L^1 solutions to the stationary Boltzmann equation in a slab.

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Résumé. On considère l'équation de Boltzmann stationnaire en une dimension d'espace et trois dimensions de vitesse, pour des conditions aux limites de réflexion maxwellienne diffuse. On montre l'existence d'une solution appartenant à L^1, en utilisant le terme de production d'entropie et un changement de variables d'espace classique.

Abstract. The stationary Boltzmann equation for pseudo-maxwellian and hard forces is considered in the slab. An L^1 existence theorem is proven in the case of diffuse reflection boundary conditions. The method of proof is based on properties of the entropy dissipation term. The approach is simplified by a classical transformation of the space variable resulting in a homogeneous equation of degree one. The case of given indata is also briefly discussed.

1 Introduction.

Consider the stationary Boltzmann equation in a slab of length L

$$\xi \frac{\partial}{\partial x} f(x,v) = Q(f,f)(x,v), \ x \in [0,L], \ v \in I\!R^3. \tag{1.1}$$

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The nonnegative function $f(x,v)$ represents the density of a rarefied gas at position x and velocity v, with ξ the velocity component in the slab direction. The boundary conditions are of diffuse reflection type,

$$f(0,v) = M_0(v)\int_{\xi'<0} \mid \xi' \mid f(0,v')dv',\ \xi>0,$$
$$f(L,v) = M_L(v)\int_{\xi'>0} \xi' f(L,v')dv',\ \xi<0, \tag{1.2}$$

where M_0 and M_L are given normalized half-space maxwellians $M_i(v) = \frac{1}{2\pi T_i^2}e^{-\frac{|v|^2}{2Ti}}$, $i\in\{0,L\}$. The collision operator Q is the classical Boltzmann operator with angular cut-off

$$\begin{aligned}Q(f,f)(x,v) &= \int_{I\!R^3}\int_{\mathcal{S}^2} B(v-v_*,\omega)[f'f'^* - ff^*]d\omega dv_*\\ &= Q^+(f,f) - Q^-(f,f),\end{aligned}$$

where Q^+ is the gain part and Q^- the loss part of the collision term, and where

$$\begin{aligned}&f^* = f(x,v_*), \quad f' = f(x,v'), \quad f'^* = f(x,v'_*),\\ &v' = v-(v-v_*,\omega)\omega, \quad v'_* = v_* + (v-v_*,\omega)\omega.\end{aligned}$$

Here, $(v-v_*,\omega)$ denotes the Euclidean inner product in $I\!R^3$. Let ω be represented by the polar angle θ (with polar axis along $v-v_*$) and the azimuthal angle ϕ. The function $B(v-v_*,\omega)$ is the kernel of the collision operator Q, and for convenience taken as $\mid v-v_* \mid^\beta b(\theta)$, with

$$0\leq\beta<2, \quad b\in L^1_+(0,2\pi), \quad b(\theta)\geq c>0,\ a.e.$$

Let us first recall that in the case of the time-dependent Boltzmann equation

$$f_t(t,x,v) + v\cdot\bigtriangledown_x f(t,x,v) = Q(f,f)(t,x,v), \quad t\in I\!R_+, \quad x\in\Omega, \quad v\in I\!R^3,$$

where Ω is a subset of $I\!R^3$, the Cauchy problem has been studied intensely, most important being the time-dependent existence proof by R.DiPerna and P.L.Lions [17], based on the use of the averaging technique and new solution concepts. For a survey and references to the time-dependent problem, see [13].

In this paper we focus on solutions to the stationary Boltzmann equation in the slab under diffuse reflection boundary conditions. Stationary solutions are of interest as candidates for the time asymptotics of evolutionary problems (cf [10], [5]). They also appear naturally in the resolution of boundary layer problems, when studying hydrodynamical limits of time-dependent solutions. However, stationary solutions cannot be obtained directly by the techniques so far used in the time-dependent case, since for the latter natural bounds on mass, energy and entropy provide an initial mathematical framework, whereas in the stationary case only bounds on *flows* of mass,energy, and entropy through the boundary are easily available. Instead our technique is based on a systematic use of suitable parts of the entropy dissipation term with its natural bounds. The range of applicability of this idea for kinetic equations goes well beyond the present problem.

A number of results are known concerning the cases of the non-linear stationary Boltzmann equation close to equilibrium, and solutions of the corresponding linearized equation. There, more general techniques - such as contraction mapping based ones - can be utilized. So e.g. in an $I\!R^n$ setting, the solvability of boundary value problems for the Boltzmann equation in situations close to equilibrium is studied in [18], [19], [21], [33]. Stationary problems in small domains for the non-linear Boltzmann equation are studied in [28], [22]. The unique solvability of internal stationary problems for the Boltzmann equation at large Knudsen numbers is established in [26]. Existence and uniqueness of stationary solutions for the linearized Boltzmann equation in a bounded domain are proven in [25], and for the linear Boltzmann equation uniqueness in [29], [31], and existence in [12] and others. A classification of the well-posedness of some boundary value problems for the linearized Boltzmann equation is made in [16]. For discrete velocity models, in particular the Broadwell model, there are a number of stationary results in two dimensions, among them [8], [9], [14],[15].

Moreover, existence results far from equilibrium have been obtained for the stationary nonlinear Povzner equation in a bounded region in $I\!R^n$ (see [6]). The Povzner collision operator ([30]) is a modified Boltzmann operator with a 'smearing' process for the pair collisions, whereas in the derivation of the Boltzmann collision operator, each separate collision between two molecules occurs at one point in space.

In the slab case mathematical results on boundary value problems with large indata for the BGK equation are presented in [32], and for the Boltzmann equation in a measure setting in [1], [11] and in an L^1 setting in [4] for cases of pseudo-maxwellian and soft forces. In the paper [4] a criterium is derived for obtaining weak L^1 compactness from the boundedness of the entropy dissipation term. It allows an existence proof for a weak L^1 solution to the Boltzmann equation in the slab when the collision kernel is truncated for small velocities. In the present paper we use the entropy dissipation term

also to get rid of such truncations, and prove an existence result for the genuine stationary Boltzmann equation with pseudo-maxwellian and hard forces in the slab .
Let us conclude this introduction by detailing our results and methods of proofs. First recall the exponential, mild and weak solution concepts in the stationary context.

Definition 1.1 *f is an exponential solution to the stationary Boltzmann problem (1.1-2), if $f \in L^1([0,L] \times I\!R^3)$, $\nu \in L^1_{loc}([0,L] \times I\!R^3)$, and if for almost all (x,v) in $[0,L] \times I\!R^3$,*

$$f(x,v) = M_0(v)\Big(\int_{\xi'<0} \mid \xi' \mid f(0,v')dv'\Big)e^{-\int_{-\frac{x}{\xi}}^{0} \nu(x+\tau\xi,v)d\tau}$$
$$+\int_{-\frac{x}{\xi}}^{0} e^{-\int_s^0 \nu(x+\tau\xi,v)d\tau} Q^+(f,f)(x+s\xi,v)ds, \quad \xi>0,$$
$$f(x,v) = M_L(v)\Big(\int_{\xi'>0} \xi' f(L,v')dv'\Big)e^{-\int_{\frac{L-x}{\xi}}^{0} \nu(x+\tau\xi,v)d\tau}$$
$$+\int_{\frac{L-x}{\xi}}^{0} e^{-\int_s^0 \nu(x+\tau\xi,v)d\tau} Q^+(f,f)(x+s\xi,v)ds, \quad \xi<0.$$

Here ν is the collision frequency defined by

$$\nu(x,v) = \int_{I\!R^3}\int_{S^2} B(v-v_*,\omega)f(x,v_*)d\omega dv_*.$$

Definition 1.2 *f is a mild solution to the stationary Boltzmann problem (1.1-2), if $f \in L^1([0,L] \times I\!R^3)$, and for almost all (x,v) in $[0,L] \times I\!R^3$,*

$$f(x,v) = M_0(v)\int_{\xi'<0} \mid \xi' \mid f(0,v')dv' + \frac{1}{\xi}\int_0^x Q(f,f)(z,v)dz, \quad \xi>0,$$
$$f(x,v) = M_L(v)\int_{\xi'>0} \xi' f(L,v')dv' - \frac{1}{\xi}\int_x^L Q(f,f)(z,v)dz, \quad \xi<0.$$

Here the integrals for Q^+ and Q^- are assumed to exist separately.

Definition 1.3 *f is a weak solution to the stationary Boltzmann problem (1.1-2), if $f \in L^1([0,L] \times I\!R^3)$, $Q^+(f,f)$ and $Q^-(f,f) \in L^1_{loc}([0,L] \times \{v \in I\!R^3; \mid \xi \mid > 0\})$, and for every test function $\varphi \in C^1_c([0,L] \times I\!R^3)$, such that φ vanishes in a neighbourhood of $\xi = 0$, and on $\{(0,v); \xi<0\} \cup \{(L,v); \xi>0\}$,*

$$\int_0^L\int_{I\!R^3}\Big(\xi f\frac{\partial\varphi}{\partial x} + Q(f,f)\varphi\Big)(x,v)dxdv = -\int_{\xi<0} \mid \xi \mid M_L(v)\varphi(L,v)dv\int_{\xi>0} \xi f(L,v)dv$$
$$-\int_{\xi>0} \xi M_0(v)\varphi(0,v)dv\int_{\xi<0} \mid \xi \mid f(0,v)dv.$$

Remark. This weak form is stronger than the mild and exponential ones. In the paper [6] the main equation, quadratic and of Povzner type in $I\!R^n$, is shown to be equivalent to a similar one but homogeneous of degree one via a transform of the *space* variables and involving the *mass*. An analogous transform involving the mass density instead of the mass was first used in radiative transfer and boundary layer studies, and later in the mid 1950 ies introduced by M. Krook [23] into gas kinetics for the BGK equation. It was recently applied by C. Cercignani [11] for measure solutions to the Boltzmann equation for pseudo-maxwellian forces in a slab. Under this transform the Boltzmann equation in the slab transforms as follows. Set

$$y(x) := \int_a^x \int_{I\!R^3} \int_{S^2} f(z, v_*) d\omega dv_* dz.$$

Let $\tilde{\Omega} = [r_1, r_2]$ be the image of $[0, L]$ under this transformation. Then, with $F(y, v) = f(x, v)$, the equation (1.1) becomes

$$\xi \frac{\partial}{\partial y} F(y,v) = \frac{\int_{I\!R^3 \times S^2} BF(y,v')F(y,v'_*) d\omega dv_*}{\int_{I\!R^3 \times S^2} F(y,v_*) d\omega dv_*}$$
$$-F(y,v) \frac{\int_{I\!R^3 \times S^2} BF(y,v_*) d\omega dv_*}{\int_{I\!R^3 \times S^2} F(y,v_*) d\omega dv_*}, \quad y \in \tilde{\Omega}, \quad v \in I\!R^3, \tag{1.3}$$

and (1.2) becomes

$$F(r_1, v) = M_0(v) \int_{\xi' < 0} \mid \xi' \mid F(r_1, v') dv', \quad \xi > 0,$$
$$F(r_2, v) = M_L(v) \int_{\xi' > 0} \xi' F(r_2, v') dv', \quad \xi < 0. \tag{1.4}$$

Any nonzero solution of (1.1-2) generates via the transform a solution to a problem of type (1.3-4). In that sense the problem (1.3-4) is a generalization of the problem (1.1-2). Reciprocally, let a slab of length L and a positive constant M be given. Choose (r_1, r_2) such that $r_2 - r_1 = M \mid S^2 \mid$. If the problem (1.3-4) has a solution F satisfying

$$\int_{r_1}^{r_2} \frac{dy}{\int F(y,v)dv} < \infty, \tag{1.5}$$

then define the function $y(x)$ by

$$\int_{r_1}^{y(x)} \frac{dz}{\int F(z,v)dv} = k \mid S^2 \mid x, \quad x \in [0, L], \tag{1.6}$$

where

$$k = \frac{1}{L \mid S^2 \mid} \int_{r_1}^{r_2} \frac{dz}{\int F(z,v)dv},$$

and define the function f by

$$f(x,v) := kF(y(x),v), \quad x \in [0,L], \quad v \in I\!R^3.$$

Then y maps $[0,L]$ into $[r_1, r_2]$, f is a solution to (1.1-2), and the total mass of f is

$$\int_0^L \int_{I\!R^3} f(x,v)dxdv = \frac{r_2 - r_1}{\mid S^2 \mid} = M.$$

Remark. In contrast to the Povzner equation, it is not obvious in the Boltzmann equation case how to extend the transform in a useful way from one to several space dimensions. On the other hand, the existence problem for (1.1-2) - in this paper solved with the above transform - can alternatively be solved via a direct approach without the transform, instead using a certain coupling between mass and boundary flow (see [7]).
The main result of this paper is the following.

