

THREE MANIFOLDS AS GEOMETRIC BRANCHED COVERINGS OF THE THREE SPHERE

G. BRUMFIEL, H. HILDEN, M.T. LOZANO*, J.M. MONTESINOS-AMILIBIA*,
E. RAMIREZ-LOSADA, H. SHORT, D. TEJADA**, M. TORO**

ABSTRACT. One method for obtaining every closed orientable 3-manifold is as a branched covering of S^3 branched over a link. There are topological results along these lines that cannot be improved upon in two respects: (A), The minimum possible number of sheets in the covering is three; (B), There are individual knots and links (universal knots and links) that can serve as branch set for every 3-manifold, M^3 . Given the growing importance of geometry in 3-manifold theory it is of interest to obtain geometrical versions of topological results (A) and (B). Twenty years ago a geometric version of result (B) was obtained using universal groups. In this paper we obtain a geometric version of result (A), also by means of universal groups.

1. INTRODUCTION

Some time ago it was established that all closed orientable 3-manifolds are threefold irregular simple branched coverings of S^3 with branch set a knot or link. ([3],[10],[13]). Both the statement and proof of this theorem were purely topological in character. Geometry played no role.

The central purpose of this paper is to introduce the idea of a geometric branched covering and then to show that every closed orientable 3-manifold M^3 is a threefold irregular simple branched covering of S^3 that is “geometric” in the following sense:

There is a universal group U ; the orbifold group of the Borromean rings with singular angle ninety degrees. Thus U is a finite covolume Kleinian group of hyperbolic isometries of H^3 .

There are finite index subgroups of U , G , and G_1 , such that $M^3 = H^3/G$, $S^3 = H^3/G_1$, and G is a non normal index three subgroup of G_1 .

The groups U , G , and G_1 induce hyperbolic orbifold structures on S^3 , M^3 , and S^3 respectively and the maps induced by group inclusions,

$$M^3 = H^3/G \longrightarrow S^3 = H^3/G_1 \longrightarrow S^3 = H^3/U,$$

are branched covering space maps with $M^3 \longrightarrow S^3$ being three to one.

The new theorem, Theorem 9 of this paper, can be considered to be the “geometrization” of the old theorem.

This work is closely related to the concepts of universal group and universal link.

A knot or link is said to be universal if every closed orientable 3-manifold occurs as a finite branched covering space of S^3 with branch set the knot or link. W. Thurston introduced this concept in his paper [16], where he also exhibited some universal links, and asked if the Whitehead link and the Figure eight knot were universal. In [6], [7] Thurston’s question in the affirmative and proved that every non toroidal rational knot or link is universal, and that the Borromean rings are

*This research was supported by grants MTM2004088080, MTM2007-67908-C02-01 and MTM2006-00825.

**This research was supported by COLCIENCIAS grant CT 436-2007.

universal. Subsequently many other universal knots and links were found: [17], [18], [15], [8], [4], [11], [9].

In a branched covering $p : M \rightarrow N$ of 3-manifold branched over a link L the preimage of a meridian disk D is a set of meridian disks D_1, \dots, D_k . For $1 \leq j \leq k$ to disk D_j is mapped in n to 1 fashion to D . If A is a set of positive integers we say that the branched covering p is of type A if n_j belongs to A for $1 \leq j \leq k$. In particular, in the sequel branched coverings of type 1, 2 and 1, 2, 4 will be particularly important.

Closely related to the concept of universal knot or link is that of universal group. A finite covolume, discrete group of hyperbolic isometries U , acting on H^3 , is said to be universal if every closed orientable 3-manifold M^3 occurs as a quotient space, $M^3 = H^3/G$, where G is a finite index subgroup of U . Such groups U must contain rotations, else all 3-manifolds including S^3 would have hyperbolic structure.

Also, given M^3 , there are infinitely many finite index G 's with $M^3 = H^3/G$ so that this doesn't give anything like a classification of 3-manifolds. It can be considered analogous to Heegaard splittings or Kirby calculus presentations. We know that $S^3 = H^3/U$ so that S^3 inherits a hyperbolic orbifold structure from U .

The existence of a universal group was demonstrated in ([5]). The group defined in ([5]), which from now on we denote by U , is a orbifold group of the Borromean rings. This group U thus induces a hyperbolic orbifold structure on S^3 with singular set the Borromean rings, and singular angle ninety degrees. Unfortunately the proof of the universality of U in ([5]) cannot be adapted to prove the geometric branched covering space theorem referred to earlier.

There is a new proof that serves our purposes. It starts out following ([5]), then follows ideas in ([6]), then follows section 5 of ([11]) in which infinitely many 2-universal links are defined, and then uses a branched covering of S^3 by S^3 with branch set the Borromean rings and branching indexes $\{1,2\}$ (branching of type $\{1,2\}$ for short) such that the "doubled" Borromean rings (2-universal) occur as a sublink of the preimage of the Borromean rings.

Rather than put the reader through the difficult task of actually finding these references in some library we prefer to give a new, relatively self-contained proof of the universality of U .

In the next section we state and prove the geometric branched covering space theorem.

2. GEOMETRIC BRANCHED COVERING SPACE THEOREM

Our point of departure is the following theorem ([3],[10],[13]).

Theorem 1. *Let M^3 be a closed orientable 3-manifold. Then M^3 is a 3 to 1 irregular simple branched cover of S^3 with branch set a knot or link (as opposed to a graph).*

Such 3 to 1 coverings with branch set a link L correspond to transitive representations $\rho : \pi_1(S^3 - L) \rightarrow \Sigma_3$ in which meridians are sent to transpositions. Years ago Ralph Fox ([2]) had the genial idea of representing transpositions by colours. Throughout the paper we shall follow this idea; Red = $R = (12)$, Yellow = $Y = (23)$, Blue = $B = (13)$.

Then, given a classical knot or link diagram, simple transitive representations to Σ_3 (Simple means meridians go to transpositions), correspond to colourings of the bridges such that at each crossing either all three bridges are the same colour or all three have different colours and at least two colours are used. (This itself is equivalent to the Wirtinger relations, which have form $xy = yz$, being satisfied.)

Given a 3-1 simple branched covering $p : M^3 \rightarrow S^3$ branched over the coloured link L , there is a move, illustrated in Figure 1 and called a Montesinos move [14],

which changes the coloured link L to a different coloured link L' . The change takes place inside a ball. Although L is changed, the topological type of M^3 is not. The reason is that the 3-fold simpler cover of a ball branched over two unknotted, unlinked arcs is a ball. Thus doing a Montesinos move on a link in S^3 is equivalent to removing a ball from M^3 and sewing it in differently. We call a sequence of Montesinos moves a Montesinos transformation.

Montesinos Move

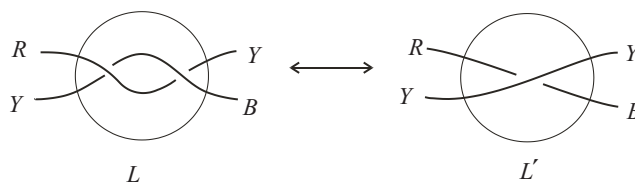


FIGURE 1

We give several examples of transformations of links using Montesinos moves and isotopies which will be useful to us in the new proof of the universality of U .

