

Extinction time and the total mass of the continuous state branching processes with competition

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Abstract

Consider a general continuous state branching process (CSBP) with additional interaction, which destroys the branching property. We give precise conditions on the interaction term, in order to decide whether the extinction time of the process remains or not bounded as the initial value tends to infinity, and similarly for the total mass of the process.

1 Introduction

Consider a continuous state branching process (CSBP), which takes the form

$$Z_t^x = x + \sigma \int_0^t \int_0^{Z_s^x} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^x} z \tilde{N}(ds, dz, du),$$

where W is space-time white noise, and \tilde{N} a compensated Poisson random measure, which models the evolution of a population. One way to introduce interactions between the individuals (and then destroy the branching property) in the reproduction mechanism is to add a nonlinear drift, leading to the SDE

$$Z_t^x = x + \int_0^t f(Z_s^x) ds + \sigma \int_0^t \int_0^{Z_s^x} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^x} z \tilde{N}(ds, dz, du). \quad (1.1)$$

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Interactions can increase the number of births, or in contrary increase the number of deaths, in particular in the case of competition for rare resources. The popular logistic competition has been considered in Le, Pardoux, Wakolbinger [10], while a much more general type of interaction appears in Ba, Pardoux [2].

We will assume that for large population size the interaction is of the type of a competition, which limits the size of the population. One may then wonder in which cases the interaction is strong enough so that the extinction time (or equivalently the height of the forest of genealogical trees) remains bounded, as the number of ancestors tends to infinity, or even such that the total mass of the forest of genealogical trees remains bounded, as the population size tends to infinity.

This question has already been addressed, in the case of processes with continuous trajectories, in the case of a polynomial interaction in Ba, Pardoux [1], and in more general cases of competition in Le, Pardoux [9]. Here we want to generalize those results to the case of processes with discontinuous paths. More precisely, suppose that $\sigma \geq 0$ is a constant, and $(r \wedge r^2)m(dr)$ is a finite measure on $(0, \infty)$. Let ψ be a function given by

$$\psi(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)m(dr), \quad \lambda \geq 0. \quad (1.2)$$

Let $W(ds, du)$ be a white noise on $(0, \infty)^2$ based on the Lebesgue measure $dsdu$, let $N(ds, dz, du)$ be Poisson random measure on $(0, \infty)^3$ with intensity $dsm(dz)du$, and $\tilde{N}(ds, dz, du) = N(ds, dz, du) - dsm(dz)du$. We will consider the CSBP with competition solution of the SDE (1.1), where the branching mechanism is specified by ψ , and the function f satisfies the following hypothesis.

Hypothesis (H1): $f \in C(\mathbb{R}_+, \mathbb{R})$, $f(0) = 0$. There exists $\theta \geq 0$ such that

$$f(x+y) - f(x) \leq \theta y \quad \forall x, y \geq 0.$$

The hypothesis (H1) implies that the function $\theta y - f(y)$ is increasing. In particular, we have

$$f(y) \leq \theta y \quad \forall y \geq 0.$$

The equation (1.1) has a unique strong solution (see Dawson, Li [4]). This SDE couples the evolution of the various $\{Z_t^x, t \geq 0\}$ jointly for all values of $x > 0$.

For $x > 0$, define T^x the extinction time of the process Z^x by

$$T^x = \inf\{t > 0, Z_t^x = 0\},$$

and S^x the total mass of Z^x by

$$S^x = \int_0^{T^x} Z_t^x dt.$$

By the same argument as in Lemma 2.3 in [11], see also Theorem 3.6 in [4], we can and do choose a version of the random field $\{Z_t^x, t \geq 0, x > 0\}$ such that $x \mapsto Z_t^x$ is a.s. increasing

for all $t \geq 0$. Consequently $x \mapsto T^x$ and $x \mapsto S^x$ are a.s. increasing. The goal of this paper is to study the limits of T^x and S^x as $x \rightarrow \infty$.

This paper is organized as follows. Section 2 studies the extinction time, while section 3 studies the total mass of the CSBP with competition. The main results are Theorem 1, 2, 3 and 4. Section 4 describes some examples to illustrate our results.

2 Extinction time of the CSBP with competition

We now study the extinction time of the process Z^x . In the logistic case where $f(y) = ay - by^2$, $b > 0$, Lambert [7] has proved the process Z^x either remains positive for ever, or is absorbed at 0 in finite time, depending solely on the branching mechanism, i.e. according to a criterion that does not involve a and b : extinction occurs with probability 1 if $\int^\infty d\lambda/\psi(\lambda) < \infty$, with probability 0 otherwise. In the case of Feller's branching diffusion with competition where $\psi(\lambda) = 2\lambda^2$ (the condition $\int^\infty d\lambda/\psi(\lambda) < \infty$ is satisfied in this case), it is showed in Le and Pardoux [9] that

$$\sup_{x>0} T^x < \infty \Leftrightarrow \int^\infty \frac{1}{|f(y)|} dy < \infty.$$

Hence we may guess that in the general case if

$$\int^\infty \frac{1}{\psi(\lambda)} d\lambda < \infty \quad \text{and} \quad \int^\infty \frac{1}{|f(y)|} dy < \infty, \quad (2.1)$$

we have $\sup_{x>0} T^x < \infty$ a.s.. In fact we will prove that condition (2.1) implies that $E[\sup_{x>0} T^x] < \infty$.

We first need the following lemma, which is Lemma 2.3 in [9].

Lemma 2.1. *Let f be a function satisfying (H1), $a \in \mathbb{R}$ be a constant. If there exists $a_0 > 0$ such that $f(y) \neq 0$, $f(y) + ay \neq 0$ for all $y \geq a_0$, then we have that*

$$\int_{a_0}^\infty \frac{1}{|f(y)|} dy < \infty \Leftrightarrow \int_{a_0}^\infty \frac{1}{|ay + f(y)|} dy < \infty,$$

and when those equivalent conditions are satisfied, we have

$$\lim_{y \rightarrow \infty} \frac{f(y)}{y} = -\infty.$$

We now establish the main results of this section

Theorem 1. *Suppose that $\int^\infty d\lambda/\psi(\lambda) = \infty$ and that f is a function satisfying (H1) and $\liminf_{y \rightarrow 0^+} \frac{f(y)}{y} > -\infty$. Then for all $x > 0$, $T^x = \infty$ a.s.*

Proof. Since f is continuous and satisfies both (H1) and $\liminf_{y \rightarrow 0^+} \frac{f(y)}{y} > -\infty$, there exists a positive constant δ such that for all $x > 0$ small enough,

$$-\delta y \leq f(y) \leq \theta y \quad \forall y \in [0, 2x].$$

Define $\tau_1 := \inf\{t > 0 : Z_t^x \geq 2x\}$, then $f(Z_t^x) \geq -\delta Z_t^x$ for all $t \in [0, \tau_1)$. By the comparison theorem (see Dawson, Li [4]) we have $Z_t^x \geq Z_t^{1,x}$ a.s. for all $t \in [0, \tau_1)$, where $Z_t^{1,x}$ solves

$$Z_t^{1,x} = x - \delta \int_0^t Z_s^{1,x} ds + \sigma \int_0^t \int_0^{Z_s^{1,x}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s^-}^{1,x}} z \tilde{N}(ds, dz, du).$$

The process $Z_t^{1,x}$ is a CSBP characterised by the branching mechanism $\psi_1(\lambda) = \psi(\lambda) + \delta\lambda$. By Lemma 2.1 we have $\int^\infty d\lambda/\psi_1(\lambda) = \infty$, so that $Z_t^{1,x}$ remains positive a.s. (see Kyprianou [6], page 279). Hence Z_t^x remains positive a.s. on $[0, \tau_1)$ and $\mathbb{P}_x(\tau_1 < \infty) = 1$.

