dv_*d\omega dtdxdv|$$
$$\leq A_n + B_n + C_n,$$

where

$$A_n = |\int \varphi(t,x,v)(f^n\chi_{n,\delta} - g_\delta)^\sharp(t,x,v)S\frac{f^n}{1+\frac{f^n}{n}}(t,x+t\xi,v_*)dv_*d\omega dtdxdv|,$$

$$B_n = |\int g^\sharp_\delta\varphi S(f^n - f)(t,x+t\xi,v_*)dv_*d\omega dtdxdv|,$$

$$C_n = \int g^\sharp_\delta\varphi S\frac{(f^n)^2}{n(1+\frac{f^n}{n})}(t,x+t\xi,v_*)dv_*d\omega dtdxdv.$$

Then, A_n tends to zero when n tends to infinity, since $\varphi(f^n\chi_{n,\delta}-g_\delta)^\sharp$ converges to zero in L^∞ weak $*$ and $(\int S\frac{f^n}{1+\frac{f^n}{n}}(t,x+t\xi,v_*)dv_*d\omega)$ converges strongly in L^1 on the support of φ. Moreover, $\lim_{n\rightarrow+\infty}B_n=0$, since $g^\sharp_\delta\varphi$ is bounded and $(\int S(f^n-f)(t,x+t\xi,v_*)dv_*d\omega)$ tends to zero in $L^1((0,T)\times(0,1)\times V)$ by the averaging lemma. Finally,

$$C_n \leq c\int S\frac{(f^n)^2}{n(1+\frac{f^n}{n})}(t,x+t\xi,v_*)dv_*d\omega dtdxdv$$
$$= c\int_{f^n(t,x+t\xi,v_*)<L} S\frac{(f^n)^2}{n(1+\frac{f^n}{n})}(t,x+t\xi,v_*)dv_*d\omega dtdxdv$$
$$+c\int_{f^n(t,x+t\xi,v_*)>L} S\frac{(f^n)^2}{n(1+\frac{f^n}{n})}(t,x+t\xi,v_*)dv_*d\omega dtdxdv$$
$$\leq c\frac{L}{n}+c\int_{f^n(t,x+t\xi,v_*)>L} f^n(t,x+t\xi,v_*)dv_*dtdxdv,$$

which tends to zero when $n\rightarrow+\infty$, by first choosing L large enough so that $\int_{f^n(t,x+t\xi,v_*)>L} f^n(t,x+t\xi,v_*)dv_*dtdxdv$ be small uniformly with respect to n, then n large enough.

Denote by

$$Q^+_n(f^n,f^n)(t,x,v)=\int S\frac{f^n}{1+\frac{f^n}{n}}(t,x,v')\frac{f^n}{1+\frac{f^n}{n}}(t,x,v'_*)dv_*d\omega.$$

Lemma 3.5 *Let $\rho>0$ be given. For any test function φ defined in Definition 1.1 and such that φ vanishes on $(0,T)\times A^c_\rho$,*

$$\underline{lim}_{\delta\rightarrow 0}\underline{lim}_{n\rightarrow+\infty}\int Q^+_n(f^n,f^n)^\sharp\chi^\sharp_{n,\delta}\varphi(t,x,v)dtdxdv$$
$$\geq\int Q^+(f,f)^\sharp\varphi(t,x,v)dv_*d\omega dtdxdv.$$

<u>Proof of Lemma 3.5.</u>

$$\int Q^+_n(f^n,f^n)^\sharp\chi^\sharp_{n,\delta}\varphi dtdxdv$$
$$=\int S\chi_{n,\delta}(t,X,v')\varphi(t,X-t\xi',v')\frac{f^n}{1+\frac{f^n}{n}}(t,X,v)\frac{f^n}{1+\frac{f^n}{n}}(t,X,v_*)dtdXdvdv_*d\omega$$
$$\geq\int S\chi_{n,\delta}(t,X,v')\varphi(t,X-t\xi',v')(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v)$$

$$(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v_*)dtdXdvdv_*d\omega$$

$$= \int S(1-\chi_{n,\delta}(t,X,v'))\varphi(t,X-t\xi',v')(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v)$$

$$(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v_*)dtdXdvdv_*d\omega$$

$$+\int S\varphi(t,X-t\xi',v')(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v)(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v_*)dtdXdvdv_*d\omega.$$

