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Absolute Continuity of the Law of the Solution of a Parabolic SPDE

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Let $\{u(t,x); t \geq 0, 0 < x < 1\}$ denote the solution of a white noise driven parabolic stochastic partial differential equation with Dirichlet boundary conditions. Using Malliavin's calculus, we give a necessary and sufficient condition for the law of the r.v. $u(t,x)$ to a possess a density.

1. INTRODUCTION

Consider the following stochastic partial differential equation,

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x;u(t,x)) \\ + g(t,x;u(t,x))\, \mathring{W}(t,x); \qquad t>0, \quad 0<x<1 \qquad (1) \\ u(0,x) = u_0(x); \qquad u(t,0) = u(t,1) = 0, \quad t \geq 0, \end{aligned}$$

where $\mathring{W}$ denotes space-time white noise, $u_0 \in C_0([0,1])$ (i.e., u is continuous and $u_0(0) = u_0(1) = 0$) is a deterministic function, $f, g: \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are measurable and locally bounded functions which are differentiable with respect to their third argument, the derivatives

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$f' = \partial f/\partial z$, $g' = \partial g/\partial z$ being locally bounded. We suppose moreover that g is jointly continuous and that there exists c such that

$$zf(t, x; z) + |g(t, x; z)|^2 \leqslant c(1 + |z|^2), \qquad (t, x, z) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}. \quad (2)$$

We normally write $f(u(s, x))$ instead of $f(s, x; u(s, x))$, and similarly with f', g, and g'.

Equation (1) is formulated rigorously as

$$\begin{aligned} u(t, x) = &\int_0^1 G_{t}(x, y)\, u_0(y)\, dy \\ &+ \int_0^t \int_0^1 G_{t-s}(x, y) f(u(s, y))\, dy\, ds \\ &+ \int_0^t \int_0^1 G_{t-s}(x, y)\, g(u(s, y))\, W(dy, ds), \end{aligned} \quad (3)$$

where $G_t(x, y)$ denotes the kernel of e^{tA}, A being the operator $\partial^2/\partial x^2$ on $(0, 1)$ with Dirichlet boundary conditions.

It is shown in Gyöngy and Pardoux [3] that under the above conditions Eq. (3) has a unique adapted solution $\{u(t, x); t \geqslant 0, 0 \leqslant x \leqslant 1\}$ with continuous paths.

The aim of this paper is to prove:

THEOREM 1.1. *Let $(t, x) \in (0, \infty) \times (0, 1)$. The law of the r.v. $u(t, x)$ is absolutely continuous with respect to Lebesgue measure iff there exists $s \in [0, t)$ such that $g(s, \cdot\,; u(s, \cdot)) \not\equiv 0$.*

This result will follow from:

THEOREM 1.2. *Suppose that $g(0, y; u_0(y)) \neq 0$ for some $y \in [0, 1]$. Then for any $t > 0$ and $0 < x < 1$, the law of the r.v. $u(t, x)$ is absolutely continuous with respect to Lebesgue measure.*

The paper is organised as follows. In Section 2, we formulate a sufficient condition given by the Malliavin Calculus for the conclusion of Theorem 1.2 to hold, and compute the Wiener space derivative of $u(t, x)$. In Section 3 we prove a result about the support of the solution of a linear SPDE, which allows us to prove Theorem 1.2, and we finally prove Theorem 1.1.

We note that there does not seem to be any other alternative approach to our result. This is a big difference with most of the results produced by the Malliavin calculus applied to "ordinary" SDEs.

2. Malliavin Calculus, and Application to Our White Noise Driven SPDE

We note that the stochastic integral in (2) is an Itô integral with respect to the Brownian sheet $\{W(t, x);\ (t, x) \in \mathbb{R}_+ \times [0, 1]\}$. Let this Brownian sheet be the canonical process defined on $\Omega = C_0(\mathbb{R}_+ \times [0, 1])$ (the space of continuous functions on $\mathbb{R}_+ \times [0, 1]$ which are zero whenever one of their arguments is zero), equipped with its Borel σ-field $\mathscr{F}$, and the "Brownian sheet measure" P.