Since the process Z_t^x has only positive jumps, on the event $\{T^x < \infty\}$, Z_t^x hits again x after time τ_1 . But from the above argument and the strong Markov property, Z_t^x cannot hit 0 before $2x$. Finally we conclude that $\mathbb{P}(T^x < \infty) = 0$. This being true for all sufficiently small $x > 0$, it is true by comparison for all $x > 0$. \square

We have moreover

Theorem 2. *Assume that f is a function satisfying (H1) and that there exists $a_0 > 0$ such that $f(y) \neq 0$ for all $y \geq a_0$. If the condition (2.1) is satisfied, we have*

$$\sup_{x>0} \mathbb{E}(T^x) < \infty.$$

Proof. From Lemma 2.1 we get

$$\lim_{y \rightarrow \infty} \frac{f(y)}{y} = -\infty,$$

then there is a constant $M > a_0$ such that $f(y) < \min\{-\theta, -1\}y$ for all $y \geq M$. We have

$$\begin{aligned} f_1(y) &:= \frac{1}{2}(\theta y - f(y)) \leq -f(y) \\ f(y) &\leq -f_1(y) \quad \forall y \geq M. \end{aligned}$$

Note that we can assume that f is decreasing on $[M, \infty]$ because without it, we can use the comparison theorem and prove the Theorem with a function h instead of f , where

$$h \in C(\mathbb{R}_+, \mathbb{R}), h(0) = 0, h(y) = -f_1(y) \quad \forall y \geq M \quad \text{and} \quad h(y) \geq f(y) \quad \forall y \geq 0.$$

Define for $x > M$,

$$T_M^x = \inf\{t > 0, Z_t^x \leq M\}.$$

For $x > M$, we can rewrite the equation (1.1) as

$$Z_t^x = x + \int_0^t f(Z_{s-}^x) ds + \sigma \int_0^t \int_0^{Z_{s-}^x} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^x} z \tilde{N}(ds, dz, du),$$

Then we have

$$\begin{aligned} dZ_t^x &= f(Z_{t-}^x) dt + \sigma \int_0^{Z_{t-}^x} W(dt, du) + \int_0^\infty \int_0^{Z_{t-}^x} z \tilde{N}(dt, dz, du), \\ \frac{dZ_t^x}{-f(Z_{t-}^x)} &= -dt + \sigma \int_0^{Z_{t-}^x} \frac{1}{-f(Z_{t-}^x)} W(dt, du) + \int_0^\infty \int_0^{Z_{t-}^x} \frac{z}{-f(Z_{t-}^x)} \tilde{N}(dt, dz, du). \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{T_M^x \wedge t} \frac{dZ_s^x}{-f(Z_{s-}^x)} &= -(T_M^x \wedge t) + \sigma \int_0^{T_M^x \wedge t} \int_0^{Z_{s-}^x} \frac{1}{-f(Z_{s-}^x)} W(ds, du) \\ &\quad + \int_0^{T_M^x \wedge t} \int_0^\infty \int_0^{Z_{s-}^x} \frac{z}{-f(Z_{s-}^x)} \tilde{N}(ds, dz, du). \end{aligned} \quad (2.2)$$

It is easy to show that

$$\mathbb{E} \left[\sup_{0 \leq r \leq t} \left| \int_0^{T_M^x \wedge r} \int_1^\infty \int_0^{Z_{s-}^x} \frac{z}{-f(Z_{s-}^x)} \tilde{N}(ds, dz, du) \right| \right] \leq 2 \mathbb{E} \left[\int_0^{T_M^x \wedge t} \frac{Z_{s-}^x}{-f(Z_{s-}^x)} ds \int_1^\infty zm(dz) \right].$$

Observe also that

$$\begin{aligned} \mathbb{E} \left[\left| \int_0^{T_M^x \wedge t} \int_0^{Z_{s-}^x} \frac{1}{-f(Z_{s-}^x)} W(ds, du) \right|^2 \right] &= \mathbb{E} \left[\int_0^{T_M^x \wedge t} \frac{Z_{s-}^x}{f(Z_{s-}^x)^2} ds \right] \\ \mathbb{E} \left[\left| \int_0^{T_M^x \wedge t} \int_0^1 \int_0^{Z_{s-}^x} \frac{z}{-f(Z_{s-}^x)} \tilde{N}(ds, dz, du) \right|^2 \right] &= \mathbb{E} \left[\int_0^{T_M^x \wedge t} \frac{Z_{s-}^x}{f(Z_{s-}^x)^2} ds \int_0^1 z^2 m(dz) \right]. \end{aligned}$$

The above expectations are finite, then

$$t \mapsto \int_0^{T_M^x \wedge t} \int_0^{Z_{s-}^x} \frac{1}{-f(Z_{s-}^x)} W(ds, du) + \int_0^{T_M^x \wedge t} \int_0^\infty \int_0^{Z_{s-}^x} \frac{z}{-f(Z_{s-}^x)} \tilde{N}(ds, dz, du)$$

is a martingale. From (2.2) we now deduce

$$\mathbb{E} \left[\int_0^{T_M^x \wedge t} \frac{dZ_s^x}{-f(Z_{s-}^x)} \right] = -\mathbb{E}(T_M^x \wedge t).$$

Now, let

$$G(y) := \int_x^y \frac{1}{-f(u)} du.$$

Then $G \in C^1[M, \infty)$ and $G'(y) = \frac{1}{-f(y)}$ is decreasing on $[M, \infty)$. By Itô's formula we have

$$G(Z_{T_M^x \wedge t}^x) = G(Z_0^x) + \int_0^{T_M^x \wedge t} G'(Z_{s-}^x) dZ_s^x + \sum_{0 \leq s \leq T_M^x \wedge t} [G(Z_s^x) - G(Z_{s-}^x) - G'(Z_{s-}^x) \Delta Z_s^x] + \frac{\sigma^2}{2} \int_0^{T_M^x \wedge t} G''(Z_s^x) Z_s^x ds.$$

By Lemma 2.2 below we obtain

$$\int_x^{Z_{T_M^x \wedge t}^x} \frac{du}{-f(u)} \leq \int_0^{T_M^x \wedge t} \frac{dZ_s^x}{-f(Z_{s-}^x)} \quad \text{a.s.}$$

Hence

$$\begin{aligned} \mathbb{E} \left[\int_x^{Z_{T_M^x \wedge t}^x} \frac{du}{-f(u)} \right] &\leq -\mathbb{E}(T_M^x \wedge t) \\ \mathbb{E}(T_M^x \wedge t) &\leq \mathbb{E} \left[\int_{Z_{T_M^x \wedge t}^x}^x \frac{du}{-f(u)} \right] \\ \mathbb{E}(T_M^x \wedge t) &\leq \int_M^\infty \frac{du}{-f(u)}. \end{aligned}$$

Taking the limit as $x \rightarrow \infty$ and $t \rightarrow \infty$ we have

$$\sup_{x > M} \mathbb{E}(T_M^x) < \infty,$$

or $\mathbb{E}(T_M) < \infty$, where $T_M := \sup_{x > M} T_M^x$.