Theorem 1.1 *Given a slab of length L, $\beta \in [0,2[$ in the collision kernel, and a positive constant M, there is a weak solution to the stationary problem (1.1-2) with $\int K_\beta(v)f(x,v)dxdv = M$ for $K_\beta(v) = (1+ \mid v \mid)^\beta$.*

Remark. S. Mischler observed in [27] that in the context of boundary conditions for the Boltzmann equation in n dimensions, the biting lemma of Brook and Chacon can be used to obtain (1.2) instead of earlier weaker alternatives (cf [20], [2], [3]). In our one dimensional case the biting lemma is not needed. Instead (1.2) follows directly via weak compactness from a control of entropy outflow.
The theorem holds with an analogous proof for velocities in $I\!R^n$, $n \geq 2$. It will be clear from the proofs that problems with given indata boundary conditions can also be treated by the methods of this paper (no singular boundary measure coming up there). The maxwellians in (1.2) can be replaced by other reentry profiles under suitable conditions on the functions replacing the maxwellians. A number of generalizations of B which take $v \in I\!R^n$, $n \geq 2$, and $-n < \beta < 2$, such as cases of $b(\theta) > 0$ a.e., or B not of the product form $\mid v - v_* \mid^\beta b(\theta)$, can also be analyzed straightforwardly by the same approach.

The second section of the paper is devoted to a crucial construction of approximated solutions to the transformed problem with a modified asymmetric collision operator. The proofs are carried out with the transformed slab for convenience equal to $[-1,1]$ throughout the paper. The asymmetry introduced in the collision operator allows monotonicity arguments which lead to uniqueness of the approximate solution. In the third section the symmetry of the collision operator is reintroduced. The weak compactness in $L^1([-1,1]\times I\!\!R^3)$ utilized for this step, is obtained by using the transformed representation to get pointwise bounds for the collision frequency, and by controlling the approximate solutions inside $[-1,1]\times I\!\!R^3$ by their values at the outgoing boundary. In the last two sections some remaining truncations in the collision operator are removed. A certain convergence in measure plays an important role. Such information is mainly extracted from the geometry of the collision process and uniform estimates for the entropy dissipation term. Throughout the paper, various constants are denoted by the letter c, sometimes with indices.

2 Approximate solutions to the transformed problem.

Let $r>0$, $\mu>0$, and $(j,m)\in I\!\!N^2$ with $\frac{1}{m}\ll r$ be given. The aim of this section is to construct via strong L^1 compactness and fixed point arguments, solutions $f^{r,\mu,j,m}$ to the following approximation of the transformed problem

$$\xi\frac{\partial f}{\partial x}=\frac{1}{\int f(x,v_*)dv_*d\omega}\int\chi^r(v,v_*,\omega)B_\mu^{j,m}(v,v_*,\omega)$$

$$\frac{f}{1+\frac{f}{j}}(x,v')\frac{f}{1+\frac{f}{j}}(x,v'_*)dv_*d\omega-f\frac{\int\chi^rB_\mu^{j,m}\frac{f}{1+\frac{f}{j}}(x,v_*)dv_*d\omega}{\int fdv_*d\omega},$$

$$(x,v)\in[-1,1]\times I\!\!R^3,$$

$$f(-1,v)=M_0(v)\frac{\int_{\xi'<0}\mid\xi'\mid f(-1,v')dv'}{\int_{\xi'<0}\mid\xi'\mid f(-1,v')dv'+\int_{\xi'>0}\xi'f(1,v')dv'},\quad\xi>0,$$

$$f(1,v)=M_L(v)\frac{\int_{\xi'>0}\xi'f(1,v')dv'}{\int_{\xi'<0}\mid\xi'\mid f(-1,v')dv'+\int_{\xi'>0}\xi'f(1,v')dv'},\quad\xi<0.\quad(2.1)$$

This problem is normalized in order that the total inflow through the boundary be one. Here, χ^r is a C_0^∞ function with range [0,1] invariant under the collision transformation J, where

$$J(v,\omega,v_*) = (v', -\omega, v'_*),$$

with χ^r also invariant under the exchange of v and v_*, and such that

$$\chi^r(v,v_*,\omega) = 1 \;\; if \;\; min(|\,\xi\,|, |\,\xi_*\,|, |\,\xi'\,|, |\,\xi'_*\,|) \geq r,$$
$$\chi^r(v,v_*,\omega) = 0 \;\; if \;\; max(|\,\xi\,|, |\,\xi_*\,|, |\,\xi'\,|, |\,\xi'_*\,|) \leq r - \frac{1}{m}.$$

The modified collision kernel $B^{j,m}_\mu$ is a positive C^∞ function approximating $max(\frac{1}{\mu}, min(B,\mu))$, when

$$\begin{aligned} v^2 + v_*^2 < \frac{\sqrt{j}}{2}, \;\; and \;\; \left|\, \frac{v-v_*}{|\,v-v_*\,|}\cdot\omega \,\right| > \frac{1}{m}, \\ and \;\; \left|\, \frac{v-v_*}{|\,v-v_*\,|}\cdot\omega \,\right| < 1 - \frac{1}{m}, \;\; and \;\; |v-v_*| > \frac{1}{m} \\ B^{j,m}_\mu(v,v_*,\omega) = 0, \;\; if \;\; v^2+v_*^2 > \sqrt{j}, \\ or \;\; \left|\, \frac{v-v_*}{|\,v-v_*\,|}\cdot\omega \,\right| < \frac{1}{2m}, \;\; or \;\; \left|\, \frac{v-v_*}{|\,v-v_*\,|}\cdot\omega \,\right| > 1 - \frac{1}{2m}, \\ or \;\; |v-v_*| < \frac{1}{2m}. \end{aligned} \tag{2.2}$$

The truncation χ^r and the boundedness of the collision kernel by μ will be removed only at the very end of the proof in Section 5, and the truncation with m will be removed together with j in Section 3. So we shall in this section skip the indices r in χ^r, μ and m in $B^{j,m}_\mu = B^j$, and write $f^{r,j,\mu,m} = f^j$. Let mollifiers in the x-variable be defined by $\varphi_k(x) := k\varphi(kx)$, where

$$\varphi \in C_0^\infty(I\!R), \quad support\; \varphi \subset (-1,1),\; \varphi \geq 0,\; \int_{-1}^{1} \varphi(x)dx = 1.$$

Let $K \times [0,1]$ be the closed and convex subset of $L^1((-1,1)\times I\!R^3)\times[0,1]$, where

$$K := \{f \in L^1((-1,1)\times I\!R^3);\;\; 0 \leq f(x,v) \leq e^j,$$
$$\int f(x,v)dv \geq c_0,\;\; a.e.\; x \in (-1,1)\},$$

where $c_0 := \frac{1}{2}e^{-2(\mu+1)} min(\int_{\xi>1} M_0(v)dv, \int_{\xi<-1} M_L(v)dv)$. For $\alpha > 0$ and $k > 0$ given and j sufficiently large, let T be the map defined on $K \times [0,1]$

by $T(f,\theta)=(F,\tilde{\theta})$, where $(F,\tilde{\theta})$ is the solution to

$$\alpha F+\xi\frac{\partial F}{\partial x}=\frac{1}{\int f*\varphi_k(x,v_*)dv_*d\omega}\int\chi B^j\frac{F}{1+\frac{F}{j}}(x,v')\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v'_*)dv_*d\omega$$
$$-\frac{F}{\int f*\varphi_k(x,v_*)dv_*d\omega}\int\chi B^j\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times I\!R^3,$$
$$F(-1,v)=\theta M_0(v),\ \xi>0,\quad F(1,v)=(1-\theta)M_L(v),\ \xi<0,$$
$$\tilde{\theta}=\int_{\xi<0}\mid\xi\mid F(-1,v)dv(\int_{\xi<0}\mid\xi\mid F(-1,v)dv+\int_{\xi>0}\xi F(1,v)dv)^{-1}.\ (2.3)$$

Denote by

$$\nu_j(x,v):=\frac{\int\chi B^j\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v_*)dv_*d\omega}{\int f*\varphi_k(x,v_*)dv_*d\omega}.$$

Lemma 2.1 *There is a positive lower bound c_0 for $\int F(x,v)dv$, with c_0 independent of $x\in(-1,1)$, $0<\alpha\leq 1$, and of $(f,\theta)\in K\times[0,1]$.*

Proof of Lemma 2.1. It follows from the exponential form of (2.3) and the boundedness from above of ν_j by μ, that

$$F(x,v)\geq\theta M_0(v)e^{-\frac{(1+x)(\mu+1)}{\xi}},\ \xi>0,\quad F(x,v)\geq(1-\theta)M_L(v)e^{\frac{(1-x)(\mu+1)}{\xi}},\ \xi<0.$$

Then, uniformly in $x\in(-1,1)$,

$$\int_{I\!R^3}F(x,v)dv\geq e^{-2(\mu+1)}[\theta\int_{\xi>1}M_0(v)dv+(1-\theta)\int_{\xi<-1}M_L(v)dv]$$
$$\geq c_0.\quad\Box$$

For $(f,\theta)\in K\times[0,1]$, one solution F of (2.3) is obtained as the strong L^1 limit of the nonnegative monotone sequence (F^l), bounded from above, defined by $F^0=0$ and

$$\alpha F^{l+1}+\xi\frac{\partial F^{l+1}}{\partial x}=\frac{1}{\int f*\varphi_k(x,v_*)dv_*d\omega}\int\chi B^j\frac{F^l}{1+\frac{F^l}{j}}(x,v')\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v'_*)dv_*d\omega$$
$$-\frac{F^{l+1}}{\int f*\varphi_k(x,v_*)dv_*d\omega}\int\chi B^j\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times I\!R^3,$$
$$F^{l+1}(-1,v)=\theta M_0(v),\ \xi>0,\quad F^{l+1}(1,v)=(1-\theta)M_L(v),\ \xi<0.$$

There is uniqueness of the solution to (2.3). Otherwise, if there were another solution G, then multiplying the equation for the difference $F-G$ by

$sign(F-G)$ and integrating with respect to (x,v) one obtains after some computations that

$$\alpha \int_{(-1,1)\times I\!R^3} \mid F-G \mid (x,v)dxdv \leq 0.$$

Consequently, $F=G$. Moreover, by Lemma 2.1, T maps $K\times[0,1]$ into itself.
Let us prove that T is continuous for the strong topology of $L^1\times[0,1]$. Let (f_l,θ_l) converge to (f,θ) and write $(F_l,\tilde{\theta}_l):=T(f_l,\theta_l)$ and $(F,\tilde{\theta}):=T(f,\theta)$. By the uniqueness of the solution of (2.3), it is enough to prove that there is a subsequence of $(F_l,\tilde{\theta}_l)$ converging to $(F,\tilde{\theta})$. By the strong L^1 convergence of (f_l) to f and the condition $\int f_l(x,v)dv \geq c_0$, the bounded sequence $(\frac{1}{\int f_l*\varphi_k(x,w)dwd\omega})$ converges in L^1 to $\frac{1}{\int f*\varphi_k(x,w)dwd\omega}$. For a suitable subsequence let $G_l:=\sup_{m\geq l} f_m$, $g_l:=\inf_{m\geq l} f_m$, $\beta_l:=\sup_{m\geq l}\theta_m$, $\gamma_l:=\inf_{m\geq l}\theta_l$, with (G_l) decreasingly converging to f, (g_l) increasingly converging to f, (β_l) decreasingly converging to θ and (γ_l) increasingly converging to θ. Let (S_l) and(s_l) be the sequences of solutions to

$$\begin{aligned}
\alpha S_l+\xi\frac{\partial S_l}{\partial x} &= \frac{1}{\int g_l*\varphi_k(x,v_*)dv_*d\omega}\int \chi B^j\frac{S_l}{1+\frac{S_l}{j}}(x,v')\frac{G_l*\varphi_k}{1+\frac{G_l*\varphi_k}{j}}(x,v'_*)dv_*d\omega \\
&\quad -\frac{S_l}{\int G_l*\varphi_k(x,v_*)dv_*d\omega}\int \chi B^j\frac{g_l*\varphi_k}{1+\frac{g_l*\varphi_k}{j}}(x,v_*)dv_*d\omega, \\
\alpha s_l+\xi\frac{\partial s_l}{\partial x} &= \frac{1}{\int G_l*\varphi_k(x,v_*)dv_*d\omega}\int \chi B^j\frac{s_l}{1+\frac{s_l}{j}}(x,v')\frac{g_l*\varphi_k}{1+\frac{g_l*\varphi_k}{j}}(x,v'_*)dv_*d\omega \\
&\quad -\frac{s_l}{\int g_l*\varphi_k(x,v_*)dv_*d\omega}\int \chi B^j\frac{G_l*\varphi_k}{1+\frac{G_l*\varphi_k}{j}}(x,v_*)dv_*d\omega, \\
&\qquad (x,v)\in(-1,1)\times I\!R^3, \\
S_l(-1,v)=\beta_l M_0(v),\ \xi>0, &\quad S_l(1,v)=(1-\gamma_l)M_L(v),\ \xi<0, \\
s_l(-1,v)=\gamma_l M_0(v),\ \xi>0, &\quad s_l(1,v)=(1-\beta_l)M_L(v),\ \xi<0.
\end{aligned}$$

