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White Noise Driven Parabolic SPDEs with Measurable Drift

V. BALLY*

Center of Mathematical Statistics, Bd Magheru 22, RO-70158 Bucharest, Roumania

I. GYÖNGY

Department of Probability Theory and Statistics, Eötvös University, Múzeum krt. 6-8, H-1088 Budapest

AND

E. PARDOUX†

Laboratoire APT, URA CNRS 225, Université de Provence, 13331 Marseille Cedex 3, France

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We prove existence and uniqueness of the solution of a white noise driven parabolic SPDE, in case the drift is measurable and satisfies a "one sided linear growth condition," and the diffusion coefficient is nondegenerate, has a locally Lipschitz derivative, and satisfies a linear growth condition. The proof combines arguments similar to those of Gyöngy and Pardoux together with an estimate of the density of the solution of the equation without drift, which is obtained with the help of the Malliavin calculus.

1. INTRODUCTION

Consider the following white noise driven nonlinear SPDE:

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+f(u)(t,x)+g(u)(t,x)\,\dot{W}(t,x),$$
$$t>0,\ 0<x<1; \tag{1.1}$$
$$u(0,x)=u_0(x),\ 0<x<1; \qquad u(t,0)=u(t,1)=0,\ t\geqslant 0;$$

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where $\{\dot{W}(t,x)\}$ is a formal expression for the "space–time white noise," $f(u)(t,x) := f(t,x;u(t,x))$ and similarly for g, with

$$f\colon \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$$

$$g\colon \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R},$$

and $u_0 \in C_0([0,1])$ ($C_0([0,1])$ stands for the set of continuous functions on $[0,1]$ which vanish at the endpoint 0 and 1). Equation (1.1) is a formal writing. One rigorous formulation of (1.1) is the "weak formulation":

$$(u(t),\varphi) = (u_0,\varphi) + \int_0^1 (u(s),\varphi'')\,ds + \int_0^t (f(u)(s),\varphi)\,ds$$

$$+ \int_0^t \int_0^1 \varphi(x)\, g(u)(s,x)\, W(ds,dx)$$

$$\forall t \geqslant 0, \qquad \varphi \in C^2(\mathbb{R}) \text{ such that } \varphi(0) = \varphi(1) = 0. \tag{1.2}$$

In (1.2), $(\cdot,\cdot)$ denotes the scalar product in $L^2(0,1)$. The second rigorous formulation is the integral formulation:

$$u(t,x) = \int_0^1 G_t(x,y)\, u_0(y)\,dy + \int_0^t \int_0^1 G_{t-s}(x,y)\, f(u)(s,y)\,dy\,ds$$

$$+ \int_0^t \int_0^1 G_{t-s}(x,y)\, g(u)(s,y)\, W(ds,dy),$$

$$t \geqslant 0,\ 0 \leqslant x \leqslant 1, \tag{1.3}$$

where $\{G_t(x,y)\}$ stands for the fundamental solution of the heat equation in $\mathbb{R}_+ \times [0,1]$ with Dirichlet boundary conditions, i.e., (see Feller [3]):

$$G_t(x,y) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left(-\frac{(2n+y-x)^2}{2t} \right) - \exp\left(-\frac{(2n+y+x)^2}{2t} \right) \right\}.$$

Note that the results given in this paper hold true also in case of homogeneous Neumann boundary conditions. The proofs are in fact simpler in that latter case.

It is shown in Walsh [11] that the two formulations (1.2) and (1.3) are equivalent, provided the random fields $f(u)$ and $g(u)$ are locally bounded.

Existence and uniqueness of a solution to the above equation is proved in [11] under Lipschitz continuity assumptions on f and g. Strong and weak existence and existence/uniqueness results under weaker assumptions can be found in Gyöngy and Pardoux [6]. In the case of a constant diffusion coefficient, it is shown in Gyöngy and Pardoux [4] that the equation has a unique strong solution in case of a measurable drift coefficient f which is locally bounded and satisfies a so-called "one sided linear growth condition." The result is extended to a class of nonnecessarily locally bounded drifts in [5]. In those two papers, the case of Neumann boundary conditions is treated in detail, and the necessary adaptations for the case of Dirichlet boundary conditions are indicated at the end.

In this paper, the assumptions concerning the drift f are those of [4], while the diffusion coefficient does not vanish, has a locally Lipschitz derivative, and satisfies a linear growth condition. It is shown that the equation has a unique strong solution. We treat only the case of Dirichlet boundary conditions, which is the hardest one.

Note that our results are close to those of Veretennikov [10] in the case of "ordinary" SDEs. However, we are not able to dispence with the Lipschitz continuity of the derivative of the diffusion coefficient. Note also that the one dimensionality of the solution is crucial in our approach.

Most of the methods of proofs are very close to those in [4], except for the first step, which consists in an L^p-estimate of the density of the solution of the equation without drift. That estimate is very easy in the case of constant g since the density is Gaussian and known explicitly. In the present situation, the estimate is obtained via Malliavin's calculus.

Malliavin's calculus has been first applied to the equation under study in Pardoux and Zhang Tusheng [9] in order to obtain the existence of a density for the law of $u(t, x)$, where $t > 0$ and $0 < x < 1$, under a very mild condition. It has been further developed in Bally and Pardoux [1] in order to establish the existence and smoothness of the density of the law of $(u(t, x_1), u(t, x_2), ..., (u(t, x_n))$, for $t > 0$, $0 < x_1 < x_2 < \cdots < x_n < 1$, under a nondegeneracy assumption. We exploit here some of the tools in [1] in order to estimate the density.

The paper is organized as follows. Section 2 is devoted to a precise statement of the assumptions, Section 3 to the estimation of the density. Section 4 contains some preparatory results for the existence theorem. Section 5 establishes the existence and uniqueness results under restrictive assumptions, which Section 6 extends to the general case, and gives a comparison and a continuity theorem.

2. Assumptions and Notations

We consider Eq. (1.1) (i.e., Eq. (1.2) or equivalently (1.3)). The "space–time white noise" $W(dt, dx)$ is defined as follows. We are given a zero mean Gaussian random field

$$\{W(B); B \in \mathscr{B}(\mathbb{R}_+ \times [0, 1])\}$$

defined on a probability space $(\Omega, \mathscr{F}, P)$, with covariance given by

$$E[W(B)\, W(C)] = \int_{B \cap C} dt\, dx.$$

For any $h \in L^2(\mathbb{R}_+ \times [0, 1])$, we can define the Wiener integral

$$\int_{\mathbb{R}_+} \int_{[0,1]} h(t, x)\, W(dt, dx).$$

Let $\mathscr{F}_t$ denote the completion of

$$\sigma\{W(B), B \in \mathscr{B}([0, t] \times [0, 1])\}.$$

We denote by $\mathscr{P}$ the σ-algebra of $\mathscr{F}_t$-progressively measurable subsets of $\Omega \times \mathbb{R}_+$. For any $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable random field $\{\varphi(t, x)\}$ satisfying

$$E \int_{\mathbb{R}_+} \int_0^1 \varphi^2(t, x)\, dt\, dx < \infty,$$

we can define the Itô integral

$$\int_{\mathbb{R}_+} \int_0^1 \varphi(t, x)\, W(dt, dx).$$

That integral can for example be considered as a stochastic integral with respect to a martingale measure; see Walsh [11].

We look for a $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable and a.s. continuous solution $\{u(t, x); (t, x) \in \mathbb{R}_+ \times [0, 1]\}$ of (1.2) (or equivalently (1.3)).

Let us now formulate three sets of assumptions on f and g. We say that the pair (f, g) satisfies the set of assumptions (A) whenever f and g are measurable mappings from $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}$ into $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ respectively, which satisfy moreover:

(Ai) f and $|g|^{-1}$ are locally bounded;

(Aii) for all $T > 0$, there exists $C(T)$ such that for any $(t, x; r) \in [0, T] \times [0, 1] \times \mathbb{R}$,

$$rf(t, x; r) \leqslant C(T)(1 + r^2)$$

$$g(t, x; r)^2 \leqslant C(T)(1 + r^2).$$

(Aiii) $g(t, x, \cdot) \in W^{2,\infty}_{\text{loc}}(\mathbb{R})$[1] for $dt \times dx$ almost all (t, x) in $\mathbb{R}_+ \times [0, 1]$; g, $\partial g/\partial r$, and $\partial^2 g/\partial r^2$ are locally bounded on $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}$.