We have just proved that the process Z comes down from infinity. For proving that $\sup_{x > 0} \mathbb{E}(T^x) < \infty$, it remains to show that the time taken by Z to descend from M to 0 is integrable, which we now establish. By the comparison theorem we have $Z_t^M \leq Z_t^{2,M}$ a.s. for all $t \geq 0$, where $Z_t^{2,M}$ solves

$$Z_t^{2,M} = M + \theta \int_0^t Z_s^{2,M} ds + \sigma \int_0^t \int_0^{Z_s^{2,M}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^{2,M}} z \tilde{N}(ds, dz, du).$$

The process $Z_t^{2,M}$ is a CSBP characterised by the branching mechanism $\psi_2(\lambda) = \psi(\lambda) - \theta\lambda$. By Lemma 2.1 we obtain $\int^\infty d\lambda/\psi_2(\lambda) < \infty$, so that $Z^{2,M}$ is absorbed at 0 in finite time with positive probability (see Kyprianou [6], page 279). Then there is a constant $T > 0$ such that $Z^{2,M}$ is absorbed at 0 before time T with positive probability. Let p denote the probability that starting from M at time $t = 0$, Z hits zero before time T . Clearly $p > 0$. Let ζ be a geometric random variable with success probability p , which is defined as follows. Let Z start from M at time 0. If Z hits zero before time T , then $\zeta = 1$. If not, we look the position Z_T of Z at time T .

If $Z_T > M$, we wait until Z goes back to M . The time needed is stochastically dominated

by the random variable T_M , which is the time needed for Z to descend to M , when starting from ∞ . If however $Z_T \leq M$, we start afresh from there, since the probability to reach zero in less than T is greater than or equal to p , for all starting points in the interval $(0, M]$. So either at time T , or at time less than $T + T_M$, we start again from a level which is less than or equal to M . If zero is reached during the next time interval of length T , then $\zeta = 2\dots$ Repeating this procedure, we see that $\sup_{x>0} T^x$ is stochastically dominated by

$$\zeta T + \sum_{i=1}^{\zeta} \eta_i,$$

where the random variables η_i are i.i.d, with the same law as T_M , globally independent of ζ . Therefore

$$\begin{aligned} \sup_{x>0} \mathbb{E}(T^x) &\leq \mathbb{E}(\zeta T + \sum_{i=1}^{\zeta} \eta_i) \\ &= \frac{T}{p} + \frac{1}{p} \mathbb{E}(T_M) \\ &< \infty. \end{aligned}$$

The result follows. \square

Lemma 2.2. *Suppose that $g \in C^1[M, \infty)$ and g' is decreasing on $[M, \infty)$. We have*

$$g(a) - g(b) - g'(b)(a - b) \leq 0 \quad \forall a, b \geq M. \quad (2.3)$$

Proof. If $a > b$ then there exists $c \in (b, a)$ such that $g(a) - g(b) = (a - b)g'(c)$. Hence

$$g(a) - g(b) - g'(b)(a - b) = (a - b)(g'(c) - g'(b)) \leq 0.$$

Similarly, (2.3) is also true in the case $a < b$. The result follows. \square

3 Total mass of the CSBP with competition

In this section, we shall assume that

Hypothesis (H2): f is a function satisfying (H1) such that

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u} = \alpha,$$

for some $-\infty < \alpha \leq \theta$, and the function $f_1(u) := \frac{f(u)}{u} - \alpha$ satisfies (H1).

3.1 The Lamperti transform

We will study the total mass S^x of the process Z^x . In this subsection we remind the reader of a celebrated result of Lamperti [8] which relates CSBP and Lévy processes with no negative jumps. This result will allow us to give a representation of a CSBP with competition in terms of spectrally positive Lévy processes with drift.

Let X be a real-valued Lévy process with no negative jumps starting from 0. Let T_0 be the first hitting time of zero by $x + X$. Then define

$$\rho_t = \int_0^{T_0 \wedge t} \frac{ds}{x + X_s} \quad t > 0,$$

and $(C_t, t > 0)$ its right-inverse. Lamperti's result then states that if

$$Y_t = x + X(C_t) \quad t > 0,$$

then Y is a CSBP with initial value $Y_0 = x$. Moreover,

$$C_t = \int_0^t Y_s ds \quad t > 0.$$

Conversely, suppose that Y is a CSBP such that $Y_0 = x > 0$. If C is defined as above, and ρ is the right-inverse of C , then $Y \circ \rho$ is a Lévy process with no negative jumps which starts at x and is killed when it hits 0.

We now time-change the CSBP with competition Z^x in Lamperti's fashion to obtain a Lévy process with drift. Consider the increasing process

$$C_t^x = \int_0^t Z_s^x ds, t \geq 0,$$

and its right-inverse ρ_t^x . We define $U^x = Z^x \circ \rho^x$. We have

Proposition 3.1. *Assume that the function f satisfies (H2). Then U^x is the unique strong solution of the following SDE*

$$dU_t^x = \frac{f(U_t^x)}{U_t^x} dt + dX_t, \quad U_0^x = x. \quad (3.1)$$

where X is a Lévy process with Laplace exponent ψ .

Proof. The process Z^x is a càdlàg homogeneous strong Markov processes (see e.g. [4]). By standard theory of Markov processes (see e.g. [5]), U^x is then a càdlàg homogeneous strong Markov process. We denote A (resp. Q, L) the infinitesimal generator of X (resp. U^x, Z^x). Using Itô's formula one can see that Z^x solves the martingale problem associated with the infinitesimal generator L given, with $D_z g(y) := g(y+z) - g(y) - g'(y)z$, by

$$\begin{aligned} Lg(y) &= \frac{1}{2} \sigma^2 y g''(y) + f(y) g'(y) + y \int_0^\infty D_z g(y) m(dz) \\ &= y A g(y) + f(y) g'(y). \end{aligned}$$

Furthermore, we deduce $Qg(y) = \frac{Lg(y)}{y}$ from the fact that for any time $t > 0$, with $r = \rho_s^x$,

$$\begin{aligned} \mathbb{E}(g(U_t^x)) &= \mathbb{E}(g(Z_{\rho_t^x}^x)) \\ &= x + \mathbb{E}\left(\int_0^{\rho_t^x} Lg(Z_r^x) dr\right) \\ &= x + \mathbb{E}\left(\int_0^t \frac{Lg(U_s^x)}{U_s^x} ds\right) \\ &= x + \mathbb{E}\left(\int_0^t Qg(U_s^x) ds\right). \end{aligned}$$

Hence

$$Qg(y) = Ag(y) + \frac{f(y)}{y}g'(y).$$

This shows that U^x is a solution of the SDE (3.1). It remains to prove uniqueness of the solution of (3.1). Suppose that $U_t^{1,x}$ and $U_t^{2,x}$ are two solutions of (3.1), we have for all $t \geq 0$,

$$U_t^{1,x} - U_t^{2,x} = \int_0^t (f_1(U_s^{1,x}) - f_1(U_s^{2,x})) ds.$$

Then by Ito's formula and Hypothesis (H1) we get

$$\begin{aligned} (U_t^{1,x} - U_t^{2,x})^2 &= \int_0^t 2(U_s^{1,x} - U_s^{2,x})(f_1(U_s^{1,x}) - f_1(U_s^{2,x})) ds \\ &\leq \int_0^t 2\theta(U_s^{1,x} - U_s^{2,x})^2 ds. \end{aligned}$$

The result follows from Gronwall's inequality. \square

Let $\tau^x := \inf\{t > 0, U_t^x = 0\}$. It is easy to see that $\rho^x(\tau^x) = T^x$, hence $S^x = \tau^x$. We next study the limits of S^x as $x \rightarrow \infty$. We want to show that under a specific assumption $S^x \rightarrow \infty$ a.s. as $x \rightarrow \infty$, and under the complementary assumption $\sup_{x>0} S^x < \infty$ a.s. Because the mapping $x \mapsto S^x$ is a.s. increasing, the result will follow for the same result proved for any collection of r.v.'s $\{S^x, x > 0\}$ which has the same monotonicity property, and has the same marginal laws as the original one. More precisely, we will consider the U^x 's solutions of (3.1) with the same X for all $x > 0$.

