Available online at www.sciencedirect.com
ScienceDirect
Mathematical and Computer Modelling 47 (2008) 515–529
MATHEMATICAL AND COMPUTER MODELLING
www.elsevier.com/locate/mcm

An existence result for a quantum BGK model

Anne Nouri*

LATP, Université d'Aix-Marseillle I, Marseille, France

Received 13 April 2007; accepted 11 May 2007

Abstract

A class of kinetic BGK models for Bose–Einstein statistics are considered for the stationary frame in a slab. Existence of bounded measure solutions is proven for Planckian diffuse boundary conditions with a given total inflow. Compactness properties are extracted from the L^1 part of the generalized Planckian distribution function and the boundary behavior.

Keywords: Quantum kinetic equation; BGK equation

1. Introduction

The experiments realizing Bose–Einstein condensates in atomic vapors [1–3] ten years ago have renewed interest in the theory of dilute quantum gases at low temperatures. Generalizing a work by Bose on photons [4], Einstein proved that a non-interacting Bose gas at sufficiently low temperature undergoes a phase transition [5,6]. Part of the particles falls into the ground state of zero momentum, to minimize the physical entropy. As pointed out by Volovik [7], 'the first quantization scheme for hydrodynamics was suggested by Landau in 1941 [8] when he developed the theory of superfluidity in liquid He_4. In his approach Landau separated liquid He_4 into two parts: the ground state (which we now call quantum vacuum) and quasi-particles — excitations above the ground state. The Landau approach was essentially different from that of Tisza [9,10], who suggested separating the liquid He_4 into the Bose condensate and the non-condensate atoms. Tisza's approach does make sense, especially for the dilute Bose gases where the condensate fraction can be easily detected. However, it is important that the dynamics of the Bose condensate and the exchange of energy and atoms between the condensated and non-condensated fractions belong to high energy microscopic physics. On the other, hand the low-energy behavior of the superfluid liquids is governed by the Landau hydrodynamic picture'.

So far, mathematical works in quantum kinetic theory have essentially focused on the time evolution of the distribution function of bosons satisfying the kinetic equation introduced by Nordheim [11], and then by Uehling and Uhlenbeck [12]. Results on the derivation of this equation from the interaction of a large system of bosons in a weak coupling regime are given in Benedetto [13]. The difficulty in solving this equation comes from the fact that the conservations of mass, momentum, and energy, and the physical entropy decrease, allow bounded measure

* Fax: +33 4 91 11 35 52.
E-mail address: anne.nouri@cmi.univ-mrs.fr.

0895-7177/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2007.05.002

solutions. Such solutions are even expected to describe Bose–Einstein condensation. It is then difficult to give a sense to the collision term containing cubic terms in the distribution function. Mild solutions to the Uehling–Uhlenbeck quantum equation are given by Lu [14] in a space homogeneous, isotropic in momentum space, for a collision kernel with cut-off. Then distributional solutions for hard spheres are derived [15]. The definition of such solutions is made possible by a Carleman representation [16], and the space homogeneous isotropic assumption. Existence results and long- time behavior are also derived by Escobedo et al. [17] for space homogeneous and isotropic solutions, with a truncation assumption on the physical kernel. The same equation, linearized around a power like steady state, is studied in Escobedo [18].

Given the mathematical difficulty for solving the Uehling–Uhlenbeck equation in a space-dependent frame, it is relevant to consider a relaxation model having similar properties. This has been done in the classical mechanics frame when studying the BGK equation [19,20]. It is also used in numerical simulations [21] due to its simplicity as compared to the Boltzmann equation. In quantum theory, Khalatnikov [22] suggested a relaxation model replacing the Uehling–Uhlenbeck collision operator by the difference $\mathcal{P} - F$, where $\mathcal{P}$ is a generalized Planckian distribution function having the same momenta as F.

In this paper, we address the problem of solving a kinetic quantum BGK model in a space-dependent stationary frame.

The rest of the paper is organized as follows. Section 2 is an introduction to quantum BGK models. In Section 3, a first such model is introduced which, however, does not capture the zero-point energy aspect of the underlying physics. The model is analyzed and a compactness result is proven. A variant of the model is introduced in Section 4, which also captures the zero-point energy aspect, and an existence theorem is proven for that case.

The method can be used for the same equation with other boundary conditions. The paper also provides a background for future developments, with work in progress for the Fermi–Dirac case and the stationary classical (Boltzmann) BGK equation.

2. The model

A gas of bosons can be described by a modified Boltzmann equation which takes quantum effects into account. It was introduced by L.W. Nordheim and by E.A. Uehling and G.E. Uhlenbeck, and writes

$$\frac{\partial F}{\partial t} + p \cdot \nabla_x F = Q(F(t, x, \cdot))(p),$$

where $F = F(t, x, p)$ is the particle density in the momentum space at time t and space x, and

$$Q(F)(p) = \int_{(\mathbb{R}^3)^3} W(p, p_*, p', p'_*) q(F) \mathrm{d}p_* \mathrm{d}p' \mathrm{d}p'_*,$$
$$q(F) = F(p')F(p'_*)(1 + \epsilon F(p))(1 + \epsilon F(p_*)) - F(p)F(p_*)(1 + \epsilon F(p'))(1 + \epsilon F(p'_*)),$$
$$W(p, p_*, p', p'_*) = w(p, p_*, p', p'_*)\delta(p + p_* - p' - p'_*)\delta(p^2 + p_*^2 - p'^2 - p'^2_*).$$

Here, $\epsilon = \frac{h^3}{8\pi^3 m^3 g}$, where h is the Planck constant, m is the mass of a particle and g the statistical weight. In the rest of this section, ϵ will be taken as equal to 1. The differential cross-section w is given, depending on the kind of interactions between particles under consideration. The collision operator Q induces the conservation with time of mass, momentum and energy, and the decrease of the physical entropy $\int H(F)(t, x, p)\mathrm{d}x\mathrm{d}p$, where

$$H(F) := F \ln F - (1 + F) \ln(1 + F).$$

Moreover, for given mass, momentum, and energy, Bose–Einstein equilibrium distribution functions minimize the entropy.

Lemma 2.1. *For any N, $E > 0$, $P \in \mathbb{R}^3$, there is a Bose–Einstein distribution function*

$$\tilde{P}(p) = \frac{1}{\mathrm{e}^{a|p - \frac{P}{N}|^2 + b^+} - 1} + b^- \delta_{\frac{P}{N}},$$

with $a \in \mathbb{R}_+$, $b \in \mathbb{R}^3$, $b^+ = \max\{b, 0\}$, $b^- = \max\{-b, 0\}$, *which satisfies*

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p - \frac{P}{N} \\ \frac{1}{2m}\left|p - \frac{P}{N}\right|^2 \end{pmatrix} F(p)\mathrm{d}p = \begin{pmatrix} N \\ 0 \\ E - \frac{P^2}{2mN} \end{pmatrix}. \tag{2.1}$$

It is the unique solution of the entropy minimization problem

$$H(\tilde{P}) = \min\{H(F);\ F \text{ satisfies } (2.1)\}.$$

For a proof of Lemma 2.1, we refer to [17].