We say that the pair (f, g) satisfies the restricted set of assumptions (RA) whenever it satisfies (A) and moreover

(RA) f, $|g|^{-1}$, g, $\partial g/\partial r$, $\partial^2 g/\partial r^2$ are bounded on $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}$.

Finally, we say that the pair (f, g) satisfies the restricted set of smoothness assumptions (RSA) whenever (f, g) satisfies (A) and moreover:

(RSAi) $f(t, x, \cdot) \in W^{2,\infty}(\mathbb{R})$ for any (t, x) in $\mathbb{R}_+ \times [0, 1]$.

(RSAii) f, $\partial f/\partial r$, $\partial^2 f/\partial r^2$, $|g|^{-1}$, g, $\partial g/\partial r$, and $\partial^2 g/\partial r^2$ are bounded on $\mathbb{R}_+ \times [0, 1] \times \mathbb{R}$.

3. An a Priori Estimate for the Density of the Law of the Solution

In this section, we assume that the pair (f, g) satisfies (RSA). Then Eq. (1.2) has a unique solution $\{u(t, x); (t, x) \in \mathbb{R}_+ \times [0, 1]\}$. Moreover it follows from Pardoux and Zhang Tusheng [9] that for any $t > 0$, $0 < x < 1$, the law of the random variable $u(t, x)$ has a density $p_{t,x}(\cdot)$ with respect to Lebesgue measure.

This section is devoted to the proof of:

THEOREM 3.1. *Assume* (RSA). *Then for any* $T > 0$ *and* $1 \leqslant q < 2$, *there exists a constant* $K_{T,q}(f, g)$, *which depends only on* T, q *and the bounds which are assumed in* (RSAii), *such that*:

$$\int_0^T \int_0^1 \int_{\mathbb{R}} |p_{t,x}(y)|^q \, dy \, dx \, dt \leqslant K_{T,q}(f, g). \tag{3.4}$$

It follows from a standard approximation argument that it suffices to prove the result with the additional assumption:

(RSAiii) $f(t, x; \cdot)$, $g(t, x; \cdot) \in C^2(\mathbb{R})$, for any $(t, x) \in \mathbb{R}_+ \times [0, 1]$.

We now fix $T > 0$, and we delete the subscript T from the constant K.

Let us now recall several elementary facts about the Malliavin calculus for the space–time white noise W. We denote $\Lambda_T = \{(t, x) : t \in [0, T],$

[1] $W^{2,\infty}_{\text{loc}}(\mathbb{R})$ denotes the space of functions from $\mathbb{R}$ into $\mathbb{R}$ which are of class C^1, the first derivative being absolutely continuous and the almost everywhere second derivative being locally bounded.

$x \in (0, 1)\}$ and $L^2(\Lambda_T)$ the L^2-space with respect to the Lebesgue measure. For $h \in L^2(\Lambda_T)$ we let

$$W(h) = \int_0^T \int_0^1 h(t, x)\, W(dx, dt).$$

The space of "smooth functionals" is

$$\mathcal{S} = \{F = f(W(h_1), ..., W(h_n)) : f \in C_b^\infty(\mathbb{R}^n, \mathbb{R}),\\ h_i \in L^2(\Lambda_T),\ 1 \leqslant i \leqslant n,\ n \in \mathbb{N}\}.$$

For $F \in \mathcal{S}$ one defines the derivatives

$$D_\alpha^{(k)} F = \sum_{i_1, ..., i_k = 1}^{n} \partial_{i_1} \cdots \partial_{i_k} f(W(h_1), ..., W(h_n))\, h_{i_1}(\alpha_1) \cdot ... \cdot h_{i_n}(\alpha_n),$$

where $\partial_i = \partial/\partial x^i$, $\alpha = (\alpha_1, ..., \alpha_n)$ and $\alpha_i = (r_i, z_i) \in \Lambda_T$, $1 \leqslant i \leqslant n$.

For $k \in \mathbb{N}$ and $p \geqslant 1$, $\mathcal{D}_{k,p}$ is the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{k,p} =: \|F\|_p + \sum_{i=1}^{k} (E(|D^{(i)} F|_i^p))^{1/p}$$

with

$$|D^{(i)} F|_i^2 =: \int_{\Lambda_T^i} |D_\alpha^{(i)} F|^2\, d\alpha \qquad (d\alpha = \text{Lebesgue measure}).$$

One also considers $L\colon \mathcal{S} \to L^2(\Omega)$ given by

$$LF = \sum_{i=1}^{n} \partial_i f(W(h_1), ..., W(h_n))\, W(h_i)\\ - \sum_{i,j=1}^{n} \partial_i \partial_j f(W(h_1), ..., W(h_n)) \langle h_i, h_j \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Lambda_T)$.

L is a closable operator and one denotes by Dom L its domain. It is well known that $\text{Dom}\, L \supseteq \mathcal{D}_\infty =: \bigcap_{p \geqslant 1} \bigcap_{k \in N} \mathcal{D}_{k,p}$. The functionals we work with are in $\mathcal{D}_\infty$. The basic formulas in this case are the following:

$$\begin{aligned} &\text{(i)} \quad L(FG) = FLG + GLF - 2\langle DF, DG \rangle \\ &\text{(ii)} \quad D\varphi(F) = \varphi'(F)\, DF \\ &\text{(iii)} \quad E(\langle DF, DG \rangle) = E(FLG) = E(GLF), \end{aligned} \tag{3.5}$$

for $F, G \in \mathcal{D}_\infty$, and $\varphi \in C^\infty(\mathbb{R})$ (actually much less regularity is needed).

One defines

$$\sigma_F = \langle DF, DF \rangle$$

and assumes that $\gamma_F = 1/\sigma_F$ exists and is in $\mathscr{D}_\infty$. Then, a standard calculation based on (3.5) (see Ikeda and Watanabe [7] for details) yields

$$E(\varphi'(F)) = E(\varphi(F)\, H_F), \tag{3.6}$$

with

$$H_F = -\gamma_F LF - \langle D\gamma_F, DF \rangle. \tag{3.7}$$

We need the evaluation

$$\|H_F\|_2 \leqslant \|\gamma_F\|_4\, (\|LF\|_4 + 2(E\, |D^2F|_2^4)^{1/4}), \tag{3.8}$$

where $\|\cdot\|_2$ is the norm in $L^2(\Omega)$.

Note that by (3.5ii), $D\gamma_F = -\gamma_F^2 D\sigma_F$ and further

$$\langle D\sigma_F, DF \rangle = 2 \int_{\Delta_T^2} D^{(2)}_{\alpha,\beta} F D^{(1)}_\alpha F D^{(1)}_\beta F\, d\alpha\, d\beta.$$

This is easy to see on simple functionals and then, by taking limits, it extends to $F \in \mathscr{D}_\infty$. Next, by Schwarz's inequality

$$\begin{aligned} |\langle D\gamma_F, DF \rangle| &\leqslant 2\, |\gamma_F|^2 \left(\int_{\Delta_T^2} (D^{(2)}_{\alpha,\beta} F)^2\, d\alpha\, d\beta \right)^{1/2} \\ &\quad \times \left(\int_{\Delta_T^2} (D^{(1)}_\alpha F)^2\, (D^{(1)}_\beta F)^2\, d\alpha\, d\beta \right)^{1/2} \\ &= 2\, |\gamma_F|\, |D^{(2)} F|_2. \end{aligned}$$

Now (3.8) follows from Schwarz's inequality.