3.2 About the Lévy process X

In this subsection we establish some preliminary results on Lévy processes which will be used later. Recall that X is a spectrally positive Lévy process with Laplace exponent ψ given by (1.2), so that for all $\lambda \geq 0$,

$$\mathbb{E}(e^{-\lambda X_t}) = e^{t\psi(\lambda)} \tag{3.2}$$

Because ψ is continuous and has a continuous derivative, $\psi(0) = 0$ and ψ is increasing on \mathbb{R}_+ so that ψ has a unique inverse ϕ which is defined and continuous on \mathbb{R}_+ and satisfies $\phi(0) = 0$. From (3.2) we get for any $t \geq 0$,

$$\mathbb{E}(X_t) = -t\psi'(0) = 0.$$

Let now η be a stopping time which is a.s. positive and integrable. It is not hard to deduce from Itô's formula that

$$e^{-\lambda X_t} = 1 + \psi(\lambda) \int_0^t e^{-\lambda X_s} ds + M_t^\lambda,$$

for for each $\lambda \geq 0$, M_t^λ is a martingale. We then deduce that

$$\mathbb{E}e^{-\lambda X_{t \wedge \eta}} = 1 + \psi(\lambda) \mathbb{E} \int_0^{t \wedge \eta} e^{-\lambda X_s} ds.$$

Since η is integrable, we can both let $t \rightarrow \infty$ in the last identity, and differentiate with respect to λ , yielding

$$\mathbb{E}(-X_\eta e^{-\lambda X_\eta}) = \psi'(\lambda) \mathbb{E} \int_0^{t \wedge \eta} e^{-\lambda X_s} ds - \psi(\lambda) \mathbb{E} \int_0^{t \wedge \eta} X_s e^{-\lambda X_s} ds.$$

We now choose $\lambda = 0$ and deduce

$$\mathbb{E}(X_\eta) = 0. \tag{3.3}$$

Furthermore we have (see Theorem 7.2 in [6])

$$\limsup_{t \rightarrow \infty} X_t = -\liminf_{t \rightarrow \infty} X_t = \infty \quad \text{a.s.}$$

So that if we define for $y > 0$,

$$\tau_y^+ = \inf\{t > 0, X_t > y\}, \quad \tau_y^- = \inf\{t > 0, X_t < -y\},$$

then τ_y^+ and τ_y^- are a.s. positive and finite. We have

Proposition 3.2. $\tau_y^+ \rightarrow \infty$ a.s. and $\tau_y^- \rightarrow \infty$ a.s. as $y \rightarrow \infty$, and for any $y > 0, \beta > 0$ we have

$$\mathbb{E}\left(\frac{1}{(\tau_y^-)^\beta}\right) < \infty.$$

Proof. Define for $t > 0$,

$$\bar{X}_t = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq s \leq t} X_s.$$

Then for all $\beta \geq 0$,

$$\mathbb{E}(e^{\beta X_{e_q}}) = \frac{\phi(q)}{\phi(q) + \beta} \quad \text{and} \quad \mathbb{E}(e^{-\beta \bar{X}_{e_q}}) = \frac{q}{\phi(q)} \frac{\phi(q) - \beta}{q - \psi(\beta)}, \tag{3.4}$$

where e_q is an independent and exponentially distributed random variable with parameter $q > 0$ (see Kyprianou [6], page 213). Letting β tend to zero in the first expression of (3.4) we see that

$$\mathbb{P}(\underline{X}_{e_q} > -\infty) = 1.$$

We have

$$\begin{aligned} \mathbb{P}(\sup_{y>0} \tau_y^- < e_q) &= \lim_{y \rightarrow \infty} \mathbb{P}(\tau_y^- < e_q) \\ &= \lim_{y \rightarrow \infty} \mathbb{P}(\underline{X}_{e_q} < -y) \\ &= 0. \end{aligned}$$

Therefore for all $t > 0, q > 0$,

$$\mathbb{P}(\sup_{y>0} \tau_y^- \leq t) \leq \mathbb{P}(\sup_{y>0} \tau_y^- < e_q) + \mathbb{P}(e_q \leq t) = 1 - e^{-qt}.$$

Then taking q to zero we get $\mathbb{P}(\sup_{y>0} \tau_y^- \leq t) = 0$ for all $t > 0$, so that

$$\mathbb{P}(\sup_{y>0} \tau_y^- = \infty) = 1.$$

Hence $\tau_y^- \rightarrow \infty$ a.s. as $y \rightarrow \infty$. Similarly, from the second expression of (3.4) we can prove $\tau_y^+ \rightarrow \infty$ a.s. as $y \rightarrow \infty$.

For proving the last result of the Proposition, it is enough to show that

$$\mathbb{E}((\tau_y^-)^{-n}) < \infty \quad \text{for all } n \in \mathbb{N}^*.$$

Note that (see [6], page 212) the process $\{\tau_y^-, y \geq 0\}$ is a subordinator with Laplace exponent ϕ , so that

$$\mathbb{E}(e^{-s\tau_y^-}) = e^{-\phi(s)y} \quad \text{for all } s > 0. \quad (3.5)$$

It is easy to see that for all $s > 0, n \in \mathbb{N}^*$

$$\mathbb{E}((\tau_y^-)^{-n} e^{-s\tau_y^-}) = F_n(s),$$

where

$$F_1(s) = \int_s^\infty e^{-\phi(u)y} du \text{ and } F_{n+1}(s) = \int_s^\infty F_n(u) du \text{ for all } n \geq 1.$$

By Lemma 3.3 below we have that for any $n \geq 1, F_n(s)$ is finite. Hence for all $n \geq 1, s \geq 0$,

$$\begin{aligned} \mathbb{E}((\tau_y^-)^{-n}) &\leq \mathbb{E}((\tau_y^-)^{-n} \mathbf{1}_{\{\tau_y^- \leq 1\}}) + 1 \\ &\leq e^s \mathbb{E}((\tau_y^-)^{-n} e^{-s\tau_y^-} \mathbf{1}_{\{\tau_y^- \leq 1\}}) + 1 \\ &\leq e^s F_n(s) + 1 \\ &< \infty. \end{aligned}$$

The result follows. □

Lemma 3.3. For $n \geq 1$, there exist positive constants $m_0^n, m_1^n, \dots, m_n^n$ which depend upon y such that

$$F_n(s) \leq e^{-\phi(s)y} (m_0^n + m_1^n \phi(s) + \dots + m_n^n \phi(s)^n), \quad s \geq 0.$$

Proof. We will prove this lemma by induction on n . It is easily seen that

$$\psi'(s) \leq b_1 s + b_0, \quad s \geq 0, \quad (3.6)$$

where

$$b_0 = \int_1^\infty r m(dr), \quad b_1 = \sigma^2 + \int_0^1 r^2 m(dr).$$

We have for any $s \geq 0$, with $u = \psi(r)$,

$$\begin{aligned} F_1(s) &= \int_s^\infty e^{-\phi(u)y} du \\ &= \int_{\phi(s)}^\infty e^{-ry} \psi'(r) dr \\ &\leq \int_{\phi(s)}^\infty e^{-ry} (b_1 r + b_0) dr. \end{aligned}$$