Let S denote the set of "simple random variables" of the form

$$F = f(W(h_1), ..., W(h_d)),$$

where $d \in \mathbb{N}$, $h_i \in L^2(\mathbb{R}_+ \times (0, 1))$, and $W(h_i)$ denotes the Wiener integral

$$W(h_i) = \int_{\mathbb{R}_+} \int_0^1 h_i(t, x)\, W(dx, dt), \qquad 1 \leqslant i \leqslant d,$$

and $f \in C_b^\infty(\mathbb{R}^d)$. For such a r.v. F, we define its "derivative"

$$\{D_{t,x}F,\ (t, x) \in \mathbb{R}_+ \times (0, 1)\}$$

by

$$D_{t,x}F = \sum_{i=1}^{d} \frac{\partial f}{\partial z_i}(W(h_i), ..., W(h_d))\, h_i(t, x)$$

and its $\|\cdot\|_{1,2}$ norm by

$$\|F\|_{1,2}^2 = E(F^2 + \|DF\|_{L^2(\mathbb{R}_+ \times (0,1))}^2).$$

We denote by $\mathbb{D}^{1,2}$ the closure of S with respect to the norm $\|\cdot\|_{1,2}$. $\mathbb{D}^{1,2}$ is a Hilbert space. It is the domain of the closure of the derivation operator D (which we still denote D).

We can also define directional derivatives as follows. For any $h \in L^2(\mathbb{R}_+ \times (0, 1))$, $F \in S$ as above

$$D_h F = \frac{d}{d\varepsilon} f(W(h_1) + \varepsilon(h, h_1), ..., W(h_d) + \varepsilon(h, h_d))|_{\varepsilon=0}.$$

D_h can be extended as a closed operator on $L^2(\Omega)$, with domain $\mathbb{D}_h \supset \mathbb{D}^{1,2}$. Furthermore, for any orthonormal basis $\{h_n, n = 1, 2, ...\}$ of $L^2(\mathbb{R}_+ \times (0, 1))$, we have that $F \in \mathbb{D}^{1,2}$ iff $F \in \mathbb{D}_{h_n}$ for each $n \geqslant 1$ and

$$\sum_{n=1}^{\infty} E[(D_{h_n}F)^2] < \infty.$$

In that case,

$$D_{t,x}F = \sum_{n=1}^{\infty} D_{h_n}Fh_n(t,x)$$
$$= \sum_{n=1}^{\infty} (h_n, DF)\, h_n(t,x),$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\mathbb{R}_+ \times (0,1))$, and for $h \in L^2(\mathbb{R}_+ \times (0,1))$,

$$D_hF = \sum_{n=1}^{\infty} (h_n, DF)(h_n, h).$$

Let $\mathscr{F}_t = \sigma\{W(s,x);\ 0 \leqslant s \leqslant t,\ 0 \leqslant x \leqslant 1\} \vee \mathscr{N}$, where $\mathscr{N}$ denotes the collection of P-null sets of $\mathscr{F}$. We say that a random field $\{v(t,x);\ t>0,\ 0 \leqslant x \leqslant 1\}$ is adapted if for each $(t,x) \in \mathbb{R}_+ \times [0,1]$, $v(t,x)$ is $\mathscr{F}_t$ measurable.

Following Bouleau and Hirsch [1] and Nualart and Pardoux [5] we state the:

DEFINITION 2.1. $\mathbb{D}^{1,2}_{loc}$ denotes the set of random variables F to which one can associate a sequence $\{(\Omega_n, F_n)\} \subset \mathscr{F} \times \mathbb{D}^{1,2}$ such that:

(i) $\Omega_n \subset \Omega_{n+1}$, $n \geqslant 1$; $\bigcup_n \Omega_n = \Omega$ a.s.