We now prove:

LEMMA 3.2. *Let $F \in \mathscr{D}_\infty$ such that γ_F exists and is in $\mathscr{D}_\infty$. Then $P \circ F^{-1}(dx) = p_F(x)\, dx$ with $p_F \in C^\infty(\mathbb{R})$ and the following evaluations hold:*

$$p_F(x) \leqslant \|H_F\|_2\, (P(|F| \geqslant |x|))^{1/2}, \tag{3.9}$$

and, for every $\rho \geqslant 1$ and $\alpha, \beta > 0$ such that $\alpha\rho < 2 < \beta\rho$

$$\int_{\mathbb{R}} |p_F(x)|^\rho\, dx \leqslant K_{\alpha,\beta}\, \|H_F\|_2^\rho\, (\|F\|_\alpha^{\alpha\rho/2} + \|F\|_\beta^{\beta\rho/2}). \tag{3.10}$$

Proof. The existence and the regularity of p_F is a consequence of Malliavin's absolute continuity theorem. Let us prove (3.9). Take $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi \geq 0$, $\int_{\mathbb{R}} \varphi(x)\,dx = 1$ and $\varphi(x) = 0$ for $|x| \geq 1$. For $\delta > 0$ one defines

$$\varphi_\delta(x) = \delta^{-1}\varphi(x/\delta)$$

$$\phi_\delta(x) = \int_{-\infty}^{x} \varphi_\delta(y)\,dy \qquad \text{and} \qquad \psi_\delta(x) = \int_{x}^{\infty} \varphi_\delta(y)\,dy.$$

One has: $0 \leq \phi_\delta, \psi_\delta \leq 1$, and $\phi'_\delta = \varphi_\delta$, $\psi'_\delta = -\varphi_\delta$. Then, by (3.6) and the continuity of p_F

$$\begin{aligned} p_F(x) &= \lim_{\delta \to 0} \int_R p_F(y)\,\varphi_\delta(y-x)\,dy \\ &= \lim_{\delta \to 0} E(\varphi_\delta(F-x)) \\ &= \lim_{\delta \to 0} E(\phi'_\delta(F-x)) \\ &= \lim_{\delta \to 0} E(\phi_\delta(F-x)\,H_F) \\ &\leq E(1_{\{F \geq x\}} H_F). \end{aligned}$$

Next, by Schwarz's inequality

$$p_F(x) \leq \|H_F\|_2\,(P(F \geq x))^{1/2}. \tag{3.11}$$

The same argument with ψ_δ instead of ϕ_δ yields

$$p_F(x) \leq \|H_F\|_2\,(P(F \leq x))^{1/2}. \tag{3.12}$$

By using (3.11) for $x \geq 0$ and (3.12) for $x \leq 0$ one gets (3.9).

Let us now prove (3.10). One writes

$$\begin{aligned} \int_R |p_F(x)|^p\,dx &\leq \|H_F\|_2^p \left(\int_{[-1,1]} (P(|F| \geq |x|))^{p/2}\,dx \right. \\ &\quad \left. + \int_{[-1,1]^c} (P(|F| \geq |x|))^{p/2}\,dx \right) \\ &\leq \|H_F\|_2^p \left(\int_{[-1,1]} \frac{1}{|x|^{\alpha p/2}} (E\,|F|^\alpha)^{p/2}\,dx \right. \\ &\quad \left. + \int_{[-1,1]^c} \frac{1}{|x|^{\beta p/2}} (E\,|F|^\beta)^{1/2}\,dx \right) \\ &\leq K_{\alpha,\beta}\,\|H_F\|_2^p\,(\|F\|_\alpha^{\alpha p/2} + \|F\|_\beta^{\beta p/2}), \end{aligned}$$

with $K_{\alpha,\beta} = \int_{[-1,1]} |x|^{-\alpha p/2}\,dx + \int_{[-1,1]^c} |x|^{-\beta p/2}\,dx$. ∎

Now we want to apply Lemma 3.2 to $F=u(t,x)$ in order to get (3.4). For that sake we need to evaluate the quantities in the RHS of (3.8), and for this we need the following evaluations on G:

LEMMA 3.3. (i) *There is a constant $C>0$ such that*

$$\int_{t-\eta}^{t}\int_0^1 G_{t-s}^2(x,y)\,dy\,ds \geqslant G\sqrt{\eta}\,\left(1-e^{-1/4\eta}-e^{-2(1-x)^2/\eta}\right)^2$$

$$\text{for } 1/2 \leqslant x < 1 \text{ and } 0<\eta\leqslant t. \tag{3.13}$$

$$\int_{t-\eta}^{t}\int_0^1 G_{t-s}^2(x,y)\,dy\,ds \geqslant C\sqrt{\eta}\,\left(1-e^{-1/4\eta}-e^{-2x^2/\eta}\right)^2$$

$$\text{for } 0<x\leqslant 1/2 \text{ and } 0<\eta\leqslant t. \tag{3.14}$$

(ii) *For each $0<q<3$ and $0\leqslant\alpha<(3-q)\wedge 1$, there exists a constant K such that*

$$\int_0^t\int_0^1 G_{t-s}^q(x,y)\,dy\,ds \leqslant K(x\wedge(1-x))^{\alpha}\,t^{(3-q-\alpha)/2}$$

$$\text{for } 0<x<1 \text{ and } t>0. \tag{3.15}$$

Proof. (i) Note first that

$$G_t(x,y) \geqslant \frac{1}{\sqrt{2\pi t}}\left(e^{-(y-x)^2/2t}-e^{-(y+x)^2/2t}-e^{-(-2+y+x)^2/2t}\right)$$

$$=\frac{1}{\sqrt{2\pi t}}\,e^{-(y-x)^2/2t}\left(1-e^{-2xy/t}-e^{-2(1-x)(1-y)/t}\right).$$

This is because

$$e^{-(2n+y-x)^2/2t}-e^{-(2n+y+x)^2/2t}\geqslant 0 \qquad \text{for} \quad n\geqslant 1,$$

and

$$e^{-(2n+y-x)^2/2t}-e^{-(2n-2+y+x)^2/2t}\geqslant 0 \qquad \text{for} \quad n\leqslant -1.$$

Take now $x\geqslant\frac{1}{2}$, $y\in[x-\frac{1}{4},x]$ and $\eta\geqslant t-s\geqslant 0$. One has

$$G_{t-s}(x,y)\geqslant\frac{1}{\sqrt{2\pi(t-s)}}\,e^{-(x-y)^2/2(t-s)}\left(1-e^{-1/4\eta}-e^{-2(1-x)^2/\eta}\right)$$

and consequently

$$\begin{aligned}
&\int_{t-\eta}^{t} \int_{x-1/4}^{x} G_{t-s}^2(x, y)\, dy\, ds \\
&\quad \geqslant (1 - e^{-1/4\eta} - e^{-2(1-x)^2/\eta})^2 \\
&\quad \times \int_{t-\eta}^{t} \int_{x-1/4}^{x} \left(\frac{1}{\sqrt{2\pi(t-s)}} e^{-(x-y)^2/2(t-s)}\right)^2 dy\, ds \\
&\quad \geqslant C\sqrt{\eta}\,(1 - e^{-1/4\eta} - e^{-2(1-x)^2/\eta})^2.
\end{aligned}$$

If $x \leqslant \frac{1}{2}$ one computes the same lower bound with $y \in [x, x+\frac{1}{4}]$ instead of $y \in [x-\frac{1}{4}, x]$ and gets the same result with x instead of $(1-x)$. So (3.13) and (3.14) are proved.

Let us now check (3.15). Denote by B^x Brownian motion starting from x and by τ^x the first time that B^x exist $[0, 1]$. Then $G_u(x, y)$ is the density of the semigroup of the Markov process obtained by killing B^x at time τ^x. So, by the Markov property

$$G_u(x, y) = E(G_{u/2}(B^x(u/2), y); \tau^x \geqslant u/2). \tag{3.16}$$

Note also that

$$\int_0^1 G_{u/2}^q(a, y)\, dy \leqslant K \int_0^1 \left[\frac{1}{\sqrt{\pi u}} e^{-(a-y)^2/u}\right]^q dy \leqslant K'/u^{(q-1)/2}. \tag{3.17}$$

By using (3.16) and Hölder's inequality first and (3.17), one gets

$$\begin{aligned}
&\int_0^1 G_{t-s}^q(x, y)\, dy \\
&\quad \leqslant E\left(\int_0^1 |G_{(t-s)/2}(B^x((t-s)/2), y)|^q\, dy; \tau^x \geqslant (t-s)/2\right) \\
&\quad \leqslant KP(\tau^x \geqslant (t-s)/2)/(t-s)^{(q-1)/2}.
\end{aligned} \tag{3.18}$$

It is well known that $P(\tau^x \geqslant u) \leqslant Ku^{-1/2}(x \wedge (1-x))$. Since $P(\tau^x \geqslant u) \leqslant 1$, one has

$$P(\tau^x \geqslant u) \leqslant (P(\tau^x \geqslant u))^\alpha \leqslant K\left(\frac{x \wedge (1-x)}{\sqrt{u}}\right)^\alpha,$$

which, together with (3.18), yields

$$\begin{aligned}
\int_0^t \int_0^1 G_{t-s}^q(x, y)\, dy\, ds &\leqslant K(x \wedge (1-x))^\alpha \int_0^t \frac{ds}{(t-s)^{(q+\alpha-1)/2}} \\
&= K(x \wedge (1-x))^\alpha\, t^{(3-q-\alpha)/2}. \quad \blacksquare
\end{aligned}$$