Similarly to the classical kinetic theory [23], a BGK type model can be introduced in the quantum case. It is the following relaxation model,

$$\frac{\partial F}{\partial t} + p \cdot \nabla_x F = \tilde{P}(F) - F, \tag{2.2}$$

where $\tilde{P}(F)$ is the Bose–Einstein distribution function having the same momenta as F, introduced in Lemma 2.1. It displays the right physical properties, i.e. the conservation of mass, momentum, and energy, together with the decrease of the entropy. Indeed, multiplying (2.2) by $\ln \frac{1+F}{F}$ and integrating on $[0, t] \times \mathbb{R}^3_x \times \mathbb{R}^3_p$ gives

$$\begin{aligned} \int H(F)(0, x, p)\mathrm{d}x\mathrm{d}p - \int H(F)(t, x, p)\mathrm{d}x\mathrm{d}p &= \int_0^t \int (\tilde{P}(F) - F) \ln \frac{1+F}{F} \mathrm{d}x\mathrm{d}p\mathrm{d}s \\ &= \int_0^t \int (\tilde{P}(F) - F)\left(\ln \frac{1+F}{F} - \ln \frac{1+\tilde{P}(F)}{\tilde{P}(F)}\right)\mathrm{d}x\mathrm{d}p\mathrm{d}s \\ &\geq 0, \end{aligned}$$

since $\ln \frac{1+\tilde{P}(F)}{\tilde{P}(F)}$ is a linear combination of 1, p and p^2.

In this paper, we consider stationary solutions to (2.2) in the slab, i.e. distribution functions $F(x, p)$ satisfying

$$p_1 \frac{\partial F}{\partial x} = \tilde{P}(F) - F, \quad x \in [-1, 1],\ p \in \mathbb{R}^3, \tag{2.3}$$

and Planckian diffuse reflexion boundary conditions

$$\begin{aligned} F(-1, p) &= P_-(p) \int_{p'_1 < 0} |p'_1| F(-1, p')\mathrm{d}p', \quad p_1 > 0, \\ F(1, p) &= P_+(p) \int_{p'_1 > 0} p'_1 F(1, p')\mathrm{d}p', \quad p_1 < 0. \end{aligned} \tag{2.4}$$

Here, $P_-(p) = \frac{1}{\mathrm{e}^{u_- p^2 + v_-} - 1}$ and $P_+(p) = \frac{1}{\mathrm{e}^{u_+ p^2 + v_{+-}} - 1}$ are given Planckian distribution functions, chosen so that

$$\int_{p_1 > 0} p_1 P_-(p)\mathrm{d}p = \int_{p_1 < 0} |p_1| P_+(p)\mathrm{d}p = 1.$$

Moreover, F is required to have a fixed total inflow (e.g. equal to one),

$$\int_{p_1 > 0} p_1 F(-1, p)\mathrm{d}p + \int_{p_1 < 0} |p_1| F(1, p)\mathrm{d}p = 1. \tag{2.5}$$

The model is further simplified by considering $F(x, p)$ such that

$$P(x) := \int pF(x, p)\mathrm{d}p = 0, \quad x \in [-1, 1].$$

The property $P_1 \equiv 0$ is justified by the integration of (2.3) with respect to p_1, and the boundary conditions (2.4). The property $P_i \equiv 0,\ 2 \le i \le 3$ is justified by the following remark. Introduce the functions $K(x, p_1) := \int p_2 F(x, p)\mathrm{d}p_2\mathrm{d}p_3$ and $L(x, p_1) := \int p_3 F(x, p)\mathrm{d}p_2\mathrm{d}p_3$. They should be solutions to

$$p_1 \frac{\partial K}{\partial x} = \frac{\pi P_2}{Na} \sum_{n\ge 1} \frac{1}{n} \mathrm{e}^{-n\left(a(p_1-\frac{P_1}{N})^2+b^+\right)} - b^- \frac{P_2}{N}\delta_{p_1=\frac{P_1}{N}} - K,$$
$$K(-1, p_1) = 0, \quad p_1 > 0, \qquad K(1, p_1) = 0, \quad p_1 < 0,$$

and a similar system for L. Since

$$P_2(x) = \int K(x, p_1)\mathrm{d}p_1, \qquad \text{resp. } P_3(x) = \int L(x, p_1)\mathrm{d}p_1,$$

a trivial solution to these systems is $K = L \equiv 0$. Choosing it leads to $P_2 = P_3 \equiv 0$.

Then the Bose–Einstein distribution $\tilde{P}(F)$ having the same mass N and energy E as F is defined by

$$\tilde{P}(F)(p) = R_{N,E}(p) + b^-\delta_{p=0},$$

where

$$R_{N,E}(p) = \frac{1}{\mathrm{e}^{(\int \frac{q^2}{\mathrm{e}^{q^2+b^+}-1}\mathrm{d}q)^{\frac{2}{5}} E^{-\frac{2}{5}} p^2+b^+} - 1},$$
$$\beta(b^+) = \frac{N}{E^{\frac{3}{5}}} \quad \text{and} \quad b^- = 0, \quad \text{if } \frac{N}{E^{\frac{3}{5}}} \le 1,$$
$$b^+ = 0 \quad \text{and} \quad b^- = N - E^{\frac{3}{5}}, \quad \text{if } \frac{N}{E^{\frac{3}{5}}} \ge 1. \tag{2.6}$$

Here, the function β is given on $\mathbb{R}^+$ by

$$\beta(s) = \frac{\int \frac{\mathrm{d}q}{\mathrm{e}^{q^2+s}-1}}{\left(\int \frac{q^2}{\mathrm{e}^{q^2+s}-1}\mathrm{d}q\right)^{\frac{3}{5}}}.$$

It is proven in [14] that β is a decreasing function from $\beta(0)$ to 0. For the sake of simplicity, $\beta(0)$ will be taken as equal to 1 when suitable in the rest of the paper.

3. Analysis of the model

We consider solutions F to (2.3)–(2.5), which can be written

$$F(x, p) = \bar{F}(x, p) + n(x)\delta_{p=0},$$

where $\bar{F}$ is a finite measure with zero measure at $p = 0$ and n a finite measure in the variable x. The purpose of this section is to identify n and give a priori bounds on $\bar{F}$. In particular, it is shown that $\bar{F}$ is an L^1 function, and that n has an arbitrary mass.

Proposition 3.1. *The $\bar{F}$ part of F satisfies*

$$p_1 \frac{\partial \bar{F}}{\partial x} = R_{\bar{N},\bar{E}} - \bar{F}, \quad \bar{N} \le \bar{E}^{\frac{3}{5}},$$
$$\bar{F}(-1, p) = P_-(p) \int_{p_1'<0} |p_1'|\bar{F}(-1, p')\mathrm{d}p', \quad p_1 > 0,$$
$$\bar{F}(1, p) = P_+(p) \int_{p_1'>0} p_1'\bar{F}(1, p')\mathrm{d}p', \quad p_1 < 0, \tag{3.1}$$
$$\int_{p_1>0} p_1\bar{F}(-1, p)\mathrm{d}p + \int_{p_1<0} |p_1|\bar{F}(1, p)\mathrm{d}p = 1,$$

where $\bar{N}(x) = \int \bar{F}(x, p)\mathrm{d}p$ *and* $\bar{E}(x) = \int p^2 \bar{F}(x, p)\mathrm{d}p$*, whereas the* $n(x)\delta_{p=0}$ *part of* F *is given by*

$$n = \chi_{\bar{N}=\bar{E}^{\frac{3}{5}}}(\max\{N, \bar{E}^{\frac{3}{5}}\} - \bar{E}^{\frac{3}{5}}), \tag{3.2}$$

for any L^1 *function* N.