Then,

$$\int S(1-\chi_{n,\delta}(t,X,v'))\varphi(t,X-t\xi',v')(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v)$$

$$(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v_*)dtdXdvdv_*d\omega$$

$$\leq \frac{c}{\delta^{\frac{2}{3}}}\int(1-\chi_{n,\delta}(t,X,v'))dtdXdvdv_*d\omega \leq c\delta^{\frac{1}{3}}.$$

Moreover, analogously to the proof of Lemma 3.4,

$$\lim_{n\to+\infty}\int S\varphi(t,X-t\xi',v')(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v)(\frac{f^n}{1+\frac{f^n}{n}}\chi_{n,\delta^{\frac{1}{3}}})(t,X,v_*)dtdXdvdv_*d\omega$$

$$= \int S\varphi(t,X-t\xi',v')g_\delta(t,X,v)g_\delta(t,X,v_*)dtdXdvdv_*d\omega,$$

which tends to

$$\int S\varphi(t,X-t\xi',v')f(t,X,v)f(t,X,v_*)dtdXdvdv_*d\omega$$

$$= \int Q^+(f,f)^\sharp\varphi(t,x,v)dtdxdv,$$

when δ tends to 0.

Lemma 3.6

$$\overline{\lim}_{\delta\to 0}\overline{\lim}_{n\to+\infty}\int Q_n^+(f^n,f^n)^\sharp\chi_{n,\delta}^\sharp\varphi dtdxdv \leq \int Q^+(f,f)^\sharp\varphi dtdxdv.$$

Proof of Lemma 3.6. Let $\eta > 0$ be given. It follows from the averaging lemma that there is a subset X_η of $(0,T)\times(0,1)\times V$, with $|X_\eta^c| < \eta$, such that

$$\int S(f^n-f)(t,X,v_*)\varphi(t,X-t\xi',v')dv_*d\omega \to 0 \text{ when } n\to+\infty, \quad (3.3)$$

uniformly with respect to $(t, X, v) \in X_\eta$, and

$$\int Sf(t, X, v_*)\varphi(t, X - t\xi', v')dv_* d\omega \text{ is bounded in } X_\eta. \tag{3.4}$$

Then,

$$\begin{aligned}
&\int Q_n^+(f^n, f^n)^\sharp \chi_{n,\delta}^\sharp \varphi dt dx dv \\
&\leq \int Sf^n(t, X, v')f^n(t, X, v'_*)\chi_{n,\delta}(t, X, v)\varphi(t, X - t\xi, v)dtdXdvdv_* d\omega \\
&\leq \int_{(t,X,v,v_*,\omega);(t,X,v')\in X_\eta} Sf^n(t, X, v')f^n(t, X, v'_*)\varphi(t, X - t\xi, v)dtdXdvdv_* d\omega \\
&+ \int_{(t,X,v,v_*,\omega);(t,X,v')\in X_\eta^c} Sf^n(t, X, v')f^n(t, X, v'_*)\chi_{n,\delta}(t, X, v)\varphi(t, X - t\xi, v)dtdXdvdv_* d\omega \\
&= \int_{(t,X,v,v_*,\omega);(t,X,v)\in X_\eta} Sf^n(t, X, v)f^n(t, X, v_*)\varphi(t, X - t\xi', v')dtdXdvdv_* d\omega \\
&+ \int_{(t,X,v,v_*,\omega);(t,X,v')\in X_\eta^c} Sf^n(t, X, v')f^n(t, X, v'_*)\chi_{n,\delta}(t, X, v)\varphi(t, X - t\xi, v)dtdXdvdv_* d\omega.
\end{aligned}$$

By (3.3-4), analogously to the proof of Lemma 3.4,

$$\begin{aligned}
&\lim_{n\to+\infty} \int_{(t,X,v,v_*,\omega);(t,X,v)\in X_\eta} Sf^n(t, X, v)f^n(t, X, v_*)\varphi(t, X - t\xi', v')dtdXdvdv_* d\omega \\
&= \int_{X_\eta} Sf(t, X, v)f(t, X, v_*)\varphi(t, X - t\xi', v')dtdXdvdv_* d\omega.
\end{aligned}$$