Let us denote by $\sigma_{t,x}$ the Malliavin covariance matrix of $u(t,x)$ and prove

LEMMA 3.4. *Assume* (RSAiii). *Then*

$$\|1/\sigma_{t,x}\|_4 \leqslant K(f,g)(1 + 1/\sqrt{t}\,(x \wedge (1-x))). \tag{3.19}$$

Proof. Consider the equation satisfied by $D^{(1)}_{(r,z)}u(t,x)$ (Eq. (3.15) in Bally and Pardoux [1]). A standard uniqueness argument shows that

$$D^{(1)}_{(r,z)}u(t,x) = g(u)(r,z)\, S_{(r,z)}(t,x),$$

where S is the solution of the equation

$$S_{(r,z)}(t,x) = G_{t-r}(x,z) + \int_r^t \int_0^1 G_{t-s}(x,y)\, g'(u)(s,y)\, S_{(r,z)}(s,y)\, W(dy,ds)$$

$$+ \int_r^t \int_0^1 G_{t-s}(x,y)\, f'(u)(s,y)\, S_{(r,z)}(s,y)\, dy\, ds,$$

where $g'(u)(s,y) = (\partial g/\partial r)(s, y; u(s,y))$, $f'(u)(s,y) = (\partial f/\partial r)(s, y; u(s,y))$.

Since $|g|$ is bounded away from zero, it follows that

$$\sigma_{t,x} \geqslant c^2 \int_0^t \int_0^1 S^2_{(r,z)}(t,x)\, dz\, dr.$$

Fix some $T > 0$, take $\varepsilon > 0$ such that $1 - 2\exp(-1/4T\varepsilon) \geqslant \frac{1}{2}$ and define $h_\varepsilon(x) = \varepsilon(x \wedge (1-x))^2$. Then, by (3.13) and (3.14)

$$\int_{t(1-h_\varepsilon(x)/k^2)}^t \int_0^1 G^2_{t-s}(x,y)\, dy\, ds \geqslant C\frac{\sqrt{\varepsilon}}{2k}\sqrt{t}\,(x \wedge (1-x))$$

for every $0 < t \leqslant T$, $x \in (0,1)$, and $k \geqslant 1$.

It follows that

$$\sigma_{t,x} \geqslant c^2 \int_{t(1-h_\varepsilon(x)/k^2)}^t \int_0^1 S^2_{(r,z)}(t,x)\, dz\, dr$$

$$\geqslant \frac{2}{3}c^2 \int_{t(1-h_\varepsilon(x)/k^2)}^t \int_0^1 G^2_{t-r}(x,z)\, dz\, dr - 2c^2 I_k(t,x) - 2c^2 J_k(t,x)$$

$$\geqslant C\frac{c^2}{3}\frac{\sqrt{\varepsilon}\sqrt{t}}{k}(x \wedge (1-x)) - 2c^2 I_k(t,x) - 2c^2 J_k(t,x),$$

with

$$I_k(t,x)=\int_{t(1-h_\varepsilon(x)/k^2)}^{t}\int_0^1\left(\int_r^t\int_0^1 G_{t-s}(x,y)\,g'(u)(s,y)\right.$$
$$\left.\times S_{(r,z)}(s,y)\,W(dy,ds)\right)^2 dz\,dr$$

$$J_k(t,x)=\int_{t(1-h_\varepsilon(x)/k^2)}^{t}\int_0^1\left(\int_r^t\int_0^1 G_{t-s}(x,y)\,f'(u)(s,y)\right.$$
$$\left.\times S_{(r,z)}(s,y)\,dy\,ds\right)^2 dz\,dr.$$

Let us denote $a=C\frac{1}{2}\cdot c^2\sqrt{\varepsilon}/3\cdot\sqrt{t}(x\wedge(1-x))$ and write

$$E((1/\sigma_{t,x})^4)\leqslant\left(\frac{1}{a}\right)^4 P(0<1/\sigma_{t,x}\leqslant 1/a)+\sum_{k=1}^{\infty}\left(\frac{k+1}{a}\right)^4 P\left(\frac{k}{a}<\frac{1}{\sigma_{t,x}}\leqslant\frac{k+1}{a}\right).$$

One dominates the first term in the RHS by $(1/a)^4$ and, for $k\geqslant 1$, one writes

$$\begin{aligned}P\left(\frac{k}{a}<\frac{1}{\sigma_{t,x}}\leqslant\frac{k+1}{a}\right)&\leqslant P(\sigma_{t,x}<a/k)\\
&\leqslant P\left(2c^2(I_k(t,x)+J_k(t,x))\geqslant\frac{a}{2k}\right)\\
&\leqslant P\left(I_k(t,x)\geqslant\frac{a}{8c^2k}\right)+P\left(J_k(t,x)\geqslant\frac{a}{8c^2k}\right)\\
&\leqslant\left(\frac{8c^2k}{a}\right)^{10}(E\,|I_k(t,x)|^{10}+E\,|J_k(t,x)|^{10}).\end{aligned}$$

By (A.6) in Bally and Pardoux [1]

$$\begin{aligned}E\,|I_k(t,x)|^{10}&\leqslant K(f,g)(th_\varepsilon(x)/k^2)^{10}\\
&=K'(f,g)\,a^{20}/k^{20},\end{aligned}$$

and the same for $J_k(t,x)$.

One concludes that

$$\sum_{k=1}^{\infty}\left(\frac{k+1}{a}\right)^4 P\left(\frac{k}{a}<\frac{1}{\sigma_{t,x}}\leqslant\frac{k+1}{a}\right)\leqslant K(f,g)\sum_{k=1}^{\infty}\left(\frac{k+1}{a}\right)^4\left(\frac{8c^2k}{a}\right)^{10}\frac{a^{20}}{k^{20}}$$

$$\leqslant K'(f,g)\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$=: K''(f,g)<\infty.$$

LEMMA 3.5. *Assume* (RSAiii). *Then, for every* $0<\alpha<1$ *and* $\varepsilon>0$

$$(E\,|D^{(2)}u(t,x)|_2^4)^{1/4}\leqslant K(f,g)(x\wedge(1-x))^{\alpha/2-\varepsilon}\,t^{(1-\alpha)/4-\varepsilon}; \tag{3.20}$$

and

$$(E\,|Lu(t,x)|^4)^{1/4}\leqslant K(f,g)(x\wedge(1-x))^{\alpha/2-\varepsilon}\,t^{(1-\alpha)/4-\varepsilon}. \tag{3.21}$$

Proof. Let $\alpha=(\bar{\alpha},\underline{\alpha})$ with $\bar{\alpha}=(\bar{r},\bar{z})$, $\underline{\alpha}=(\underline{r},\underline{z})$, $0<\underline{r}<\bar{r}\leqslant t$ and $\bar{z},\underline{z}\in(0,1)$. Then, the equation (Eq. (3.16) in Bally and Pardoux [1]) satisfied by $D^{(2)}u$ is

$$\begin{aligned}D_\alpha^{(2)}u(t,x)&=G_{t-\bar{r}}(x,\bar{z})\,g'(u)(\bar{\alpha})\,D_{\underline{\alpha}}^{(1)}u(\bar{\alpha})\\&+\int_{\bar{r}}^{t}\int_0^1 G_{t-s}(x,y)\,f''(u)(s,y)\,D_{\underline{\alpha}}^{(1)}u(s,y)\,D_{\bar{\alpha}}^{(1)}u(s,y)\,dy\,ds\\&+\int_{\bar{r}}^{t}\int_0^1 G_{t-s}(x,y)\,g''(u)(s,y)\,D_{\underline{\alpha}}^{(1)}u(s,y)\,D_{\bar{\alpha}}^{(1)}u(s,y)\,W(dy,ds)\\&+\int_{\bar{r}}^{t}\int_0^1 G_{t-s}(x,y)\,f'(u)(s,y)\,D_\alpha^{(2)}u(s,y)\,dy\,ds\\&+\int_{\bar{r}}^{t}\int_0^1 G_{t-s}(x,y)\,g'(u)(s,y)\,D_\alpha^{(2)}u(s,y)\,W(dy,ds)\\&=\sum_{i=1}^{5} I_\alpha^{(i)}(t,x).\end{aligned}$$

We shall prove that each $I_\alpha^{(i)}(t,x)$, $1\leqslant i\leqslant T$, satisfies (3.20). The proof is similar for each i, so we check (3.20) for $I_\alpha^{(5)}(t,x)$ only.