Proof of Proposition 3.1. Since $p_1 F = p_1 \bar{F}$, Eq. (2.3) writes

$$\frac{\partial}{\partial x}(p_1 \bar{F}) = R_{N,E} - \bar{F} + (b^- - n)\delta_{p=0}. \tag{3.3}$$

Let φ_1 be a test function in $C^1([-1, 1])$. Let a sequence (φ_2^j) of C^1 functions with compact support in $\mathbb{R}^3$ satisfy $\varphi_2^j(p) = 1$ for $|p_1| \le \frac{1}{j} - \frac{1}{j^2}$ and $\varphi_2^j(p) = 0$ for $|p_1| \ge \frac{1}{j}$. Let $\varphi^j(x, p) = \varphi_1(x)\varphi_2^j(p)$. Multiply (3.3) by φ^j and integrate on $[-1, 1] \times \mathbb{R}^3$. First,

$$\left\langle \frac{\partial}{\partial x}(p_1 \bar{F}), \varphi^j \right\rangle = -\langle p_1 \bar{F}, \varphi_1'(x)\varphi_2^j(p)\rangle \to_{j\to+\infty} 0,$$

since $\bar{F}$ is of zero measure at $p = 0$. Analogously,

$$\lim_{j\to+\infty} \int (R_{N,E} - \bar{F})\varphi^j \mathrm{d}x\mathrm{d}p = 0.$$

Then,

$$\langle (b^- - n)\delta_{p=0}, \varphi^j\rangle = \langle b^- - n, \varphi_1\rangle.$$

And so,

$$\langle b^- - n, \varphi_1\rangle = 0, \quad \varphi_1 \in C^1([-1, 1]).$$

It is then a classical argument to conclude that $b^- - n = 0$. Consequently, F gives

$$F = \bar{F} + \chi_{N\ge E^{\frac{3}{5}}}(N - E^{\frac{3}{5}})\delta_{p=0},$$

with $\bar{F}$ of zero measure at $p = 0$. Integrating the previous equation with respect to p implies that

$$\bar{N} = N \quad \text{if } N \le E^{\frac{3}{5}},$$

and

$$\bar{N} = E^{\frac{3}{5}} \quad \text{if } N \ge E^{\frac{3}{5}}.$$

Since $E = \bar{E}$, this shows that $\bar{N} \le \bar{E}^{\frac{3}{5}}$. Finally, $b^+_{N,E} = \bar{b}^+_{\bar{N},\bar{E}}$. Indeed, either $N \le E^{\frac{3}{5}}$, then $\beta(\bar{b}^+) = \frac{\bar{N}}{\bar{E}^{\frac{3}{5}}} = \frac{N}{E^{\frac{3}{5}}} = \beta(b^+)$. Or $N \ge E^{\frac{3}{5}}$, then $b^+ = 0$ and $\bar{N} = E^{\frac{3}{5}}$, so that $\bar{b}^+ = 0$. This proves that $R_{N,E} = R_{\bar{N},\bar{E}}$, so that $\bar{F}$ should be solution to the closed system (3.1). Then the sets $\{N \ge E^{\frac{3}{5}}\}$ and $\{\bar{N} = \bar{E}^{\frac{3}{5}}\}$ are equal. Indeed, it has just been seen that $\{N \ge E^{\frac{3}{5}}\} \subset \{\bar{N} = \bar{E}^{\frac{3}{5}}\}$. Reciprocally, $N < \bar{E}^{\frac{3}{5}}$ implies that $\bar{N} = N < \bar{E}^{\frac{3}{5}}$, so that $\bar{N} \ne \bar{E}^{\frac{3}{5}}$. And so, $F = \bar{F} + \chi_{\bar{N}=\bar{E}^{\frac{3}{5}}}(\max\{N, \bar{E}^{\frac{3}{5}}\} - \bar{E}^{\frac{3}{5}})\delta_{p=0}$. □

Remark. Solving (3.1) provides the location in space of the Dirac part of F at $p = 0$, namely

$$\{x \in [-1, 1]; \bar{N}(x) = \bar{E}^{\frac{3}{5}}(x)\}.$$

This does not capture the zero point energy aspect of the Dirac part coming from the Heisenberg uncertainty relation. In Section 4, a variant of the model is introduced that includes a zero point energy aspect. Also, the restriction of N to the set $\{\bar{N} = \bar{E}^{\frac{3}{5}}\}$ remains undetermined.

The condition $\bar{N} \le \bar{E}^{\frac{3}{5}}$ reflects a phase transition in terms of a critical energy.

Proposition 3.2. *Let (A^j) be a given sequence of $L^1([-1,1])$ functions. Let (B^j) be a bounded sequence in $L^1([-1,1])$, and (α^j), (β^j) (resp. (γ^j), (η^j)) be uniformly equi-integrable sequences in $L^1((0,+\infty))$ (resp. $L^1((-\infty,0))$). Let (G^j) (resp. (H^j)) be solutions to*

$$
\begin{aligned}
&p_1\frac{\partial G^j}{\partial x} = \int R_{A^j,B^j}\,\mathrm{d}p_2\mathrm{d}p_3 - G^j, \quad x\in[-1,1],\ p_1\in\mathbb{R},\\
&G^j(-1,p_1) = \alpha^j(p_1), \quad p_1>0,\\
&G^j(1,p_1) = \gamma^j(p_1), \quad p_1<0,
\end{aligned}
$$

(resp.

$$
\begin{aligned}
&p_1\frac{\partial H^j}{\partial x} = \int (p_2^2+p_3^2)R_{A^j,B^j}\,\mathrm{d}p_2\mathrm{d}p_3 - H^j, \quad x\in[-1,1],\ p_1\in\mathbb{R},\\
&H^j(-1,p_1) = \beta^j(p_1), \quad p_1>0,\\
&H^j(1,p_1) = \eta^j(p_1), \quad p_1<0.
\end{aligned}
$$

Then (G^j) (resp. (H^j)) is weakly compact in $L^1_{\mathrm{loc}}([-1,1]\times\mathbb{R})$, and for every ψ in $L^\infty(\mathbb{R})$ with compact support, $(\int\psi(p)G^j(x,p_1)\mathrm{d}p_1)$ (resp. $(\int\psi(p)H^j(x,p_1)\mathrm{d}p_1)$) is relatively compact in $L^1([-1,1])$.

Proof of Proposition 3.2. Denote by

$$
\begin{aligned}
&S^j(x,p_1) = \int R_{A^j,B^j}\,\mathrm{d}p_2\mathrm{d}p_3, \qquad T^j(x,p_1) = \int (p_2^2+p_3^2)R_{A^j,B^j}\,\mathrm{d}p_2\mathrm{d}p_3,\\
&a^j(x) = \left(\int \frac{u^2\mathrm{d}u}{\mathrm{e}^{u^2+b^{+j}(x)}-1}\right)^{\frac{2}{5}}(B^j(x))^{-\frac{2}{5}},\\
&b^{+j}=0 \quad \text{if } A^j\ge (B^j)^{\frac{3}{5}}, \qquad \beta(b^{+j}) = \frac{A^j}{(B^j)^{\frac{3}{5}}} \text{ else.}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&S^j(x,p_1)\le c(B^j(x))^{\frac{2}{5}}\sum_{n\ge1}\frac{1}{n}\mathrm{e}^{-na^j(x)p_1^2}, \quad \text{a.a. } x\in[-1,1];\, b^{+j}(x)\le 1,\ \text{a.a. } p_1\in\mathbb{R},\\
&S^j(x,p_1)\le c(B^j(x))^{\frac{2}{5}}, \quad \text{a.a. } x\in[-1,1];\, b^{+j}(x)>1,\ \text{a.a. } p_1\in\mathbb{R},\\
&T^j(x,p_1)\le c(B^j(x))^{\frac{4}{5}}, \quad \text{a.a. } x\in[-1,1],\ p_1\in\mathbb{R}.
\end{aligned}
\tag{3.4}
$$

Indeed,

$$
S^j(x,p_1) = \frac{1}{2a^j}\sum_{n\ge1}\frac{1}{n}\mathrm{e}^{-n(a^jp_1^2+b^{+j})}.
$$

Hence, the first equation of (3.4) holds. Then

$$
\begin{aligned}
S^j(x,p_1) &\le \frac{1}{2a^j}\sum_{n\ge1}\mathrm{e}^{-nb^{+j}} = \frac{1}{2a^j(\mathrm{e}^{b^{+j}}-1)}\\
&= \frac{(B^j(x))^{\frac{2}{5}}}{2\left(\int\frac{u^2\mathrm{d}u}{\mathrm{e}^{u^2+b^{+j}}-1}\right)^{\frac{2}{5}}(\mathrm{e}^{b^{+j}}-1)}.
\end{aligned}
$$