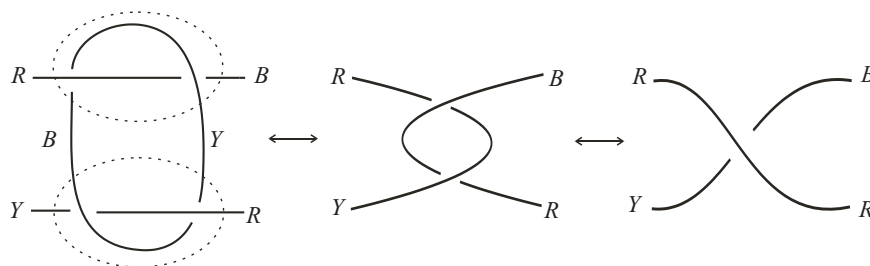


FIGURE 2

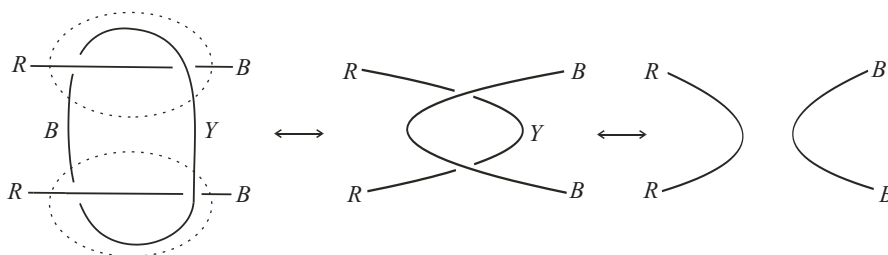


FIGURE 3

Now we are ready to prove the universality of U . Let M^3 be a closed orientable 3-manifold and let $p : M^3 \rightarrow S^3$ be a simple 3-fold covering branched over the link L . We shall apply a Montesinos transformation to L to obtain a link L' which suits our purposes. For definitiveness we shall think of S^3 as $E^3 \cup \{\infty\}$, assume L is contained in E^3 and use cylindrical coordinates in E^3 .

As every link is a closed braid we can assume L is a closed braid. This means that each component of L has a parametrization $(r(t), \theta(t), z(t))$ in which $\theta(t)$ is strictly increasing and the projection on the plane $z = 0$ is "nice". (i.e., there are no triple points.)

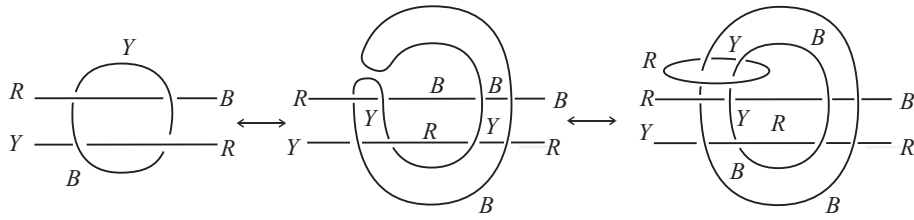


FIGURE 4

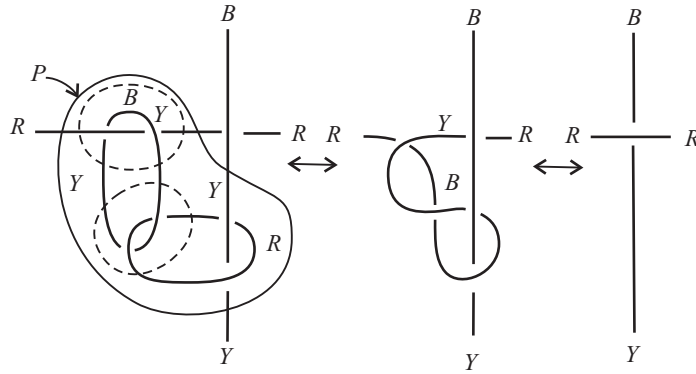


FIGURE 5

Using an isotopy of the type illustrated in Figure 7 we can assume every crossing has 3 colours.

Then, using Montesinos moves as in Figure 1, we can assume all crossings are “positive” (See Figure 8)

We replace each crossing with a new small circle component, as in the left hand side of Figure 2. After an isotopy we can assume that our link L has two types of components; “braid” or “horizontal” components that lie in the plane $z = 0$ and have equations $r = \text{constant}; \theta = \text{arbitrary}, z = 0$, and “small circle” components, whose projection on the plane $z = 0$ appears as in the left side of Figure 2.

Next, using the Montesinos transformation of Figure 4, (which is validated by using the Montesinos transformation of Figure 3.), we replace each small circle component by three components as in the right hand side of Figure 4. Now our link L has three types of components; horizontal components, big circle components and small circle components as illustrated in the right hand side of Figure 4. Each small circle component links two big circle components.

We isotope out link L so that each big circle component appears as in the left hand side of Figure 9, that is, it extends over the top and bottom of all the horizontal components.

Then we isotope the small circle components, one at a time, so that they become braid or horizontal components. As we do this to a particular small circle component “ c ” as in the left hand side of Figure 9 it becomes, briefly, the topmost braid component.

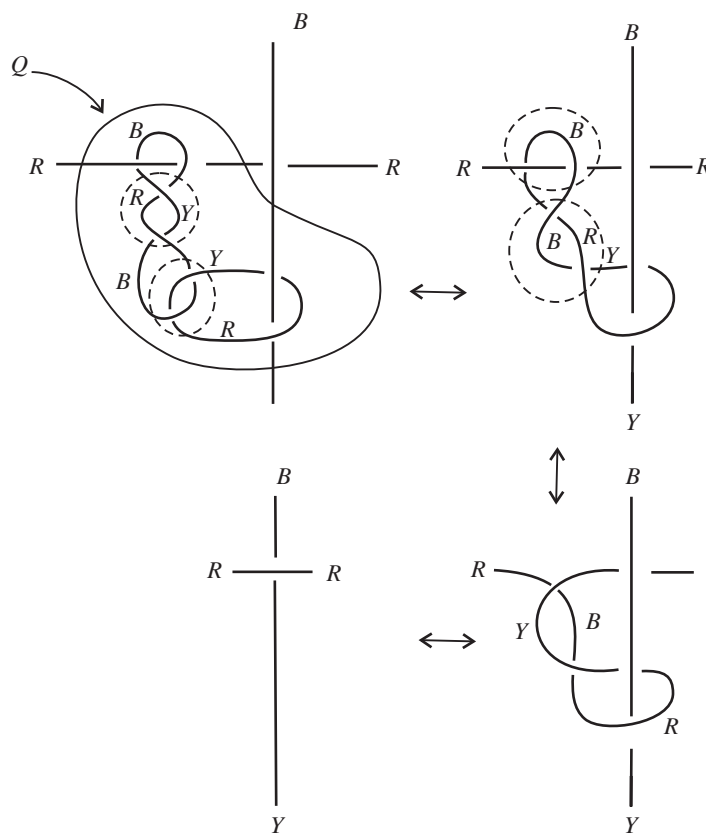


FIGURE 6

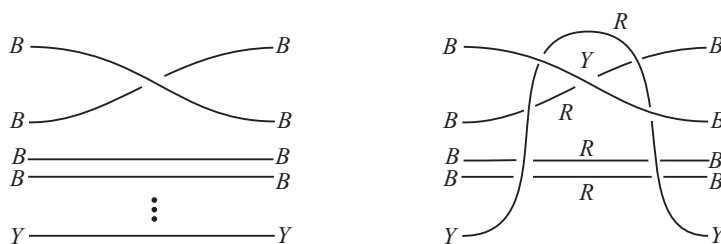


FIGURE 7



FIGURE 8

Now our link L has two types of components, horizontal components which lie in the plane $z = 0$ and have equation of form $r = \text{constant}$, $\theta = \text{arbitrary}$, and $z = 0$ and large circle components, which we now call vertical components,

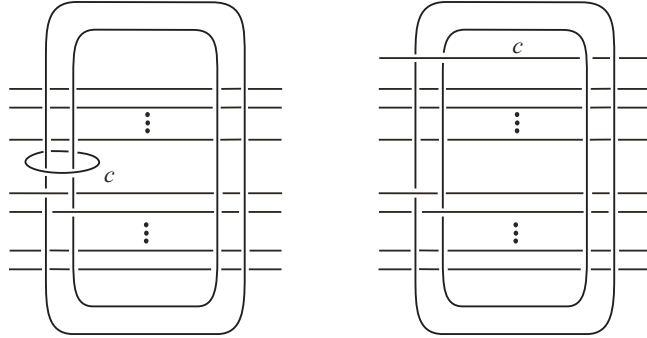


FIGURE 9

whose projections on the plane $z = 0$ are rectangles in the (r, θ) coordinate system. Crossings are called horizontal or vertical as indicated in Figure 10.