(S_l) is a non-increasing sequence, and (s_l) is a non-decreasing sequence. Moreover,

$$s_l \leq F_l \leq S_l. \tag{2.4}$$

But (S_l) decreasingly converges in L^1 to some S and(s_l) increasingly converges in L^1 to some s which are solutions to

$$\alpha S+\xi\frac{\partial S}{\partial x}=\frac{1}{\int f(x,v_*)dv_*d\omega}\int \chi B^j\frac{S}{1+\frac{S}{j}}(x,v')\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v'_*)dv_*d\omega$$

$$-\frac{S}{\int f(x,v_*)dv_*d\omega}\int \chi B^j\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times I\!R^3,$$

$$S(-1,v)=\theta M_0(v),\ \xi>0,\quad S(1,v)=(1-\theta)M_L(v),\ \xi<0,$$

and

$$\alpha s+\xi\frac{\partial s}{\partial x}=\frac{1}{\int f(x,v_*)dv_*d\omega}\int \chi B^j\frac{s}{1+\frac{s}{j}}(x,v')\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v'_*)dv_*d\omega$$

$$-\frac{s}{\int f(x,v_*)dv_*d\omega}\int \chi B^j\frac{f*\varphi_k}{1+\frac{f*\varphi_k}{j}}(x,v_*)dv_*d\omega,\quad (x,v)\in(-1,1)\times I\!R^3,$$

$$s(-1,v)=\theta M_0(v),\ \xi>0,\quad s(1,v)=(1-\theta)M_L(v),\ \xi<0.$$

By the uniqueness of the solution to such systems, $S=s=F$. It follows from (2.4) that (F_l) converges to F.
Let us prove that T is compact for the strong topology of L^1. Let (f_l,θ_l) be a bounded sequence in $L^1\times[0,1]$ and $(F_l,\tilde{\theta}_l)=T(f_l,\theta_l)$. F_l can be written in exponential form $F_l=G_l+H_l$, where for $\xi>0$,

$$G_l(x,v)=\theta_l M_0(v)$$

$$e^{-\int_{-\frac{1+x}{\xi}}^0\frac{1}{\int f_l*\varphi_k(x+\tau\xi,v_*)dv_*d\omega}\int\chi B^j\frac{f_l*\varphi_k}{1+\frac{f_l*\varphi_k}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau-\alpha\frac{1+x}{\xi}},$$

and

$$H_l(x,v)=\int_{-\frac{1+x}{\xi}}^0\frac{1}{\int f_l*\varphi_k(x+s\xi,v_*)dv_*d\omega}$$

$$e^{\alpha s-\int_s^0\frac{1}{\int f_l*\varphi_k(x+\tau\xi,v_*)dv_*d\omega}\int\chi B^j\frac{f_l*\varphi_k}{1+\frac{f_l*\varphi_k}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau}$$

$$\int\chi B^j\frac{F_l}{1+\frac{F_l}{j}}(x+s\xi,v')\frac{f_l*\varphi_k}{1+\frac{f_l*\varphi_k}{j}}(x+s\xi,v'_*)dv_*d\omega ds,$$

and similarly for $\xi<0$. The sequence (G_l) is strongly compact because of the convolution of f_l with φ_k. Namely, we can pick a subsequence so that $\int f_l*\varphi_k dv_*$ as well as $\int\chi B_j\frac{f_l*\varphi_k}{1+\frac{f_l*\varphi_k}{j}}dv_*d\omega$ are strongly convergent. For the same reason, for proving the strong compactness of (H_l), it is enough to prove it for

$$K_l(x,v) := \int \chi B^j \frac{F_l}{1+\frac{F_l}{j}}(x,v') \frac{f_l * \varphi_k}{1+\frac{f_l*\varphi_k}{j}}(x,v'_*)dv_* d\omega.$$

The argument is similar to the corresponding one in the limit when k tends to infinity on next page, where details are given.
So, T is a continuous and compact map from the closed and convex subset $K \times [0,1]$ of $L^1 \times [0,1]$ into itself. It follows from the Schauder theorem that T has a fixed point $F^{k,j,\alpha}$ solution to

$$\alpha F^{k,j,\alpha} + \xi \frac{\partial F^{k,j,\alpha}}{\partial x} = \frac{1}{\int F^{k,j,\alpha} * \varphi_k dv_* d\omega} \int \chi B^j \frac{F^{k,j,\alpha}}{1+\frac{F^{k,j,\alpha}}{j}}(x,v') \frac{F^{k,j,\alpha} * \varphi_k}{1+\frac{F^{k,j,\alpha}*\varphi_k}{j}}(x,v'_*)dv_* d\omega$$

$$- \frac{F^{k,j,\alpha}}{\int F^{k,j,\alpha} * \varphi_k dv_* d\omega} \int \chi B^j \frac{F^{k,j,\alpha} * \varphi_k}{1+\frac{F^{k,j,\alpha}*\varphi_k}{j}}(x,v_*)dv_* d\omega, \; (x,v) \in (-1,1) \times I\!\!R^3,$$

$$F^{k,j,\alpha}(-1,v) = M_0(v) \frac{\int_{\xi'<0} \mid \xi' \mid F^{k,j,\alpha}(-1,v')dv'}{\int_{\xi'<0} \mid \xi' \mid F^{k,j,\alpha}(-1,v')dv' + \int_{\xi'>0} \xi' F^{k,j,\alpha}(1,v')dv'}, \; \xi > 0,$$

$$F^{k,j,\alpha}(1,v) = M_L(v) \frac{\int_{\xi'>0} \xi' F^{k,j,\alpha}(1,v')dv'}{\int_{\xi'<0} \mid \xi' \mid F^{k,j,\alpha}(-1,v')dv' + \int_{\xi'>0} \xi' F^{k,j,\alpha}(1,v')dv'}, \; \xi < 0. \tag{2.5}$$

Keeping α and j fixed, let us write $F^{k,j,\alpha} = F^k$ and study the passage to the limit when k tends to infinity. The sequence of mappings

$$(x,v_*) \to \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x,v_*), \quad k \in I\!\!N,$$

is uniformly bounded by j, hence is weakly compact in L^1. Moreover,

$$\xi \frac{\partial}{\partial x}\left(\frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}\right) = \frac{1}{(1+\frac{F^k*\varphi_k}{j})^2}$$

$$(-\alpha F^k * \varphi_k + \int \frac{1}{\int F^k * \varphi_k(x-y,v_*)dv_* d\omega} \int \chi B^j \frac{F^k}{1+\frac{F^k}{j}}(x-y,v')$$

$$\frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x-y,v'_*)dv_* d\omega \varphi_k(y)dy$$

$$- \int \frac{F^k(x-y,v)}{\int F^k * \varphi_k(x-y,v_*)dv_* d\omega} \int \chi B^j \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x-y,v_*)dv_* d\omega \varphi_k(y)dy).$$

Here the right-hand side is uniformly bounded with respect to x, v, k, hence weakly compact in L^1. Using the first equation in (2.5), and that $B^j \in C^1_c$, it follows that

$$\int \chi B^j(v, v_*, \omega) \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x, v_*) dv_* d\omega \tag{2.6}$$

is strongly compact in $L^1((-1,1)\times I\!\!R^3_v)$. Analogously, $(\int F^k * \varphi_k(x,w) dw d\omega)$ is strongly compact in $L^1((-1,1))$. Finally let us recall the argument from [24] that

$$Q^+_k(x,v) := \int \chi B^j(v, v_*, \omega) \frac{F^k}{1+\frac{F^k}{j}}(x, v') \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x, v'_*) dv_* d\omega$$

is strongly compact in L^1. For $\delta > 0$, let ρ_δ be a mollifier in the v-variable. There is a function $R \in L^1$ such that for any $\delta > 0$, a subsequence of $(Q^+_k * \rho_\delta)$ strongly converges in L^1 to $R * \rho_\delta$. Indeed,

$$Q^+_k * \rho_\delta(x,v) = \int Q^+_k(x,w)\rho_\delta(v-w) dw$$

$$= \int \chi B^j(w, w_*, \omega) \frac{F^k}{1+\frac{F^k}{j}}(x, w') \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x, w'_*) \rho_\delta(v-w) dw dw_* d\omega,$$

so that, by the change of variables $(w, w_*) \to (w', w'_*)$,

$$Q^+_k * \rho_\delta(x,v) = \int (\int \chi B^j(w, w_*, \omega) \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x, w_*) \rho_\delta(v-w') dw_* d\omega) \frac{F^k}{1+\frac{F^k}{j}}(x,w) dw.$$

As above for (2.6), up to subsequences,

$$\int \chi B^j(w, w_*, \omega) \frac{F^k * \varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x, w_*) \rho_\delta(v-w') dw_* d\omega$$

strongly converges in L^1 to some

$$\int \chi B^j(w, w_*, \omega) \tilde{F}(x, w_*) \rho_\delta(v-w') dw_* d\omega,$$

and

$$\int (\int \chi B^j(w, w_*, \omega) \tilde{F}(x, w_*) \rho_\delta(v-w') dw_* d\omega) \frac{F^k}{1+\frac{F^k}{j}}(x,w) dw$$

strongly converges in L^1 to some function

$$\int \Big(\int \chi B^j(w, w_*, \omega) \tilde{F}(x, w_*) \rho_\delta(v - w') dw_* d\omega \Big) G(x, w) dw.$$

Hence $(Q_k^+ * \rho_\delta)$ converges in L^1 to $R * \rho_\delta$, for any $\delta > 0$, where

$$R(x, v) := \int \chi B^j(w, w_*, \omega) \tilde{F}(x, w'_*) G(x, w') dw_* d\omega.$$

Let us prove that $Q_k^+ * \rho_\delta - Q_k^+$ tends to zero in L^1 when δ tends to zero, uniformly with respect to k. If $\hat{g}$ denotes the Fourier transform of a function $g(x, v)$ with respect to the variable v, then for any $x \in (-1, 1)$,

$$\begin{aligned}
\| Q_k^+ * \rho_\delta - Q_k^+ \|^2_{L^2(I\!R^3_v)} = & \int | \hat{Q}_k^+(x, \xi) |^2 \, (1 - \hat{\rho}_\delta(\xi))^2 d\xi \\
= & \int | \hat{Q}_k^+(x, \xi) |^2 | \, 1 - \hat{\rho}(\delta\xi) \, |^2 \, d\xi \\
\leq & \int_{|\xi| \leq \lambda} | \hat{Q}_k^+(x, \xi) |^2 | \, 1 - \hat{\rho}(\delta\xi) \, |^2 \, d\xi \\
& + \frac{1}{\lambda^2} \int_{|\xi| \geq \lambda} | \hat{Q}_k^+(x, \xi) |^2 \, \xi^2 d\xi \\
\leq & \int_{|\xi| \leq \lambda} | \hat{Q}_k^+(x, \xi) |^2 | \, 1 - \hat{\rho}(\delta\xi) \, |^2 \, d\xi \\
& + \frac{1}{\lambda^2} \| D_v Q_k^+ \|^2_{L^2(I\!R^3_v)} \, .
\end{aligned}$$

But $D_v Q_k^+$ satisfies

$$\| D_v Q_k^+ \|_{L^2(I\!R^3)} \leq c \, \| \frac{F_k}{1 + \frac{F_k}{j}} \|_{L^1} \| \frac{F_k * \varphi_k}{1 + \frac{F_k * \varphi_k}{j}} \|_{L^2} \leq c. \tag{2.7}$$

(cf [24], [34]), so that $\frac{1}{\lambda^2} \, \| \, D_v Q_k^+ \, \|^2_{L^2(I\!R^3_v)}$ tends to zero when λ tends to infinity. Finally, for any λ,

$$\lim_{\delta \to 0} \int_{|\xi| \leq \lambda} | \hat{Q}_k^+(x, \xi) |^2 | \, 1 - \hat{\rho}(\delta\xi) \, |^2 \, d\xi = 0,$$

since $\hat{\rho}(0) = 1$, $\hat{\rho}$ is uniformly continuous on $| \, \xi \, | \leq 1$, and $\int | \, \hat{Q}_k^+(x, \xi) \, |^2 \, d\xi < c$. This ends the proof of the strong L^1-compactness of (Q_k^+).