For x such that $b^{+j}(x)>1$, it holds that

$$
\left(\int\frac{u^2\mathrm{d}u}{\mathrm{e}^{u^2+b^{+j}}-1}\right)^{\frac{2}{5}}(\mathrm{e}^{b^{+j}}-1) \ge \left(\int u^2\mathrm{e}^{-u^2}\mathrm{d}u\right)^{\frac{2}{5}}(\mathrm{e}^{\frac{3}{5}b^{+j}}-\mathrm{e}^{-\frac{2}{5}b^{+j}}) \ge c.
$$

Hence,

$$S^j(x,p_1) \le c(B^j(x))^{\frac{2}{5}}, \quad \text{a.a. } x \in [-1,1];\ b^{+j}(x) > 1,\ \text{a.a. } p_1 \in \mathbb{R}.$$

Analogously,

$$\begin{aligned} T^j(x,p_1) &= \sum_{n\ge 1} \mathrm{e}^{-n(a^j p_1^2 + b^{+j})} \int (p_2^2+p_3^2)\mathrm{e}^{-na^j(p_2^2+p_3^2)} \mathrm{d}p_2 \mathrm{d}p_3 \\ &= \frac{1}{2(a^j)^2} \sum_{n\ge 1} \frac{1}{n^2} \mathrm{e}^{-n(a^j p_1^2 + b^{+j})} \le \frac{c}{(a^j)^2} \mathrm{e}^{-b^{+j}} \\ &= c \frac{(B^j(x))^{\frac{4}{5}} \mathrm{e}^{-b^{+j}}}{\left(\int \frac{u^2 \mathrm{d}u}{\mathrm{e}^{u^2+b^{+j}}-1}\right)^{\frac{4}{5}}} \le c \frac{(B^j(x))^{\frac{4}{5}} \mathrm{e}^{-\frac{1}{5}b^{+j}}}{\left(\int u^2 \mathrm{e}^{-u^2} \mathrm{d}u\right)^{\frac{4}{5}}} \\ &\le c(B^j(x))^{\frac{4}{5}}. \end{aligned}$$

Then, multiplying the equation satisfied by G^j (resp. H^j) by $(G^j)^{\frac{1}{4}}$ (resp. $(H^j)^{\frac{1}{4}}$) and integrating it over $(-1,1) \times [-\mu,\mu]$, for a fixed positive μ leads to

$$\begin{aligned} \int_{|p_1|<\mu} (G^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 &= \int_{|p_1|<\mu} S^j (G^j)^{\frac{1}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 \\ &\le \frac{4}{5}\int_{|p_1|<\mu} (S^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 + \frac{1}{5}\int_{|p_1|<\mu} (G^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1, \end{aligned}$$

and analogously for H^j. Hence,

$$\begin{aligned} \int_{|p_1|<\mu} (G^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 &\le \int_{|p_1|<\mu} (S^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 \\ &\le \int_{|p_1|<\mu, b^{+j}(x)>1} (B^j(x))^{\frac{1}{2}} \mathrm{d}x\mathrm{d}p_1 + \int_{b^{+j}(x)<1} (B^j(x))^{\frac{1}{2}} \\ &\quad \times \int \left(\sum_{n\ge 1} \frac{1}{n} \mathrm{e}^{-na^j(x)p_1^2}\right)^{\frac{5}{4}} \mathrm{d}p_1 \mathrm{d}x \\ &\le c\mu + c\int (B^j(x))^{\frac{7}{10}} \mathrm{d}x \int \left(\sum_{n\ge 1} \frac{1}{n} \mathrm{e}^{-nq_1^2}\right)^{\frac{5}{4}} \mathrm{d}q_1 \le c, \end{aligned}$$

since

$$\int \left(\sum_{n\ge 1} \frac{1}{n} \mathrm{e}^{-nq_1^2}\right)^{\frac{5}{4}} \mathrm{d}q_1 \le \int_{|q_1|<1} \frac{\mathrm{d}q_1}{\sqrt{|q_1|}} \left(\sum_{n\ge 1} \frac{1}{n^{\frac{6}{5}}}\right)^{\frac{5}{4}} + \int_{|q_1|>1} \frac{\mathrm{d}q_1}{(\mathrm{e}^{q_1^2}-1)^{\frac{5}{4}}} \le c.$$

Similarly,

$$\int_{|p_1|<\mu} (H^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 \le \int_{|p_1|<\mu} (T^j)^{\frac{5}{4}}(x,p_1)\mathrm{d}x\mathrm{d}p_1 \le c.$$

And so, (G^j) and (H^j) are locally equi-integrable with respect to the variable p_1. Moreover, (G^j) and $(p_1 \frac{\partial G^j}{\partial x})$ (resp. (H^j) and $(p_1 \frac{\partial H^j}{\partial x})$) are bounded in $L^1_{\mathrm{loc}}([-1,1]\times\mathbb{R})$. The use of an averaging lemma [24] ends the proof of Proposition 3.2. □

4. An existence theorem for a kinetic quantum BGK equation.

In this section, we consider the slab $[-\frac{\eta}{2}, \frac{\eta}{2}]$, and denote by I_F (resp. O_F) the total influx (resp. outflux) of a distribution function F at the boundaries, i.e.

$$I_F = \int_{p_1>0} p_1 F\left(-\frac{\eta}{2}, p\right) \mathrm{d}p + \int_{p_1<0} |p_1| F\left(\frac{\eta}{2}, p\right) \mathrm{d}p,$$
$$O_F = \int_{p_1<0} |p_1| F\left(-\frac{\eta}{2}, p\right) \mathrm{d}p + \int_{p_1>0} p_1 F\left(\frac{\eta}{2}, p\right) \mathrm{d}p.$$

We move the Dirac part in the Planckian distribution function in Sections 2 and 3 from $p = 0$ to $p = \pm(\frac{1}{n}, 0, 0)$. For any positive r, denote by $\mathcal{P}^r$

$$\mathcal{P}^r(p) = \frac{1}{\mathrm{e}^{ap^2+b^+} - 1} + \frac{1}{2} b^- \left(\delta_{p_1=r, p_2=p_3=0} + \delta_{p_1=-r, p_2=p_3=0}\right),$$

with a, b^+ and b^- defined as in (2.6). Hence,

$$\int \mathcal{P}^r(p)(1, p^2)\mathrm{d}p = (N, E + r^2(N - E^{\frac{3}{5}})\chi_{N \geq E^{\frac{3}{5}}}). \tag{4.1}$$

Theorem 4.1. *For some n_0 and any $n > n_0$, and for η small enough, there is a distribution function $F \in \mathcal{M}([-\frac{\eta}{2}, \frac{\eta}{2}] \times \mathbb{R}^3)$ is a solution to*

$$p_1 \frac{\partial F}{\partial x} = \mathcal{P}^{\frac{1}{n}}_{N_F, E_F - \frac{1}{n^2}(N_F - (E_F)^{\frac{3}{5}})\chi_{N_F \geq (E_F)^{\frac{3}{5}}}} - F, \quad x \in \left[-\frac{\eta}{2}, \frac{\eta}{2}\right], \; p \in \mathbb{R}^3,$$
$$F\left(-\frac{\eta}{2}, p\right) = P_-(p) \int_{p_1'<0} |p_1'| F\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 > 0,$$
$$F\left(\frac{\eta}{2}, p\right) = P_+(p) \int_{p_1'>0} p_1' F\left(\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 < 0,$$
$$I_F = 1.$$

Remark. Theorem 4.1 states the existence of a solution to the quantum BGK equation with Planckian diffuse boundary conditions and total inflow equal to one. This fixed total inflow is a driving mechanism in the proof of the theorem.

Proof of Theorem 4.1. The proof of Theorem 4.1 requires two preliminary lemmas.