We deduce that the lemma holds for $n = 1$ from the fact that for all $a > 0, m \geq 1$,

$$\int_a^\infty e^{-ry} r^m dr = \frac{1}{y} e^{-ay} a^m + \frac{m}{y} \int_a^\infty e^{-ry} r^{m-1} dr. \quad (3.7)$$

Assume that the lemma holds for $n = k$. Hence for any $s \geq 0$, with $u = \psi(r)$,

$$\begin{aligned} F_{k+1}(s) &\leq \int_s^\infty e^{-\phi(u)y} (m_0^k + m_1^k \phi(u) + \dots + m_k^k \phi(u)^k) du \\ &= \int_{\phi(s)}^\infty e^{-ry} (m_0^k + m_1^k r + \dots + m_k^k r^k) \psi'(r) dr \\ &\leq \int_{\phi(s)}^\infty e^{-ry} (m_0^k + m_1^k r + \dots + m_k^k r^k) (b_1 r + b_0) dr, \end{aligned}$$

where we have used (3.6) for the last inequality. From (3.7) we now conclude that the lemma holds for $n = k + 1$. The result follows. \square

Lemma 3.4. For $t \geq 0$, define $\Gamma_t = \inf\{s \geq 0, X_s - s < -t\}$. We have $\mathbb{E}(\Gamma_t) = t$.

Proof. Note that $X_s - s$ is a spectrally positive Lévy process with Laplace exponent $\psi_0(\lambda) = \psi(\lambda) + \lambda$. Denote ϕ_0 the unique inverse of ψ_0 . It is well known that the process $\{\Gamma_t, t \geq 0\}$ is a subordinator with Laplace exponent ϕ_0 , hence

$$\mathbb{E}(e^{-s\Gamma_t}) = e^{-\phi_0(s)t} \quad \text{for all } s \geq 0, t \geq 0.$$

Therefore $\mathbb{E}(\Gamma_t) = \phi_0'(0)t$. The result follows from the fact that

$$\psi_0'(0) = 1 \quad \text{and} \quad \psi_0'(0)\phi_0'(0) = 1.$$

\square

Lemma 3.5. *Assume that (H) the paths of X are of infinite variation a.s. Then for all positive constants a, b we have*

$$\mathbb{P}(a + \inf_{[0,b]} X_t \leq 0) > 0.$$

Proof. According to [3] (Corollary VII.5), assumption (H) holds iff

$$\lim_{\lambda \rightarrow \infty} \frac{\psi(\lambda)}{\lambda} = \infty. \quad (3.8)$$

Note that (3.8) happens iff at least one of the following two conditions is satisfied: $\sigma > 0$, or

$$\int_0^1 rm(dr) = \infty.$$

If $\mathbb{P}(a + \inf_{[0,b]} X_s \leq 0) = 0$, we have $\tau_a^- \geq b$ a.s. Hence $\mathbb{E}(e^{-s\tau_a^-}) \leq e^{-bs}$ for all $s \geq 0$. By (3.5) we get

$$\begin{aligned} e^{-\phi(s)a} &\leq e^{-bs} \\ \phi(s)a &\geq bs. \end{aligned} \quad (3.9)$$

Let $s = \psi(r)$ in (3.9) we obtain $ar \geq b\psi(r)$ for all $r > 0$. This contradicts (3.8), so that

$$\mathbb{P}(a + \inf_{[0,b]} X_t \leq 0) > 0.$$

□

3.3 Main results

We now establish the main results of this section

Theorem 3. *Suppose that f is a function satisfying (H2) and that there exists $a_0 > 0$ such that $f(u) \neq 0$ for all $u \geq a_0$. If $\int_{a_0}^{\infty} \frac{u}{|f(u)|} du = \infty$, then*

$$S^x \rightarrow \infty \quad \text{a.s. as } x \rightarrow \infty.$$

Proof. Let γ be a constant such that $f_2(u) := \gamma u - f_1(u)$ is a positive and increasing function (we can choose e.g. $\gamma > \theta$, by Hypothesis (H2)). We can rewrite the SDE (3.1) as

$$dU_t^x = (\alpha + \gamma U_t^x - f_2(U_t^x))dt + dX_t, \quad U_0^x = x,$$

Setting $V_t^x = U_t^x - X_t$, then V_t^x solves the ODE

$$\frac{dV_t^x}{dt} = \alpha + \gamma(V_t^x + X_t) - f_2(V_t^x + X_t), \quad V_0^x = x.$$

Let $\{x_n, n \geq 1\}$ be an increasing sequence of positive real numbers such that $x_n \rightarrow \infty$ as $n \rightarrow \infty$. For any $y > 0$, there exists $n_y > 0$ such that $x_n > 2y$ for all $n \geq n_y$. Define

$$R_y^n := \inf\{t > 0, V_t^{x_n} < 2y\} \quad \text{for any } y > 0, n \geq n_y.$$

For $n \geq n_y$, we have on the time interval $[0, R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2}]$,

$$\begin{aligned} -y &\leq X_t \leq y \\ y &\leq V_t^{x_n} + X_t \leq V_t^{x_n} + y \\ \frac{dV_t^{x_n}}{dt} &\geq -|\alpha| - f_2(V_t^{x_n} + y) \\ -t &\leq \int_0^t \frac{dV_s^{x_n}}{|\alpha| + f_2(V_s^{x_n} + y)} \\ t &\geq \int_{V_t^{x_n} + y}^{x_n + y} \frac{du}{|\alpha| + f_2(u)}. \end{aligned} \tag{3.10}$$

Consider now the integral $\int^\infty \frac{du}{|\alpha| + f_2(u)}$. If $\int^\infty \frac{du}{|\alpha| + f_2(u)} < \infty$, then by Lemma 2.1 we have

$$\lim_{u \rightarrow \infty} \frac{|\alpha| + f_2(u)}{u} = \infty.$$

We deduce that there exists a constant $a_1 > a_0$ such that

$$-\frac{f(u)}{u} \geq \alpha + |\alpha| + \gamma u \quad \text{for all } u \geq a_1.$$

Therefore

$$0 < \int_{a_1}^\infty \frac{-u}{2f(u)} du \leq \int_{a_1}^\infty \frac{du}{\alpha + |\alpha| + \gamma u - \frac{f(u)}{u}} = \int_{a_1}^\infty \frac{du}{|\alpha| + f_2(u)} < \infty.$$

This contradicts our standing assumption. Consequently $\int^\infty \frac{du}{|\alpha| + f_2(u)} = \infty$. Now (3.10) implies that

$$\begin{aligned} \inf_{0 \leq s \leq t \wedge R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2}} V_s^{x_n} &\geq \Phi(x_n, t), \quad \text{where} \\ \Phi(x_n, t) &= \inf \left\{ a > 0, \int_{a+y}^{x_n+y} \frac{du}{|\alpha| + f_2(u)} \leq t \right\} \\ &\rightarrow \infty, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Consequently

$$\lim_{n \rightarrow \infty} \inf_{0 \leq s \leq t} V_s^{x_n} = \infty \text{ a.s. for all } t \in \left[0, R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2} \right].$$

Hence

$$\lim_{n \rightarrow \infty} R_y^n \geq \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.}$$

Moreover, because $U_t^{x_n} = V_t^{x_n} + X_t > 0$ a.s. for all $t \in [0, R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2}]$, then

$$\begin{aligned} S^{x_n} = \tau^{x_n} &\geq R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.} \\ \lim_{n \rightarrow \infty} S^{x_n} = \lim_{n \rightarrow \infty} \tau^{x_n} &\geq \lim_{n \rightarrow \infty} R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.} \\ &\geq \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.} \end{aligned}$$

Letting y tend to infinity, the result follows from Proposition 3.2. \square

We next consider the case $\int_{a_0}^{\infty} \frac{u}{|f(u)|} du < \infty$. We will see that in this case $\sup_{x>0} S^x < \infty$ a.s. Indeed, we can prove that it has some finite moments.