Writing (2.5) in exponential form implies that, for $\xi > 0$,

$$F^k(x,v) = \theta_k M_0(v) e^{-\alpha\frac{1+x}{\xi} - \int_{-\frac{1+x}{\xi}}^{0} \frac{1}{\int F^k*\varphi_k(x+\tau\xi,v_*)dv_*d\omega}\int \chi B^j \frac{F^k*\varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau}$$

$$+\int_{-\frac{1+x}{\xi}}^{0} e^{\alpha s - \int_s^0 \frac{1}{\int F^k*\varphi_k(x+\tau\xi,v_*)dv_*d\omega}\int \chi B^j \frac{F^k*\varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x+\tau\xi,v_*)dv_*d\omega d\tau} Q_k^+(x+s\xi,v)ds.$$

Here $(\theta_k) := (\frac{\int_{\xi<0}|\xi|F^k(-1,v)dv}{\int_{\xi<0}|\xi|F^k(-1,v)dv+\int_{\xi>0}\xi F^k(1,v)dv})$ is a bounded sequence of $[0,1]$, so converges up to a subsequence. Then, from the strong compactness of

$$(\int F^k*\varphi_k(x,v_*)dv_*d\omega), \quad (\int B_j(v,v_*,\omega)\frac{F^k*\varphi_k}{1+\frac{F^k*\varphi_k}{j}}(x,v_*)dv_*d\omega),$$

and (Q_k^+), as well as the boundedness from above of $\frac{1}{\int F^k*\varphi_k(x,v_*)dv_*}$, it follows that (F^k) strongly converges in L^1 to some F. Passing to the limit when k tends to infinity in (2.5) implies that $F := F^{\alpha,j}$ is a solution to

$$\begin{aligned}
\alpha F^{\alpha,j} + \xi\frac{\partial F^{\alpha,j}}{\partial x} &= \frac{1}{\int F^{\alpha,j}(x,v_*)dv_*d\omega}\int \chi B^j \frac{F^{\alpha,j}}{1+\frac{F^{\alpha,j}}{j}}(x,v')\frac{F^{\alpha,j}}{1+\frac{F^{\alpha,j}}{j}}(x,v'_*)dv_*d\omega \\
&\quad - \frac{F^{\alpha,j}}{\int F^{\alpha,j}(x,v_*)dv_*d\omega}\int \chi B^j \frac{F^{\alpha,j}}{1+\frac{F^{\alpha,j}}{j}}(x,v_*)dv_*d\omega, \quad (x,v)\in(-1,1)\times I\!R^3, \\
F^{\alpha,j}(-1,v) &= M_0(v)\frac{\int_{\xi'<0}\mid\xi'\mid F^{\alpha,j}(-1,v')dv'}{\int_{\xi'<0}\mid\xi'\mid F^{\alpha,j}(-1,v')dv'+\int_{\xi'>0}\xi' F^{\alpha,j}(1,v')dv'}, \quad \xi>0, \\
F^{\alpha,j}(1,v) &= M_L(v)\frac{\int_{\xi'>0}\xi' F^{\alpha,j}(1,v')dv'}{\int_{\xi'<0}\mid\xi'\mid F^{\alpha,j}(-1,v')dv'+\int_{\xi'>0}\xi' F^{\alpha,j}(1,v')dv'}, \quad \xi<0.
\end{aligned} \tag{2.8}$$

The passage to the limit in (2.8) when α tends to zero is similar, and implies that the limit F of $F^{\alpha,j}$ is a solution to (2.1), which was the aim of the present section.
Remark. The construction so far also holds for $\Omega\subset I\!R^n$.
The solution F of (2.1) depends on the parameters j,r,μ, $F=F^{j,r,\mu}$. The following lemma gives an estimate of its boundary fluxes independent of j and r.

Lemma 2.2 *Let $F = F^{j,r,\mu}$ denote a solution to the approximate problem (2.1). Set*

$$\rho(1) := \int_{\xi>0} \xi F(1,v)dv, \quad \rho(-1) := \int_{\xi<0} \mid \xi \mid F(-1,v)dv,$$

$$\sigma(1) := \frac{\rho(1)}{\rho(1)+\rho(-1)}, \quad \sigma(-1) := \frac{\rho(-1)}{\rho(1)+\rho(-1)}.$$

Then

$$Min(\sigma(1), \sigma(-1)) \geq c_1 > 0,$$

with c_1 only depending on M_i, $i \in \{0, L\}$ but not on j, r.

Proof of Lemma 2.2. $\sigma(1) + \sigma(-1) = 1$, so one of them is bigger than or equal to $\frac{1}{2}$, say $\sigma(-1) \geq \frac{1}{2}$. From the exponential form

$$\rho(1) = \int_{\xi>0} \xi F(1,v)dv \geq \sigma(-1) \int_{\xi>0} \xi e^{-\frac{2\mu}{\xi}} M_0(v)dv$$
$$\geq \frac{1}{2} e^{-2\mu} \int_{\xi>1} \xi M_0(v)dv = c_1' > 0.$$

Moreover, integrating (2.1) on $(-1,1) \times I\!\!R^3$ implies, by Green's formula

$$\rho(1) + \rho(-1) \leq 1.$$

Hence $\sigma(1) \geq c_1'$. Then

$$\rho(-1) = \int_{\xi<0} \mid \xi \mid F(-1,v)dv \geq \sigma(1) \int_{\xi<0} \mid \xi \mid e^{\frac{2\mu}{\xi}} M_L(v)dv$$
$$\geq c_1' e^{-2\mu} \int_{\xi<-1} \mid \xi \mid M_L(v)dv = c_1.$$

□

3 Reintroduction of the gain-loss symmetry.

In this section the asymmetry between the gain and the loss terms will be removed by taking the limit $j \to \infty$. The smoothness of $\chi^r B_\mu^{j,m}$ was needed in Section 2 for the Radon transform argument in the proof of (2.7). That smoothness will now be removed from $B_\mu^{j,m}$ and χ^r by keeping r and μ fixed, but letting $\chi^r B_\mu^{j,m}$ converge to $max(\frac{1}{\mu}, min(B,\mu))$ times the characteristic function for the set where χ^r equals one, when $m = j \to \infty$. We start with a $j(=m)$-independent estimate of the ξ-flux of F^j.

Lemma 3.1 *If F^j is a solution to (2.1), then*

$$\int_{|\xi|\geq r} \xi^2 F^j(x,v)dv \leq c_r, \quad a.a.\ x \in (-1,1), \quad j \in \mathbb{N}. \tag{3.1}$$

Proof of Lemma 3.1. Multiplying (2.1) by 1 and $\mid v \mid^2$ respectively, and integrating it over $(-1,1)\times \mathbb{R}^3$ implies that

$$\int_{\xi>0} \xi F^j(1,v)dv + \int_{\xi<0} \mid \xi \mid F^j(-1,v)dv \leq 1, \tag{3.2}$$

and

$$\begin{aligned} &\int_{\xi>0} \xi \mid v \mid^2 F^j(1,v)dv + \int_{\xi<0} \mid \xi \mid\mid v \mid^2 F^j(-1,v)dv \\ &\leq \int_{\xi<0} \mid \xi \mid\mid v \mid^2 F^j(1,v)dv + \int_{\xi>0} \xi \mid v \mid^2 F^j(-1,v)dv \\ &= \sigma^j(1)\int_{\xi<0} \mid \xi \mid\mid v \mid^2 M_L(v)dv + \sigma^j(-1)\int_{\xi>0} \xi \mid v \mid^2 M_0(v)dv < c. \end{aligned} \tag{3.3}$$

By the exponential representation of F^j, for $\mid \xi \mid\geq r$

$$F^j(x,v) \leq c_r F^j(1,v),\ \xi>0, \quad F^j(x,v)\leq c_r F^j(-1,v),\ \xi<0,$$

with c_r independent of j. But

$$F^j(\pm 1,v)\xi^2 \leq F^j(\pm 1,v) \mid \xi \mid (1+\mid v \mid^2),$$

so (3.2-3) imply (3.1).□

Lemma 3.2 *The sequence of solutions (F^j) to (2.1) is weakly precompact in $L^1((-1,1)\times \mathbb{R}^3)$.*

Proof of Lemma 3.2. Let us prove that $\int_{-1}^1\int F^j log F^j(x,v)dxdv$ is uniformly in j bounded from above. By the truncation χ^r,

$$\begin{aligned} &\int_{-1}^1\int_{|\xi|\leq r-\frac{1}{j}} F^j log F^j(x,v)dxdv = \\ &\int_{-1}^1\int_{0<\xi\leq r-\frac{1}{j}} M_0(v)\sigma^j(-1)log(M_0(v)\sigma^j(-1))dxdv \\ &+\int_{-1}^1\int_{-r+\frac{1}{j}<\xi\leq 0} M_L(v)\sigma^j(1)log(M_L(v)\sigma^j(1))dxdv \leq c, \end{aligned}$$

since $0 \leq \sigma^j(-1) \leq 1$ and $0 \leq \sigma^j(1) \leq 1$. Take $j \geq \frac{2}{r}$. By Lemma 3.1, $(\int_{|\xi|\geq r-\frac{1}{j}} F^j(x,v)dxdv)$ is uniformly bounded from above. Denote by

$$\nu^j(x,v) := \frac{\int \chi B^j \frac{F^j}{1+\frac{F^j}{j}}(x,v_*)dv_*d\omega}{\int F^j(x,v_*)dv_*d\omega}.$$

It follows from the exponential form of (2.1) that

$$F^j(x,v) \leq F^j(1,v)e^{\int_0^{\frac{1-x}{\xi}} \nu^j(x+\tau\xi,v)d\tau}, \quad (x,v) \in (-1,1)\times I\!R^3, \quad \xi > 0,$$

$$F^j(x,v) \leq F^j(-1,v)e^{\int_0^{\frac{-1-x}{\xi}} \nu^j(x+\tau\xi,v)d\tau}, \quad (x,v) \in (-1,1)\times I\!R^3, \quad \xi < 0.$$

Hence, for j large enough

$$\int_{-1}^1 \int_{|\xi|\geq r-\frac{1}{j}} F^j log F^j(x,v)dxdv =$$

$$\int_{\xi>r-\frac{1}{j}} \xi(\int_{-\frac{2}{\xi}}^0 F^j log F^j(1+s\xi,v)ds)dv + \int_{\xi<-r+\frac{1}{j}} \mid\xi\mid (\int_{\frac{2}{\xi}}^0 F^j log F^j(-1+s\xi,v)ds)dv$$

$$\leq \frac{8\mu}{r^2}e^{\frac{4\mu}{r}}[\int_{\xi>r-\frac{1}{j}} \xi F^j(1,v)dv + \int_{\xi<-r+\frac{1}{j}} \mid\xi\mid F^j(-1,v)dv]$$

$$+\frac{4}{r}e^{\frac{4\mu}{r}}[\int_{\xi>r-\frac{1}{j}} \xi F^j log F^j(1,v)dv + \int_{\xi<-r+\frac{1}{j}} \mid\xi\mid F^j log F^j(-1,v)dv]. \quad (3.4)$$

By (3.1), the first two terms to the right are uniformly in j bounded. As for the last two terms, the following estimates are chosen also with a view to next lemma.

Denote by

$$e(F^j,F^j) = \frac{1}{4}\int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j(x,v_*)dv_*d\omega}\chi B^j(\frac{F^j}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(x,v'_*)$$

$$-\frac{F^j}{1+\frac{F^j}{j}}(x,v)\frac{F^j}{1+\frac{F^j}{j}}(x,v_*))log\frac{\frac{F^j}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(x,v'_*)}{\frac{F^j}{1+\frac{F^j}{j}}(x,v)\frac{F^j}{1+\frac{F^j}{j}}(x,v_*)}dxdvdv_*d\omega.$$

Multiplying (2.1) by $log\frac{F^j}{1+\frac{F^j}{j}}$ and integrating over $(-1,1)\times I\!R^3$ implies

$$\int_{\xi<0} \mid\xi\mid (F^j log F^j - j(1+\frac{F^j}{j})log(1+\frac{F^j}{j}))(-1,v)dv$$

$$+\int_{\xi>0}\xi(F^jlogF^j-j(1+\frac{F^j}{j})log(1+\frac{F^j}{j}))(1,v)dv+e(F^j,F^j)$$

$$=\int_{\xi>0}\xi(F^jlogF^j-j(1+\frac{F^j}{j})log(1+\frac{F^j}{j}))(-1,v)dv$$

$$+\int_{\xi<0}\mid\xi\mid(F^jlogF^j-j(1+\frac{F^j}{j})log(1+\frac{F^j}{j}))(1,v)dv$$

$$-\frac{1}{j}\int\frac{\chi B^j}{\int F^j(x,v_*)dv_*d\omega}\frac{(F^j)^2}{(1+\frac{F^j}{j})}(x,v)\frac{F^j}{1+\frac{F^j}{j}}(x,v_*)log\frac{F^j}{1+\frac{F^j}{j}}(x,v)dxdvdv_*d\omega \quad (3.5)$$

Then the two first terms of the right-hand side are bounded because $0\leq\sigma^j(-1)\leq 1$ and $0\leq\sigma^j(1)\leq 1$. Moreover, with $log^-x:=logx$, if $x\leq 1$, $log^-x:=0$ otherwise, then for the third term

$$-\frac{1}{j}\int\frac{\chi B^j}{\int F^j(x,v_*)dv_*d\omega}\frac{(F^j)^2}{(1+\frac{F^j}{j})}(x,v)\frac{F^j}{1+\frac{F^j}{j}}(x,v_*)log\frac{F^j}{1+\frac{F^j}{j}}(x,v)dxdvdv_*d\omega$$

$$\leq\frac{1}{j}\int_{F^j(x,v)\leq\frac{j}{j-1}}\frac{\chi B^j}{\int F^j(x,v_*)dv_*d\omega}\frac{F^j}{1+\frac{F^j}{j}}(x,v_*)\frac{(F^j)^2}{1+\frac{F^j}{j}}\mid log^-\frac{F^j}{1+\frac{F^j}{j}}\mid(x,v)dxdvdv_*d\omega$$

$$\leq\frac{1}{je}\int_{F^j(x,v)\leq\frac{j}{j-1}}\frac{\chi B^j}{\int F^j(x,v_*)dv_*d\omega}F^j(x,v)\frac{F^j}{1+\frac{F^j}{j}}(x,v_*)dxdvdv_*d\omega\leq cj^{-\frac{1}{4}},$$