(ii) $F|_{\Omega_n} = F_n|_{\Omega_n}$, $n \geqslant 1$.

Such a sequence $\{(\Omega_n, F_n)\}$ is called a localizing sequence for F.

The following result, which follows from the local property of D, is proved in Bouleau and Hirsch [1] and Nualart and Pardoux [5]:

PROPOSITION 2.2. *Let $F \in \mathbb{D}^{1,2}_{loc}$. There exists a unique measurable function of (t,x,ω) DF such that for any localizing sequence $\{(\Omega_n, F_n)\}$, $\mathbf{I}_{\Omega_n}DF = \mathbf{I}_{\Omega_n}DF_n$, $dt\,dx\,dP$* a.e.

The following result is an immediate consequence of Proposition 7.1.4 in Bouleau and Hirsch [1]:

PROPOSITION 2.3. *Let F be a (real valued) random variable. A sufficient condition for the law of F to be absolutely continuous with respect to Lebesgue measure is that*

(i) $F \in \mathbb{D}^{1,2}_{loc}$

(ii) $\|DF\|_{L^2(\mathbb{R}_+ \times (0,1))} > 0$ *a.s.*

We now turn to the solution u of Eq. (3). Let $t>0$ and $0<x<1$. We first need to show that $u(t,x)\in \mathbb{D}^{1,2}_{loc}$ and compute $D_h u(t,x)$ for $h\in L^2(\mathbb{R}_+\times(0,1))$.

PROPOSITION 2.4. *For any $(t,x)\in(0,+\infty)\times(0,1)$, $u(t,x)\in\mathbb{D}^{1,2}_{loc}$, and for any $h\in L^2(\mathbb{R}_+\times(0,1))$, $D_h u(t,x)$ is the unique solution of the following SPDE*:

$$\begin{aligned} D_h u(t,x) &= \int_0^t\int_0^1 G_{t-s}(x,y)\,g(u(s,y))\,h(s,y)\,dy\,ds \\ &\quad+\int_0^t\int_0^1 G_{t-s}(x,y)\,f'(u(s,y))\,D_h u(s,y)\,dy\,ds \\ &\quad+\int_0^t\int_0^1 G_{t-s}(x,y)\,g'(u(s,y))\,D_h u(s,y)\,W(dy,ds); \\ &\qquad t\geqslant 0,\quad x\in[0,1]. \end{aligned}$$

Proof. Note that the equation for $D_h u$ formally reads

$$\begin{aligned} \frac{\partial}{\partial t}D_h u(t,x) &= \frac{\partial^2}{\partial x^2}D_h u(t,x)+f'(u(t,x))\,D_h u(t,x) \\ &\quad+g'(u(t,x))\,D_h u(t,x)\,\mathring{W}(t,x) \\ &\quad+g(u(t,x))\,h(t,x),\qquad t\geqslant 0,\ 0<x<1; \\ D_h u(0,x) &= D_h u(t,0)=D_h u(t,1)=0,\qquad 0\leqslant x\leqslant 1,\quad t\geqslant 0. \end{aligned}$$

Under our standing assumptions, we have (see [3]) that for any $T>0$, $p\geqslant 1$,

$$E\Big(\sup_{0\leqslant t\leqslant T,\,0\leqslant x\leqslant 1}|u(t,x)|^p\Big)<\infty.$$

Hence, since f, f', g, g' are locally bounded, it follows from Definition 2.1 and Proposition 2.2 that it suffices to prove the result (with the reinforced statement $u(t,x)\in\mathbb{D}^{1,2}$) under the additional assumption that f, f', g, and g' are bounded, which we assume for the rest of this proof.

Let $\{e_i, i=1,2,...\}$ be an orthonormal basis of $L^2(0,1)$ consisting of smooth functions which vanish at 0 and at 1. We first show that for $\rho\in L^2(\mathbb{R}_+)$ and $h(t,x)=\rho(t)\,e_l(x)$, $u(t,x)\in\mathbb{D}_h$, and derive the equation for $D_h u$.