Lemma 4.1. *Let $(\alpha, \gamma) \in L^1([-1, 1]) \times L^\infty([-1, 1])$ be given, such that*

$$\alpha_* \leq \alpha(x), \qquad \gamma_* \leq \gamma(x) \leq M,$$

where

$$\alpha_* = \min\left\{\int_{p_1>0} P_-(p)\mathrm{e}^{-\frac{\eta}{p_1}}\mathrm{d}p, \int_{p_1<0} P_+(p)\mathrm{e}^{\frac{\eta}{p_1}}\mathrm{d}p\right\},$$
$$\gamma_* = \min\left\{\int_{p_1>0} p^2 P_- \mathrm{e}^{-\frac{\eta}{p_1}}\mathrm{d}p, \int_{p_1<0} p^2 P_+ \mathrm{e}^{\frac{\eta}{p_1}}\mathrm{d}p\right\},$$

and $M > 0$ is given. Then there is a distribution function F and a positive number λ such that

$$p_1 \frac{\partial F}{\partial x} = \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha, \lambda^{\frac{5}{3}}\gamma} - F, \quad x \in \left[-\frac{\eta}{2}, \frac{\eta}{2}\right], \; p \in \mathbb{R}^3,$$

$$F\left(-\frac{\eta}{2}, p\right) = P_-(p)\int_{p_1'<0} |p_1'| F\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 > 0,$$
$$F\left(\frac{\eta}{2}, p\right) = P_+(p)\int_{p_1'>0} p_1' F\left(\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 < 0,$$
$$O_F = 1.$$

Moreover,

$$\lambda_* \le \lambda \le \lambda^*,$$

with λ_* (resp λ^*) only depending on α_*, $\int \alpha(x)\mathrm{d}x$, γ_* and M.

Proof of Lemma 4.1. Let $\lambda > 0$ be given. Let $F_0 = 0$. Define $(F_l)_{l\ge 1}$ by induction as the solutions to

$$p_1 \frac{\partial F_l}{\partial x} = \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha, \lambda^{\frac{5}{3}}\gamma} - F_l,$$
$$F_l\left(-\frac{\eta}{2}, p\right) = P_-(p)\int_{p_1'<0} |p_1'| F_{l-1}\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 > 0,$$
$$F_l\left(\frac{\eta}{2}, p\right) = P_+(p)\int_{p_1'>0} p_1' F_{l-1}\left(\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 < 0.$$

It defines a monotone Cauchy sequence in L^1, since $G_l := F_l - F_{l-1}$ satisfies

$$p_1 \frac{\partial G_l}{\partial x} = -G_l,$$
$$G_l\left(-\frac{\eta}{2}, p\right) = P_-(p)\int_{p_1'<0} |p_1'| G_{l-1}\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 > 0,$$
$$G_l\left(\frac{\eta}{2}, p\right) = P_+(p)\int_{p_1'>0} p_1' G_{l-1}\left(\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 < 0,$$

with

$$\int_{p_1'<0} |p_1'| G_l\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p' = \int_{p_1>0} p_1 \mathrm{e}^{-\frac{\eta}{p_1}} P_-(p)\mathrm{d}p \int_{p_1'>0} p_1' G_{l-1}\left(\frac{\eta}{2}, p'\right) \mathrm{d}p',$$
$$\int_{p_1'>0} p_1' G_l\left(\frac{\eta}{2}, p'\right) \mathrm{d}p' = \int_{p_1<0} |p_1| \mathrm{e}^{\frac{\eta}{p_1}} P_+(p)\mathrm{d}p \int_{p_1'<0} |p_1'| G_{l-1}\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p',$$

hence being convergent series. And so, (F_l) converges in L^1 to some F_λ, solution to

$$p_1 \frac{\partial F_\lambda}{\partial x} = \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha, \lambda^{\frac{5}{3}}\gamma} - F_\lambda,$$
$$F_\lambda\left(-\frac{\eta}{2}, p\right) = P_-(p)\int_{p_1'<0} |p_1'| F_\lambda\left(-\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 > 0,$$
$$F_\lambda\left(\frac{\eta}{2}, p\right) = P_+(p)\int_{p_1'>0} p_1' F_\lambda\left(\frac{\eta}{2}, p'\right) \mathrm{d}p', \quad p_1 < 0.$$

Let

$$\tilde{\mu}_\lambda = \int_{p_1>0} p_1 P_-(p)\mathrm{e}^{-\frac{\eta}{p_1}} \mathrm{d}p \int_{p_1<0} |p_1| F_\lambda\left(-\frac{\eta}{2}, p\right) \mathrm{d}p$$
$$+ \int_{p_1<0} |p_1| P_+(p)\mathrm{e}^{\frac{\eta}{p_1}} \mathrm{d}p \int_{p_1>0} p_1 F_\lambda\left(\frac{\eta}{2}, p\right) \mathrm{d}p,$$

and

$$Y=\left\{y\in\left[-\frac{\eta}{2},\frac{\eta}{2}\right];\ \frac{\alpha}{\gamma^{\frac{3}{5}}}(y)\geq 1\right\}.$$

$\tilde{\mu}_\lambda$ is a continuous strictly increasing function of λ, since it holds for the first iterate O_{G_1} and the scheme is monotone. Then the total outflow of F_λ is given by

$$O_{F_\lambda}=\tilde{\mu}_\lambda+\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int_{p_1>0}\frac{\mathrm{e}^{\frac{y-\frac{\eta}{2}}{p_1}}+\mathrm{e}^{-\frac{\frac{\eta}{2}+y}{p_1}}}{\mathrm{e}^{(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2+b^+}-1})^{\frac{2}{5}}\lambda^{-\frac{2}{3}}\gamma^{-\frac{2}{5}}p^2+b^+}-1}\mathrm{d}p\mathrm{d}y$$
$$+\lambda\int_Y(\mathrm{e}^{n\lambda^{-\frac{1}{3}}(y-\frac{\eta}{2})}+\mathrm{e}^{-n\lambda^{-\frac{1}{3}}(y+\frac{\eta}{2})})(\alpha-\gamma^{\frac{3}{5}})\mathrm{d}y,$$

with b^+ independent of λ. Hence, it is a continuous and strictly monotone function of λ, such that $O_{F_0}<1$ and $\lim_{\lambda\to+\infty}O_{F_\lambda}=+\infty$. Consequently, $O_{F_\lambda}=1$ for a unique λ. Moreover,

$$\beta(b^+)=\frac{\alpha}{\gamma^{\frac{3}{5}}}\geq\frac{\alpha_*}{M^{\frac{3}{5}}},$$

so that $b^+\leq b^*$ for some b^*. Then,

$$a=\lambda^{-\frac{2}{3}}\left(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2+b^+}-1}\right)^{\frac{2}{5}}\gamma^{-\frac{2}{5}}\in\left[\lambda^{-\frac{2}{3}}\left(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2+b^*}-1}\right)^{\frac{2}{5}}M^{-\frac{2}{5}},\lambda^{-\frac{2}{3}}\left(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2}-1}\right)^{\frac{2}{5}}\gamma_*^{-\frac{2}{5}}\right].$$

The property $O_{F_\lambda}=1$ implies that

$$\int_Y\int_{p_1>0}\frac{\mathrm{e}^{-\frac{\frac{\eta}{2}-y}{p_1}}}{\mathrm{e}^{a(y)p^2}-1}\mathrm{d}p\mathrm{d}y+\int_{Y^c}\int_{p_1>0}\frac{\mathrm{e}^{-\frac{\frac{\eta}{2}-y}{p_1}}}{\mathrm{e}^{a(y)p^2+b^*}-1}\mathrm{d}p\mathrm{d}y\leq 1.$$

