# Some isoperimetric inequalities for kernels of free extensions

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**Abstract.** If $G$ is a hyperbolic group (resp. synchronously or asynchronously automatic group) and $G = H \rtimes_\phi \mathbb{Z}$, with $H$ finitely presented and $\phi$ an automorphism of $H$, then $H$ satisfies a polynomial isoperimetric inequality (resp. exponential isoperimetric inequality).

**Keywords:** finitely presented group, word metric, van Kampen diagram, radius, area, isoperimetric function, hyperbolic group, automatic group

**Primary 20E07:** Secondary 20F05, 20F32, 57M07

*For John Stallings on his 65th birthday.*

## 1. Introduction

There is an extensive literature on hyperbolic groups, beginning with Gromov's fundamental paper [12] and its exegeses, for instance [11], [6], [1], [2]. However, remarkably little is known about subgroups of general hyperbolic groups; for example, it is unknown how distorted the word metrics of finitely generated subgroups can be and how distorted the areas of finitely presented subgroups [8] can be.[1] Recall that, for a given finite presentation, the area of a relation is the minimum number of terms expressing the relation as a product of conjugates of relators. An isoperimetric function (or isoperimetric inequality) for a given finite

[1] An open test question is whether a finitely generated subgroup of a hyperbolic group can be distorted more than exponentially in its word metric. Is even exponential distortion possible in area for a finitely presented subgroup? We mention in this context a remarkable recent result of P. Papasoglu that the area distortion function of a finitely presented subgroup of a finitely presented group is *always* recursive.

presentation gives an upper bound for the area of the relations as a function of their length. The growth of this function is an invariant of the group (see below), and a group is hyperbolic if and only if this growth is linearly bounded.

Here is a brief sketch of what is known about such subgroups. A finitely generated subgroup of a hyperbolic group $G$ has a solvable word problem (since $G$ itself has a solvable word problem) and a subgroup of $G$ has finite rational cohomological dimension (since $G$ itself has finite rational cohomological dimension). From the action of $G$ on the Gromov boundary one knows that solvable (and more generally amenable) subgroups of $G$ are virtually cyclic, and if a subgroup is not virtually solvable then it contains a nonabelian free subgroup [12]. Furthermore Rips showed there can exist finitely generated subgroups of hyperbolic groups which are not finitely presentable [17] (*i.e.* in general hyperbolic groups are not *coherent*). N. Brady gave one example of a hyperbolic group $G = H \rtimes_\phi \mathbb{Z}$ where $H$ is finitely presented but not of type $FP_3$ [3]; since hyperbolic groups are of type $FP_\infty$, $H$ is not hyperbolic. In his example, $G$ is of cohomological dimension 3; this contrasts with the result of [10] that a finitely presented subgroup of a hyperbolic group of cohomological dimension 2 is hyperbolic. Finally, there is a universal bound on the order of a finite subgroup of a hyperbolic group $G$ that depends only on the number of generators and the Rips constant $\delta$ for that set of generators [4].

The main application of our general result is the following:

THEOREM A.

*Let $G$ be a split extension of a finitely presented group $K$ by a finitely generated free group $F$, so one has the short exact sequence*

$$1 \to K \to G \to F \to 1.$$

*(1) If $G$ is hyperbolic, then $K$ satisfies a polynomial isoperimetric inequality.*
*(2) If $G$ is (either synchronously or asynchronously) automatic, then $K$ satisfies an exponential isoperimetric inequality.*

It follows from conclusion (1) that in Brady's example of the preceding paragraph, $H$ satisfies a polynomial isoperimetric inequality. This answers a question raised in [3], and improves the result of [8], where the first author showed that $H$ had an exponential isoperimetric function.

Concerning conclusion (2), there is an example due to M. Bridson *et al* (see [5]), which was discussed in [8] section 4, of a homomorphism of a synchronously automatic group onto $\mathbb{Z}$ with kernel finitely presentable and having an optimal exponential isoperimetric function (*i.e.*

the Dehn function of the kernel is exponential). Thus conclusion (2) is best possible in this generality.