It follows from this, that (3.5) becomes

$$\int_{\partial\Omega^-}\mid\xi\mid(F^jlogF^j-j(1+\frac{F^j}{j})log(1+\frac{F^j}{j}))+e(F^j,F^j)<c. \quad (3.6)$$

Here

$$\partial\Omega^-:=\{(-1,v),\ \xi<0\}\cup\{(1,v),\ \xi>0\}.$$

But

$$\frac{3}{4}tlnt-j(1+\frac{t}{j})ln(1+\frac{t}{j})\geq 0,\quad J>16,\quad j>J,\quad t\in(J,j^3). \quad (3.7)$$

Since F^j is bounded by j^3 for j large enough, it follows from (3.6) that

$$\int_{\partial\Omega^-}\mid\xi\mid F^jlogF^j+e(F^j,F^j)\leq logJ\int_{\partial\Omega^-,F^j\leq J}\mid\xi\mid F^j$$

$$+4\int_{\partial\Omega^-,F^j\in[J,j^3]}\mid\xi\mid[F^jlogF^j-j(1+\frac{F^j}{j})log(1+\frac{F^j}{j})](x,v)dxdv$$

$$+4e(F^j,F^j)$$

$$\leq c-4\int_{\partial\Omega^-,F^j\leq J}\mid\xi\mid[F^jlogF^j-j(1+\frac{F^j}{j})log(1+\frac{F^j}{j})](x,v)dxdv.$$

Also,

$$-\int_{\partial\Omega^-, F^j\le J} \mid \xi \mid F^j log F^j \le -\int_{\partial\Omega^-, F^j\le 1} \mid \xi \mid F^j log F^j$$
$$= \int_{\partial\Omega^-, F^j\le e^{-v^2}} \mid \xi \mid F^j \mid log F^j \mid + \int_{\partial\Omega^-, e^{-v^2}\le F^j\le 1} \mid \xi \mid F^j \mid log F^j \mid$$
$$\le \frac{2}{e}\int_{\partial\Omega^-} \mid \xi \mid e^{-\frac{1}{2}v^2} + \int_{\partial\Omega^-} \mid \xi \mid v^2 F^j \le c,$$

by (3.3). Moreover,

$$\int_{\partial\Omega^-, F^j\le J} \mid \xi \mid j(1+\frac{F^j}{j})log(1+\frac{F^j}{j})$$
$$= \int_{\partial\Omega^-, F^j\le J} \mid \xi \mid F^j(1+\frac{F^j}{j})log[(1+\frac{F^j}{j})^{\frac{j}{F^j}}]$$
$$\le \int_{\partial\Omega^-, F^j\le J} \mid \xi \mid F^j(1+\frac{F^j}{j}) \le 2\int_{\partial\Omega^-, F^j\le J} \mid \xi \mid F^j \le c, \quad j > J.$$

Consequently,

$$\int_{\partial\Omega^-} \mid \xi \mid F^j log F^j + e(F^j, F^j) < c. \tag{3.8}$$

Hence, the remaining term of the right-hand side of (3.4) is uniformly bounded from above, thus also the entropy of (F^j). From here the desired precompactness holds, since the mass is uniformly bounded from above (cf. (3.1)), and the contribution to the integral from large velocities can be made arbitrarily small by using a comparison with outgoing boundary values. This ends the proof of Lemma 3.2.□

Lemma 3.3 *The sequence* $(\frac{1}{\int F^j(x,v_*)dv_*d\omega}\int_{S^2\times I\!R^3} \chi B^j \frac{F^j}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(x,v'_*)d\omega dv_*)$ *is weakly precompact in* $L^1((-1,1)\times I\!R^3)$.

<u>Proof of Lemma 3.3.</u> The weak L^1 precompactness of

$$(Q_j^-(F^j)(x,v)) := (F^j(x,v)\frac{\int \chi B^j \frac{F^j}{1+\frac{F^j}{j}}(x, v_* dv_* d\omega)}{\int F^j(x,v_*)dv_*d\omega})$$

follows from the weak precompactness of (F^j). Then the weak L^1 precompactness of $(Q_j^+(F^j))$, the corresponding gain terms, is a consequence of the weak L^1 precompactness of $(Q_j^-(F^j))$ and the boundedness from above of

$(e(F^j, F^j))$, which is a consequence of (3.8). $\square$
We are now in a position to remove the asymmetry between the gain and the loss term by taking the limit $j \to \infty$. It is enough to consider the weak formulation of (2.1); for F^j and test functions $\zeta \in C_c^1([-1,1] \times I\!R^3)$,

$$\int_{(-1,1)\times I\!R^3} \xi \frac{\partial \zeta}{\partial x} F^j(x,v)dxdv$$
$$+\int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j dv_* d\omega} \chi B^j \Big(\frac{F^j}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(x,v'_*)$$
$$-F^j(x,v)\frac{F^j}{1+\frac{F^j}{j}}(x,v_*)\Big)\zeta(x,v)dxdvdv_*d\omega$$
$$= \int_{\xi>0} \xi F^j \zeta(1,v)dv - \int_{\xi<0} \xi F^j \zeta(-1,v)dv$$
$$+\sigma^j(1)\int_{\xi<0} \xi M_L(v)\zeta(1,v)dv - \sigma^j(-1)\int_{\xi>0} \xi M_0(v)\zeta(-1,v)dv. \qquad (3.9)$$

First,

$$\lim_{j\to+\infty} \int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j(x,v_*)dv_* d\omega} \chi B^j \frac{F^j}{1+\frac{F^j}{j}}(x,v')\frac{F^j}{1+\frac{F^j}{j}}(x,v'_*)\zeta(x,v)dxdvdv_*d\omega =$$
$$\lim_{j\to+\infty} \int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j(x,v_*)dv_* d\omega} \chi B^j F^j(x,v')F^j(x,v'_*)\zeta(x,v)dxdvdv_*d\omega,$$

by the weak L^1-compactness of (F^j). Then, by the change of variables $(v, v_*) \to (v', v'_*)$,

$$\int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j(x,v_*)dv_* d\omega} \chi B^j F^j(x,v')F^j(x,v'_*)\zeta(x,v)dxdvdv_*d\omega =$$
$$\int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j(x,v_*)dv_* d\omega} \chi B^j F^j(x,v)F^j(x,v_*)\zeta(x,v')dxdvdv_*d\omega.$$

(F^j), as well as $(\xi \frac{\partial F^j}{\partial x})$ are weakly compact in $L^1((-1,1)\times I\!R^3)$ by Lemmas 3.2-3. Consequently, $(\int_{I\!R^3} F^j(x,v_*)\zeta(x,v')dv_*d\omega)$ is compact in $L^1((-1,1)\times I\!R^3)$ and converges (for a subsequence) to $\int_{I\!R^3} F(x,v_*)\zeta(x,v')dv_*d\omega$, where F is a weak L^1 limit of (F^j). Hence

$$\lim_{j\to+\infty} \int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F^j(x,v_*)dv_* d\omega} \chi B^j F^j(x,v)F^j(x,v_*)\zeta(x,v')dxdvdv_*d\omega =$$
$$\int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F(x,v_*)dv_* d\omega} \chi B F(x,v)F(x,v_*)\zeta(x,v')dxdvdv_*d\omega.$$

Moreover, $(\gamma^{\pm} F^j)$ converges weakly (for a subsequence) to $(\gamma^{\pm} F)$, since (F^j) and $(\xi \frac{\partial F^j}{\partial x})$ are weakly compact in $L^1((-1,1)\times I\!R^3)$. Here $\gamma^{\pm} F^j$ denote the traces of F^j on

$$\partial\Omega^+ = \{(-1,v),\ \xi>0\} \cup \{(1,v),\ \xi<0\},$$

and on $\partial\Omega^-$ defined above. Hence we can pass to the limit when $j\to+\infty$ in (3.9) and obtain

$$\begin{aligned}\int_{(-1,1)\times I\!R^3} \xi \frac{\partial\zeta}{\partial x} F(x,v)dxdv + \int_{(-1,1)\times I\!R^6\times S^2} \frac{1}{\int F(x,v_*)dv_*d\omega} \chi B(F(x,v')F(x,v'_*)\\ -F(x,v)F(x,v_*))\zeta(x,v)dxdvdv_*d\omega = \int_{\xi>0} \xi F\zeta(1,v)dv - \int_{\xi<0}\xi F\zeta(-1,v)dv\\ +\sigma(1)\int_{\xi<0}\xi M_L(v)\zeta(1,v)dv - \sigma(-1)\int_{\xi>0}\xi M_0(v)\zeta(-1,v)dv,\end{aligned}$$

which means that $F := F^{r,\mu}$ is a weak solution to the stationary transformed problem

$$\begin{aligned}\xi\frac{\partial F^{r,\mu}}{\partial x} = \int \frac{1}{\int F^{r,\mu}(x,v_*)dv_*d\omega}\chi^r max(\frac{1}{\mu}, min(B(v-v_*,\omega),\mu))(F^{r,\mu}(x,v')F^{r,\mu}(x,v'_*)\\ -F^{r,\mu}(x,v)F^{r,\mu}(x,v_*))dv_*d\omega, \quad (x,v)\in(-1,1)\times I\!R^3, \quad (3.10)\\ F^{r,\mu}(-1,v) = M_0(v)\sigma(-1),\ \xi>0, \quad F^{r,\mu}(1,v)=M_L(v)\sigma(1),\ \xi<0.\end{aligned}$$

Integrating (3.10) on $(-1,1)\times I\!R^3$ implies that

$$\int_{\xi>0}\xi F^{r,\mu}(-1,v)dv + \int_{\xi<0} |\ \xi\ |\ F^{r,\mu}(1,v)dv = 1, \quad (3.11)$$

so that the boundary conditions satisfied by $F^{r,\mu}$ are indeed

$$\begin{aligned}F^{r,\mu}(-1,v) = M_0(v)\int_{\xi'<0} |\ \xi'\ |\ F^{r,\mu}(-1,v')dv', \quad \xi>0,\\ F^{r,\mu}(1,v) = M_L(v)\int_{\xi'>0}\xi' F^{r,\mu}(1,v')dv', \quad \xi<0. \quad (3.12)\end{aligned}$$

And so the aim of this section has been achieved, to obtain a solution for an approximate equation with gain and loss terms of the same type, with the truncation χ^r a characteristic function, and with total inflow one through the boundary.

4 Removal of the small velocity truncation; some preparatory lemmas.

In the previous section solutions $F^{r,\mu}$ to (3.10-12) were obtained corresponding to the approximations involving χ^r and B_μ. Writing $F^r := F^{r,\mu}$, we shall in this section make some necessary preparations to remove the small velocity truncation χ^r, while keeping $1<\mu$ fixed. As in the previous section we start with some estimates independent of the relevant parameter, here r.

Lemma 4.1 *There are $c>0$, $\bar{c}>0$, and for $\delta>0$, constants $c_\delta>0$ and $\bar{c}_\delta>0$, such that uniformly with respect to r*

$$\int \xi^2 F^r(x,v)dv \le c, \quad x\in(-1,1), \tag{4.1}$$

$$F^r(x,v)\ge \bar{c}_\delta,\ x\in(-1,1),\ \mid \xi \mid \ge \delta, \mid v \mid \le \frac{1}{\delta}, \quad \int F^r(x,v)dv \ge \bar{c}, \quad x\in(-1,1),$$

$$\int_{|\xi|\ge\delta} \mid v \mid^2 F^r(x,v)dxdv \le c_\delta.$$

Proof of Lemma 4.1 (4.1) follows from Green's formula. By the exponential form of (3.10), and by Lemma 2.2,

$$F^r(x,v)\ge F^r(-1,v)e^{-\frac{(1+x)\mu}{\xi}} \ge e^{-\frac{2\mu}{\delta}}M_0(v)\sigma^r(-1)\ge \bar{c}_\delta, \quad \xi>\delta, \mid v \mid \le \frac{1}{\delta}$$
$$F^r(x,v)\ge F^r(1,v)e^{\frac{(1-x)\mu}{\xi}} \ge e^{-\frac{2\mu}{\delta}}M_L(v)\sigma^r(1)\ge \bar{c}_\delta, \quad \xi<-\delta, \mid v \mid \le \frac{1}{\delta}.$$

Hence

$$\int F^r(x,v)dv \ge \bar{c},$$

for some $\bar{c}$ independent of r. Then

$$\int_{|\xi|\ge\delta} \mid v \mid^2 F^r(x,v)dxdv = \int_{\xi>\delta} \xi \mid v \mid^2 \int_{-\frac{2}{\xi}}^0 F^r(1+s\xi,v)dsdv$$
$$+\int_{\xi<-\delta} \mid \xi \mid\mid v \mid^2 \int_{\frac{2}{\xi}}^0 F^r(-1+s\xi,v)dsdv$$
$$\le \frac{2}{\delta}e^{\frac{2\mu}{\delta}}\Big(\int_{\xi>\delta} \xi \mid v \mid^2 F^r(1,v)dv + \int_{\xi<-\delta} \mid \xi \mid\mid v \mid^2 F^r(-1,v)dv\Big) \le ce^{\frac{3\mu}{\delta}},$$

the last step by using (3.3). $\square$
¿From (4.1), it follows that for any δ,

$$\sup_{x\in(-1,1),r>0}\int_{|\xi|>\delta}F^r(x,v)dv\leq\frac{c}{\delta^2}. \tag{4.2}$$