Let us approximate the SPDE for u by a sequence of SDEs.

Let $\{u_i^{n,m}(t);\ 1 \leqslant i \leqslant n,\ t \geqslant 0\}$ denote the solution of the SDE

$$du_i^{n,m}(t) = -\sum_{j=1}^{n} (e_i', e_j')\, u_j^{n,m}(t)\, dt + \left(f\left(\sum_{j=1}^{n} u_j^{n,m}(t)\, e_j\right), e_i\right) dt$$
$$+ \chi_i^m \left(g\left(\sum_{j=1}^{n} u_j^{n,m}(t)\, e_j\right), e_i\right) dW_t^i$$
$$u_i^{n,m}(0) = (u_0, e_i), \qquad 1 \leqslant i \leqslant n;$$

with $\chi_i^m = 1$ if $i \leqslant m$ and $\chi_i^m = 0$ if $i > m$. Since $\{u_i^{n,m}(t)\}$ is the solution of an SDE with C_b^1 coefficients, driven by the finite dimensional Wiener process $\{W_t^i;\ 1 \leqslant i \leqslant m,\ t \geqslant 0\}$, it follows from the results in Watanabe [8] that $u_i^{n,m}(t) \in \mathbb{D}^{1,2}$,

$$D_h u_i^{n,m}(t) = 0 \qquad \text{for} \quad m < l,$$

and for $m \geqslant l$, $\{D_h u_i^{n,m}(t);\ t \geqslant 0,\ 1 \leqslant i \leqslant n\}$ in the unique solution of the SDE

$$dD_h u_i^{n,m}(t) = \left(g\left(\sum_{j=1}^{n} u_j^{n,m}(t)\, e_j\right), e_l\right) \rho(t)\, dt - \sum_{j=1}^{n} (e_i', e_j')\, D_h u_j^{n,m}(t)\, dt$$
$$+ \left(f'\left(\sum_{j=1}^{n} u_j^{n,m}(t)\, e_j\right) \sum_{k=1}^{n} D_h u_k^{n,m}(t)\, e_k, e_i\right) dt$$
$$+ \chi_i^m \left(g'\left(\sum_{j=1}^{n} u_j^{n,m}(t)\, e_j\right) \sum_{k=1}^{n} D_h u_k^{n,m}(t)\, e_k, e_i\right) dW_t^i,$$
$$D_h u_i^{n,m}(0) = 0, \qquad 1 \leqslant i \leqslant n.$$

Now we have that

$$\left(\sum_{i=1}^{n} u^{n,m}(t, i)\, e_i(x),\ \sum_{i=1}^{n} D_h u^{n,m}(t, i)\, e_i(x)\right) \to (u(t, x), v(t, x))$$

in $L^2(\Omega)$, if we let first $n \to \infty$, then $m \to \infty$, where v satisfies the SPDE

$$v(t, x) = \int_0^t \int_0^1 G_{t-s}(x, y)\, g(u(s, y))\, h(s, y)\, dy\, ds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, y)\, f'(u(s, y))\, v(s, y)\, dy\, ds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, y)\, g'(u(s, y))\, v(s, y)\, W(dy, ds).$$

Indeed, the convergence as $n \to \infty$ to a pair of SPDEs driven by the m dimensional Wiener process $(W_t^1, \ldots, W_t^m)$ is the well-known convergence of the Galerkin method, see, e.g., Pardoux [6] (the pointwise convergence in (t, x) follows from the integral representation of the equations), and then the convergence as $m \to \infty$ follows from Lemma 2.1 in [2].