To prove Theorem A, we use the diagrammatic methods of [8] with some important minor adjustments to obtain a more general result. Consider a split extension $1 \to K \to G \to F \to 1$ with $K$ finitely presented and $F$ a finitely generated free group. We obtain an isoperimetric inequality for $K$ in terms of a combined area/radius function (an "AR" pair) for $G$. The precise result is stated below as Theorem B.

## 2. Isoperimetric functions and AR pairs

Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of the group $G$. We use $F(X)$ to denote the free group on the set $X$, $\overline{X}$ to denote the set $X \cup X^{-1}$, and $\overline{X}^*$ to denote the free semigroup on $\overline{X}$. For $v, w \in \overline{X}^*$, we write $v =_G w$ to mean that these words represent the same element of the group presented.

We recall that if $w \in F(X)$ is a relation for $\mathcal{P}$, *i.e.* $w =_G 1$ ($w$ is in the normal closure of the set $R$) then there is a van Kampen diagram for $w$ over $\mathcal{P}$ (or a $\mathcal{P}$–diagram for $w$): briefly, this can be thought of as a connected, oriented, labelled, planar graph, such that all of the bounded regions, or interior faces, of its complement have boundaries labelled by words in $R$ (read from an appropriate starting point, with an appropriate orientation), and the unbounded region has boundary label $w$ (for more about such diagrams see for instance [15, Chap.V].) We consider here also diagrams for relations in $\overline{X}^*$.

An *area function* for $\mathcal{P}$ is a function $f : \mathbb{N} \to \mathbb{R}$ such that for each relation $w \in \overline{X}^*$ of length at most $n$, there is a collection of $N \leq f(n)$ words $p_i \in F(X)$ and choices $r_i \in R \cup R^{-1}$, such that $w = \prod_{i=1}^{N} p_i r_i p_i^{-1}$ in the free group $F(X)$. The number $N$ is the number of interior faces of the associated van Kampen diagram. A *radius function* for $\mathcal{P}$ is a function $g : \mathbb{N} \to \mathbb{R}$ such that for each relation $w \in F(X)$ of length at most $n$, there is a *not necessarily* reduced (in the sense of [15, Chap.V]) van Kampen diagram $D_{\mathcal{P}}(w)$ such that from each vertex there is a path in the 1–skeleton to the boundary $\partial D_{\mathcal{P}}(w)$ of length at most $g(n)$. Notice that reduction of a diagram reduces the area, but may in general increase the radius; there is an obvious diagram of area $N$ and of radius zero for the unreduced word $\prod_{i=1}^{N} p_i r_i p_i^{-1}$ (this is the so-called lollypop construction, which is illustrated in [15, Chap.V] p. 237 Fig. 1.1).

An *area–radius pair*, or AR pair, for $\mathcal{P}$, is a pair $(f, g)$ of functions $f, g : \mathbb{N} \to \mathbb{R}$ such that for each relation $w \in F(X)$ of length at most $n$, there is a not necessarily reduced diagram $D_{\mathcal{P}}(w)$ whose area is bounded by $f(n)$ and whose radius is bounded by $g(n)$, *i.e.* $\mathrm{Area}(D_{\mathcal{P}}(w)) \leq f(n)$ and $\mathrm{radius}(D_{\mathcal{P}}(w)) \leq g(n)$.

Our main theorem is :

THEOREM B.

*Let $G$ be a split extension of a finitely presented group $K$ by a finitely generated free group $F$, so one has the short exact sequence*

$$1 \to K \to G \to F \to 1.$$

*If $(f, g)$ is an AR pair for $G$, then there is a constant $A > 1$ such that $A^g f$ is an isoperimetric inequality for $K$.*

We define two equivalence relations on functions $\mathbb{N} \to \mathbb{R}$; we say that $f \simeq \overline{f}$ when there are integer constants $A, B, C, D$ such that, for all $n \in \mathbb{N}$, both $f(n) \leq A\overline{f}(Bn)+Cn+D$ and $\overline{f}(n) \leq Af(Bn)+Cn+D$. Notice that all constant functions are equivalent to a linear function in this relation.

Also we have an equivalence relation $g \cong \overline{g}$ if there are integer constants $A, B, C$ such that, for all $n \in \mathbb{N}$, $g(n) \leq A\overline{g}(Bn) + C$ and $\overline{g}(n) \leq Ag(Bn) + C$. Notice that the zero function is here equivalent to any constant function, but is *not* equivalent to a non–constant linear function. This finer equivalence relation is required in order to consider the radius function for hyperbolic groups.