We make the convention $D_\alpha^{(2)}u(s,y)=0$ for $\bar{r}\geqslant s$. For every $(s,y)\in R_+\times[0,1]$, α runs in Λ_T^2 so we may consider

$$M(h,\cdot)=\int_0^h\int_0^1 G_{t-s}(x,y)\,g'(u)(s,y)\,D^{(2)}u(s,y)\,W(dy,ds),\ 0\leqslant h\leqslant t$$

as an $L^2(\Lambda_T^2)$-valued martingale. Its increasing process is

$$\langle M\rangle(h)=\int_0^h\int_0^1 \|G_{t-s}(x,y)\,g'(u)(s,y)\,D_\cdot^{(2)}u(s,y)\|^2_{L^2(\Lambda_T^2)}\,dy\,ds$$

$$\leqslant \|g'\|_\infty^2\left(\int_0^h\int_0^1 G_{t-s}^{2q}(x,y)\,dy\,ds\right)^{1/q}$$

$$\times\left(\int_0^h\int_0^1 \|D_\cdot^{(2)}u(s,y)\|^{2p}_{L^2(\Lambda_T^2)}\,dy\,ds\right)^{1/p},$$

where $p, q>1$ are conjugate numbers.

Then, by Burholder's inequality for Hilbert space valued martingales (see, e.g., Métivier [8, E.2. p. 212])

$$\left(E\left|\int_{\Lambda_T^2}|I_\alpha^{(5)}(t,x)|^2\,d\alpha\right|^2\right)^{1/4}=(E\,\|M(t,\cdot)\|^4_{L^2(\Lambda_T^2)})^{1/4}$$

$$\leqslant K(E\,|\langle M\rangle(t)|^2)^{1/4}$$

$$\leqslant K(f,g)\left(1+\int_0^t\int_0^1 E\,\|D_\cdot^{(2)}u(s,y)\|^{2p}_{L^2(\Lambda_T^2)}\,dy\,ds\right)^{1/4}$$

$$\times\left(\int_0^t\int_0^1 G_{t-s}^{2q}(x,y)\,dy\,ds\right)^{1/2q}. \tag{3.22}$$

The same argument as in [1] show that

$$\int_0^t\int_0^1 E\,\|D_\cdot^{(2)}u(s,y)\|^{2p}_{L^2(\Lambda_T^2)}\,dy\,ds<\infty$$

and (3.15) yields

$$\left(\int_0^t\int_0^1 G_{t-s}^{2q}(x,y)\,dy\,ds\right)^{1/2q}\leqslant K(x\wedge(1-x))^{\alpha/2q}\,t^{(3-2q-\alpha)/4q}.$$

Now, by taking q sufficiently close to 1, (3.20) is proved.

Let us prove (3.21). Since $u(t,x)\in\mathscr{D}_\infty$, it follows that $u(t,x)\in\operatorname{Dom} L$. On the other hand, it is easy to prove (one uses the discretized Eq. (3.4) in [1] and the definition of L on smooth functionals) that $Lu(t,x)$ is the solution of the equation

$$
\begin{aligned}
Lu(t,x) = & \int_0^t \int_0^1 G_{t-s}(x,y)\, g(u)(s,y)\, W(dy,ds) \\
& - \int_0^t \int_0^1 G_{t-s}(x,y)\, f''(u)(s,y)\, \sigma_{s,y}\, dy\, ds \\
& - \int_0^t \int_0^1 G_{t-s}(x,y)\, g''(u)(s,y)\, \sigma_{s,y}\, W(dy,ds) \\
& + \int_0^t \int_0^1 G_{t-s}(x,y)\, f'(u)(s,y)\, Lu(s,y)\, dy\, ds \\
& + \int_0^t \int_0^1 G_{t-s}(x,y)\, g'(u)(s,y)\, Lu(s,y)\, W(dy,ds) \\
=: & \sum_{i=1}^{5} I^{(i)}(t,x).
\end{aligned}
\tag{3.23}
$$

Let $\lambda > 1$. Burholder's and Hölder's inequalities yield

$$
\begin{aligned}
(E\,|I^{(3)}(t,x)|^{\lambda})^{1/\lambda} & \leqslant K\, \|g'\|_{\infty} \left(E \left| \int_0^t \int_0^1 G_{t-s}^2(x,y)\, \sigma_{s,y}^2\, dy\, ds \right|^{\lambda/2} \right)^{1/\lambda} \\
& \leqslant K\, \|g'\|_{\infty} \left(\int_0^t \int_0^1 G_{t-s}^{2q}(x,y)\, dy\, ds \right)^{1/2q} \\
& \quad \times \left(E \left| \int_0^t \int_0^1 \sigma_{s,y}^{2p}\, dy\, ds \right|^{\lambda/2p} \right)^{1/\lambda} \\
& \leqslant K(f,g)(x \wedge (1-x))^{\alpha/2-\varepsilon}\, t^{(1-\alpha)/4-\varepsilon},
\end{aligned}
\tag{3.24}
$$

the last inequality holding for q sufficiently close to 1. The same inequality holds for $I^{(1)}$ and $I^{(2)}$.

Now, by using (3.23), (3.24), and Burkholder's inequality, one gets

$$
\begin{aligned}
E\,|Lu(t,x)|^{\lambda} & \leqslant K + K' E \left| \int_0^t \int_0^1 G_{t-s}^2(x,y)\, |Lu(s,y)|^2\, dy\, ds \right|^{\lambda/2} \\
& \leqslant K + K' \left(\int_0^t \int_0^1 G_{t-s}^{2\lambda'}(x,y)\, dy\, ds \right)^{\lambda/2\lambda'} \\
& \quad \cdot \int_0^t \int_0^1 E\,|Lu(s,y)|^{\lambda}\, dy\, ds,
\end{aligned}
$$

where λ' is the conjugate of $\lambda/2$. If $\lambda > 6$ then $2\lambda' < 3$ and so $\int_0^t \int_0^1 G_{t-s}^{2\lambda'}(x, y)\, dy\, ds < \infty$. Then, a Gronwall type argument permits us to conclude that

$$\sup_{t,x} E\,|Lu(t,x)|^{\lambda} < \infty, \qquad \forall \lambda > 1.$$

So the reasoning in (3.24) holds for $I^{(4)}$ and $I^{(5)}$ also and (3.21) is proved. ▮

Proof of Theorem 3.1. By Lemmas 3.4 and 3.5

$$\begin{aligned}\|H_{u(t,x)}\|_2 &\leqslant K(f, g)(1 + 1/t^{1/2}(x \wedge (1-x)))\, t^{(1-\alpha)/4-\varepsilon}(x \wedge (1-x))^{\alpha/2-\varepsilon}\\ &\leqslant K(f, g)(1 + t^{-(\varepsilon+(1+\alpha)/4)}(x \wedge (1-x)))^{-(\varepsilon+(2-\alpha)/2)}.\end{aligned}$$

Since $\sup_{t,x} \|u(t,x)\|_p < \infty$, $\forall p \geqslant 1$, (3.10) yields

$$\begin{aligned}&\int_0^T \int_0^1 \int_R |p_{t,x}(y)|^{\rho}\, dy\, dx\, dt\\ &\quad\leqslant K(f, g)\left(1 + \int_0^T \frac{dt}{t^{\rho(\varepsilon+(1+\alpha)/4)}} \cdot \int_0^1 \frac{dx}{(x \wedge (1-x))^{\rho(\varepsilon+(2-\alpha)/2)}}\right). \qquad (3.25)\end{aligned}$$

If $\rho < 2$ one may choose $\alpha < 1$ sufficiently close to 1 and $\varepsilon > 0$ sufficiently close to 0 in order to get $\rho(\varepsilon + (1+\alpha)/4) < 1$, $\rho(\varepsilon + (2-\alpha)/2) < 1$, so that the two integrals in the RHS of (3.25) are finite and so the proof is complete. ▮

Remark 3.6. Under the additional assumption $\sup u_0 \subseteq [a, 1-a] \subset (0, 1)$, $0 < a < \frac{1}{2}$, one would get that (3.4) holds for any $q < 2.3$. That is because $|G_t(x, u_0) - u_0(x)| \leqslant Kt^{1/2}(x \wedge (1-x))$ for (t, x) sufficiently close to the corners of the domain $[0, \infty) \times (0, 1)$. Then one applies Lemma 3.2 to $F(t,x) = u(t,x) - u_0(x)$ which verifies

$$\begin{aligned}F(t,x) = (G_t(x, u_0) - u_0(x)) &+ \int_0^t \int_0^1 G_{t-s}(x, y)\, g(u)(s, y)\, W(dy, ds)\\ &+ \int_0^t \int_0^1 G_{t-s}(x, y)\, f(u)(s, y)\, dy\, ds.\end{aligned}$$

The same arguments as in Lemma 3.4 show that $\|F(t,x)\|_p \leqslant Kt((1-\alpha)/4)(x \wedge (1-x))^{\alpha/2}$. Coming back to (3.10) and taking advantage in this majoration also yields (3.25) with $t^{-(\rho(1+\alpha)-(1-\alpha))/4}$ and $(x \wedge (1-x))^{-(\rho(2-\alpha)-\alpha)/2}$. If one takes $\alpha = \frac{4}{5}$, then $\rho = \frac{7}{3}$.