It is easy to see that in this case $\frac{f(u)}{u} \rightarrow -\infty$ as $u \rightarrow \infty$, so that there exists a constant $a_2 > a_0$ such that $\frac{f(u)}{u} \leq -2|\alpha|$ for all $u \geq a_2$. Hence

$$\int_{a_2}^{\infty} \frac{1}{|f_1(u)|} du = \int_{a_2}^{\infty} \frac{1}{\alpha - \frac{f(u)}{u}} du \leq \int_{a_2}^{\infty} \frac{2u}{-f(u)} du < \infty.$$

By Lemma 2.1 we have

$$\int_{a_2}^{\infty} \frac{1}{f_2(u)} du = \int_{a_2}^{\infty} \frac{1}{\gamma u - f_1(u)} du < \infty. \quad (3.11)$$

Let $g(y) := \int_y^{\infty} \frac{1}{f_2(u)} du$ for $y \geq a_0$. Then g is decreasing and $g(y) \rightarrow 0$ as $y \rightarrow \infty$. We suppose that the following hypothesis holds:

Hypothesis (H3): The function f_2 is C^1 on (a_0, ∞) and there exist some constants $d > 0, c > a_0$ such that

$$g(y)f_2'(y) \geq 1 + d \quad \text{for all } y \geq c.$$

Remark 3.6. Consider the particular case where $f(u) = -\alpha u^k$, with $\alpha, k > 0$. The main assumption of the next Theorem 4 will require in this case that $k > 2$, which we assume from now on. Now choose $\delta \in (0, (k-2)^{-1})$. Let c_δ be such that $\frac{\alpha}{\alpha y^{k-1}} + \frac{\gamma}{\alpha y^{k-2}} \leq \delta$ for any $y \geq c_\delta$. We have that $g(y) > \frac{1}{\alpha(k-2)(1+\delta)} y^{-k+2}$ for $y \geq c_\delta$. Since $f_2'(y) = \gamma + \alpha(k-1)y^{k-2}$, we deduce that

$$g(y)f_2'(y) > \frac{\gamma}{\alpha(k-2)(1+\delta)} y^{-k+2} + \frac{k-1}{(k-2)(1+\delta)}.$$

It is plain that **Hypothesis (H3)** is satisfied with $c = c_\delta$ and $d = \frac{k-1}{(k-2)(1+\delta)} - 1$. In this particular case, this new assumption is satisfied as soon as the other main assumption of the next Theorem 4 is satisfied.

Define the function $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ as follows.

$$h(y) = \begin{cases} \frac{1}{g(c)^d}, & 0 \leq y \leq c \\ \frac{1}{g(y)^d}, & y > c. \end{cases}$$

Then h is increasing and is C^2 on (c, ∞) , $h(y) \rightarrow \infty$ as $y \rightarrow \infty$, and

$$h''(y) = \frac{-d[g(y)f_2'(y) - d - 1]}{f_2(y)^2 g(y)^{d+2}} \leq 0 \quad \text{for all } y > c.$$

Therefore $h'(y)$ is decreasing on (c, ∞) . From the fact that for $y > 2c$, there exists $\xi \in (2c, y)$ such that

$$h(y) - h(2c) = h'(\xi)(y - 2c) \leq h'(2c)(y - 2c) \leq h'(2c)y,$$

we easily deduce that

$$h(y) \leq h(2c) + h'(2c)y \quad \text{for all } y \geq 0. \quad (3.12)$$

We have

Lemma 3.7. *There exists a positive constant c_1 such that*

$$h(a + b) \leq h(a) + h(b) + c_1 \quad \text{for all } a, b \geq 0.$$

Proof. For all $0 \leq a, b \leq 2c$ we have $h(a + b) \leq h(4c)$. Define the function $h_1 \in C((c, \infty), \mathbb{R}_+)$ by

$$h_1(y) = h(y + b) - h(y).$$

We have $h_1'(y) = h'(y + b) - h'(y) \leq 0$ for all $y > c$. Then for $a \geq 2c$,

$$\begin{aligned} h(a + b) - h(a) &= h_1(a) \leq h_1(2c) = h(2c + b) - h(2c) \leq h(2c + b) \\ h(a + b) - h(a) - h(b) &\leq h(2c + b) - h(b). \end{aligned}$$

But $h(2c + b) \leq h(4c)$ for $0 \leq b < 2c$, and $h(2c + b) - h(b) \leq h(4c) - h(2c)$ for $b \geq 2c$, again since h' is decreasing. The result follows by choosing $c_1 = h(4c)$. \square

Theorem 4. *Suppose that there exists $a_0 > 0$ such that $f(u) \neq 0$ for all $u \geq a_0$ and that (H2), (H3) hold. If*

$$\int_{a_0}^{\infty} \frac{u}{|f(u)|} du < \infty \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \frac{\psi(\lambda)}{\lambda} = \infty,$$

then

$$\mathbb{E}\left(h\left(\sup_{x>0} S^x\right)\right) < \infty.$$

Proof. From (3.11) and Lemma 2.1 we deduce that

$$\lim_{u \rightarrow \infty} \frac{f_2(u)}{u} = \infty.$$

Therefore there exists a constant $M > a_0$ such that $f_2(u) \geq 2\gamma u + 2\alpha$ for all $u \geq M$. Let $\{x_n, n \geq 1\}$ be an increasing sequence of positive real numbers such that $x_n \rightarrow \infty$ as $n \rightarrow \infty$. There exists $n_0 > 0$ such that $x_n > 2M$ for all $n \geq n_0$. Hence

$$R_M^n = \inf\{t > 0, V_t^{x_n} < 2M\} > 0 \text{ a.s. for any } n \geq n_0.$$

For $n \geq n_0$, we have on the time interval $[0, R_M^n \wedge \tau_M^-]$,

$$\begin{aligned} -M &\leq X_t \\ M &\leq \frac{1}{2}V_t^{x_n} \leq V_t^{x_n} + X_t \\ \alpha + \gamma(V_t^{x_n} + X_t) - f_2(V_t^{x_n} + X_t) &\leq -\frac{1}{2}f_2(V_t^{x_n} + X_t) \leq -\frac{1}{2}f_2\left(\frac{1}{2}V_t^{x_n}\right) \\ \frac{dV_t^{x_n}}{dt} &\leq -\frac{1}{2}f_2\left(\frac{1}{2}V_t^{x_n}\right) \\ \int_0^t \frac{dV_s^{x_n}}{f_2\left(\frac{1}{2}V_s^{x_n}\right)} &\leq -\frac{1}{2}t \\ \int_{\frac{1}{2}V_t^{x_n}}^{\frac{1}{2}x_n} \frac{du}{f_2(u)} &\geq \frac{1}{4}t \\ g\left(\frac{1}{2}V_t^{x_n}\right) &\geq \frac{1}{4}t. \end{aligned} \tag{3.13}$$

Now, for proving $\mathbb{E}(h(\sup_{x>0} S^x)) < \infty$ we follow the following five steps:

Step 1. We first show that for all $n \geq n_0$,

$$R_M^n \wedge \tau_M^- \leq 4g(M) \text{ a.s.}$$

Indeed, if $R_M^n \wedge \tau_M^- > 4g(M)$ then

$$V_{4g(M)}^{x_n} > 2M.$$

So that

$$g\left(\frac{1}{2}V_{4g(M)}^{x_n}\right) < g(M),$$

because g is decreasing. This contradicts (3.13).