The $\simeq$–equivalence class of the area function for $\mathcal{P}$ is an invariant of the group called an isoperimetric function (see for instance [1], [2], [11], [6]). The equivalence of the radius functions is more subtle, requiring *a priori* the consideration of non–reduced diagrams, and gives an invariance of the $\cong$–equivalence class of the radius functions.

Notice that if $(f, g)$ is an AR pair for the finite presentation $\mathcal{P}$, then:

(i) $f$ is an isoperimetric inequality for $\mathcal{P}$, and $g$ is a radius function for $\mathcal{P}$, but they may well not be simultaneously best possible;

(ii) $(f, g)$ is also an AR pair for the unreduced words.

The equivalence class of an AR pair for $\mathcal{P}$ is an invariant of the group so presented, in the following sense:

PROPOSITION 2.1. *Let $\mathcal{P}, \mathcal{Q}$ be finite presentations for the group $G$.*

*If $(f, g)$ is an AR pair for $\mathcal{P}$, then there is an AR pair $(\tilde{f}, \tilde{g})$ for $\mathcal{Q}$, such that $f \simeq \tilde{f}$ and $g \cong \tilde{g}$.*

*Proof.* Let $\mathcal{P} = \langle X \mid R\rangle$, $\mathcal{Q} = \langle Y \mid S\rangle$, and let $w \in F(Y)$ be a relation. Rewrite each generator $y_i \in Y$ as a word $v_i(X) \in F(X)$, such that $y_i =_G v_i$. Replacing each letter $y_i$ in $w$ by the corresponding word $v_i$, gives a not necessarily reduced word $w' \in \overline{Y}^*$. Let $\beta_1$ be the maximum of the lengths of the words $v_i$. Let $D = D_{\mathcal{P}}(w')$ be a not necessarily reduced diagram for this relation satisfying the AR pair $(f, g)$, *i.e.* $\mathrm{Area}(D_{\mathcal{P}}(w') \leq f(\ell(w'))$ and $\mathrm{radius}(D_{\mathcal{P}}(w')) \leq g(\ell(w'))$.

Now translate this $\mathcal{P}$–diagram into a diagram over $\mathcal{Q}$ as follows. For each $x_j \in X$, there is a word $z_j \in F(Y)$ such that $x_j =_G z_j$ in $G$. Let $\beta_2$ be the maximum of the lengths of these words $z_j$. After subdivision as necessary, replace each edge of $D_{\mathcal{P}}(w')$ labelled $x_j$ by edges labelled by the word $z_j$, to give a planar labelled graph $D'$. Each interior region of $\overline{\mathbb{R}^2 - D'}$ which in $D$ was labelled $r_k \in R$, is now labelled by a not necessarily reduced word ${r_k}' \in \overline{Y}^*$. The unbounded region of the plane is labelled by a word $w'' \in \overline{Y}^*$ which is equal to $w$ in $G$.

For each $r_k \in R$, choose a van Kampen diagram $D_{\mathcal{Q}}({r_k}')$ for ${r_k}'$, whose interior faces are labelled in $S$. Let $\alpha_1$ be the maximum of the areas of these $\mathcal{Q}$–diagrams, and let $\rho_1$ be the maximum of their radii. Inserting these diagrams into the corresponding regions of $D'$ gives a not necessarily reduced van Kampen diagram $D''$ over the presentation $\mathcal{Q}$ for the relation $w''$. The radius of $D''$ is at most $\rho_1 + \beta_2 \mathrm{radius}(D)$.

In this procedure, each $Y$–letter $y_i$ appearing in the original word $w$ is first replaced by an $X$–word $v_i$, which is then rewritten as a word $Y_i$ in $\overline{Y}^*$. As before, for each $y_i \in Y$ choose a $\mathcal{Q}$–diagram for the relation $Y_i =_G y_i$; let $\alpha_2$ be the maximum of the areas of these diagrams, and let $\rho_2$ be the maximum of their radii. Adding these diagrams to the boundary of $D''$ gives a not necessarily reduced diagram $\overline{D}_{\mathcal{Q}}(w)$ for the original word $w$.