Hence,

$$\lambda\left(\frac{\gamma_*}{\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2}-1}}\right)^{\frac{3}{5}}\int\int_{q_1>0}\frac{\mathrm{e}^{-\left(\frac{1}{\gamma_*}\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2}-1}\right)^{\frac{1}{5}}\lambda^{-\frac{1}{3}}\frac{\frac{\eta}{2}-y}{q_1}}}{\mathrm{e}^{q^2+b^*}-1}\mathrm{d}q\mathrm{d}y\leq 1.$$

And so, λ is bounded from above by a constant λ^* only depending on γ_* and M. Analogously,

$$\begin{aligned}\frac{1}{2}\left(1-\min\left\{\int_{p_1>0}p_1P_-(p)\mathrm{d}p,\int_{p_1<0}|p_1|P_+(p)\mathrm{d}p\right\}\right)&\leq\frac{1-\tilde{\mu}_\lambda}{2}\\&\leq\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int_{p_1>0}\frac{1}{\mathrm{e}^{ap^2}-1}\mathrm{d}p\mathrm{d}y+\lambda\int_Y(\alpha-\gamma^{\frac{3}{5}})(x)\mathrm{d}x\\&\leq\lambda\left(\gamma_*^{\frac{3}{5}}+\int\alpha(x)\mathrm{d}x\right).\end{aligned}$$

And so, λ is bounded from below by a constant λ_* only depending on $\int\alpha(x)\mathrm{d}x$ and γ_*.

Let

$$\Lambda=\left\{\left(\alpha_1,\alpha_2,\gamma+\frac{1}{n^2}(\alpha_1+\alpha_2-\gamma^{\frac{3}{5}})\chi_{\alpha_1+\alpha_2\geq\gamma^{\frac{3}{5}}}\right);\alpha_*\leq(\alpha_1+\alpha_2)(x),\int\alpha_1(x)\mathrm{d}x\leq 1,\right.$$
$$\left.\alpha_2(x)\leq M,\gamma(x)\leq M\right\},$$

where the constant M will be fixed during the proof of Lemma 4.2. Let $\mathcal{T}$ be the map that maps $(\alpha_1, \alpha_2, \gamma + \frac{1}{n^2}(\alpha_1 + \alpha_2 - \gamma^{\frac{3}{5}})\chi_{\alpha_1+\alpha_2\geq\gamma^{\frac{3}{5}}}) \in \Lambda$ into

$$(\tilde{\alpha}_1(x), \tilde{\alpha}_2(x), \tilde{\gamma}_1(x)) = \left(\int_{|p_1|<\delta} F(x,p)\mathrm{d}p, \int_{|p_1|>\delta} F(x,p)\mathrm{d}p, \int p^2 F(x,p)\mathrm{d}p\right),$$

where δ will be fixed during the proof of Lemma 4.2, and F is the solution of

$$\begin{aligned}
&p_1\frac{\partial F}{\partial x} = \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda(\alpha_1+\alpha_2),\lambda^{\frac{5}{3}}\gamma} - F,\\
&F\left(-\frac{\eta}{2}, p\right) = P_-(p)\int_{p_1'<0}|p_1'|F\left(-\frac{\eta}{2}, p'\right)\mathrm{d}p', \quad p_1>0,\\
&F\left(\frac{\eta}{2}, p\right) = P_+(p)\int_{p_1'>0}p_1'F\left(\frac{\eta}{2}, p'\right)\mathrm{d}p', \quad p_1<0,\\
&O_F = 1,
\end{aligned}$$

as defined in Lemma 4.1. □

Lemma 4.2. *There are constants δ and M such that $\mathcal{T}$ maps Λ into itself.*

Proof of Lemma 4.2. Consider constants δ strictly smaller than $\frac{\lambda_*^{\frac{1}{3}}}{n}$. Then

$$\begin{aligned}
\int \tilde{\alpha}_1(x)\mathrm{d}x &= \int_{|p_1|<\delta} \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}(x,p)\mathrm{d}x\mathrm{d}p + \int_{|p_1|<\delta} p_1 F\left(-\frac{\eta}{2}, p\right)\mathrm{d}p - \int_{|p_1|<\delta} p_1 F\left(\frac{\eta}{2}, p\right)\mathrm{d}p\\
&\leq \int_{|p_1|<\delta} \frac{\mathrm{d}x\mathrm{d}p}{\mathrm{e}^{a(x)p^2}-1} + c_\delta,
\end{aligned}$$

with

$$c_\delta = \int_{0<p_1<\delta} p_1 P_-(p)\mathrm{d}p + \int_{-\delta<p_1<0}|p_1|P_+(p)\mathrm{d}p.$$

Moreover,

$$\begin{aligned}
\int_{|p_1|<\delta}\frac{\mathrm{d}x\mathrm{d}p}{\mathrm{e}^{a(x)p^2}-1} &= \int a^{-\frac{3}{2}}(x)\int_{|q_1|<\sqrt{a(x)}\delta}\frac{\mathrm{d}q}{\mathrm{e}^{q^2+b^+}-1}\mathrm{d}x\\
&\leq \eta\lambda^* M^{\frac{3}{5}}\left(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2+b^*}-1}\right)^{-\frac{3}{5}}\int_{|q_1|<\lambda_*^{-\frac{1}{3}}(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2}-1})^{\frac{1}{5}}\gamma_*^{-\frac{1}{5}}\delta}\frac{\mathrm{d}q}{\mathrm{e}^{q^2}-1}.
\end{aligned}$$

Hence

$$\int\tilde{\alpha}_1(x)\mathrm{d}x \leq c_\delta + c_1\eta\lambda^* M^{\frac{3}{5}}\tilde{c}_\delta, \tag{4.2}$$

with c_δ and $\tilde{c}_\delta$ tending to zero when δ tends to zero. Then

$$\int\tilde{\alpha}_2(x)\mathrm{d}x \leq \frac{1}{\delta^2}\int p^2 F(x,p)\mathrm{d}p. \tag{4.3}$$

The control of $\int p^2F(x,p)\mathrm{d}p$ is done by studying $\int p_1^2 F(x,p)\mathrm{d}p$ and $\int(p_2^2+p_3^2)F(x,p)\mathrm{d}p$ separately. First,

$$\int p_1 F(x,p)\mathrm{d}p = \int_{-\frac{\eta}{2}}^{x}\int \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}(x,p)\mathrm{d}x\mathrm{d}p - \int_{-\frac{\eta}{2}}^{x}\int F(x,p)\mathrm{d}x\mathrm{d}p,$$

so that

$$\left|\int p_1 F(x,p)\mathrm{d}p\right| \le \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \int \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}(x,p)\mathrm{d}x\mathrm{d}p.$$

Then,

$$\begin{aligned}\int p_1^2 F(x,p)\mathrm{d}p &= \int p_1^2 F\left(-\frac{\eta}{2},p\right)\mathrm{d}p - \int_{-\frac{\eta}{2}}^{x}\int p_1 F(y,p)\mathrm{d}y\mathrm{d}p \\ &\le \int p_1^2 F\left(-\frac{\eta}{2},p\right)\mathrm{d}p + \eta\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \int \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}(x,p)\mathrm{d}x\mathrm{d}p.\end{aligned}$$

Moreover,

$$\int p_1^2 F\left(-\frac{\eta}{2},p\right)\mathrm{d}p \le \int_{p_1>0} p_1^2 P_-(p)\mathrm{d}p + \int_{-1<p_1<0}|p_1|F\left(-\frac{\eta}{2},p\right)\mathrm{d}p + \int_{p_1<-1}|p_1|^3 F\left(-\frac{\eta}{2},p\right)\mathrm{d}p.$$