Step 2. We show that for all $n \geq n_0$,

$$\mathbb{E}(h(V_{R_M^n \wedge \tau_M^-}^{x_n})) < \infty.$$

Note that on the set $\{R_M^n < \tau_M^-\}$, $V_{R_M^n \wedge \tau_M^-}^{x_n} = 2M$. Consequently,

$$V_{R_M^n \wedge \tau_M^-}^{x_n} = 2M \mathbf{1}_{\{R_M^n < \tau_M^-\}} + V_{\tau_M^-}^{x_n} \mathbf{1}_{\{R_M^n \geq \tau_M^-\}}.$$

Therefore, from Lemma 3.7 and (3.13),

$$\begin{aligned} h(V_{R_M^n \wedge \tau_M^-}^{x_n}) &= h(2M) \mathbf{1}_{\{R_M^n < \tau_M^-\}} + h(V_{\tau_M^-}^{x_n}) \mathbf{1}_{\{R_M^n \geq \tau_M^-\}} \\ &\leq h(2M) + 2h\left(\frac{1}{2} V_{\tau_M^-}^{x_n}\right) \mathbf{1}_{\{R_M^n \geq \tau_M^-\}} + c_1 \\ &\leq h(2M) + \frac{2^{2d+1}}{(\tau_M^-)^d} + c_1. \end{aligned}$$

Hence

$$\mathbb{E}(h(V_{R_M^n \wedge \tau_M^-}^{x_n})) \leq h(2M) + c_1 + 2^{2d+1} \mathbb{E}\left(\frac{1}{(\tau_M^-)^d}\right). \quad (3.14)$$

Step 2 now follows from Proposition 3.2.

Step 3. We show that for all $n \geq n_0$,

$$\mathbb{E}(h(U_{R_M^n \wedge \tau_M^-}^{x_n})) < \infty.$$

From (3.3) and Step 1 we get

$$\begin{aligned} \mathbb{E}(X_{R_M^n \wedge \tau_M^-}) &= 0 \\ \mathbb{E}(X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}} + X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} \leq 0\}}) &= 0 \\ \mathbb{E}(X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}} - M) &\leq 0 \\ \mathbb{E}(X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}}) &\leq M. \end{aligned} \quad (3.15)$$

We have

$$\begin{aligned} h(U_{R_M^n \wedge \tau_M^-}^{x_n}) &= h(V_{R_M^n \wedge \tau_M^-}^{x_n} + X_{R_M^n \wedge \tau_M^-}) \\ &\leq h(V_{R_M^n \wedge \tau_M^-}^{x_n} + X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}}) \\ &\leq h(V_{R_M^n \wedge \tau_M^-}^{x_n}) + h(X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}}) + c_1 \\ &\leq h(V_{R_M^n \wedge \tau_M^-}^{x_n}) + h'(2c) X_{R_M^n \wedge \tau_M^-} \mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}} + h(2c) + c_1, \end{aligned}$$

where we have used Lemma 3.7 and (3.12) for the last two inequalities. Hence by (3.15) and Step 2

$$\begin{aligned} \mathbb{E}(h(U_{R_M^n \wedge \tau_M^-}^{x_n})) &\leq \mathbb{E}(h(V_{R_M^n \wedge \tau_M^-}^{x_n})) + h'(2c)M + h(2c) + c_1 \\ &< \infty. \end{aligned} \quad (3.16)$$

Step 4. We show that for $n \geq n_0$,

$$\mathbb{E}(h(\tau_M^n)) < \infty,$$

where

$$\tau_M^n := \inf\{t > 0, U_t^{x_n} \leq M\}.$$

Note that we can choose M large enough such that

$$\frac{f(u)}{u} \leq -1 \quad \text{for all } u \geq M.$$

Consequently, $U_{R_M^n \wedge \tau_M^- + r}^{x_n} \leq Y_r$, for all $0 \leq r \leq \tau_M^n - R_M^n \wedge \tau_M^-$ a.s., where Y solves

$$dY_r = -dr + dX_r^{n,M}, \quad Y_0 = U_{R_M^n \wedge \tau_M^-}^{x_n},$$

where $X_r^{n,M} = X_{R_M^n \wedge \tau_M^- + r} - X_{R_M^n \wedge \tau_M^-}$. Let $A_M^n := \inf\{r > 0, Y_r \leq M\}$. Clearly

$$\tau_M^n \leq R_M^n \wedge \tau_M^- + A_M^n \leq 4g(M) + A_M^n.$$

We deduce from Lemma 3.7 that

$$h(\tau_M^n) \leq h(4g(M)) + h(A_M^n) + c_1, \quad (3.17)$$

We now prove that $\mathbb{E}(h(A_M^n)) < \infty$, from which Step 4 will follow. Indeed, we have for $t > 0$ (recall that $\Gamma_t^{n,M} = \inf\{s \geq 0, X_s^{n,M} - s < -t\}$)

$$\begin{aligned} \mathbb{P}(h(A_M^n) > t) &= \mathbb{P}(A_M^n > h^{-1}(t)) \\ &= \mathbb{P}(Y_0 + \inf_{[0, h^{-1}(t)]} (X_s^{n,M} - s) > M) \\ &\leq \mathbb{P}(\inf_{[0, h^{-1}(t)]} (X_s^{n,M} - s) > -Y_0) \\ &= \mathbb{P}(\Gamma_{Y_0}^{n,M} > h^{-1}(t)) \\ &= \mathbb{P}(h(\Gamma_{Y_0}^{n,M}) > t). \end{aligned}$$

Hence

$$\mathbb{E}(h(A_M^n)) = \int_0^\infty \mathbb{P}(h(A_M^n) > t) dt \leq \int_0^\infty \mathbb{P}(h(\Gamma_{Y_0}^{n,M}) > t) dt = \mathbb{E}(h(\Gamma_{Y_0}^{n,M})). \quad (3.18)$$

Furthermore, since h is a concave function on (c, ∞) , we can use Jensen's inequality and Lemma 3.4 to get for all $t > 0$,

$$\begin{aligned} \mathbb{E}(h(\Gamma_t \vee 2c)) &\leq h(\mathbb{E}(\Gamma_t \vee 2c)) \\ &\leq h(\mathbb{E}(\Gamma_t) + 2c) \\ &= h(t + 2c). \end{aligned}$$

Therefore, since $(\Gamma_t^{n,M}, t \geq 0)$ has the same law as $(\Gamma_t, t \geq 0)$ and is independent of Y_0 ,

$$\begin{aligned} \mathbb{E}(h(\Gamma_{Y_0}^{n,M})) &\leq \mathbb{E}(h(\Gamma_{Y_0}^{n,M} \vee 2c)) \\ &= \mathbb{E}[\mathbb{E}(h(\Gamma_{Y_0}^{n,M} \vee 2c)|Y_0)] \\ &\leq \mathbb{E}(h(Y_0 + 2c)) \\ &\leq \mathbb{E}(h(Y_0)) + h(2c) + c_1, \end{aligned} \tag{3.19}$$

where we have used Lemma 3.7 for the last inequality. From Step 3, this last right-hand side is finite. Step 4 now follows from (3.17), (3.18) and (3.19).