The $\mathcal{Q}$–diagram $\overline{D}_{\mathcal{Q}}(w)$ now satisfies:

$$\begin{aligned}
\mathrm{Area}(\overline{D}_{\mathcal{Q}}(w)) \qquad & \leq \alpha_1 \mathrm{Area}(D_{\mathcal{P}}(w')) + \alpha_2 \ell(w) \\
& \leq \alpha_1 f(\ell(w')) + \alpha_2 \ell(w) \\
& \leq \alpha_1 f(\beta_1 \ell(w)) + \alpha_2 \ell(w)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{radius}(\overline{D}_{\mathcal{Q}}(w)) & \leq \beta_2(\rho_1 + \mathrm{radius}(D_{\mathcal{P}}(w'))) + (\rho_2 + \beta_1\beta_2) \\
& \leq \beta_2 g(\ell(w')) + \beta_2(\rho_1 + \rho_2) \\
& \leq \beta_2 g(\beta_1 \ell(w)) + \beta_2(\rho_1 + \rho_2)\ .
\end{aligned}$$

The result now follows, using the two equivalence relations.

It is clear that for any finitely presented group satisfying an isoperimetric function $f$, there is an AR pair $(f, \rho f)$, where $\rho$ is one half of the maximum of the lengths of the relations.

Hyperbolic groups have an AR pair with logarithmic radius function, as is shown in [1]:

LEMMA 2.2. *Let $\mathcal{P} = \langle X; R \rangle$ be a finite presentation of a group $G$. If $G$ is hyperbolic, then there are constants $A, B > 0$ such that, for any relation $w \in \overline{X}^*$ with $\ell(w) \geq 1$, there is a $\mathcal{P}$–diagram of area at most $A\ell(w)\log_2(\ell(w))$ and of radius at most $B\log_2(\ell(w))$.*

*Proof.* This is proved in [1] (for relations in $F(X)$), using the definition of a hyperbolic group as one where geodesic triangles in the Cayley graph are $\delta$–thin. It suffices to subdivide a loop labelled $w$, viewed as a loop in the Cayley graph of $G$. Subdivide by a shortest arc between vertices, cutting the loop into two (almost) equal parts. Continually subdividing the outside boundary (i.e. the intersection of each with the original loop labelled $w$) of the two new loops obtained, subdivides the loop into geodesic triangles, each of which is $\delta$–thin. After at most $\log_2(\ell(w))$ steps, the original loop has been subdivided into edges (arcs of length 1). The uniform thinness of these geodesic triangles gives a decomposition of each triangle into subloops of length at most $3\delta + 3$. There at most $\ell(w)(\log_2(\ell(w) + 1))$ such subloops, and so the area of this diagram is at most $A\ell(w)\log_2(\ell(w))$ (where $A$ is the maximum area aver the minimal $\mathcal{P}$–diagrams for the relations of length at most $3\delta$. The radius of the diagram is easily seen to be at most $\delta(\log_2(\ell(w)) + 1) + C \leq B\log_2 \ell(w)$, by moving towards the boundary through the different subdivisions, where $C$ is the maximum radius over the minimal $\mathcal{P}$–diagrams for the relations of length at most $3\delta$.

In the same way , it is not hard to see from the proofs in [7, pages 52 and 152] that if $G$ is a synchronously (respectively asynchronously) automatic group then there are constants $A, B > 0$ (resp. $C > 1,\ D > 0$) such that $(Ax^2, Bx)$ (resp. $(C^x, Dx)$) is an AR pair for $G$.

*Remark*. All minimal area van Kampen diagrams in a finite presentation of a hyperbolic group have a uniform logarithmic upper bound on their radii. This is asserted in [13] p. 100 and can be proved by the methods of [14] section 5.

## 3. New diagram from an automorphism

Let $\mathcal{P} = \langle X \mid R \rangle$ be a finite presentation of the group $G$, and let $\phi : G \to G$ be an automorphism.