Moreover, for $j \geq 2$,

$$\begin{aligned}
&\int_{(t,X,v,v_*,\omega);(t,X,v')\in X_\eta^c} Sf^n(t, X, v')f^n(t, X, v'_*)\chi_{n,\delta}(t, X, v)\varphi(t, X - t\xi, v)dtdXdvdv_* d\omega \\
&\leq j \int_{(t,X,v,v_*,\omega);(t,X,v')\in X_\eta^c} S(f^n\chi_{n,\delta})(t, X, v)f^n(t, X, v_*)\varphi(t, X - t\xi, v)dtdXdvdv_* d\omega + \frac{c}{lnj} \\
&\leq \frac{cj}{\delta} \int_{(t,X,v,v_*,\omega);(t,X,v')\in X_\eta^c} Sf^n(t, X, v_*)dtdXdvdv_* d\omega + \frac{c}{lnj} \\
&\leq \frac{cj}{\delta} o(\eta) + \frac{c}{lnj}.
\end{aligned}$$

And so, Lemma 3.6 holds since, given $\tilde{\epsilon} > 0$, one can choose j big enough, then $o(\eta) \leq \frac{\delta\tilde{\epsilon}}{2cj}$, finally δ small enough, so that $\frac{cj}{\delta} o(\eta) + \frac{c}{lnj} \leq \tilde{\epsilon}$ holds.

<u>End of the passage to the limit.</u>

It remains to pass to the limit when ρ tends to zero. Let φ be a test function as defined in Definition 1.1. Let $t \in (0,T)$ be given and

$$E_\rho = \{(x,v) \in A_\rho; \xi > 0 \text{ and } \int_{0\vee -\frac{x}{\xi}}^{t\wedge \frac{1-x}{\xi}} Q(f,f)^\sharp \varphi(s,x,v)ds \geq 0\}$$
$$\cup\{(x,v) \in A_\rho; \xi < 0 \text{ and } \int_{0\vee \frac{1-x}{\xi}}^{t\wedge -\frac{x}{\xi}} Q(f,f)^\sharp (s,x,v)ds \geq 0\}.$$

Then the iterated form enounced in Definition 1.1 holds for $f\varphi\chi_{(0,T)\times E_\rho}$, so that

$$\begin{aligned}
&\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0} \chi_{E_\rho}(x,v)\Big(\int_{0\vee -\frac{x}{\xi}}^{t} h^\sharp(s,x,v)ds\Big)dxdv \\
&+\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0} \chi_{E_\rho}(x,v)\Big(\int_{0\wedge \frac{1-x}{\xi}}^{t} h^\sharp(s,x,v)ds\Big)dxdv \\
&= -\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0} (f\varphi\chi_{E_\rho})^\sharp(0\vee -\frac{x}{\xi},x,v)dxdv \\
&-\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0} (f\varphi\chi_{E_\rho})^\sharp(0\vee \frac{1-x}{\xi},x,v)dxdv \\
&-\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi>0} \int_{0\vee -\frac{x}{\xi}}^{t} \Big(f\chi_{E_\rho}\frac{\partial\varphi}{\partial s}+2\pi g\chi_{E_\rho}\varphi\Big)^\sharp(s,x,v)dsdxdv \\
&-\int_{(x,v);(x+t\xi,v)\in(0,1)\times V,\xi<0} \int_{0\wedge \frac{1-x}{\xi}}^{t} \Big(f\chi_{E_\rho}\frac{\partial\varphi}{\partial s}+2\pi g\chi_{E_\rho}\varphi\Big)^\sharp(s,x,v)dsdxdv. \qquad (3.5)
\end{aligned}$$

The left-hand side of (3.6) is non negative by definition of E_ρ, increasing when ρ decreases to zero, and equal to the right-hand side that has a limit when ρ tends to zero. And so, the iterated form holds for $\varphi\chi_E$, where

$$E = \{(x,v); \xi > 0 \text{ and } \int_{0\vee -\frac{x}{\xi}}^{t\wedge \frac{1-x}{\xi}} h^\sharp(s,x,v)ds \geq 0\}$$
$$\cup\{(x,v); \xi < 0 \text{ and } \int_{0\vee \frac{1-x}{\xi}}^{t\wedge -\frac{x}{\xi}} h^\sharp(s,x,v)ds \geq 0\}.$$

The same argument can be used for $\varphi\chi_{E^c}$. And so, the iterated form holds for $f\varphi(\chi_E+\chi_{E^c}) = f\varphi$. Analogously, the iterated form holds for $g\psi$, where ψ is a test function for g as defined in Definition 1.1. This ends the proof of Theorem 1.1.

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