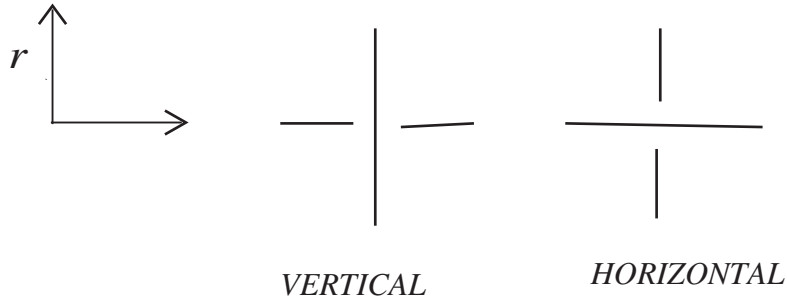


FIGURE 10

We observe, and this is crucial in what follows, that every horizontal crossing is 3-coloured. (The vertical crossings may not be.) Also, a particular vertical component links a particular horizontal component if and only if the left crossing on the right hand side of Figure 9 is horizontal. The next step will be to eliminate horizontal crossings. We refer the reader to Figures 5 and 6.

We use the Montesinos transformations illustrated in either Figures 5 or 6 (both are useful) to replace each horizontal crossing by a vertical crossing. In the course of doing this, new components, as indicated in the left sides of Figures 5 and 6, are introduced. These are contained in the “peanut shaped” balls indicated by a “ P ” or “ Q ” in Figures 5 and 6.

After a slight isotopy our link L has three types of components; horizontal (with equation $r = \text{constant}$, $z = 0$, $\theta = \text{arbitrary}$); vertical (lying in the plane $z = \varepsilon > 0$, whose projection in the plane $z = 0$ is a rectangle in (r, θ) coordinates); and “special”. Each special component is contained in a “peanut shaped” topological ball lying in the region $-\varepsilon \leq z \leq 2\varepsilon$ and having projection on the $z = 0$ plane as indicated in either the left or right hand sides of Figure 11. (We can use one or the other but never both in the same proof.)

Thus we have shown that every closed orientable 3-manifold M^3 is a 3-fold simple branched covering of a link of very special type. We shall isotope the link some more and then state a theorem. We shall use cylindrical coordinates in $S^3 = E^3 \cup \{\infty\}$ to better describe the link. At the moment our link is contained in the region $[-\varepsilon \leq z \leq 2\varepsilon, 0 < r_0 < r_1 \leq R_0, \theta = \text{arbitrary}]$

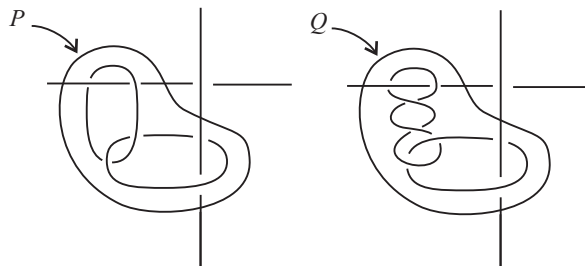


FIGURE 11

We isotope the region $-\varepsilon \leq z \leq 2\varepsilon, r_0 \leq r \leq R_0$ without changing the θ coordinate, so that the link lies in the thickened toroidal region $1 - \delta \leq z^2 + (r - 2)^2 \leq 1 + \delta$ and so that the horizontal components lie in the torus $z^2 + (r - 2)^2 = 1$, have equations of form $[r = \text{constant}, z = \text{constant}, \theta = \text{arbitrary}]$, and are “evenly spaced”. (This means that after the isotopy the images of the horizontal components intersect the circle $z^2 + (r - 2)^2 = 1, \theta = \theta_0$, in n evenly spaced points on the circle.)

Next we isotope the vertical components so that their images lie on the torus $z^2 + (r - 2)^2 = 1 - \delta$, have equations $\theta = \text{constant}, z^2 + (r - 2)^2 = 1 - \delta$, and are “evenly spaced” which means that if there are m vertical components the constants referred to are $\{2\pi j/m; 0 \leq j \leq m - 1\}$.

It is convenient to introduce new “toroidal” coordinates (ρ, θ, φ) which are well defined in a neighborhood of the torus $z^2 + (r - 2)^2 = 1 - \delta$. These are given by equations $\theta = \theta; \rho = \sqrt{z^2 + (r - 2)^2} = \text{distance of a point from the circle } [z = 0, r = 2, \theta = \text{arbitrary}]; \sin \varphi = z/\rho$ and $\cos \varphi = r - 2/\rho$. Using these “toroidal” coordinates we can define projections of $E^3 - \{(z \text{ axis}) \cup (\text{circle } [r = 2; z = 0])\}$ onto the torus $z^2 + (r - 2)^2 = 1$ or the torus $z^2 + (r - 2)^2 = 1 - \delta$ by $(\rho, \theta, \varphi) \rightarrow (\theta, \varphi)$.

We summarize the results of the preceding isotopies in a theorem.

Theorem 2. *Let M^3 be a closed oriented 3-manifold. Then there is a 3-fold simple branched covering $p : M^3 \rightarrow S^3$ branched over a link L .*

The link L has three types of components.

a. Horizontal. These lie in the torus $\rho = 1$ or $z^2 + (r - 2)^2 = 1$ and have equations $[\theta = \text{arbitrary}; \varphi = 2\pi j/n, 0 \leq j \leq n - 1, \rho = 1]$

b. Vertical. These lie in the torus $\rho = .99$ or $z^2 + (r - 1)^2 = (.99)^2$ and have equations $\varphi = \text{arbitrary}, \theta = 2\pi j/m, 0 \leq j \leq m - 1, \rho = .99$

c. Special. These have local projections on the torus $\rho = 1$ as in either the left or right hand side of Figure 11. The vertical coordinate is φ , the horizontal coordinate is θ .

In this proof we will use all left hand side or all right hand side of Figure 11. Both are useful.

Now we find it useful to define two rotations T_1 and T_2 of $S^3 = E^3 \cup \{\infty\}$. The rotation T_1 is simply the m -fold rotation about the z -axis given by $T_1 : (r, \theta, z) \rightarrow (r, \theta + 2\pi/m, z)$ the rotation T_1 leaves invariant the set of horizontal and the set of vertical components of the link L . The rotation T_2 is more difficult to describe in coordinates and we shall not attempt to do so. Instead we indicate its important properties. The rotation T_2 has as its axis the circle $z = 0, r = 2, \theta = \text{arbitrary}$; it has order n and leaves the θ coordinate unchanged. It leaves the set of horizontal and the set of vertical components of L invariant. It cyclically permutes the horizontal components and it sends each vertical component to itself. Its restriction to a vertical component is just the usual n -fold rotation of a circle.

At this point we must decide whether to use the branch set in the left or right hand side of Figure 11. We choose the left for purposes of illustration.