For each $x_j \in X$, choosing a word in $F(X)$ representing $\phi(x_j)$ induces a semigroup homomorphism $\Phi : \overline{X}^* \to \overline{X}^*$ such that $\Phi(x_j) =_G \phi(x_j)$, and $\Phi(x_j^{-1}) =_G \phi(x_j^{-1}) =_G (\phi(x_j))^{-1}$. As $\phi$ is an *auto*morphism, there is also a semigroup homomorphism $\Psi : \overline{X}^* \to \overline{X}^*$ such that $\Psi(\Phi(x_j^{\pm 1})) =_G x_j^{\pm 1} =_G \Phi(\Psi(x_j^{\pm 1}))$. Rapaport's theorem, used in [8], says that there is a finite set of generators $Z$ for $G$, such that $\Phi$ can be chosen to be an automorphism of $F(Z)$. We shall not make use of Rapaport's theorem in this note.

Let $D_{\mathcal{P}}(w)$ be a $\mathcal{P}$–diagram for the relation $w$. It is a straightforward matter to obtain a not necessarily reduced diagram $\Phi(D_{\mathcal{P}}(w))$ over $\mathcal{P}$ for the not necessarily reduced word $\Phi(w)$, as follows:

(i) Subdivide and relabel the 1–skeleton of $D_{\mathcal{P}}(w)$ so that an edge previously labelled $x_i$ is subdivided and relabelled $\Phi(x_i)$.

This gives a connected, planar, labelled, oriented 1–complex $D'$ in the plane, whose outside boundary is labelled by the not necessarily freely reduced word $\Phi(w)$. The compact regions are labelled by words $\Phi(r)$ where $r \in R$. Each $\Phi(r)$ is a relation in $G$; choose a diagram $\Phi(D_{\mathcal{P}}(r))$ over $\mathcal{P}$ for each $\Phi(r)$.

(ii) Insert the diagrams $\Phi(D_{\mathcal{P}}(r))$ into the corresponding faces of the planar graph $D'$ to obtain $\Phi(D_{\mathcal{P}}(w))$.

We have thus shown that:

LEMMA 3.1.

*There is a positive constant $S$ such that if $D_{\mathcal{P}}(w)$ is a $\mathcal{P}$–diagram for the relation $w$, then there is a $\mathcal{P}$–diagram $\Phi(D_{\mathcal{P}}(w))$ such that* $\mathrm{Area}(\Phi(D_{\mathcal{P}}(w))) \leq S\mathrm{Area}(D_{\mathcal{P}}(w))$.

It suffices to take $S$ to be the maximum of the areas of the diagrams $\Phi(D_{\mathcal{P}}(r))$, $r \in R$.

The "inverse" map on diagrams is slightly more complicated:

LEMMA 3.2. *There are positive constants $S', S''$ such that if $D_{\mathcal{P}}(\Phi(w))$ is a diagram for the relation $\Phi(w)$, then there is a diagram $D_{\mathcal{P}}(w)$ for the relation $w$ such that* $\mathrm{Area}(D_{\mathcal{P}}(w)) \leq S'\mathrm{Area}(D_{\mathcal{P}}(\Phi(w))) + S''\ell(w)$.

*Proof.* As before there is a diagram $\Psi(D_{\mathcal{P}}(\Phi(w))$ for the relation $\Psi(\Phi(w))$, whose area is at most $S'$ times the area of the diagram $D_{\mathcal{P}}(\Phi(w))$, where $S'$ is the maximum of the areas of the diagrams $\Psi(D_{\mathcal{P}}(r))$.

If $w = a_1 a_2 \ldots$, where $a_i \in \overline{X}$, then $\Psi(\Phi(w)) = \Psi(\Phi(a_1))\Psi(\Phi(a_2))\ldots$ in $\overline{X}^*$. For each $x_i \in X$, choose a diagram for the $\mathcal{P}$ relation $x_i =_G \Psi(\Phi(x_i))$; let $S''$ be the maximum area of these diagrams. Adding these to the boundary of $\Psi(D_{\mathcal{P}}(\Phi(w)))$ gives a not necessarily reduced diagram for $w$ over $\mathcal{P}$ with the required area.