Hence from the closedness property of the operator D_h,

$$v(t, x) = D_h u(t, x).$$

It remains to show that if $\{h_n\}$ is an orthonormal basis of $L^2(\mathbb{R}_+ \times (0, 1))$, each h_n being of the same form as above,

$$\sum_{n=1}^{\infty} E(|D_{h_n} u(t, x)|^2) < \infty.$$

But

$$E(|D_{h_n} u(t, x)|^2) \leqslant cE \int_0^t \int_0^1 G_{t-s}^2(x, y)(D_{h_n} u(s, y))^2 \, dy \, ds$$
$$+ cE\left[\left(\int_0^t \int_0^1 G_{t-s}(x, y) \, g(u(s, y)) \, h_n(s, y) \, dy \, ds\right)^2\right].$$

Hence, if $V_m(t) \triangleq \sup_x E \sum_{n=1}^m |D_{h_n} u(t, x)|^2$, c denoting a constant whose value may vary from one line to another, but which is independent of m,

$$\begin{aligned}
V_m(t) &\leqslant c \int_0^t \int_0^1 G_{t-s}^2(x, y) \, V_m(s) \, dy \, ds + c \int_0^t \int_0^1 G_{t-s}^2(x, y) \, dy \, ds \\
&\leqslant c\left(1 + \int_0^t \frac{V_m(s)}{\sqrt{t-s}} \, ds\right) \\
&\leqslant c\left(1 + \int_0^t \int_0^s \frac{V_m(u)}{\sqrt{s-u}} \, du \frac{ds}{\sqrt{t-s}}\right) \\
&\leqslant c\left(1 + \int_0^t \int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} \, V_m(u) \, du\right) \\
&\leqslant c\left(1 + \int_0^t V_m(u) \, du\right),
\end{aligned}$$

and

$$V_m(t) \leqslant ce^{ct},$$

for all $m \in \mathbb{N}$. Hence $\sup_x E \sum_{n=1}^{\infty} |D_{h_n} u(t, x)|^2 < \infty$. We have proved that $u(t, x) \in \mathbb{D}^{1,2}$. The equation for $D_h u$ when h is not of the form considered in the first part of this proof follows by linearity and continuity with respect to h. ▮

3. Proof of Theorems 1.1 and 1.2

We first prove Theorem 1.2. In view of Proposition 2.3 and 2.4 all we need to show is that the condition $g(0, y; u_0(y)) \neq 0$ for some $y \in [0, 1]$ implies that $\|Du(t, x)\| > 0$ a.s., for any given $t > 0$ and $0 < x < 1$.

Suppose for instance that for some $0 < y < 1$, $g(0, y; u_0(y)) > 0$. Since $(s, z) \to g(s, z; u(s, z))$ is a.s. continuous, there exists $\varepsilon > 0$ and a stopping time τ s.t. $0 < \tau \leqslant t$ a.s. and

$$g(\theta, z; u(\theta, z)) > 0, \qquad y - \varepsilon \leqslant z \leqslant y + \varepsilon, \quad 0 \leqslant \theta \leqslant \tau.$$

Note that

$$\|Du(t, x)\| > 0 \Leftrightarrow \int_0^t \int_0^1 |D_{\theta, z} u(t, x)| \, dz \, d\theta > 0$$

and a sufficient condition for this is that

$$\int_0^\tau \int_{y-\varepsilon}^{y+\varepsilon} |D_{\theta, z} u(t, x)| \, dz \, d\theta > 0. \tag{4}$$

However, it follows easily from the comparison theorem in Donati-Martin and Pardoux [2] that for any $h \in L^2(\Omega \times \mathbb{R}_+ \times [0, 1]; dP \times dt \times dx)$ such that $h(s, y)$ is $\mathscr{F}_s$ measurable for each (s, y) and $\mathrm{supp}(h) \subset \{(s, y); g(s, y; u(s, y)) \geqslant 0\}$, $D_h u(t, x) \geqslant 0$. Hence $D_{\theta, z} u(t, x) \geqslant 0$ a.s., for any $(\theta, z) \in [0, \tau] \times [y - \varepsilon, y + \varepsilon]$.