Multiplying the equation satisfied by F by p^2 and integrating it leads to

$$\int_{p_1<0}|p_1|p^2 F\left(-\frac{\eta}{2},p\right)\mathrm{d}p \le c_2 + \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int p^2\mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}\mathrm{d}x\mathrm{d}p,$$

for some constant c_2. And so,

$$\int p_1^2 F(x,p)\mathrm{d}p \le c_3 + \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int (p^2+\eta)\mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}\mathrm{d}x\mathrm{d}p,$$

for some constant c_3. Then,

$$\begin{aligned}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int p^2\mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}\mathrm{d}x\mathrm{d}p &= \lambda^{\frac{5}{3}}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\left(\gamma + \frac{1}{n^2}(\alpha_1+\alpha_2-\gamma^{\frac{3}{5}})\right)(x)\mathrm{d}x \\ &\le \lambda^{\frac{5}{3}}\left(2\eta M + \frac{1}{n^2}\right),\end{aligned}$$

and

$$\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\int \mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}\mathrm{d}x\mathrm{d}p = \lambda\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}(\alpha_1+\alpha_2)(x)\mathrm{d}x \le \lambda(1+\eta M).$$

And so,

$$\int p_1^2 F(x,p)\mathrm{d}p \le c_3 + \frac{1}{n^2}(\lambda^*)^{\frac{5}{3}} + \eta\left(2(\lambda^*)^{\frac{5}{3}}M + \lambda^*(1+\eta M)\right). \tag{4.4}$$

Let

$$H(x,p_1) := \int (p_2^2+p_3^2)F(x,p)\mathrm{d}p_2\mathrm{d}p_3,$$

$$T(x,p_1) := \int (p_2^2+p_3^2)\mathcal{P}^{\frac{\lambda^{\frac{1}{3}}}{n}}_{\lambda\alpha,\lambda^{\frac{5}{3}}\gamma}(x,p)\mathrm{d}p_2\mathrm{d}p_3.$$

Then

$$p_1\frac{\partial H}{\partial x} = T(x,p_1) - H,$$

and (cf. Proposition 3.2)

$$T(x,p_1) \le c_4(\lambda^{\frac{5}{3}}\gamma(x))^{\frac{4}{5}}, \quad \text{a.a. } x\in\left(-\frac{\eta}{2},\frac{\eta}{2}\right),\ p_1\in\mathbb{R}.$$

Hence,

$$\begin{aligned}\int_{p_1>0} H(x,p_1)\mathrm{d}p_1 &\le c_5 + \int_{-\frac{\eta}{2}}^{x}\int_{p_1>0}\frac{1}{p_1}\mathrm{e}^{\frac{y-x}{p_1}}T(y,p_1)\mathrm{d}y\mathrm{d}p_1\\ &= c_5 + \int_{-\frac{\eta}{2}}^{x}\int_{p_1>\epsilon}\frac{1}{p_1}\mathrm{e}^{\frac{y-x}{p_1}}T(y,p_1)\mathrm{d}y\mathrm{d}p_1 + \int_{-\frac{\eta}{2}}^{x}\int_0^{\epsilon}\frac{1}{p_1}\mathrm{e}^{\frac{y-x}{p_1}}T(y,p_1)\mathrm{d}y\mathrm{d}p_1\\ &\le c_5 + \frac{1}{\epsilon}\int_{-\frac{\eta}{2}}^{x}\int_{p_1>0}T(y,p_1)\mathrm{d}y\mathrm{d}p_1 + c_4\lambda^{\frac{4}{3}}M^{\frac{4}{5}}\epsilon.\end{aligned}$$

And so,

$$\int H(x,p_1)\mathrm{d}p_1 \le c_6 + \frac{\eta}{\epsilon}(\lambda^*)^{\frac{5}{3}}M + 2c_4(\lambda^*)^{\frac{4}{3}}M^{\frac{4}{5}}\epsilon. \tag{4.5}$$

Taking (4.4–4.5) into account, Lemma 4.2 is proven by first choosing $\delta \in]0,1[$ such that $c_\delta < \frac{1}{2}$. So then $M = \frac{10}{\delta^2}(c_3+c_6)$, and then $n > \frac{(\lambda^*)^{\frac{5}{6}}}{\delta}\sqrt{\frac{10}{M}}$; thus
$\epsilon < \frac{M^{-\frac{4}{5}}\delta^2}{20c_4}(\lambda^*)^{-\frac{4}{3}}$, and finally $\eta \in]0,1[$ such that

$$\eta < \min\left\{\frac{\delta^2}{(\lambda^*)^{\frac{5}{3}}\left(\frac{1}{\epsilon}+2+(\lambda^*)^{-\frac{2}{3}}(M^{-1}+1)\right)}, \frac{1}{2c_1\lambda^* M^{\frac{3}{5}}\tilde{c}_\delta}\right\}. \quad \square$$

End of the proof of Theorem 4.1. The map $\mathcal{T}$ maps the closed, bounded, and convex set Λ into itself. Its continuity for the L^1 topology follows from the continuity of the map that maps $(\alpha_1,\alpha_2,\gamma)$ into λ_1 such that $O_{F_{\lambda_1}} = 1$ and the continuity of λ_1 into $(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\gamma}_1)$. The continuity of λ_1 with respect to $(\alpha_1,\alpha_2,\gamma)$ follows by the strict monotonicity of O_{F_λ} and its continuity with respect to λ. The map $\mathcal{T}$ is compact for the L^1 topology. Indeed, let $(\alpha_1^i,\alpha_2^i,\gamma^i)$ be a sequence in Λ. First,

$$\tilde{\alpha}_1^i = \beta^i + \bar{\beta}^i + \mu^i + \bar{\mu}^i + e_b,$$

where

$$\begin{aligned}\beta^i(x) &= \int_{0<p_1<\epsilon}\int_{-\frac{\eta}{2}}^{x}\frac{\mathrm{e}^{\frac{y-x}{p_1}}}{p_1(\mathrm{e}^{a(y)p^2+b^+(y)}-1)}\mathrm{d}y\mathrm{d}p,\\ \mu^i(x) &= \int_{p_1>\epsilon}\int_{-\frac{\eta}{2}}^{x}\frac{\mathrm{e}^{\frac{y-x}{p_1}}}{p_1(\mathrm{e}^{a(y)p^2+b^+(y)}-1)}\mathrm{d}y\mathrm{d}p,\\ e_b(x) &= \int_{p_1>0}P_-(p)\mathrm{e}^{-\frac{\frac{\eta}{2}+x}{p_1}}\mathrm{d}p\int_{p_1<0}|p_1|F\left(-\frac{\eta}{2},p\right)\mathrm{d}p\\ &\quad + \int_{p_1<0}P_+(p)\mathrm{e}^{\frac{\frac{\eta}{2}-x}{p_1}}\mathrm{d}p\int_{p_1>0}p_1F\left(\frac{\eta}{2},p\right)\mathrm{d}p,\end{aligned}$$

and $\bar{\beta}^i$ (resp. $\bar{\mu}^i$) defined analogously to β^i (resp. μ^i) for $-\epsilon < p_1 < 0$ (resp. $p_1 < -\epsilon$). For ϵ small enough, $\int \beta^i(x)\mathrm{d}x$ can be made arbitrarily small, uniformly with respect to i, since

$$\int \beta^i(x)\mathrm{d}x \le \lambda^*\left(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2+b^*}-1}\right)^{-\frac{3}{5}}M^{\frac{3}{5}}\int_{0<q_1<(\lambda^*)^{-\frac{1}{3}}\left(\int\frac{q^2\mathrm{d}q}{\mathrm{e}^{q^2}-1}\right)^{\frac{1}{5}}\gamma_*^{-\frac{1}{5}}\epsilon}\frac{q_1\mathrm{d}q}{\mathrm{e}^{q^2}-1}.$$