## 4. Proof of the theorem B

Let $\mathcal{P}_K = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_k \rangle$ be a finite presentation of the group $K$, and let $F_n$ be the free group of rank $n$, generated by elements $\{t_1, \ldots, t_n\}$. For a split extension $1 \to K \to G \to F_n \to 1$, there is an automorphism $\phi_i : K \to K$ given by conjugation by $t_i$ restricted to $K$ for each $i$. Let $\Phi_i$ be a lift of $\phi_i$ to a semigroup endomorphism and let $\Psi_i$ be a lift of the inverse, as in the preceding section. Then $G$ has a presentation

$$\mathcal{P}_G = \langle x_1, \ldots, x_m, t_1, \ldots, t_n \mid r_1, \ldots r_k, \{t_i^{-1} x_j t_i \Phi_i(x_j)^{-1}\} \rangle \quad .$$

In a diagram over $\mathcal{P}_G$ for a relation $w$ which does not involve any of the stable letters $t_i$ for the HNN-extension $G$, the regions corresponding to relations $t_i^{-1} x_j t_i \Phi_i(x_j)^{-1}$ form "$t$–rings" (as in [8, §5]); each $t$–ring is an annular region and is associated to one of the stable letters.

Let $w \in \overline{X}^*$ be a relation over $\mathcal{P}_G$, and let $D = D_{\mathcal{P}}(w)$ be a $\mathcal{P}_G$–diagram for $w$. Let $A$ be a $t$–ring, and let $D_0, D_1$ be the two components of $\overline{D - A}$, where $D_1$ is the outer component, containing the boundary of $D$, and $D_0$ is the inner component. Let $u_0$ be the label on the boundary of $D_0$ (the inner boundary of $A$), and let $u_1$ be the label on the outer boundary component of $A$; each is a word in $\overline{X}^*$ which is a relation in $\mathcal{P}_G$ (and in $\mathcal{P}_K$). The subdiagram $D_0$ is a diagram for $u_0$ over $\mathcal{P}_G$.

In a $\mathcal{P}_G$–diagram for $w$, the $t$–rings are nested at most as deep as the radius of the diagram. Consider an innermost $t$–ring $A$ in $D$, where the $t$–edges in $A$ are labelled $t_i$; in this case, $D_0$ is a $\mathcal{P}_K$–diagram for $u_0$.

There are two cases to consider according to the orientation the edge $t_i$ in $A$: either $u_1 = \Phi_i(u_0)$, or $u_0 = \Phi_i(u_1)$.

In case 1, applying $\Phi_i$ to the $P_K$–diagram $D_0$, lemma 3.1 says that there is a $\mathcal{P}_K$–diagram $D(u_1)$ for $u_1$ of area at most $S.\mathrm{Area}(D_0)$. Thus replacing $A \cup D_0$ in $D$ by $D(u_1)$, and simultaneously doing the analogous procedure on all such innermost $t$–rings, at worst multiplies the area of $D$ by $S$.

In case 2, lemma 3.2 shows how to obtain a $\mathcal{P}_K$–diagram $D(u_1)$ of area at most $S'\mathrm{Area}(D_0) + S''\ell(u_1)$. Each edge in $D$ occurs in at most two $t$–rings, so twice the number of edges in $D$ gives a bound on $\ell(u_1)$; thus $\ell(u_1) \leq \ell(w) + 2\rho\ \mathrm{Area}(D)$, where $\rho$ is the maximum length of a relation in $R \cup \{t_i x_j {t_i}^{-1} \Phi_i(x_j)^{-1}\}$. Thus replacing $D_0 \cup A$ in $D$ by $D(u_1)$, and doing this on all innermost $t$–rings simultaneously, at worst multiplies the area by $S'$, and adds on $S''(\ell(w) + 2\rho\ \mathrm{Area}(D)) \leq Mf(\ell(w))$, for some positive constant $M$, independent of $w$. Choose $M \geq \max(S, 1)$.

These changes must be performed at most $g(\ell)$ times (where we let $\ell = \ell(w)$), as the nesting of $t$–rings is bounded by the radius. Then removing all $t$–rings gives a $\mathcal{P}_K$–diagram for $w$ of area at most

$$\underbrace{M(\ldots(M(Mf(\ell)+Mf(\ell))+Mf(\ell))+\ldots+Mf(\ell))}_{\text{at most } g(\ell) \text{ times}} \leq M^{g(\ell)+1}f(\ell).$$

Theorem B follows by replacing $M$ by an even larger constant $A > 1$ ($A > M^2$ will work).

Theorem A now follows from applying Theorem B to the AR pair in Lemma 2.2.

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