A weaker assumption which permits to improve Theorem 3.1 is that u_0 be Hölder continuous at $x=0$ and at $x=1$. In that case, the theorem holds with any $q>\beta$, and β depending on the Hölder exponent for u_0, $2<\beta<2.3$.

4. Preliminary Results

In this section, we assume that the set of "restricted assumption" (RA) are in force.

Let $\{u(t,x); (t,x)\in\mathbb{R}_+\times[0,1]\}$ be a continuous and $\mathscr{P}\otimes\mathscr{B}([0,1])$ measurable solution of Eq. (1.1). We have the:

Proposition 4.1. *For any $T>0$ and $p>2$, there exists a constant $K(T,p)$ (which depends also on $\|f\|_\infty$, $\|g\|_\infty$, $\|\partial g/\partial r\|_\infty$, $\|\partial^2 g/\partial r^2\|_\infty$, and $\|g^{-1}\|_\infty$) such that for any measurable $h\colon \mathbb{R}_+\times[0,1]\times\mathbb{R}\to\mathbb{R}_+$,*

$$E\int_0^T\int_0^1 h(t,x;u(t,x))\,dx\,dt \leqslant K(T,p)\left(\int_0^T\int_0^1\int_{\mathbb{R}} h(t,x;z)^p\,dz\,dx\,dt\right)^{1/p}.$$

Proof. We fix $T>0$ throughout the proof. Let $\tilde{f}(t,x;z)=f(t,x;z)/g(t,x;z)$.

Define

$$Z=\exp\left(-\int_0^T\int_0^1 \tilde{f}(u)(t,x)\,W(dt,dx)-\tfrac{1}{2}\int_0^T\int_0^1 \tilde{f}^2(u)(t,x)\,dx\,dt\right)$$

and a new measure $\tilde{P}$ on $(\Omega,\mathscr{F})$ by $d\tilde{P}=Z\,dP$.

Since $\tilde{f}$ is bounded, it follows from Girsanov's theorem applied to space–time white noise (see, e.g., Gyöngy and Pardoux [6]) that $\tilde{P}$ is a probability measure under which

$$\tilde{W}(dt,dx)=\tilde{f}(u)(t,x)\,dt\,dx+W(dt,dx)$$

is a space–time white noise. Clearly, u solves the SPDE

$$u(t,x)=\int_0^1 G_t(x,y)\,u_0(y)\,dy+\int_0^t\int_0^1 G_{t-s}(x,y)\,g(u(s,y))\,\tilde{W}(ds,dy).$$

Hence, from Theorem 3.1, under $\tilde{P}$, $u(t,x)(t>0, 0<x<1)$ has a density $\tilde{p}_{t,x}(\cdot)$ which satisfies (3.4).

Then, if $r, r', q, q' > 1$, $r^{-1} + r'^{-1} = 1$, $q^{-1} + q'^{-1} = 1$,

$$\begin{aligned}
E \int_0^T \int_0^1 & h(t, x; u(t, x))\, dx\, dt \\
&= \tilde{E}\left(Z^{-1} \int_0^T \int_0^1 h(t, x; u(t, x))\, dx\, dt\right) \\
&\leqslant T^{1/r'} (\tilde{E} Z^{-r'})^{1/r'} \left(\tilde{E} \int_0^T \int_0^1 h^r(t, x; u(t, x))\, dx\, dt\right)^{1/r} \\
&\leqslant C(T, r) \left(\int_0^T \int_0^1 \int_{\mathbb{R}} h^r(t, x; z)\, \tilde{p}_{t,x}(z)\, dr\, dx\, dt\right)^{1/r} \\
&\leqslant C(T, r) \left(\int_0^T \int_0^1 \int_{\mathbb{R}} \tilde{p}_{t,x}(z)^{q'}\, dz\, dx\, dt\right)^{1/q'} \\
&\quad \times \left(\int_0^T \int_0^1 \int_{\mathbb{R}} h^{rq}(t, x; z)\, dz\, dx\, dt\right)^{1/rq}.
\end{aligned}$$

The result follows from (3.4) by choosing some q with $2 < q < p$ (hence $q' < 2$) and $r = p/q$. ∎

Let $\{f_n(t, x, z); t \geqslant 0, x \in [0, 1], z \in \mathbb{R}\}$, $n = 1, 2, \ldots$ and $\{f(t, x; r), r \geqslant 0, x \in [0, 1], r \in \mathbb{R}\}$ be measurable mappings which satisfy:

(A) f_n is bounded, uniformly with respect to n.

(B) $f_n \to f$ in $L^p((0, T) \times (0, 1) \times (-R, R))$ as $n \to \infty$, for some $p > 2$ and any $T, R > 0$.

We assume moreover that

(C) for any $n \in \mathbb{N}$, there exists a continuous and $\mathcal{P} \otimes \mathcal{B}([0, 1])$ measurable solution u_n to $Eq(f_n, g)$, and for every $(t, x) \in \mathbb{R}_+ \times [0, 1]$,

$$u_n(t, x) \to u(t, x) \text{ a.s.}, \qquad \text{as} \quad n \to \infty,$$

where u is an $\mathbb{R}$-valued random field.

The aim of this section is to prove

THEOREM 4.2. *Assume* (A), (B), *and* (C), *and moreover that for any* $T > 0$

$$\sup_n \sup_{(t,x) \in [0,T] \times [0,1]} |u_n(t, x)| < \infty \text{ a.s.}$$

Then u solves $Eq(f, q)$.

We first establish:

LEMMA 4.3. *Assume* (A) *and* (C). *Then* u *satisfies Proposition* 4.1.

Proof. We sketch the argument, which is identical to that of Proposition 3.2 in [4]. If h is continuous with respect to r, the result follows by taking the limit in the results for u_n, as $n \to \infty$. The general case follows from the monotone class theorem. ∎

PROPOSITION 4.4. *Assume* (A), (B), (C). *Then*

$$E\int_0^T \int_0^1 |f_n(u_n)(t, x) - f(u)(t, x)|\, dx\, dt \to 0, \qquad \textit{as} \quad n \to \infty.$$

Proof. The proof is again very similar to that of Proposition 3.3 in [4]. Let $T > 0$ be fixed throughout the proof. Define $\kappa\colon \mathbb{R} \to [0, 1]$ be a smooth function such that $\kappa(r) = 0$ for $|r| \geqslant 1$ and $\kappa(0) = 1$. Given $\varepsilon > 0$, let $R > 0$ be such that

$$E\int_0^T \int_0^1 |1 - \kappa(u(t, x)/R)|\, dx\, dt \leqslant \varepsilon.$$

Since the set $\{f_n\}_{n \in \bar{\mathbb{N}}}$ (where $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$, $f_\infty = f$) is relatively compact in $L^p((0, T) \times (0, 1) \times (-R, R))$, there exists $N \in \mathbb{N}$ and bounded smooth functions $H_1, ..., H_N$ from $[0, T] \times [0, 1] \times \mathbb{R}$ into $\mathbb{R}$ such that

$$\sup_{n \in \bar{\mathbb{N}}} \inf_{1 \leqslant i \leqslant N} \left(\int_0^T \int_0^1 \int_{-R}^R |f_n(t, x, r) - H_i(t, x, r)|^p\, dr\, dx\, dt \right)^{1/p} \leqslant \varepsilon$$

$$\begin{aligned} I(n) &:= E\int_0^T \int_0^1 |f_n(u_n)(t, x) - f(u)(t, x)|\, dx\, dt \\ &\leqslant I_1(n) + I_2(n) + I_3(n) + I_4(n), \end{aligned}$$