¿From Lemma 4.1 it also follows that the contribution to this integral from large v's, uniformly in r can be made arbitrarily small. Also for $\mid\xi\mid\geq\delta$, by the exponential form

$$F^r(x,v)\leq c_\delta F^r(1,v),\ \xi>\delta,\quad F^r(x,v)\leq c_\delta F^r(-1,v),\ \xi<-\delta.$$

By a change of variables,

$$\int_{-1}^{1}\int_{|\xi|\geq\delta}F^r logF^r(x,v)dxdv=$$

$$\int_{\xi>\delta}\xi\int_{-\frac{2}{\xi}}^{0}F^r logF^r(1+s\xi,v)dsdv+\int_{\xi<-\delta}\mid\xi\mid\int_{\frac{2}{\xi}}^{0}F^r logF^r(-1+s\xi,v)dsdv$$

$$\leq\tilde{c_\delta}[\int_{\xi>\delta}\xi F^r(1,v)dv+\int_{\xi<-\delta}\mid\xi\mid F^r(-1,v)dv]$$

$$+\bar{c_\delta}[\int_{\xi>\delta}\xi F^r logF^r(1,v)dv+\int_{\xi<-\delta}\mid\xi\mid F^r logF^r(-1,v)dv]\leq c_\delta$$

with c_δ independent of r. In the last step we used Green's formula applied to (3.10) successively multiplied by 1 and $logF^r$ and integrated on $(-1,1)\times\mathbb{R}^3$. Using Lemma 4.1, it follows that

$$\int_{-1}^{1}\int_{|\xi|\geq\delta}F^r\mid logF^r\mid(x,v)dxdv\leq c_\delta.$$

Hence

Lemma 4.2 *For $\delta>0$, the family $(F^r)_{0<r\leq r_0}$ is weakly precompact in $L^1((-1,1)\times\{v\in\mathbb{R}^3;\mid\xi\mid\geq\delta\})$.*

Take (r_j) with $\lim_{j\to\infty}r_j=0$, so that F^{r_j} converges weakly in $L^1((-1,1)\times\{v\in\mathbb{R}^3;\mid\xi\mid\geq\delta\})$ to F for each $\delta>0$. Write $F^j:=F^{r_j}$, $\chi^j:=\chi^{r_j}$. We shall next prove the

Lemma 4.3

$$\int Q^{\pm}(F)(x,v)dv<\infty,\quad\sup_j\int Q_j^{\pm}(F^j)(x,v)dv<\infty\ for\ a.e.\ x\in(-1,1).$$

Proof of Lemma 4.3. Assume that, given $\eta > 0$, there is $j_0 \in I\!N$ such that for all $j \in I\!N, j \geq j_0$,

$$meas\{x \in (-1,1); \int_{|\xi| \leq \frac{1}{j_0}} F^j(x,v)dv > j_0^3\} \leq \frac{\eta}{8}. \tag{4.3}$$

(The property (4.3) of (F^j) is proved on next page.) Since the $Q^{\pm}(F)$ integrals are equal, we discuss the $Q^-(F)$ case. The $Q_j^{\pm}(F^j)$ case follows from the same proof. Suppose that for some $\eta > 0$, $\int Q^-(F)(x,v)dv = \infty$ on $S \subset [-1,1]$ with $\mid S \mid \geq \eta > 0$. Then there is a subset $S_1 \subset S$ with $\mid S_1 \mid = \frac{\eta}{2}$ such that $\lim_{\delta \to 0} \int_{\delta \leq |\xi| \leq \frac{1}{j_0}} F(x,v)dv = \infty$, uniformly with respect to $x \in S_1$. Hence there is a sequence (δ_k) with $\lim_{k \to \infty} \delta_k = 0$ such that

$$\int_{\delta_k \leq |\xi| \leq \frac{1}{j_0}} F(x,v)dv > k^3, \quad x \in S_1. \tag{4.4}$$

Multiplying (3.10) by $logF^j$ and integrating it on $(-1,1) \times I\!R^3$ implies

$$\int_{-1}^{1} \frac{1}{\int F^j(x,v_*)dv_*d\omega} \int \chi^j B(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))$$
$$log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}dxdvdv_*d\omega \leq$$
$$\int \xi F^j logF^j(-1,v)dv - \int \xi F^j logF^j(1,v)dv.$$

But for all j

$$\int \xi F^j logF^j(-1,v)dv$$
$$\leq \int_{\xi<0} \xi F^j log^- F^j(-1,v)dv + (\int_{\xi>0} \xi M_0 logM_0(v)dv) \int_{\xi'<0} \mid \xi' \mid F^j(-1,v')dv'$$
$$+(\int_{\xi>0} \xi M_0(v)dv)(\int_{\xi'<0} \mid \xi' \mid F^j(-1,v')dv')log \int_{\xi'<0} \mid \xi' \mid F^j(-1,v')dv' \leq c \tag{4.5}$$

since $\int_{\xi'<0} \mid \xi' \mid F^j(-1,v')dv' \in [0,1]$. Similarly, $-\int \xi F^j logF^j(1,v)dv$ is uniformly in j bounded. Hence uniformly with respect to j,

$$\int_{-1}^{1} \frac{1}{\int F^j(x,v_*)dv_*d\omega} \int \chi^j B(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))$$
$$log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}dxdvdv_*d\omega < c < \infty. \tag{4.6}$$

The collision frequency is bounded by μ, so Lemma 4.2 implies the same compactness for the loss term $(Q^-(F^j))$ as for (F^j). That together with the bound (4.6) implies this compactness property for $(Q^+(F^j))$. From Egoroff's theorem and a Cantor diagonalization argument, there is a subset $S_2 \subset S_1$, with $\mid S_2 \mid = \frac{\eta}{4}$ such that for all $k \in I\!N$ and uniformly with respect to $x \in S_2$,

$$\lim_{j\to\infty} \int_{\delta_k<|\xi|\leq\frac{1}{j_0}} F^j(x,v)dv = \int_{\delta_k<|\xi|\leq\frac{1}{j_0}} F(x,v)dv.$$

By (4.4) this contradicts (4.3) for $k > j_0$.

Let us prove (4.3) by contradiction. If (4.3) does not hold, there is $\eta > 0$ and a subsequence of (F^j), still denoted by (F^j), such that $\mid S_j \mid \geq \frac{\eta}{8}$, where

$$S_j := \{x \in (-1,1); \int_{|\xi|\leq\frac{1}{j}} F^j(x,v)dv \geq j^3\}.$$

By (4.6), there is a subset S'_j of S_j, with $\mid S'_j \mid \geq \frac{\eta}{16}$, such that

$$\int \chi^j B(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}dvdv_*d\omega$$
$$\leq c\int F^j(x,v_*)dv_*d\omega \leq 2c\int_{|\xi|\leq\frac{1}{j}} F^j(x,v_*)dv_*. \quad (4.7)$$

The last inequality holds for j large enough, since by Lemma 4.1 and the definition of S_j,

$$\int_{|\xi|\geq\frac{1}{j}} F^j(x,v)dv \leq cj^2 \leq j^3 \leq \int_{|\xi|\leq\frac{1}{j}} F^j(x,v)dv, \quad x \in S_j.$$

Let us estimate from above the right-hand side of

$$1 = \frac{X_j}{\int_{|\xi|\leq\frac{1}{j}} F^j(x,v)dv} + \frac{Y_j}{\int_{|\xi|\leq\frac{1}{j}} F^j(x,v)dv}, \quad x \in S'_j,$$

where

$$X_j = \int_{|\xi|\leq\frac{1}{j},|\rho|\leq j^{\frac{1}{3}}} F^j(x,v)dv, \quad Y_j = \int_{|\xi|\leq\frac{1}{j},|\rho|\geq j^{\frac{1}{3}}} F^j(x,v)dv.$$

Here

$$\rho := \sqrt{\eta^2+\zeta^2}, \quad v = (\xi,\eta,\zeta).$$

Either $X_j \leq j^2$, and then

$$\frac{X_j}{\int_{|\xi|\leq\frac{1}{j}} F^j(x,v)dv} \leq \frac{1}{j}.$$

Or $X_j \geq j^2$, and then

$$\int_{|\xi|\leq\frac{1}{j},|\rho|\leq j^{\frac{1}{3}},F^j(x,v)\geq\frac{1}{16}j^{\frac{7}{3}}} F^j(x,v)dv \geq \frac{1}{2}j^2.$$

Given v, let

$$V_* := \{v_* \in I\!R^3; \frac{1}{10} \leq\mid \xi_* \mid\leq 1, \mid \rho_* \mid\leq 100, \mid \rho - \rho_* \mid> 10\}.$$

By Lemma 4.1,

$$F^j(x,v_*) \geq c, \quad v_* \in V_*.$$

Then, from the geometry of the velocities involved, and from $\int_{|\xi|\geq 1} F^j(x,v)dv \leq c$, given v with $F^j(x,v) \geq \frac{1}{16}j^{\frac{7}{3}}$, it holds for v_* in a half volume of V_* and given (v,v_*) for ω in a subset $U_j(v,v_*)$ of S^2 with measure a small fixed fraction of the measure of S^2, that

$$\mid \xi' \mid\geq 1, \ \mid \xi'_* \mid\geq 1, \ F^j(x,v') \leq \tilde{c}, \ F^j(x,v'_*) \leq \tilde{c}.$$

It follows for some $c > 0$, for v, $v_* \in V_*$, $\omega \in U_j(v,v_*)$, for j large, and with c independent of v, v_*, ω, j, that

$$cF^j(x,v) \leq F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*),$$
$$\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)} \geq cj^{\frac{7}{3}}.$$

By (4.7), in this case,

$$\frac{\int_{|\xi|\leq\frac{1}{j},|\rho|\leq j^{\frac{1}{3}},F^j(x,v)\geq\frac{1}{16}j^{\frac{7}{3}}} F^j(x,v)dv}{\int_{|\xi|\leq\frac{1}{j}} F^j(x,v)dv} \leq \frac{c}{logj}, \quad x \in S'_j.$$

Moreover,

$$\int_{|\xi|\leq\frac{1}{j},|\rho|\leq j^{\frac{1}{3}},F^j(x,v)\leq\frac{1}{16}j^{\frac{7}{3}}} F^j(x,v)dv \leq cj^2.$$

Hence for $x \in S'_j$

$$\frac{X_j}{\int_{|\xi|\leq \frac{1}{j}} F^j(x,v)dv} \leq \frac{c}{logj}.$$

Let us bound Y_j from above. By Lemma 4.1,

$$F^j(x,v_*) \geq c, \quad | v_* |\leq 10, \quad | \xi_* |\geq \frac{1}{10}.$$

For (v,v_*) such that $| \xi |\leq \frac{1}{j}$, $| \rho |\geq j^{\frac{1}{3}}$, $v_* \in V_*$, we have $| v - v_* |\geq cj^{\frac{1}{3}}$ for j large. Hence, for a set Ω of ω with a small fixed fraction of the total area of S^2, it holds that $| \xi' |\geq cj^{\frac{1}{3}}$, $| \xi'_* |\geq cj^{\frac{1}{3}}$. From

$$\int_{|\xi|\geq cj^{\frac{1}{3}}} F^j(x,v)dv \leq \frac{c}{j^{\frac{2}{3}}},$$

it follows that

$$\int_{|\xi'|\geq cj^{\frac{1}{3}}, |\xi'_*|\geq cj^{\frac{1}{3}}} F^j(x,v')F^j(x,v'_*)dvdv_*d\omega \leq \frac{c}{j^{\frac{4}{3}}}.$$

Hence

$$\begin{aligned}
c_1 \int_{r_j\leq|\xi|\leq \frac{1}{j}, |\rho|\geq j^{\frac{1}{3}}} F^j(x,v)dv &\leq \int_{r_j\leq|\xi|\leq \frac{1}{j}, |\rho|\geq j^{\frac{1}{3}}, v_*\in V_*, \omega\in\Omega} B_j F^j(x,v)F^j(x,v_*)dvdv_*d\omega \\
&\leq Kc_2 \int_{|\xi'|\geq cj^{\frac{1}{3}}, |\xi'_*|\geq cj^{\frac{1}{3}}} B_j F^j(x,v')F^j(x,v'_*)dv'dv'_* \\
&+ \frac{2}{logK} \int B_j (F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*)) log \frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)} dvdv_*d\omega \\
&\leq \frac{c_3 K}{j^{\frac{4}{3}}} + \frac{c_3}{logK} \int_{|\xi|\leq \frac{1}{j}} F^j(x,v)dv.
\end{aligned}$$

Then, for some $c > 0$ independent of j,

$$1 = \frac{X_j}{\int_{|\xi|\leq \frac{1}{j}} F^j(x,v)dv} + \frac{Y_j}{\int_{|\xi|\leq \frac{1}{j}} F^j(x,v)dv} \leq c(\frac{r_j}{j^3} + \frac{1}{j} + \frac{1}{logj} + \frac{K}{j^{\frac{4}{3}}} + \frac{1}{logK}).$$

Choosing K large enough, this gives a contradiction for j large. $\square$

Remark. We have proven that for any $\eta > 0$, there is $j_0 \in I\!N$ such that for all $j \in I\!N$, $j \geq j_0$, (4.3) holds.