Using the projection $(\rho, \theta, \varphi) \rightarrow (\theta, \varphi)$ a portion of the image of the link L appears as in Figure 12. Some of the “peanut shaped” balls contain two component links and arcs from a vertical and horizontal component, others contain only arcs from a vertical and horizontal component.

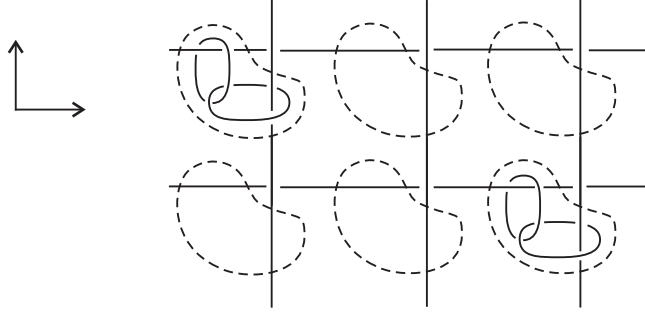


FIGURE 12

We can and shall assume that both rotations T_1 and T_2 leave the “expanded” link L invariant.

Now consider the map $f_1 : S^3 \rightarrow S^3/T_1 = S^3$ which is an m -fold cyclic branched covering $S^3 \rightarrow S^3$ with branch set the trivial knot or z -axis. The branch set for the composite map $f_1 \circ p : M^3 \rightarrow S^3$ consists of the union of the branch set for f_1 and the image f_1 (branch set for p) = z -axis $\cup f_1(L)$.

The branching is of type $\{1, 2\}$ on $f_1(L)$ and of type $\{m\}$ on the z -axis $\cup \{\infty\}$. This means that any disc which belongs to the preimage of a meridian disk D for $f_1(L)$ is either mapped homeomorphically $(1-1)$ to D or mapped to D as a $2-1$ branched covering. The preimage of a meridian disk D for the z -axis is mapped to D as an m -fold branched covering. The overall branching is of type $\{1, 2, m\}$ for the map $f_1 \circ p$ and the “ m ” is bad for our purposes. Shortly we shall show how to change the map f_1 to a map f so that the branching for $f \circ p$ is of type $\{1, 2\}$ but first we observe that the part of the branch set $f_1(L)$ for $f_1 \circ p$ contains n horizontal components, n “peanut” components but only one vertical component.

Consider the connected k -fold branched coverings of a disk D^2 with two branch points A and B in the interior of D^2 . These are determined by transitive representations ρ of $\pi_1(D^2 - \{A, B\})$ (which is free on the two meridian generators, call them x and y , pictured in Figure 13) into Σ_k .

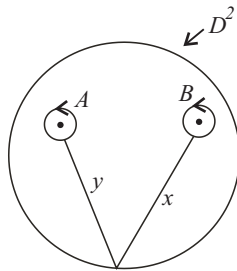


FIGURE 13

We are interested in a particular dihedral representation. Let P be a regular k -gon with its vertices labeled 1 through k as in Figure 14. We map x to the reflection in the axis l_1 and y to the reflection in the axis l_2 of Figure 14.

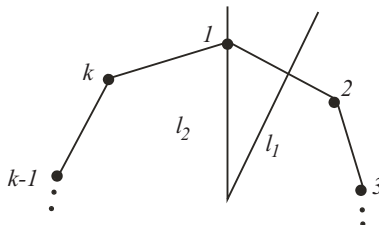


FIGURE 14

The reflections induce permutations of vertices and elements of Σ_k . Then $\rho(x) = (1, 2)(3, k)(4, k-1)\cdots$ and $\rho(y) = (2, k)(3, k-1)(4, k-2)\cdots$ are both products of disjoint transpositions and as xy is a counterclockwise rotation by $2\pi/k$, $\rho(xy)$ is the k -cycle $(1, k, k-1, \dots, 3, 2)$.

Let $q : X \rightarrow D^2$ be the branched covering induced by ρ . Then xy is a generator of $\pi_1(S^1) = Z$ where $S^1 = \text{boundary } D^2$ and restricting q to boundary X we see that $q : \partial X \rightarrow S^1$ is just the usual k -fold unbranched cover of S^1 . Using the branching data we compute the Euler characteristic of X which is one. Thus ρ induces a k -fold branched cover $q : D^2 \rightarrow D^2$ with branch set two points in interior D^2 and branching type $\{1, 2\}$. The restriction of q to $S^1 = \partial X = \partial D^2$ is the usual k -fold unbranched covering of S^1 by S^1 . It is convenient to summarize the preceding in a proposition.

Proposition 3. *Let $q_1 : D^2 \rightarrow D^2$ be the usual k -fold cyclic covering of D^2 ; the one given by $q_1 : z \rightarrow z^n$ in complex coordinates if D^2 is the unit disc in \mathbb{C} . Then there is another k -fold branched covering $q : D^2 \rightarrow D^2$ such that the branch set is two points in interior D^2 , the branching is of type $\{1, 2\}$ and $q = q_1$ on $S^1 = \text{boundary } D^2$.*

By “crossing” with S^1 the next proposition follows easily.

Proposition 4. *Let $q_1 : S^1 \times D^2 \rightarrow S^1 \times D^2$ be the usual k -fold cyclic covering of $S^1 \times D^2$ by $S^1 \times D^2$; the one given by $q_1 : (e^{i\theta}, z) \rightarrow (e^{i\theta}, z^n)$ in natural coordinates for $S^1 \times D^2$. Then there is another k -fold branched covering $q : S^1 \times D^2 \rightarrow S^1 \times D^2$ such that the branch set equals $S^1 \times \{A, B\}$ where A and B are points in the interior of D^2 ; the branching is of type $\{1, 2\}$, and $q = q_1$ on boundary $(S^1 \times D^2)$.*

We return to our map f_1 which is an m -fold cyclic covering of S^3 by S^3 with branch set and preimage of branched set the z -axis. We choose a natural solid torus neighborhood of the z -axis in S^3 and its preimage and we coordinatize this neighborhood so that f_1 is the usual m -fold cyclic covering of $S^1 \times D^2$ by $S^1 \times D^2$ as in Proposition 4. This neighborhood should be small enough so that it doesn’t intersect the link L which is the branch set of the map $p : M^3 \rightarrow S^3$ defined earlier. Then we use Proposition 4 to define a new map $f : S^3 \rightarrow S^3$ where outside the solid torus neighborhood $f = f_1$ and within the solid torus neighborhood f is like q of Proposition 4. Thus $f \circ p$ is a $3m$ to 1 branched covering of S^3 by M^3 with branch set $S^1 \times \{A, B\} \cup f(L)$.

The part of the branch set $f(L) = f_1(L)$ has one vertical component and n -horizontal components and n -“peanut” components. Via an isotopy, if necessary, we may assume that the rotation T_2 leaves $S^1 \times \{A\}$ and $S^1 \times \{B\}$ invariant and that its restriction to either component is just the usual n -fold rotation. The relevant

part of the branch set for $f \circ p$ is depicted in Figure 15. The branching is type $\{1, 2\}$.

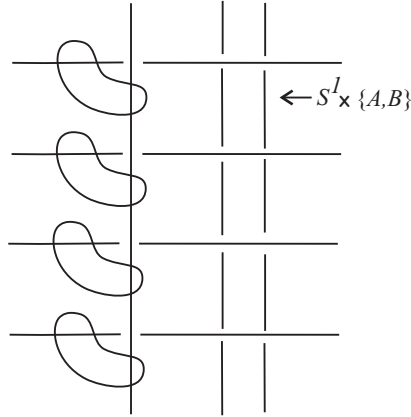


FIGURE 15

Next we let g_1 be the map $S^3 \rightarrow S^3/T_2 = S^3$. Then $g_1 \circ f \circ p$ is a branched covering of S^3 by M^3 which is $3mn$ to 1 and has branch set equal to $g_1(\text{branch set } (f \circ p)) \cup \{\text{the circle } [z = 0, r = 2, \theta = \text{any}]\}$. The branching is of type $\{1, 2\}$ on $g_1(\text{branch set } (f \circ p))$ and of type $\{m\}$ on the circle $[z = 0, r = z, \theta = \text{any}]$.