where

$$I_1(n) = E\int_0^T \int_0^1 |f_n(u_n)(t, x) - H_{i_n}(u_n)(t, x)|\, dx\, dt$$

$$I_2(n) = \sum_{i=1}^N E\int_0^T \int_0^1 |H_i(u_n)(t, x) - H_i(u)(t, x)|\, dx\, dt$$

$$I_3(n) = E\int_0^T \int_0^1 |f_n(u)(t, x) - H_{i_n}(u)(t, x)|\, dx\, dt$$

$$I_4(n) = E\int_0^T \int_0^1 |f_n(u)(t, x) - f(u)(t, x)|\, dx\, dt,$$

where $i_n = \operatorname{Arg\,min}_i \|f_n - H\|_p$. We now estimate successively each of the above terms. For any $1 \leqslant i \leqslant N$, by Proposition 4.1,

$$\begin{aligned} I_1(n) &\leqslant E\int_0^T\int_0^1 \kappa(u_n/R)\,|f_n(u_n) - H_{i_n}(u_n)|\,dx\,dt \\ &\quad + E\int_0^T\int_0^1 (1-\kappa(u_n/R))\,|f_n(u_n) - H_{i_n}(u_n)|\,dx\,dt \\ &\leqslant K\left(\int_0^T\int_0^1\int_{-R}^R |f_n(t,x;r) - H_{i_n}(t,x;r)|^p\,dr\,dx\,dt\right)^{1/p} \\ &\quad + KE\int_0^T\int_0^1 |1-\kappa(u_n/R)|\,dx\,dt. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_n I_1(n) &\leqslant K\left(\varepsilon + E\int_0^T\int_0^1 |1-\kappa(u/R)|\,dx\,dt\right) \\ &\leqslant 2K\varepsilon, \end{aligned}$$

and this holds for any $\varepsilon > 0$.

By continuity and boundedness of the H_i's, $I_2(n) \to 0$ as $n \to \infty$. By arguments similar to those use for estimating $I_1(n)$, we deduce that $I_3(n) \to 0$, as $n \to \infty$. Finally, from Lemma 4.3,

$$\begin{aligned} I_4(n) &= E\int_0^T\int_0^1 \kappa(u/R)\,|f_n(u) - f(u)|\,dx\,dt \\ &\quad + E\int_0^T\int_0^1 (1-\kappa(u/R))\,|f_n(u) - f(u)|\,dx\,dt \\ &\leqslant K\left(\int_0^T\int_0^1\int_{-R}^R |f_n(t,x;r) - f(t,x;r)|^p\,dr\,dx\,dt\right)^{1/p} + K\varepsilon, \end{aligned}$$

hence $I_4(n) \to \infty$ as $n \to \infty$, from (B). ∎

We can now proceed with the

Proof of Theorem 4.2. It suffices to take the limit in the identity (with $\varphi \in C^2([0,1]) \cap C_0([0,1])$):

$$\begin{aligned} \int_0^1 u_n(t,x)\,\varphi(x)\,dx &= \int_0^1 u_0(x)\,\varphi(x)\,dx + \int_0^t\int_0^1 u_n(s,x)\,\varphi''(x)\,dx\,ds \\ &\quad + \int_0^t\int_0^1 \varphi(x)\,f_n(u_n)(s,x)\,dx\,ds \\ &\quad + \int_0^t\int_0^1 \varphi(x)\,g(u_n)(s,x)\,W(ds\,dx). \end{aligned}$$

We can take the limit in this identity as $n \to \infty$ (for the term involving $f_n(u_n)$, we use Proposition 4.4), yielding

$$\int_0^1 u(t,x)\,\varphi(x)\,dx = \int_0^1 u_0(x)\,\varphi(x)\,dx + \int_0^1 u(s,x)\,\varphi''(x)\,dx\,ds$$
$$+ \int_0^t \int_0^1 \varphi(x)\,f(u)(s,x)\,dx\,ds$$
$$+ \int_0^t \int_0^1 \varphi(x)\,g(u)(s,x)\,W(ds,dx).$$

The result is proved. ∎

5. Existence and Uniqueness under the "Restrictive Assumptions"

In this section, we assume again that the "restrictive assumptions" (RA) are satisfied. In particular, f, g, and g^{-1} are bounded.

Theorem 5.1. *There exists a continuous and* $\mathscr{P} \otimes \mathscr{B}([0,1])$ *measurable solution of* $Eq(f, g)$.

Proof. The proof is very similar to that of Theorem 4.1 in [4]. Let $\rho \in C_c^\infty(\mathbb{R}; \mathbb{R}_+)$, with $\int_{\mathbb{R}} \rho(z)\,dz = 1$. For $j \in \mathbb{N}$, define

$$\tilde{f}_j(t,x;r) = j \int_{\mathbb{R}} f(t,x,z)\,\rho(j(r-z))\,dz,$$

for $n \leqslant k$,

$$f_{n,k} = \bigwedge_{j=n}^{k} \tilde{f}_j$$

and for $n \in \mathbb{N}$,

$$f_n = \bigwedge_{j=n}^{\infty} \tilde{f}_j.$$

Clearly, $f_{n,k}$ is Lipschitz in r, uniformly with respect (t,x), $f_{n,k}(t,x;r) \downarrow f_n(t,x;r)$ as $k \uparrow \infty$ and $f_n(t,x;r) \uparrow f(t,x;r)$ as $n \uparrow \infty$, dr a.e., for each (t,x).

We first note that for each $n<k$, $Eq(f_{n,k}, g)$ has a unique solution $u_{n,k}$. From the comparison theorem for SPDEs (see, e.g., Theorem 2.1 in Donati-Martin and Pardoux [2]), the sequence $\{u_{n,k}, k=n, n+1, ...\}$ is decreasing; hence

$$u_{n,k}(t, x) \downarrow u_n(t, x), \qquad \forall (t, x) \in \mathbb{R}_+ \times [0, 1], \text{ a.s.}$$

Since moreover $u_{n,k}(t, x)$ is bounded above by the solution of $Eq(K, g)$ and below by the solution of $Eq(-K, g)$ for some $K>0$, all the assumptions of Theorem 4.2 are in force. Hence u_n solves $Eq(f_n)$. Now, since $u_{n,k} \leqslant u_{n+1,k}$, $k \geqslant n+1$,

$$u_n(t, x) \downarrow u(t, x)$$

and from the same argument as above, u solves $Eq(f, g)$. u is $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable by construction. Its continuity follows from Walsh [11]. ▮

THEOREM 5.2. *$Eq(f, g)$ has at most one continuous and $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable solution.*

Proof. Let u denote the solution constructed in Theorem 5.1, and v another solution. Define $\varphi(t, x) := f(v)(t, x)$. φ is $\mathscr{P} \otimes \mathscr{B}([0, 1])$ measurable and bounded. Note that v solves the two following SPDEs:

$$(*) \quad \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \varphi + g(v) \cdot W$$

$$(**) \quad \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f_n(v) \vee \varphi + g(v) \cdot W,$$

where $n \in \mathbb{N}$ is fixed. For $k \geqslant n$, let v_k denote the unique solution of

$$\frac{\partial v_k}{\partial t} = \frac{\partial^2 v_k}{\partial x^2} + f_{n,k}(v_k) \vee \varphi + g(v_k) \cdot W$$

with initial $v_k(0, x) = u_0(x)$, $x \in [0, 1]$, and Dirichlet boundary conditions. Note that

$$r \rightarrow f_{n,k}(t, x; r) \vee \varphi(t, x)$$

is uniformly Lipschitz, and also this coefficient is random, the usual theory of existence and uniqueness as in Walsh [11] applies. Now, v is the unique solution of Eq. $(*)$, which is an equation with Lipschitz coefficients. Hence from the comparison theorem (Theorem 2.1 in [2]),

$$v_k(t, x) \geqslant v(t, x),$$

and the sequence $\{v_k(t,x); k=1,2,...\}$ is decreasing. It follows from Theorem 4.2 (which again can be adapted to the present situation of random coefficients) that

$$\bar{v}(t,x) := \lim_{k\to\infty} v_k(t,x)$$

solves Eq. (**). Moreover,

$$\bar{v}(t,x) \geqslant v(t,x).$$

But from Girsanov's theorem, the law of the solution of (**) is unique. Hence

$$\bar{v} \equiv v.$$

Finally, again by the same comparison theorem,

$$v_k(t,x) \geqslant u_{n,k}(t,x).$$

Hence

$$v(t,x) \geqslant u_n(t,x), \qquad \forall n \in \mathbb{N},$$

and

$$v(t,x) \geqslant u(t,x).$$

But again uniqueness in law applies to $Eq(f,g)$. Consequently, $u \equiv v$. ∎

6. Existence and Uniqueness in the General Case

In this section, we assume that the set of assumptions (A) is satisfied.