Step 5. We will now conclude the proof of the Theorem. From (3.17), (3.18), (3.19), (3.16) and (3.14) we deduce that for all $n \geq n_0$,

$$\mathbb{E}(h(\tau_M^n)) \leq h(2M) + h(4g(M)) + 2h(2c) + h'(2c)M + 4c_1 + 2^{2d+1}\mathbb{E}\left(\frac{1}{(\tau_M^n)^d}\right).$$

Hence

$$\mathbb{E}(h(\tau_M)) < \infty, \quad \text{where } \tau_M := \sup_{n > n_0} \tau_M^n.$$

Let T be a positive constant. Let p denote the probability that starting from M at time $t = 0$, U hits zero before time T . There exists a constant $K > 0$ such that

$$\frac{f(u)}{u} \leq K \quad \text{for all } u \geq 0.$$

We have

$$p \geq \mathbb{P}(M + KT + \inf_{[0,T]} X_t \leq 0) > 0,$$

by Lemma 3.5. Using the same argument used in the proof of Theorem 2 we obtain that $\sup_{x>0} \tau^x$ is stochastically dominated by

$$\zeta T + \sum_{i=1}^{\zeta} \eta_i,$$

where ζ is a geometric random variable with success probability p , the random variables η_i are i.i.d, with the same law as τ_M , globally independent of ζ . Therefore

$$\begin{aligned} \mathbb{E}(h(\sup_{x>0} S^x)) &= \mathbb{E}(h(\sup_{x>0} \tau^x)) \leq \mathbb{E}(h(\zeta T + \sum_{i=1}^{\zeta} \eta_i)) \\ &\leq \mathbb{E}(\zeta h(T) + \sum_{i=1}^{\zeta} h(\eta_i) + (2\zeta - 1)c_1) \\ &\leq \frac{h(T)}{p} + \frac{1}{p}\mathbb{E}(h(\tau_M)) + \left(\frac{2}{p} - 1\right)c_1 \\ &< \infty, \end{aligned}$$

where we have used Lemma 3.7 for the second inequality. The result follows. \square

4 Some examples

In this section we will discuss some special cases to illustrate our results.

Example 4.1. An important example is the case of a logistic interaction where

$$f(u) := au - bu^2, \quad a \in \mathbb{R}, b > 0.$$

It is easily seen that f satisfies (H2). There exists a positive constant a_0 such that $f(u) < 0$ for all $u \geq a_0$, and

$$\int_{a_0}^{\infty} \frac{1}{|f(u)|} du = \int_{a_0}^{\infty} \frac{1}{bu^2 - au} du < \infty, \quad \int_{a_0}^{\infty} \frac{u}{|f(u)|} du = \int_{a_0}^{\infty} \frac{u}{bu^2 - au} du = \infty.$$

Hence in this case, from Theorem 1, 2 and 3 we have

$$\begin{cases} \sup_{x>0} T^x = \infty & \text{a.s. if } \int^{\infty} d\lambda/\psi(\lambda) = \infty \\ \mathbb{E}(\sup_{x>0} T^x) < \infty & \text{if } \int^{\infty} d\lambda/\psi(\lambda) < \infty, \end{cases}$$

and

$$\sup_{x>0} S^x = \infty \quad \text{a.s.}$$

Example 4.2. We consider the case of a polynomial interaction where

$$f(u) := au - bu^{\beta}, \quad a \in \mathbb{R}, b > 0, \beta > 1.$$

Then f satisfies (H2) and there exists a positive constant a_0 such that $f(u) < 0$ for all $u \geq a_0$. Since

$$\int_{a_0}^{\infty} \frac{1}{|f(u)|} du = \int_{a_0}^{\infty} \frac{1}{bu^{\beta} - au} du < \infty,$$

from Theorem 1 and 2 we have

$$\begin{cases} \sup_{x>0} T^x = \infty & \text{a.s. if } \int^{\infty} d\lambda/\psi(\lambda) = \infty \\ \mathbb{E}(\sup_{x>0} T^x) < \infty & \text{if } \int^{\infty} d\lambda/\psi(\lambda) < \infty. \end{cases}$$

Concerning the total mass we note that

$$\int_{a_0}^{\infty} \frac{u}{|f(u)|} du = \int_{a_0}^{\infty} \frac{u}{bu^{\beta} - au} du \quad \begin{cases} = \infty, & \text{if } \beta \leq 2 \\ < \infty, & \text{if } \beta > 2. \end{cases}$$

Hence $\sup_{x>0} S^x = \infty$ a.s. for $\beta \leq 2$, by Theorem 3. For $\beta > 2$, we can choose

$$f_2(u) := bu^{\beta-1}.$$

Therefore for all $u \geq a_0$,

$$g(u) = \int_u^{\infty} \frac{1}{f_2(r)} dr = \frac{1}{b(\beta-2)u^{\beta-2}}.$$

Since for all $u \geq a_0$,

$$g(u)f_2'(u) = \frac{\beta - 1}{\beta - 2},$$

(H3) holds for $d = \frac{1}{\beta-2}$. So that for $u > a_0$,

$$h(u) = \frac{1}{g(u)^d} = (b(\beta - 2))^{\beta-2}u.$$

Hence from Theorem 4, if $\lim_{\lambda \rightarrow \infty} \frac{\psi(\lambda)}{\lambda} = \infty$ we have

$$\mathbb{E}(\sup_{x>0} S^x) < \infty.$$

Example 4.3. We consider the case where

$$f(u) := -ue^u.$$

In this case U^x is the solution of the following SDE

$$dU_t^x = -e^{U_t^x} dt + dX_t, \quad U_0^x = x, \quad (4.1)$$

where X is a Lévy process with Laplace exponent ψ . It is easily seen that an explicit formula for the unique strong solution of (4.1) is

$$U_t^x = x + X_t - \log\left(1 + \int_0^t e^{X_s+x} ds\right).$$

We have for $x > 0$,

$$\tau^x := \inf\{t > 0, U_t^x = 0\} = \inf\{t > 0, e^{X_t} = e^{-x} + \int_0^t e^{X_s} ds\}.$$

Therefore

$$\sup_{x>0} \tau^x = \inf\{t > 0, e^{X_t} = \int_0^t e^{X_s} ds\}.$$

Now, we can choose the function f_1 satisfying (H2) and

$$\begin{aligned} f_1(u) &\geq -ue^u && \forall u \geq 0, \\ f_1(u) &= au - bu^\beta && \forall u \geq c \text{ for some constant } c > 0, \end{aligned}$$

where $a \in \mathbb{R}, b > 0, \beta > 2$. Let $Z_t^{1,x}$ be the solution of the SDE (1.1) with the function f_1 instead of f , and let S_1^x be the total mass of $Z^{1,x}$. By the comparison theorem we have

$$\begin{aligned} Z_t^x &\leq Z_t^{1,x} \quad \text{a.s.} \quad \forall t \geq 0, \\ S^x &\leq S_1^x \quad \text{a.s.} \end{aligned}$$

We already know that if ψ satisfies the condition

$$\lim_{\lambda \rightarrow \infty} \frac{\psi(\lambda)}{\lambda} = \infty,$$

then

$$\mathbb{E}(\sup_{x>0} S_1^x) < \infty.$$

Hence we deduce in this case

$$\mathbb{E}(\sup_{x>0} \tau^x) = \mathbb{E}(\sup_{x>0} S^x) < \infty.$$

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