Lemma 4.4 *Given $\eta > 0$, there is j_0 such that for $j > j_0$ and outside a j-dependent set in x of measure less than η, $\int_{|\rho|>\lambda} F^j(x,v)dv$ converges to zero when $\lambda \to +\infty$, uniformly with respect to x and j.*

Proof of Lemma 4.4. It follows from the geometry of the velocities involved and the inequality

$$\int_{|\xi|\geq\bar{c}\lambda} \mid \xi \mid^{\beta} F^j(x,v)dv \leq \frac{c}{\lambda^{2-\beta}},$$

that for each (v, v_*) with $\rho \geq \lambda \gg 10$, and v_* in

$$V_* := \{v_* \in I\!R^3; \mid \xi_* \mid \geq \frac{1}{10}, \ \mid v_* \mid \leq 10\},$$

there is a subset of $\omega \in S^2$ with measure (say) $\frac{1}{100} \mid S^2 \mid$ such that

$$\bar{c}\rho \leq \mid v' \mid \leq c \mid \xi' \mid, \quad \bar{c}\rho \leq \mid v'_* \mid \leq c \mid \xi'_* \mid .$$

Moreover,

$$F^j(x, v_*) \geq c, \quad \mid \xi_* \mid \geq \frac{1}{10}, \quad \mid v_* \mid \leq 10.$$

Hence for $r_j \leq \mid \xi \mid \leq \bar{c}\lambda$,

$$cF^j(x,v) \leq F^j(x,v)F^j(x,v_*) \leq KB_j(v-v_*,\omega)F^j(x,v')F^j(x,v'_*)$$
$$+\frac{2}{logK}B_j(v-v_*,\omega)(F^j(x,v)F^j(x,v_*)-F^j(x,v')F^j(x,v'_*))log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}.$$

Let us integrate this inequality on the above set of (v, v_*, ω), so that

$$\int_{|\rho|>\lambda} F^j(x,v)dv \leq \frac{cr_j}{\lambda}+\frac{cK}{\lambda^{4-\beta}}+\frac{c}{logK}\int \chi^j B_j(F^j(x,v)F^j(x,v_*)-F^j(x,v')F^j(x,v'_*))$$
$$log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}dvdv_*d\omega.$$

Given $\eta > 0$, by (4.3) there is j_0 such that for $j > j_0$, outside of a set in x of measure $\frac{\eta}{8}$, it holds that

$$\int F^j(x,v)dv \leq cj_0^2 + j_0^3.$$

By (4.6),

$$\int \chi^j B(F^j(x,v)F^j(x,v_*)-F^j(x,v')F^j(x,v'_*))$$
$$log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}dvdv_*d\omega \leq c_\eta \int F^j(x,v)dv$$

outside of a set of measure (say) $\frac{7\eta}{8}$, so that

$$\int_{|\rho|>\lambda} F^j(x,v)dv \leq \frac{cr_j}{\lambda} + \frac{cK}{\lambda^{2-\beta}} + \frac{c_\eta(cj_0^2 + j_0^3)}{logK},$$

for x outside of a set of measure η. Choosing K so that $\frac{1}{K}$ is small enough and then taking λ so that $\frac{cK}{\lambda^{4-\beta}}$ is small enough, implies that $\int_{|\rho|>\lambda} F^j(x,v)dv$ tends to zero uniformly outside of $j-$dependent sets of measure bounded by η. $\Box$

Lemma 4.5 *Given $\lambda > 0$ and $\eta > 0$, there is j_0 such that for $j > j_0$ and outside of a j-dependent set in x of measure less than η, $\int_{|\rho|\leq\lambda,|\xi|\leq\frac{1}{i}} F^j(x,v)dv$ converges to zero when $i \to +\infty$, uniformly with respect to x and j.*

Proof of Lemma 4.5. Given $\eta > 0$, $0 < \epsilon < \frac{1}{\mu}$ and x, j, either

$$\int_{|\rho|\leq\lambda,|\xi|\leq\frac{1}{i}} F^j(x,v)dv \leq \epsilon^2 < \epsilon,$$

or

$$\int_{|\rho|\leq\lambda,|\xi|\leq\frac{1}{i}} F^j(x,v)dv > \epsilon^2.$$

In the latter case

$$\int_{|\rho|\leq\lambda,|\xi|\leq\frac{1}{i},F^j(x,v)\leq\frac{\epsilon^2}{4\mu\lambda^2\pi}i} F^j(x,v)dv \leq \frac{\epsilon^2}{2\mu} < \epsilon,$$

and

$$\int_{|\rho|\leq\lambda,|\xi|\leq\frac{1}{i},F^j(x,v)\geq\frac{\epsilon^2}{4\mu\lambda^2\pi}i} F^j(x,v)dv \geq \frac{\epsilon^2}{2}.$$

For each (x,v) such that $F^j(x,v) \geq \frac{\epsilon^2}{4\mu\lambda^2\pi}i$, take v_* in

$$V_* := \{v_* \in I\!R^3; \frac{1}{10} \leq\mid \xi_* \mid\leq 1, \mid \rho_* \mid\leq 100, \mid \rho - \rho_* \mid > 10\}.$$

Then $F^j(x,v_*) \geq c > 0$ for $v_* \in V_*$. Given v we may take v_* in a half volume of V_* and ω in a subset of S^2 of measure (say) $\frac{1}{100} \mid S^2 \mid$, so that $v' = v - (v - v_*, \omega)\omega$ and $v'_* = v_* + (v - v_*, \omega)\omega$ satisfy

$$\mid \xi' \mid\geq 1,\ \mid \xi'_* \mid\geq 1,\ F^j(x,v') \leq \tilde{c},\ F^j(x,v'_*) \leq \tilde{c},$$

with $\tilde{c}$ independent of j. Hence, for such x, v, v_* and ω,

$$F^j(x,v) \leq cF^j(x,v)F^j(x,v_*)$$
$$\leq \frac{c}{logi} B_j(v-v_*,\omega)(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)},$$

if $\mid \xi \mid \geq r_j$. Since the integral

$$\frac{1}{\int F^j(x,v)dv} \int B_j(v-v_*,\omega)(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))$$
$$log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)} dvdv_*d\omega$$

is bounded outside of a set of measure $\frac{\eta}{2}$ in x, it follows using (4.3) that outside of a set of measure $\frac{5}{8}\eta$ in x,

$$\int_{|\rho|\leq\lambda, |\xi|\leq\frac{1}{i}} F^j(x,v)dv \leq \frac{c_1(j_o^3 + cj_o^2)}{logi} + \epsilon + \frac{c_1}{i} < 2\epsilon,$$

for i large enough. $\Box$

5 Proof of the main theorem.

In this section the small velocity truncation will first be removed while keeping $0 < \mu$ fixed. The bounds from below of the approximations by their boundary values imply that the condition (1.5) holds in the limit, and that the function $y(x)$ from (1.6) is well defined. This will prove Theorem 1.1 in the pseudo-maxwellian case, i.e. when $\beta = 0$. In a final step the generalization to hard forces will be treated, using generalizations of the previous approach.

Lemma 5.1 *There is a subsequence of* (F^j) *that converges to a weak solution of*

$$\xi\frac{\partial F}{\partial x} = \frac{1}{\int F dv_* d\omega} \int max(\frac{1}{\mu}, min(B(v-v_*,\omega),\mu))(F'F'^* - FF^*)dv_*d\omega,$$
$$F(-1,v) = c_0 M_0(v) \int_{\xi'<0} \mid \xi' \mid F(-1,v')dv', \quad \xi > 0,$$
$$F(1,v) = c_L M_L(v) \int_{\xi'>0} \xi' F(1,v')dv', \quad \xi < 0,$$

with $c_0 \geq 1$, $c_L \geq 1$, *and*

$$\int_{\xi>0} \xi F(-1,v)dv + \int_{\xi<0} \mid \xi \mid F(1,v)dv = 1.$$

<u>Remark.</u> This proves Theorem 1.1 in the pseudo-maxwellian case.

<u>Proof of Lemma 5.1.</u> Let φ be a test function vanishing for $\mid \xi \mid \leq \delta$ and for $\mid v \mid \geq \frac{1}{\delta}$. By Lemma 4.2, there is a measurable function F, such that (F^j) weakly converges to F in $L^1([-1,1] \times \{v \in I\!R^3; \mid \xi \mid \geq \delta\})$. Hence $\int_{(-1,1)\times I\!R^3} \xi F^j \frac{\partial \varphi}{\partial x}(x,v)dxdv$ converges to $\int_{(-1,1)\times I\!R^3} \xi F \frac{\partial \varphi}{\partial x}(x,v)dxdv$ when j tends to infinity. Let us prove that $\int Q_j^-(F^j)\varphi(x,v)dxdv$ converges to $\int Q^-(F)\varphi(x,v)dxdv$ when j tends to infinity. $(Q_j^{\pm}(F^j))$ are weakly compact in $L^1([-1,1] \times \{v \in I\!R^3; \mid \xi \mid \geq \delta\})$, since $0 \leq Q^-(F^j) \leq cF^j$,

$$Q_j^+(F^j) \leq KQ_j^-(F^j) + \frac{2}{logK \int F^j dv_* d\omega} \int \chi^j B_\mu (F^j F_*^j - F^{j'} F_*^{j'}) log \frac{F^j F_*^j}{F^{j'} F_*^{j'}} dv_* d\omega,$$

and the integral of the entropy dissipation term is bounded uniformly with respect to j. Consequently for any $\alpha > 0$ and $\lambda > 0$

$$\int_{|\xi_*| \geq \alpha, |\rho_*| \leq \lambda} \chi^j B_\mu(v - v_*, \omega) F^j(x, v_*) dv_* d\omega$$

and

$$\int_{|\xi_*| \geq \alpha, |\rho_*| \leq \lambda} F^j(x, v_*) dv_* d\omega$$

converge strongly in $L^1([-1,1] \times \{v \in I\!R^3; \mid v \mid \leq c\})$, hence uniformly outside of certain arbitrarily small sets, to

$$\int_{|\xi_*| \geq \alpha, |\rho_*| \leq \lambda} B_\mu(v - v_*, \omega) F(x, v_*) dv_* d\omega$$

and

$$\int_{|\xi_*| \geq \alpha, |\rho_*| \leq \lambda} F(x, v_*) dv_* d\omega$$

respectively, when j tends to infinity. By Lemmas 4.4 and 4.5, uniformly with respect to $j \geq j_0$ and $\mid v \mid \leq c$,

$$\int_{|\xi_*| \leq \alpha} \chi^j B_\mu(v - v_*, \omega) F^j(x, v_*) dv_* d\omega$$

and

$$\int_{|\xi_*|\leq\alpha} F^j(x,v_*)dv_*d\omega$$

tend to zero in measure when α tends to zero, and

$$\int_{|\rho_*|\geq\lambda} \chi^j B_\mu(v-v_*,\omega)F^j(x,v_*)dv_*d\omega$$

and

$$\int_{|\rho_*|\geq\lambda} F^j(x,v_*)dv_*d\omega$$

tend to zero in measure when λ tends to infinity, uniformly with respect to j and $\mid v \mid\leq c$. Together with the weak $L^1((-1,1)\times\{v\in I\!\!R^3; \mid \xi \mid\geq\delta\})$ compactness of (F^j), this implies that

$$\int Q_j^-(F^j)\varphi(x,v)dxdv$$

converges to $\int Q^-(F)\varphi(x,v)dxdv$ when $j\to\infty$. Performing the change of variables $(v,v_*)\to(v',v'_*)$ in $\int Q_j^+(F^j)\varphi(x,v)dxdv$, and using similar arguments, we obtain that $(\int Q_j^+(F^j)\varphi(x,v)dxdv)$ converges to $\int Q^+(F)\varphi(x,v)dxdv$ when j tends to infinity. Finally, using the arguments leading up to $F^{r,\mu}$ satisfying (3.11-12) together with Lemma 4.4, we may conclude that F satisfies (3.11). For (3.12) we also notice that (4.5) and convexity imply that the present weak limits F^j satisfy

$$\int_{\xi<0} |\xi| F^j \log^+ F^j(-1,v)dv + \int_{\xi>0} \xi F^j \log^+ F^j(1,v)dv \leq c,$$

uniformly in j. It follows that $(\gamma^\pm F^j)$ converges weakly (for a subsequence) to $\gamma^\pm F$, so that (3.12) holds. $\Box$