THEOREM 6.1. *There is one and only one continuous and $\mathscr{P}\otimes\mathscr{B}([0,1])$ measurable solution of $Eq(f,g)$.*

Proof. *Uniqueness.* Let u and v be two solutions. For $R>0$, let

$$\tau_R = \inf\{t; \sup_{0\leqslant x\leqslant 1} |u(t,x)| \vee |v(t,x)| \geqslant R\},$$

$$f_R(t,x;r) = f(t,x;(r\wedge R)\vee(-R)),$$

$$g_R(t,x;r) = g(t,x;(r\wedge R)\vee(-R)).$$

f_R and g_R satisfy the (RA) assumptions. But the restrictions of u and v to $[0, \tau_R] \times [0, 1]$ are restrictions of the unique solution to $Eq(f_R, g_R)$. Hence, they coincide. The result follows, since $\tau_R \to \infty$ as $R \to \infty$.

Existence. For any $R > 0$, let f_R and g_R be as defined in the proof of uniqueness, and u_R denote the solution of $Eq(f_R, g_R)$. Let

$$\tau_R = \inf\{t \geqslant 0; \sup_x |u_R(t, x)| \geqslant R\}$$

and $\tau = \lim_{R \to \infty} \tau_R$. Clearly, we can define a solution u of $Eq(f, g)$ on $[0, \tau) \times [0, 1]$ by

$$u(t, x) = u_R(t, x) \qquad \text{on} \quad [0, \tau_R] \times [0, 1], \qquad R > 0.$$

It remains to show that $\tau = +\infty$ p.s. This follows from

$$\sup_{R>0} E\left(\sup_{(s,x) \in [0,t] \times [0,1]} |u_R(s, x)|^k \right) < \infty$$

for any $t > 0$ and some $k > 0$.

This last estimate is proved exactly as in Theorem 5.2.5 of Gyöngy and Pardoux [6]. ▮

We conclude this section with the proofs of two auxiliary results. We begin with a comparison theorem.

THEOREM 6.2. *Let two pairs of coefficients (f, g) and (F, g) satisfy the set of assumptions* (A). *Suppose moreover that*

$$f(t, x; r) \leqslant F(t, x; r) \; dt \times dx \times dr \; a.e.$$

Let u (resp. v) denote the solution of $Eq(f, g)$ (resp. of $Eq(F, g)$). Then $u(t, x) \leqslant v(t, x)$, $\forall (t, x) \in \mathbb{R}_+ \times [0, 1]$, a.s.

Proof. It suffices to prove the results on $[0, \tau_R] \times [0, 1]$, where

$$\tau_R = \inf\{t; \sup_x |u(t, x)| \vee |v(t, x)| \geqslant R\};$$

hence it suffices to prove the result in case the coefficients satisfy the set of restricted assumptions (RA). u and v are approximated by the double sequences $u_{n,k}$ and $v_{n,k}$ as in Theorem 5.1. But $u_{n,k} \leqslant v_{n,k}$ follows from the comparison theorem in Donati–Martin and Pardoux [2]. ▮

We finally want to show that the solution u depends continuously on the data (u_0, f, g).

THEOREM 6.3. *Let* $(u_{0n}, f_n, g_n)_{\mathbb{N}}$ *denote a sequence of initial conditions and coefficients which satisfy*:

(i) $u_{0n} \in C_0([0,1])$, $n \in \mathbb{N}$ *and* $u_{0n}(x) \to u_0(x)$ *uniformly with respect to* $x \in [0,1]$, *as* $n \to \infty$;

(ii) *for each* $n \in \mathbb{N}$, (f_n, g_n) *satisfies the set of assumptions* (A), *and moreover*:

(a) *the constants* $\{C(T), T>0\}$ *in* (Aii) *do not depend on* n,

(b) *for each* $T, R > 0$,

$$\sup_{n\in\mathbb{N}} \sup_{(t,x,r)\in[0,T]\times[0,1]\times[-R,R]} |h_n(t,x,r)| < \infty$$

with f_n, g_n^{-1}, g_n, $\partial g_n/\partial r$, *and* $\partial^2 g_n/\partial r^2$ *in place of* h_n;

(c) *as* $n \to \infty$,

$$f_n(t,x,r) \to f(t,x,r) \; dt\, dx\, dr \text{ a.e.}$$

$$g_n(t,x,r) \to g(t,x,r) \qquad \text{for each} \quad r \in \mathbb{R},\ dt\,dx \text{ a.e.},$$

with (f, g) *satisfying* (A).

Then if u_n *denotes the unique solution of Eq* (u_{0n}, f_n, g_n) *and* u *the unique solution of* $Eq(u_0, f, g)$, *for any* $T>0$,

$$\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x) - u(t,x)| \to 0$$

in probability, as $n \to \infty$.

Proof. The same arguments as those used to establish (28) in Donati–Martin and Pardoux [2] lead to

$$\sup_x E\Big(\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x)|^p\Big) < \infty,$$

for each $T>0$, $p \geqslant 1$. But from the local boundedness of f_n and g_n uniformly in n, we deduce by the arguments in Lemma 6.1 of [2] that for any $T>0$, $K>0$, there exists a constant $C(p, T, K)$ such that for all (t,x) and (s,y) in $[0,T]\times[0,1]$,

$$E\Big(|u_n(t,x) - u_n(s,y)|^p;\ \sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x)| \leqslant K\Big)$$

$$\leqslant C(p,T,K)\,|(x,t)-(y,s)|^{p/4-6}$$

It now follows that the sequence (u_n, u) is tight in $C([0, T] \times [0, 1]; \mathbb{R}^2)$, for all $T > 0$. Now for each $n \in \mathbb{N}$, $\varphi, \psi \in C^2(0, 1) \cap C_0([0, 1])$,

$$(u_n(t), \varphi) + (u(t), \psi) = (u_0, \varphi + \psi) + \int_0^t [(u_n(s), \varphi'') + (u(s), \psi'') + (f_n(u_n)(s), \varphi) + (f(u)(s), \psi)] \, ds + M^{n, \varphi, \psi}(t),$$

where $\{M^{n, \varphi, \psi}(t), t \geqslant 0\}$ is a martingale with the increasing process

$$\langle M^{n, \varphi, \psi} \rangle(t) = \int_0^t \int_0^1 [g_n(u_n)(s, x) \varphi(x) + g(u)(s, x) \psi(x)]^2 \, dx \, ds.$$

Combining classical arguments with those in the proof of Proposition 4.4, on can show that for any weakly converging subsequence $(u_{n'}, u) \to (\bar{u}, u)$, the limit $(\bar{u}, u)$ solves a similar martingale problem; hence there exists a space–time white noise $\tilde{W}$ (possibly defined on some enlarged probability space) such that

$$\frac{\partial \bar{u}}{\partial t}(t, x) = \frac{\partial^2 \bar{u}}{\partial x^2}(t, x) + f(\bar{u})(t, x) + g(\bar{y})(t, x) \dot{\tilde{W}}(t, x)$$

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + f(u)(t, x) + g(u)(t, x) \dot{\tilde{W}}(t, x)$$

$$\bar{u}(0, x) = u(0, x) = u_0(x), \; x \in [0, 1];$$

$$\bar{u}(t, 0) = \bar{u}(t, 1) = u(t, 0) = u(t, 1) = 0, \; t \geqslant 0.$$

Therefore from the uniqueness part of Theorem 6.1, $\bar{u} \equiv u$ a.s.

Consequently, for each $T > 0$,

$$E\Big(\sup_{(t,x) \in [0, T] \times [0, 1]} |u_n(t, x) - u(t, x)| \wedge 1\Big) \to 0,$$

as $n \to \infty$.

Remark 6.4. In the case where $g_n \equiv g$ for each n, one can prove that

$$\sup_{(t,x) \in [0, T] \times [0, 1]} |u_n(t, x) - u(t, x)| \to 0 \text{ a.s.},$$

as $n \to \infty$. The proof of a.s. convergence exploits again heavily the comparison theorem, in exactly the same way as it is done in Gyöngy and Pardoux [5].

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