Now (4) is equivalent to

$$\int_0^\tau \int_{y-\varepsilon}^{y+\varepsilon} D_{\theta, z} u(t, x) \, dz \, d\theta > 0,$$

and a sufficient condition for the conclusion of Theorem 1.2 to hold is that

$$\int_{y-\varepsilon}^{y+\varepsilon} D_{\theta, z} u(t, x) \, dz > 0 \qquad \text{a.s.,} \quad \forall 0 \leqslant \theta \leqslant \tau. \tag{5}$$

But

$$v(\theta; t, x) \triangleq \int_{y-\varepsilon}^{y+\varepsilon} D_{\theta, z} u(t, x) \, dz$$

is the unique solution of the SPDE

$$v(\theta; t, x) = \int_{y-\varepsilon}^{y+\varepsilon} G_{t-\theta}(x, z)\, g(u(\theta, z))\, dz$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, z)\, f'(u(s, z))\, v(\theta; s, z)\, dz\, ds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x, z)\, g'(u(s, z))\, v(\theta; s, z)\, W(dz, ds). \qquad (6)$$

Indeed, for any $\rho \in L^2(\mathbb{R}_+)$, $v_\rho(t, x) \triangleq \int_0^t \rho(\theta)\, v(\theta; t, x)\, d\theta$ coincides with $D_h u(t, x)$, where $h(t, x) = \rho(t)\, \mathbf{I}_{[y-\varepsilon, y+\varepsilon]}(x)$, since the two quantities solve the same equation.

Hence the result will follow from

PROPOSITION 3.1. *Under the above assumptions,*

$$v(\theta; t, x) > 0, \qquad \forall t > \theta, \quad 0 < x < 1; \quad \text{a.s.}$$

Proof. Our proof is inspired by that of a similar result in Mueller [4]. However, we rely on a cruder (and simpler) estimate. Since $v(\theta; \cdot)$ is a.s. continuous, it suffices to prove that for any fixed $t > \theta$, $0 < x < 1$, $v(\theta; t, x) > 0$ a.s.

Using a standard localization procedure, one can easily see that it suffices to prove the result under the assumption that f', g, and g' are bounded. For the sake of simplifying the notations we choose $\theta = 0$, and write $v(t, x)$ for $v(\theta; t, x)$. We note that v solves the linear SPDE

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f'(u)\, v + g'(u)\, v \dot{W}$$
$$v(0) = \varphi,$$

where

$$\varphi(z) = \mathbf{I}_{[y-\varepsilon, y+\varepsilon]}(z)\, g(u(0, z))$$
$$\geqslant \beta \mathbf{I}_{[y-\varepsilon, y+\varepsilon]}(z)$$

for some $\beta > 0$. From the comparison theorem in Donati-Martin and Pardoux [2], it suffices to prove the result with

$$\varphi(z) = \beta \mathbf{I}_{[y-\varepsilon, y+\varepsilon]}(z)$$

and by linearity we can as well choose $\beta = 1$. Hence we are reduced to prove our result with

$$\varphi(z) = \mathbf{I}_{[a,b]}(z),$$

where $0 \leqslant a < b \leqslant 1$ are arbitrary. Moreover, $\bar{v}(t, x) = e^{ct} v(t, x)$ satisfies the same equation as v, but with $f'(u)$ replaced by $f'(u) + c$. Hence we can assume that $f'(u) \geqslant 0$, and again from the same comparison theorem, it suffices to prove the result with $f'(u) \equiv 0$. Hence

$$v(t, x) = \int_0^1 G_t(x, y)\, \varphi(y)\, dy + \int_0^t \int_0^1 G_{t-s}(x, y)\, g'(u(s, y))\, v(s, y)\, W(dy, ds). \tag{7}$$

Suppose for instance that

$$a \leqslant x < 1$$

(the case $0 < x \leqslant a$ is treated analogously). Let $m \in \mathbb{N}$. For $k = 1, 2, ..., m$, consider the event

$$E_k = \left\{ v\left(\frac{kt}{m}, \cdot\right) \geqslant \alpha^k \mathbf{I}_{[a, b + dk/m]}(\cdot) \right\},$$

where $d > 0$ is such that $x \leqslant b + d < 1$, and $\alpha > 0$ is given by

$$\alpha = \tfrac{1}{2} \inf_{1 \leqslant k \leqslant m} \inf_{a \leqslant y \leqslant b + dk/m} \int_a^{b + d(k-1)/m} G_{t/m}(z, y)\, dz.$$