We choose a solid torus neighborhood of the circle $z = 0, r = z, \theta = \text{any}$, small enough so that it does not intersect $g_1(\text{branch set } (f \circ p))$, and so that it can be coordinatized as $S^1 \times D^2$ with $g_1 = q_1$ as in Proposition 4.

Then we define $g : S^3 \rightarrow S^3$ so that $g = g_1$ outside the torus neighborhood and g behaves like q of Proposition 4 inside the torus neighborhood. The $3mn$ to 1 branched covering $g \circ f \circ p : M^3 \rightarrow S^3$ has branch set with three vertical components and three horizontal components and one “peanut” component as pictured in Figure 16. All branching is of type $\{1, 2\}$.

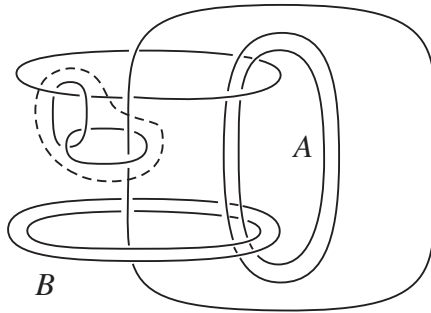


FIGURE 16

The link in Figure 16 can be isotoped to the link in Figure 17. To help the reader see this we have labeled the corresponding components in Figures 16 and 17. There are two obvious annuli in Figures 16 and 17 labeled A and B . The other components are labeled $\gamma, \delta, \epsilon,$ and ζ .

We may add four components, $\gamma_1, \delta_1, \epsilon_1, \zeta_1$, to the link of Figure 17 so that there are annuli $C, D, E,$ and F with boundaries $\gamma \cup \gamma_1, \delta \cup \delta_1, \epsilon \cup \epsilon_1$ and $\zeta \cup \zeta_1$ respectively in such a way that the new link has a 3-fold rational symmetry. Let T_3 be this

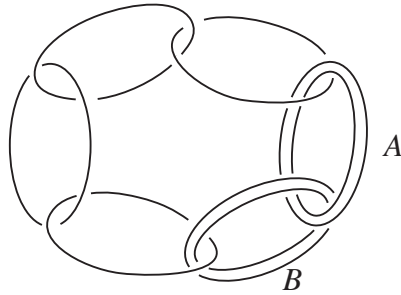


FIGURE 17

3-fold rotation and let $h_1 : S^3 \rightarrow S^3 = S^3/T_3$ be the resulting branched covering. The map $h_1 \circ g \circ f \circ p$ is a $9mn$ to 1 branched covering of S^3 by M^3 with branching of type $\{1, 2\}$ on the part of the branch set h_1 (branch set $g \circ f \circ p$) and branching of type $\{3\}$ on h_1 (axis T_3).

This branch set is depicted in Figure 18.

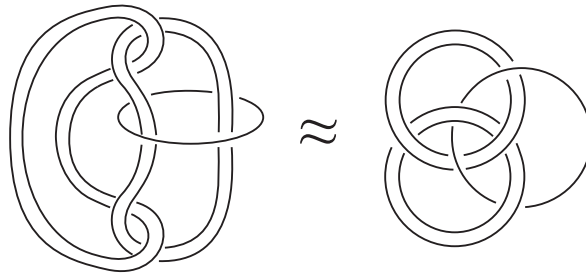


FIGURE 18

As before we replaced f_1 by f and g_1 by g we now replace h_1 by h using Proposition 4 where $h_1 = h$ except in solid torus neighbourhood of the axis of rotation of h but the branching of the map $h : S^3 \rightarrow S^3$ is of type $\{1, 2\}$. The new branch set is displayed in Figure 19.

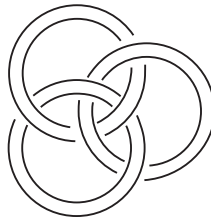


FIGURE 19

We call the branch set of Figure 19 the “doubled Borromean rings”. We summarize this result in the form of a theorem [11].

Theorem 5. *Let M^3 be a closed orientable 3-manifold. Then M^3 is a branched covering of S^3 with branch set the doubled Borromean rings, and with branching of type $\{1, 2\}$. That is, the doubled Borromean rings are 2-universal.*

If we use the right hand side of Figure 11 instead of the left hand side we obtain the following theorem [11].

Theorem 6. *Let M^3 be a closed orientable 3-manifold. Then M^3 is a branched covering of S^3 with branch set the doubled Whitehead link and with all branching of type $\{1, 2\}$. That is, the doubled Whitehead link is 2-universal.*

The doubled Whitehead link is depicted in Figure 20.

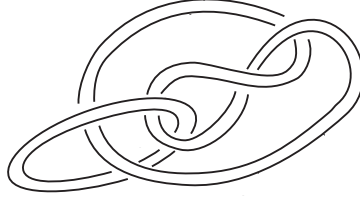


FIGURE 20

Theorem 5 and Theorem 6 show that the doubled Borromean rings and doubled Whitehead link are 2-universal. In ([11]) two co-authors of this paper prove this result and additionally give infinitely many examples of 2-universal links. The three component link in the right bottom part of Figure 5.10 of [11] is a minimal hyperbolic 2-universal link. In fact any proper sublink of it is either a split link or a toroidal link.

We note, in passing, that 2-universal knots are known to exist ([9]) but so far there are no easy examples.

Our next task, which is the new idea of this paper, will be to define a branched covering $k : S^3 \rightarrow S^3$ with branch set the Borromean rings for which the doubled Borromean rings occur as a sublink of the preimage of the branch set.

We begin by tessellating E^3 by $2 \times 2 \times 2$ cubes all of whose vertices have odd integer coordinates. Let \widehat{U} be the group generated by 180° rotations in the axes a , b , and c displayed in Figure 21. The cube there is centered at the origin.

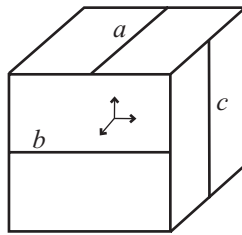


FIGURE 21

The group \widehat{U} is a well known Euclidean crystallographic group that preserves the tessellation. A fundamental domain for \widehat{U} is the $2 \times 2 \times 2$ cube of Figure 21 centered at the origin. The map $E^3 \rightarrow E^3/\widehat{U} \approx S^3$ is a branched cover of S^3 by E^3 with branch set the Borromean rings. This gives S^3 the structure of a Euclidean orbifold with singular set the Borromean rings and singular angle 180° . We can see that E^3/\widehat{U} equals S^3 with singular set the Borromean rings by making face identifications in the fundamental domain of Figure 21. We do this in Figure 22.