<u>Proof of Theorem 1.1 for hard forces.</u>

The solution procedure in the pseudo-maxwellian case can be applied in the same way to prove the existence of a solution to

$$\xi\frac{\partial F^\mu}{\partial x} = \frac{1}{\int K_\mu(v_*)F^\mu(x,v_*)dv_*}\Big[\int B_\mu(v-v_*,\omega)F^\mu(x,v')F^\mu(x,v'_*) \\ -F^\mu(x,v)\int B_\mu(v-v_*,\omega)F^\mu(x,v_*)dv_*d\omega\Big], \tag{5.1}$$

with boundary conditions (1.7) and

$$\int_{\xi>0} \xi F^\mu(-1,v)dv + \int_{\xi<0} \mid\xi\mid F^\mu(1,v)dv = 1.$$

Here, $K_\mu(v) := min(\mu, (1+ \mid v \mid)^\beta)$. We shall prove Theorem 1.1 in the hard force case by passing to the limit in this equation when μ tends to infinity. Similarly to the corresponding proof in Lemma 4.1, uniformly in μ

$$\int K_\mu(v_*)F^\mu(x,v_*)dv_* \geq c > 0.$$

For any $\delta > 0$, the family $(F^\mu)_{\mu\geq\mu_0}$ is weakly precompact in $L^1((-1,1)\times\{v \in \mathbb{R}^3; \mid \xi \mid \geq \delta, \mid v \mid \leq \frac{1}{\delta}\})$. Indeed,

$$\frac{\int B_\mu(v-v_*,\omega)F^\mu(x,v_*)dv_*d\omega}{\int K_\mu(v_*)F^\mu(x,v_*)dv_*} \leq c_\delta, \quad \mid v \mid \leq \frac{1}{\delta}, \tag{5.2}$$

so that

$$F^\mu(x,v) \leq c_\delta F^\mu(1,v), \quad \xi > \delta, \quad \mid v \mid \leq \frac{1}{\delta},$$
$$F^\mu(x,v) \leq c_\delta F^\mu(-1,v), \quad \xi < -\delta, \quad \mid v \mid \leq \frac{1}{\delta},$$

and

$$F^\mu logF^\mu(x,v) \leq c_\delta F^\mu logF^\mu(1,v) + c_\delta logc_\delta F^\mu(1,v), \quad \xi > \delta, \quad \mid v \mid \leq \frac{1}{\delta},$$
$$F^\mu(x,v) \leq c_\delta F^\mu logF^\mu(-1,v) + c_\delta logc_\delta F^\mu(-1,v), \quad \xi < -\delta, \quad \mid v \mid \leq \frac{1}{\delta}.$$

The weak precompactness of (F^μ) implies by (5.2) the weak precompactness of $(Q^-_\mu(F^\mu))$ in $L^1((-1,1)\times\{v \in \mathbb{R}^3; \mid \xi \mid \geq \delta, \mid v \mid \leq \frac{1}{\delta}\})$. But the entropy dissipation estimate (4.6) holds in this case uniformly in μ with $\int K_\mu(v_*)F^\mu(x,v_*)dv_*$ as denominator,

$$\int \Big(\frac{1}{\int K_\mu(v_*)F^\mu(x,v_*)dv_*}\int B_\mu(F^{\mu'}F^{\mu'}_* - F^\mu F^\mu_*)log\frac{F^{\mu'}F^{\mu'}_*}{F^\mu F^\mu_*}dvdv_*d\omega\Big)dx \leq c.$$

Also, for $k \geq 2$,

$$Q^+_\mu(F^\mu) \leq kQ^-_\mu(F^\mu) + \frac{1}{logk\int K_\mu(v_*)F^\mu(x,v_*)dv_*}\int B_\mu(F^{\mu'}F^{\mu'}_* - F^\mu F^\mu_*)log\frac{F^{\mu'}F^{\mu'}_*}{F^\mu F^\mu_*}dv_*d\omega.$$

Hence $(Q^+_\mu(F^\mu))$ is weakly precompact in $L^1((-1,1)\times\{v \in \mathbb{R}^3; \mid \xi \mid \geq \delta,$ $\mid v \mid \leq \frac{1}{\delta}\})$. And so $(\int F^\mu\varphi(x,v)dv)_{\mu\geq\mu_0}$ is compact in $L^1(-1,1)$ for any test function φ vanishing on $\mid \xi \mid \leq \delta$ and $\mid v \mid \geq \frac{1}{\delta}$.
To end the proof, the following three lemmas will be needed.

Lemma 5.2 *Given $\eta > 0$, there are μ_0, j_0 and constants c_0 and $\bar{c}$, such that for $\mu \geq \mu_0$,*

$$meas\{x \in (-1,1); \int_{|\xi| \leq \frac{1}{j_0}} K_\mu(v) F^\mu(x,v) dv \geq c_0 j_0 e^{j_0 \bar{c}}\} \leq \eta.$$

<u>Proof of Lemma 5.2.</u> By the exponential form of (5.1), there is a constant $\tilde{c}$ such that

$$\begin{aligned} F^\mu(x,v) &\leq F^\mu(1,v)e^{j\tilde{c}}, \quad \xi \geq \frac{1}{j}, \quad | v | \leq 101, \\ F^\mu(x,v) &\leq F^\mu(-1,v)e^{j\tilde{c}}, \quad \xi \leq -\frac{1}{j}, \quad | v | \leq 101. \end{aligned} \tag{5.3}$$

We shall prove Lemma 5.2 for $\bar{c} = 2\tilde{c}$ by contradiction. If the lemma does not hold, then for some $\eta > 0$ there are sequences $(B_j)_{j \in I\!N}$, and $(F^j)_{j \in I\!N}$, $(\mu_j)_{j \in I\!N}$ such that $\lim_{j \to \infty} \mu_j = \infty$, $F^j = F^{\mu_j}$, $B_j = B_{\mu_j}$, and $(S_j)_{j \in I\!N}$ with $| S_j | \geq \eta$, where

$$S_j := \{x \in (-1,1); \int_{|\xi| \leq \frac{1}{j}} K_j(v) F^j(x,v) dv \geq j e^{j\bar{c}}\}.$$

Recall that

$$\int \xi^2 F^j(x,v) dv \leq c, \quad j \in I\!N, \quad x \in (-1,1). \tag{5.4}$$

This implies that

$$\int_{|\xi| \geq \frac{1}{j}, \rho \leq 100} K_j(v) F^j(x,v) dv \leq c j e^{j\tilde{c}}. \tag{5.5}$$

Also, by the exponential form of (5.1),

$$\begin{aligned} F^j(x,v_*) &\geq c_1 F^j(-1,v_*), \quad \xi_* \geq \frac{1}{10}, \quad | v_* | \leq 10, \\ F^j(x,v_*) &\geq c_1 F^j(1,v_*), \quad \xi_* \leq -\frac{1}{10}, \quad | v_* | \leq 10. \end{aligned}$$

For (say) $\rho \geq 100$, and v_* such that $| \xi_* | \geq \frac{1}{10}$, $| v_* | \leq 10$, there is a set of $\omega \in S^2$ of measure (say) $\frac{1}{100} | S^2 |$ such that

$$| \xi' | \sim | \xi'_* | \sim | v | .$$

Then

$$K_j(v)F^j(x,v) \leq c_2 K_j(v)F^j(x,v)F^j(x,v_*)$$
$$\leq c_3 k(| \xi' |^\beta + | \xi'_* |^\beta) F^j(x,v')F^j(x,v'_*)$$
$$+\frac{c_4}{logk} B_j(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)}.$$

For x in half of S_j,

$$\int B_j(F^j(x,v)F^j(x,v_*) - F^j(x,v')F^j(x,v'_*))log\frac{F^j(x,v)F^j(x,v_*)}{F^j(x,v')F^j(x,v'_*)} dvdv_*d\omega$$
$$\leq c \int K_j(v)F^j(x,v)dvd\omega.$$

Hence

$$\int_{\rho \geq 100} K_j(v)F^j(x,v)dv$$
$$\leq kc_5(\int_{|\xi|\geq 1} F^j(x,v)dv)(\int_{|\xi|\geq 1} | \xi |^\beta F^j(x,v)dv) + \frac{c_6}{logk}\int K_j(v)F^j(x,v)dv. \tag{5.6}$$

Choose k so that $\frac{c_6}{logk} \leq \frac{1}{2}$. For the above x-es, it follows from (5.5) and (5.6) that

$$\int K_j(v)F^j(x,v)dv \leq \int_{|\xi|\leq\frac{1}{j}} K_j(v)F^j(x,v)dv + kc_7 + \frac{1}{2}\int K_j(v)F^j(x,v)dv + c_8 j e^{j\tilde{c}},$$

so that

$$\int K_j(v)F^j(x,v)dv \leq 2\int_{|\xi|\leq\frac{1}{j}} K_j(v)F^j(x,v)dv + c_9 j e^{j\tilde{c}} \leq 3\int_{|\xi|\leq\frac{1}{j}} K_j(v)F^j(x,v)dv,$$

by the definition of S_j. From here the proof follows the lines of the proof of (4.3) in Lemma 4.3, and using a variant of (5.6) for Y_j. Again the assumption $| S_j | \geq \eta$ for $j \in I\!N$ leads to a contradiction. This completes the proof of the lemma. $\Box$

Lemma 5.3 *Given $c > 0$ and $\eta > 0$, there is μ_0 such that for $\mu > \mu_0$ and outside a μ-dependent set in x of measure less than η,*

$$\int_{|\rho_*|>\lambda} B_\mu F^\mu(x,v_*)dv_*d\omega$$

tends to zero when $\lambda \to \infty$, uniformly with respect to $| v | \leq c$, x and $\mu > \mu_0$.

Proof of Lemma 5.3. For $| v | \leq c$, $B_\mu(v - v_*)$ is of the same magnitude as $K_\mu(v_*)$. Then the proof of Lemma 5.3 follows the lines of the proof of Lemma 4.4. $\Box$

Lemma 5.4 *Given $c > 0, \lambda > 0$, and $\eta > 0$, there is μ_0 such that for $\mu > \mu_0$ and outside a μ-dependent set in x of measure less than η,*

$$\int_{|\rho_*| \leq \lambda, |\xi_*| \leq \frac{1}{j}} B_\mu F^\mu(x, v_*) dv_* d\omega$$

tends to zero when $j \to \infty$, uniformly with respect to $| v | \leq c$, x and $\mu > \mu_0$.

Proof of Lemma 5.4. The proof follows the lines of the proof of Lemma 4.5, after noticing that

$$B_\mu(v - v_*, \omega) \leq cb(\theta), \quad | v | \leq c, \quad | \rho_* | \leq \lambda, \quad | \xi_* | \leq \frac{1}{j}, \quad \mu \geq \mu_0. \ \Box$$

End of proof for hard forces. Using the weak L^1 compactness of (F^{μ_n}), $(Q^{\pm}(F^{\mu_n}))$, (5.4), and Lemma 5.3, it follows for some sequence μ_n tending to $+\infty$ with n, that

$$\int_{|\rho_*| \leq \lambda, |\xi_*| \geq \frac{1}{j}} B_{\mu_n} F^{\mu_n}(x, v_*) dv_* \to \int_{|\rho_*| \leq \lambda, |\xi_*| \geq \frac{1}{j}} BF(x, v_*) dv_*$$

in $L^1((-1, 1) \times \{v \in I\!R^3; | v | \leq c\})$ for $c > 0$. This convergence, together with the results from Lemma 5.2-4, imply that for $| v | \leq c$,

$$\frac{\int B_{\mu_n} F^{\mu_n}(x, v_*) dv_* d\omega}{\int K_{\mu_n}(v_*) F^{\mu_n}(x, v_*) dv_*} \to \frac{\int BF(x, v_*) dv_* d\omega}{\int K(v_*) F(x, v_*) dv_*} (\leq c_0),$$

in measure on $[-1, 1]$ when $n \to \infty$. Together with the weak compactness in $L^1([-1, 1] \times \{v \in I\!R^3; | \xi | \geq \delta, | v | \leq \frac{1}{\delta}\})$, the convergence in measure implies that if φ is a test function in $C^1([-1, 1], L^\infty(I\!R^3))$ vanishing for $| \xi | \leq \delta$ and for $| v | \geq \frac{1}{\delta}$, then

$$\int Q_n^-(F^{\mu_n}) \varphi(x, v) dx dv \to \int Q^-(F) \varphi(x, v) dx dv, \quad n \to \infty.$$

The above argument holds for a subsequence of (μ_n) if, instead of B_{μ_n}, we use $\varphi(x, v') B_{\mu_n}$ throughout. And so for a subsequence of (μ_n),

$$\int Q_n^+(F^{\mu_n}) \varphi(x, v) dx dv \to \int Q^+(F) \varphi(x, v) dx dv, \quad n \to \infty.$$

As in the pseudo-maxwellian case, we may conclude that F satisfies (3.11-12). This implies that F is a weak solution to the stationary Boltzmann equation with maxwellian diffuse reflection boundary conditions in the hard force case (for test functions having compact support and vanish for ξ small). That in turn implies that F is a mild solution. On the other hand, the integrability properties of $Q^{\pm}(F,F)$ in the above weak solutions, are stronger than what is required for a mild solution. Hence the present solutions are somewhat stronger than mild solutions.$\square$

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