Let $\delta > 0$ be an arbitrarily small number. We show that for m large enough, $0 \leqslant k \leqslant m - 1$,

$$P(E^c_{k+1} / E_k \cap \cdots \cap E_1) \leqslant \frac{\delta}{m}. \tag{8}$$

Suppose for a moment that (8) is true. Then

$$P(E_{k+1} / E_k \cap \cdots \cap E_1) \geqslant 1 - \frac{\delta}{m}$$

and

$$P(E_1 \cap \cdots \cap E_m) \geqslant \left(1 - \frac{\delta}{m}\right)^m \geqslant 1 - \delta.$$

Hence

$$P(v(t, x) > 0) \geqslant P(v(t, \cdot) \geqslant \alpha^m \mathbf{I}_{[a, b+d]}(\cdot)) \\ \geqslant 1 - \delta,$$

and since δ is arbitrary,

$$P(v(t, x) > 0) = 1.$$

It remains to prove (8). It suffices in fact to show that

$$P(E_1^c) \leqslant \frac{\delta}{m}. \tag{9}$$

From the definition of α and (7), it is easily seen that (9) is a consequence of the

LEMMA 3.2. *For any $\delta > 0$, there exists $m_0 \in \mathbb{N}$ such that for any $m \geqslant m_0$,*

$$P\left(\sup_{0 \leqslant y \leqslant 1} \left| v_2\left(\frac{t}{m}, y\right) \right| > \alpha\right) \leqslant \frac{\delta}{m},$$

where $v_2(t, y) \triangleq \int_0^t \int_0^1 G_{t-s}(y, z) g'(u(s, z)) v(s, z) W(dz, ds)$.

Proof. It suffices to show that there exists $n, p > 1$, and c such that

$$E(\sup_{0 \leqslant y \leqslant 1} |v_2(t, y)|^n) \leqslant c t^p. \tag{10}$$

In order to prove (10), we first note that

$$E(|v_2(t, y)|^n) \leqslant c \left(\int_0^t \int_0^1 G_{t-s}^2(y, z)\, dz\, ds \right)^{n/2},$$

since g' is bounded and $E(\sup_{0 \leqslant s \leqslant 1, 0 \leqslant y \leqslant 1} |v(s, y)|^n) \leqslant \bar{c}$. Hence

$$E(|v_2(t, y)|^n) \leqslant c \left(\int_0^t \int_0^1 G_{t-s}^r(y, z)\, dz\, ds \right)^{n/2} t^{n/q},$$

where $2/r + 2/q = 1$. Provided $r < 3$, hence $q > 6$, we deduce

$$E(|v_2(t, y)|^n) \leqslant c t^{t/q}. \tag{11}$$

Similarly, we obtain, following the computations in Corollary 3.4 of Walsh [7],

$$E(|v_2(t, x) - v_2(t, y)|^n) \leqslant c |x - y|^{n/2 - 1} t^{n/q}. \tag{12}$$

Inequality (10) now follows from (11), (12), and Corollary 1.2 in [7], if we choose $n > q > 6$. ∎

It remains to prove that Theorem 1.1 follows from Theorem 1.2. First note that if $g(s, y; u(s, y)) \equiv 0$ on $[0, t] \times [0, 1]$, then $u(t, x)$ is deterministic. If that is not the case, then there exists a random time T such that $0 \leqslant T < t$ a.s. and a $\mathscr{F}_T$ measurable random element $X \in [0, 1]$ such that $g(T, X; u(T, X)) \neq 0$. By conditioning by $\mathscr{F}_T$, the result now follows from Theorem 1.2.

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