Next we consider a tessellation of E^3 by $6 \times 6 \times 6$ cubes with integer coordinates that are odd multiples of three. Let \widetilde{U} be the group generated by 180° rotations in

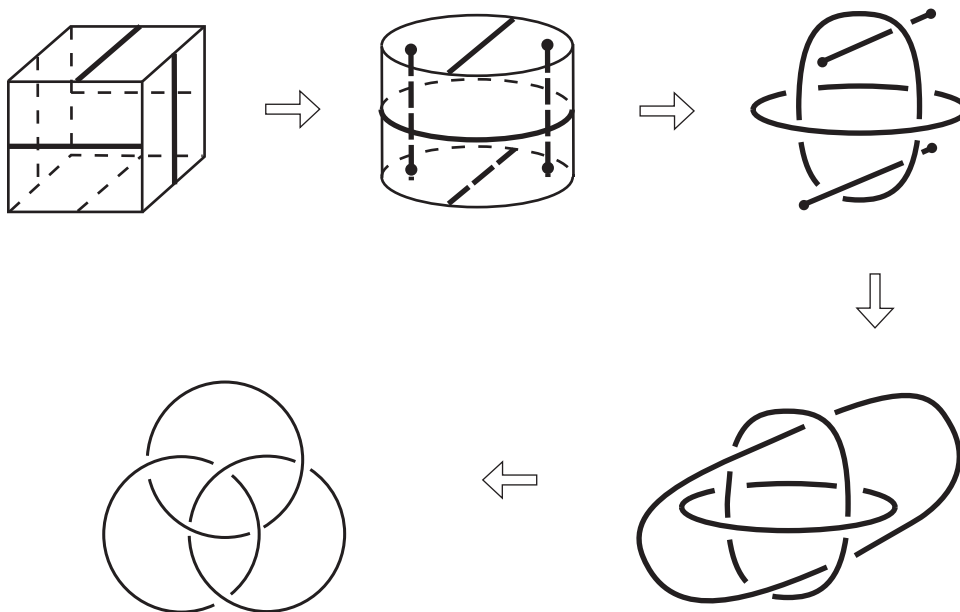


FIGURE 22

the axes a' , b' , c' where $a' = (t, 0, 3)$, $b' = (3, t, 0)$ and $c' = (0, 3, t)$; $-\infty < t < \infty$. We can envision a fundamental domain for \tilde{U} by looking at Figure 21 and imagining prime accents over a, b , and c . Of course $E^3/\tilde{U} = S^3$ and the map $E^3 \rightarrow E^3/\tilde{U}$ is a branched covering of S^3 by E^3 . As the rotations about a' , b' and c' belong to \hat{U} we see that $\tilde{U} \subset \hat{U}$ and we see that $[\hat{U} : \tilde{U}] = 27$ by comparing the size of fundamental domains. \tilde{U} is not a normal subgroup of \hat{U} . We are in fact really interested in the map $t : S^3 = E^3/\tilde{U} \rightarrow E^3/\hat{U} = S^3$ induced by the inclusion of \tilde{U} in U .

Consider the following commutative diagram

$$\begin{array}{ccc}
 E^3 & \xrightarrow{id} & E^3 \\
 \downarrow & & \downarrow \\
 S^3 = E^3/\tilde{U} & \longrightarrow & E^3/\hat{U} = S^3.
 \end{array}$$

The maps $E^3 \rightarrow E^3/\tilde{U}$ and $E^3 \rightarrow E^3/\hat{U}$ are both branched covers of S^3 by E^3 with all branching of type $\{2\}$. (The map $E^3 \rightarrow S^3$ is infinite to one but the notion of branched covering generalizes naturally from finite to one maps to infinite to one maps, at least in this special case.) We see, by considering images of meridian discs in E^3 , that the map $S^3 = E^3/\tilde{U} \rightarrow E^3/\hat{U} = S^3$ is also a branched covering space map with branching of type $\{1, 2\}$.

The set of points in $S^3 = E^3/\tilde{U}$ in the preimage of the Borromean rings branch set in $E^3/\hat{U} = S^3$ for which the branching of type $\{2\}$ is called the upper branch set. The set of points in $S^3 = E^3/\tilde{U}$ in the preimage of the Borromean rings branch set in $E^3/\hat{U} = S^3$ for which the branching is of type $\{1\}$ is called the upper pseudo branch set.

We can compute the preimage of the branch set in $S^3 = E^3/\tilde{U}$ from a fundamental domain for \tilde{U} , which consists of $27 \cdot 2 \times 2 \times 2$ cubes.

A point in this fundamental domain belongs to the upper branch set if and only if it belongs to an axis of rotation for \widehat{U} but does not belong to an axis of rotation for \widetilde{U} . A point in this fundamental domain belongs to the upper pseudo branch set if and only if it belongs to an axis of rotation for \widetilde{U} . We do not need to compute the full preimage of the Borromean rings in $S^3 = E^3/\widetilde{U}$, (which turns out to be a 15 component link), but only a certain sublink.

In Figure 23 we give a $6 \times 6 \times 6$ cube fundamental domain for \widetilde{U} and display only those axes of rotation of \widehat{U} that lie in the faces of the cube.

The axes a', b' and c' (and their analogues on the invisible faces) are axes for \widetilde{U} and so give rise to the upper pseudo branch set. The axes \tilde{a}, \tilde{b} , and \tilde{c} (and their analogues on the invisible faces) are axes for \widehat{U} , but not \widetilde{U} , and so give rise to a sublink of the upper branch set.

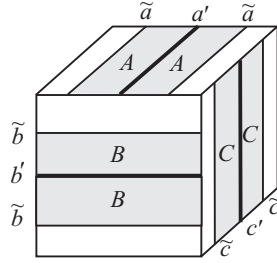


FIGURE 23

Also, axes a' and \tilde{a} (resp. b' and \tilde{b} ; c' and \tilde{c}) lie on the boundary of a rectangle A (resp. B ; C). Something similar occurs on the invisible faces. After identifications are made in the faces of the cubes these rectangles become annuli. In Figure 24 we make the identifications and show that the doubled Borromean rings appear.

We summarize all this in the next proposition.

Proposition 7. *The map $t : S^3 = E^3/\widetilde{U} \longrightarrow E^3/\widehat{U} = S^3$ is a 27 to 1 irregular branched covering of S^3 by S^3 with branch set the Borromean rings.*

The doubled Borromean rings occur as a sublink of the preimage of the branch set.

The doubled Borromean rings consist of three pairs of components. Each pair bounds an annulus disjoint from the other pairs. Each pair is mapped to the same component of the Borromean rings. And each pair contains one component of the upper branch set and one component of the upper pseudo branch set.

The 27 to 1 map t of Proposition 7 can be decomposed into a composition of three 3 to 1 branched coverings of S^3 by S^3 with branch set a trivial 2-component link. Each of these maps is easier to understand than t . We thank the referee for pointing this out to us.

Thus far, starting with an arbitrary closed orientable 3-manifold we have defined a series of branched covering space maps.

$$(1) \quad M^3 \xrightarrow[p]{3-1} S^3 \xrightarrow[f]{m-1} S^3 \xrightarrow[g]{n-1} S^3 \xrightarrow[h]{3-1} S^3 \xrightarrow[t]{27-1} S^3.$$

The maps f and g only depend on M^3 in a superficial way; i.e. they depend on the number of vertical and horizontal components in the link that is the branch set for p , but not on the branching itself and the maps h and t don't depend on M^3 at all.