The sequence (μ^i) is bounded in $W^{1,1}([-1,1])$, and hence compact in L^1. Consequently, the sequence $(\tilde{\alpha}_1^i)$ is compact in L^1. Analogously, $(\tilde{\alpha}_2^i)$ and $(\int p_1^2F^i(x,p)\mathrm{d}p)$ are compact in L^1. Then the L^1 compactness of $(\int_{|p_1|<\lambda}(p_2^2+p_3^2)F^i(x,p)\mathrm{d}p)$ for any $\lambda > 0$ follows from Proposition 3.2. Finally, $(\int_{|p_1|>\lambda}(p_2^2+p_3^2)F^i(x,p)\mathrm{d}p)$ is compact in L^1. Indeed, for any $l \in]0, \frac{\eta}{2}[$, it holds that

$$\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}-l} \left| \int_{p_1>\lambda} (p_2^2+p_3^2)\left(F^i(x+l,p)-F^i(x,p)\right)\mathrm{d}p \right| \mathrm{d}x$$

$$\leq \eta \int_{p_1>\lambda} (p_2^2+p_3^2)F^i\left(-\frac{\eta}{2},p\right)(1-\mathrm{e}^{-\frac{l}{p_1}})\mathrm{d}p + \frac{1}{\lambda}\int (p_2^2+p_3^2)\left(\int_x^{x+l} \frac{\mathrm{d}y}{\mathrm{e}^{a^i(y)p^2+b^{+i}(y)}-1}\right)\mathrm{d}x\mathrm{d}p$$

$$+\frac{\eta}{\lambda}\int (p_2^2+p_3^2)\left(\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \frac{\mathrm{d}y}{\mathrm{e}^{a^i(y)p^2+b^{+i}(y)}-1}\right)(1-\mathrm{e}^{-\frac{l}{p_1}})\mathrm{d}p$$

$$\leq cl.$$

A similar treatment of the integral over $p_1 < -\lambda$ can be carried out.

A fixed point of $\mathcal{T}$ is a solution F to Theorem 4.1, since the integration w.r.t. $(x,p) \in (-\frac{\eta}{2}, \frac{\eta}{2}) \times \mathbb{R}^3$ of the equation satisfied by F, and the Planckian diffuse reflection boundary conditions satisfied by F, imply that $\lambda = 1$. Moreover, it follows from the integration of (4.2) on $(-\frac{\eta}{2}, \frac{\eta}{2}) \times \mathbb{R}^3$ that $I_F = O_F = 1$. □

Remark. Using analogous techniques as in the proofs of Lemma 4.2 and Theorem 4.1, the limit when $n \to +\infty$ can be performed in Theorem 4.1 for the L^1 part of the distribution function. If the solution F^n to Theorem 4.1 is split into

$$F^n(x,p) = \bar{F}^n(x,p) + \alpha^n(x)\delta_{p_1=\frac{1}{n},p_2=p_3=0} + \beta^n(x)\delta_{p_1=-\frac{1}{n},p_2=p_3=0},$$

it can be proven that $(\bar{F}^n)$ is weakly compact in $L^1_{\mathrm{loc}}((-\frac{\eta}{2}, \frac{\eta}{2}) \times \mathbb{R}^3)$, and that its limit $\bar{F}$ is solution to

$$p_1\frac{\partial \bar{F}}{\partial x} = \bar{R} - \bar{F}, \quad x \in \left[-\frac{\eta}{2}, \frac{\eta}{2}\right], \; p \in \mathbb{R}^3,$$

$$\bar{F}\left(-\frac{\eta}{2},p\right) = P_-(p)\int_{p_1'<0} |p_1'|\bar{F}\left(-\frac{\eta}{2},p'\right)\mathrm{d}p', \quad p_1 > 0,$$

$$\bar{F}\left(\frac{\eta}{2},p\right) = P_+(p)\int_{p_1'>0} p_1'\bar{F}\left(\frac{\eta}{2},p'\right)\mathrm{d}p', \quad p_1 < 0,$$

$$I_{\bar{F}} = 1,$$

for some function $\bar{R}$ of type (2.6) satisfying $\int \bar{R}\mathrm{d}x\mathrm{d}p = \int \bar{F}\mathrm{d}x\mathrm{d}p$. However, the effect on the L^1 part of the $\mathcal{P}^{\frac{1}{n}}$ from the Dirac part of F^n, might still be present in the limit $\bar{R}$. The Heisenberg uncertainty relation is also captured in that way.

References

[1] M.H. Anderson, et al., Observation of Bose–Einstein condensation in a dilute atomic vapor, Science 269 (1995) 198–201.
[2] C.C. Bradley, C.A. Sackett, J.J. Tollet, R.G. Hulet, Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions, Phys. Rev. Lett. 75 (1995) 1687–1690.
[3] K.B. Davies, et al., Bose–Einstein condensation in a gas of sodium atoms, Phys. Rev. Lett. 75 (1995) 3969–3973.
[4] S.N. Bose, Plancks Gestez und Lichtquantenhypothese, Z. Phys. 26 (1924) 178–181.
[5] A. Einstein, Quantentheorie des einatomigen idealen gases I, Sitz. Kgl. Preuss. Akad. Wiss. (1924) 261–267.
[6] A. Einstein, Quantentheorie des einatomigen idealen gases II, Sitz. Kgl. Preuss. Akad. Wiss. (1925) 3–14.
[7] G. Volovik, From quantum hydrodynamics to quantum gravity, Los Alamos, 2006. Archive. gr-qc 0612134.
[8] L.D. Landau, Theory of superfluid helium II, Phys. Rev. II Ser. 60 (1941) 356–358.
[9] L. Tisza, Sur la théorie des liquides quantiques. Application à l'hélium liquide, J. Phys. Radium VIII Ser. 1 (1940) 164–172.
[10] L. Tisza, Sur la théorie des liquides quantiques. Application à l'hélium liquide, J. Phys. Radium VIII Ser. 1 (1940) 350–358.
[11] L.W. Nordheim, On the kinetic method in the new statistics and its applications in the electron theory of conductivity, Proc. R. Soc. Lond. Ser. A 119 (1928) 689–698.
[12] E.A. Uehling, G.E. Uhlenbeck, Transport phenomena in Einstein–Bose and Fermi–Dirac gases, Phys. Rev. 43 (1933) 552–561.
[13] D. Benedetto, F. Castella, R. Esposito, M. Pulvirenti, On the weak coupling limit for bosons and fermions, Math. Models Methods Appl. Sci. 15 (2005) 1811–1843.
[14] X. Lu, A modified Boltzmann equation for Bose–Einstein particles: Isotropic solutions and long time behavior, J. Statist. Phys. 98 (2000) 1335–1394.

[15] X. Lu, On isotropic distributional solutions to the Boltzmann equation for Bose–Einstein particles, J. Statist. Phys. 116 (5–6) (2004) 1597–1649.
[16] T. Carleman, Problèmes mathématiques dans la théorie cinétique des gaz, Almqvistand Wiksell, Uppsala, 1957.
[17] M. Escobedo, S. Mischler, M.A. Valle, Homogeneous Boltzmann equation in quantum relativistic kinetic theory, in: Monograph (4 Southwest-Texas State University, San Marcos, TX), Electron. J. Differential Equations (2003).
[18] M. Escobedo, S. Mischler, J.J.L. Velasquez, On the fundamental solution of a linearized Uehling–Uhlenbeck equation, Preprint.
[19] B. Perthame, Global existence to the BGK model of Boltzmann equation, J. Differential Equations 82 (1989) 191–205.
[20] B. Perthame, M. Pulvirenti, Weighted L^{∞} bounds and uniqueness for the Boltzmann BGK model, Arch. Ration. Mech. Anal. 125 (1993) 289–295.
[21] K. Aoki, Y. Sone, Rarefied gas dynamics, Jpn. Soc. Fluid Mech. (1998).
[22] I.M. Khalatnikov, An Introduction to the Theory of Superfluidity, W.A. Benjamin, New York, 1965.
[23] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, in: Applied Mathematical Sciences, vol. 106, Springer-Verlag, New York, 1988.
[24] F. Golse, L. Saint-Raymond, Velocity averaging in L^1 for the transport equation, C. R. Acad. Math. Sér. I 334 (2002) 557–562.