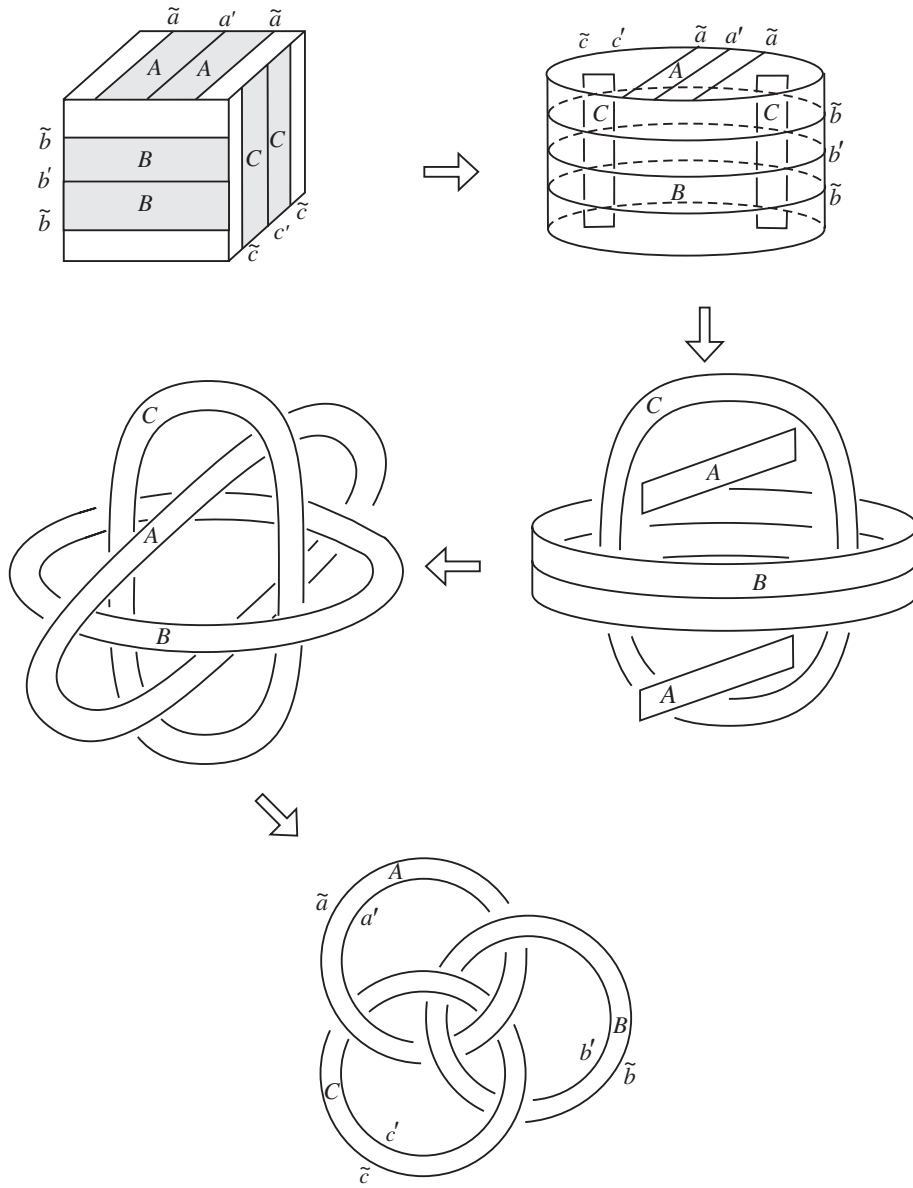


FIGURE 24

In general, when one composes branched covering space maps $a : X^3 \rightarrow Y^3$, $b : Y^3 \rightarrow Z^3$ one obtains a branched covering space map $b \circ a : X^3 \rightarrow Z^3$.

In the sequel we abbreviate branch set by BS . Thus $BS(b \circ a) = BS(b) \cup b(BS(a))$. In our case $BS(f) \cap f(BS(p)) = \emptyset$; $BS(g) \cap g(BS(f \circ p)) = \emptyset$; $BS(h) \cap h(BS(g \circ f \circ p)) = \emptyset$. And $h \circ g \circ f \circ p$ is a branched covering space map of S^3 by M^3 , of branching type $\{1, 2\}$ with branch set the doubled Borromean rings. Unlike the previous three compositions $BS(t) = t(BS(h \circ g \circ f \circ p))$ and the map $t \circ h \circ g \circ f \circ p$ is a branched covering space map of S^3 by M^3 of branching type $\{1, 2, 4\}$.

We summarize this as the next theorem

Theorem 8. *Let M^3 be a closed orientable 3-manifold. There is a branched covering space map $q : M^3 \rightarrow S^3$ with branching type $\{1, 2, 4\}$ and branch set the Borromean rings.*

There is a 3-fold simple branched covering space map $p : M^3 \rightarrow S^3$ branched over a link, and there is a branched covering space map $\phi : S^3 \rightarrow S^3$ with branch set the Borromean rings such that $q = \phi \circ p$.

We remark that Theorem 8 implies that the Borromean rings are 4-universal. That is, that every closed orientable 3-manifold is a upper branched set of S^3 with branch set the Borromean rings. This was shown for the first time in [6]. The proof here refines the proof of [6] in the sense that bounds are given on the branching indices. The branching is of type $\{1, 2, 4\}$. The refinement consists in introducing the map $t : S^3 \rightarrow S^3$ whose definition is based on the Euclidean orbifold structure of S^3 with singular set the Borromean rings.

Part of Theorem 8, the first two sentences, was first proven in ([5]). But the branched covering space map $q : M^3 \rightarrow S^3$ whose existence was proven there did not factor as in the rest of the statement in Theorem 8. This factorization is crucial to the coming proof of Theorem 9, the principal result of this paper. Observe that Theorem 8 is totally topological in character; hyperbolic geometry appears nowhere in its statement. On the other hand, hyperbolic geometry in the form of the group of hyperbolic isometries U dominates the statement of Theorem 9 below.

The sphere S^3 has a hyperbolic orbifold structure with singular set the Borromean rings and singular angle 90° . The maps t, h, g, f , and p are used to pull back the hyperbolic orbifold structure on S^3 to hyperbolic orbifold structures on S^3, S^3, S^3, S^3 , and M^3 respectively. Thus there is a sequence of groups, orbifold groups, $G \subset G_1 \subset G_2 \subset G_3 \subset G_4 \subset U$ such that $M^3 = H^3/G$, $S^3 = H^3/G_1$, $S^3 = H^3/G_2$, $S^3 = H^3/G_3$, $S^3 = H^3/G_4$, $S^3 = H^3/U$ and the following diagram is commutative. The vertical arrows are homeomorphisms.

$$\begin{array}{ccccccccc}
 H^3/G & \longrightarrow & H^3/G_1 & \longrightarrow & H^3/G_2 & \longrightarrow & H^3/G_3 & \longrightarrow & H^3/G_4 & \longrightarrow & H^3/U \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M^3 & \xrightarrow{p} & S^3 & \xrightarrow{f} & S^3 & \xrightarrow{g} & S^3 & \xrightarrow{h} & S^3 & \xrightarrow{t} & S^3
 \end{array}$$

In particular we see that $[G_1 : G] = 3$.

If T is any rotation contained in G_1 but not contained in G then $G_1 = \langle G, T \rangle$, as $[G_1 : G] = 3$ and 3 is prime. We summarize these remarks in the main theorem of this paper.

Theorem 9. *(Geometric branched covering space theorem.) Let M^3 be a closed orientable 3-manifold. Then there are subgroups G and G_1 of the universal group U such that $[G_1 : G] = 3$ and $[U : G] < \infty$ and $M^3 = H^3/G$ and $S^3 = H^3/G_1$.*

The map induced by the inclusion of groups $H^3/G \rightarrow H^3/G_1$ is a 3-fold simple branched covering of S^3 by M^3 .

We recall ([5]) that if M^3 is as in the statement of Theorem 9 then $\pi_1(M^3) \cong G/\text{TOR}(G)$ where $\text{TOR}(G)$ is the subgroup of G generated by rotations. In particular M^3 is simply connected if and only if $G = \text{TOR}(G)$. Then G_1 is generated by G and any one rotation not in G .

An interesting problem is to classify the finite index subgroups of U that are generated by rotations. We intend to return to this theme in a future paper.

Applying the theory of associated regular coverings to the above situation we obtain an interesting property on the involved groups that restricts their study to a subclass of the class of subgroups of the universal group U defining the same variety. Next we explain this.

In general, given a covering $p : M \rightarrow N$ branched over L , with monodromy $\omega : \pi_1(N - L) \rightarrow \Sigma_n$, the *associated regular covering* $q : X \rightarrow N$ is the branched covering determined by the monodromy $\eta \circ \omega : \pi_1(N - L) \rightarrow \Sigma_{\sharp \text{Im}(\omega)}$, where η is the regular representation of the group $\text{Im}(\omega)$. Recall that $q = u \circ p$ where $u : X \rightarrow M$ is a regular (branched or unbranched) covering. Actually $(q|_{X-q^{-1}(L)})_*(\pi_1(X - q^{-1}(L))) = \text{Ker}(\omega)$.

The monodromy of $p : M = H^3/G \rightarrow S^3 = H^3/G_1$ is a homomorphism $\omega : \pi_1(S^3 - L) \rightarrow \Sigma_3$ where the image of every meridian element of L is a transposition (i, j) , $1 \leq i < j \leq 3$. Therefore, $\text{Im}(\omega) = \Sigma_6$, and the monodromy $\eta \circ \omega : \pi_1(S^3 - L) \rightarrow \Sigma_6$ sends every meridian element of L to the product of three different transpositions. Thus, $u : X \rightarrow M$ is a 2-fold covering branched over the upper pseudobranch set of $p : M = H^3/G \rightarrow S^3 = H^3/G_1$. The covering $q = u \circ p : X \rightarrow S^3 = H^3/G_1$ is a regular 6-fold covering branched over L with all branching indexes equal to 2. Actually, the map u can be used to pull back the hyperbolic orbifold structure on M , so that there exist a normal subgroup $G_0 \triangleleft G$, such that $[G : G_0] = 2$ and

$$u : X = H^3/G_0 \rightarrow M = H^3/G$$

is an orbifold covering. Observe that G_0 is a normal subgroup of G_1 .

The following diagram of orbifold coverings is commutative, where $G_0 \subset G' \subset G_1$ and $[G_1 : G'] = 2$, $[G' : G_0] = 3$.

$$\begin{array}{ccc} H^3/G_0 & \xrightarrow[2:1]{u} & M = H^3/G \\ 3:1 \downarrow p' & & p \downarrow 3:1 \\ H^3/G' & \xrightarrow[2:1]{u'} & S^3 = H^3/G_1 \end{array}$$

The covering p' is unbranched and u' is the cyclic 2-fold covering branched over L . Observe that G_0 is a normal subgroup of G_1 . We summarize these remarks in the following theorem.

Theorem 10. *Let M^3 be a closed orientable 3-manifold. Let G and G_1 be the subgroups of the universal group U given in Theorem 9. Then there exist a subgroup G_0 of index 2 of G such that G_0 is a normal subgroup of G_1 .*

Recall that every finite index subgroup G of the universal group U gives rise to a 3-manifold $M = H^3/G$, but infinitely many finite index G 's produce the same manifold. The above theorem restricts the class of subgroups of U to consider in order to construct all closed 3-manifolds.

REFERENCES

- [1] I. Bernstein, A. Edmonds. On the construction of branched coverings of low dimensional manifolds. *Trans. Amer. Math. Soc.*, 247:87–123, 1979.
- [2] R.H. Crowell and R.H. Fox. *Introduction to knot theory*. Based upon lectures given at Haverford College under the Philips Lecture Program. Ginn and Co., Boston, Mass., 1963.
- [3] H. M. Hilden. Every closed orientable 3-manifold is a 3-fold branched covering space of S^3 . *Bull. Amer. Math. Soc.*, 80:1243–1244, 1974.
- [4] H. M. Hilden, M. T. Lozano, and J. M. Montesinos. Nonsimple universal knots. *Math. Proc. Cambridge Philos. Soc.*, 102(1):87–95, 1987.
- [5] H. M. Hilden, M. T. Lozano, J. M. Montesinos, and W. C. Whitten. On universal groups and three-manifolds. *Invent. Math.*, 87(3):441–456, 1987.
- [6] H.M. Hilden, M.T. Lozano, and J. M. Montesinos. The Whitehead link, the Borromean rings and the knot 9_{46} are universal. *Collect. Math.*, 34(1):19–28, 1983.
- [7] H.M. Hilden, M.T. Lozano, and J. M. Montesinos. On knots that are universal. *Topology*, 24(4):499–504, 1985.

- [8] H.M. Hilden, M.T. Lozano, and J.M. Montesinos. Universal knots. In *Knot theory and manifolds (Vancouver, B.C., 1983)*, volume 1144 of *Lecture Notes in Math.*, pages 25–59. Springer, Berlin, 1985.
- [9] H.M. Hilden, M.T. Lozano, and J. M. Montesinos-Amilibia. On 2-universal knots. *Bol. Soc. Mat. Mexicana (3)*, 10(Special Issue):239–253, 2004.
- [10] U. Hirsch. Über offene Abbildungen auf die 3-Sphäre. *Math. Z.*, 140:203–230, 1974.
- [11] M.T. Lozano and J.M. Montesinos-Amilibia. Geodesic flows on hyperbolic orbifolds, and universal orbifolds. *Pacific J. Math.*, 177(1):109–147, 1997.
- [12] J. M. Montesinos. A note on a theorem of Alexander, *Rev. Mat. Hisp.-Amer. (4)*, 32:167–187, 1972
- [13] J. M. Montesinos. A representation of closed orientable 3-manifolds as 3-fold branched coverings of S^3 . *Bull. Amer. Math. Soc.*, 80:845–846, 1974.
- [14] J.M. Montesinos. Sobre la conjetura de poincaré y los recubridores ramificados sobre un nudo. *Ph.D. Thesis*, 1971.
- [15] V. Núñez and J. Rodríguez-Viorato Dihedral coverings of Montesinos knots. *Bol. Soc. Mat. Mexicana (3)*, 10(Special Issue):423-449, 2004.
- [16] W. P. Thurston. Universal links. *Preprint*, 1982.
- [17] Y. Uchida. Universal chains. *Kobe J. Math.*, 8(1):55–65, 1991.
- [18] Y. Uchida. Universal pretzel links. In *Knots 90 (Osaka, 1990)*, pages 241–270. de Gruyter, Berlin, 1992.

(G. Brumfiel) STANFORD UNIVERSITY, STANFORD, CA, USA
E-mail address, G. Brumfiel: brumfield@math.stanford.edu

(H. Hilden) UNIVERSITY OF HAWAII, HONOLULU, HI, USA
E-mail address, H. Hilden: mike@math.hawaii.edu

(M.T. Lozano) UNIVERSIDAD DE ZARAGOZA, ZARAGOZA, SPAIN
E-mail address, M.T. Lozano: tlozano@unizar.es

(J.M. Montesinos) UNIVERSIDAD COMPLUTENSE, MADRID, SPAIN
E-mail address, J.M. Montesinos: montesin@mat.ucm.es

(E. Ramirez) CIMAT, MEXICO
E-mail address: kikis@ciimat.mx

(H. Short) UNIVERSITE DE PROVENCE, MARSEILLE, FRANCE
E-mail address: hamish.short@cmi.uni-mrs.fr

(D. Tejada, M. Toro) UNIVERSIDAD NACIONAL DE COLOMBIA, MADELLIN, COLOMBIA
E-mail address: dtejada@unalmed.edu.co, mmtoro@unalmed.edu.co