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Branching processes with interaction and a generalized Ray–Knight Theorem

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Abstract. We consider a discrete model of population dynamics with interaction between individuals, where the birth and death rates are nonlinear functions of the population size. We obtain the large population limit of a renormalization of our model as the solution of the SDE

$$Z_t^x = x + \int_0^t f\big(Z_s^x\big)\,\mathrm{d}s + 2\int_0^t \int_0^{Z_s^x} W(\mathrm{d}s, \mathrm{d}u),$$

where $W(\mathrm{d}s, \mathrm{d}u)$ is a time space white noise on $[0, \infty)^2$.

We give a Ray–Knight representation of this diffusion in terms of the local times of a reflected Brownian motion H with a drift that depends upon the local time accumulated by H at its current level, through the function $f'/2$.

Résumé. Nous considérons un modèle d'évolution d'une population avec intercation entre les individus, où les taux de naissance et de mort sont fonction de la taille de la population. Nous obtenons la limite en grande population après renormalisation, qui est solution de l'EDS

$$Z_t^x = x + \int_0^t f\big(Z_s^x\big)\,\mathrm{d}s + 2\int_0^t \int_0^{Z_s^x} W(\mathrm{d}s, \mathrm{d}u),$$

où $W(\mathrm{d}s, \mathrm{d}u)$ est un bruit blanc sur $[0, \infty)^2$.

Nous donnons une représentation de cette diffusion à la Ray–Knight, en fonction des temps locaux d'un mouvement brownien réfléchi H avec une dérive qui dépend du temps local accumulé par H à son niveau courant, à travers la fonction $f'/2$.

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Introduction

Consider a population evolving in continuous-time with m ancestors at time $t = 0$, in which each individual, independently of the others, gives birth to children at a constant rate μ (in our model each birth event means the birth of a single individual), and dies after an exponential time with parameter λ. For each individual we superimpose additional birth and death rates due to interactions with others at a certain rate which depends upon the size of the total population. For instance, we might decide that each individual dies because of competition at a rate equal to γ times the number of presently alive individuals in the population, which amounts to add a global death rate equal to $\gamma(X_t^m)^2$, if X_t^m denotes the total number of alive individuals at time t.

If we consider this population with $m = [Nx]$ ancestors at time $t = 0$, weigh each individual with the factor $1/N$, and choose $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma/N$, then it is shown in Le, Pardoux and Wakolbinger [7] (in the above particular case of a quadratic competition term) that the "total population mass process" converges weakly to the solution of the Feller SDE with logistic drift

$$\mathrm{d}Z_t^x = \left[\theta Z_t^x - \gamma\left(Z_t^x\right)^2\right]\mathrm{d}t + 2\sqrt{Z_t^x}\,\mathrm{d}W_t, \quad Z_0^x = x. \tag{0.1}$$

The diffusion Z^x is called Feller diffusion with logistic growth and models the evolution of the size of a large population with competition. In this model θ represents the supercritical branching parameter while γ is the rate at which each individual is killed by any member of her generation. This model has been studied in Lambert [5], who shows in particular that its extinction time is finite almost surely.

We generalize the logistic model by replacing the quadratic function $\theta z - \gamma z^2$ by a more general nonlinear function f of the population size. We then obtain in the continuous setting a diffusion which is the solution of the SDE

$$Z_t^x = x + \int_0^t f\left(Z_s^x\right)\mathrm{d}s + 2\int_0^t \int_0^{Z_s^x} W(\mathrm{d}s, \mathrm{d}u), \tag{0.2}$$

where the function f satisfies the following hypothesis.

Hypothesis A. $f \in C(\mathbb{R}_+; \mathbb{R})$, $f(0) = 0$ *and there exists* $\beta \geq 0$ *such that*

$$f(x+y) - f(x) \leq \beta y, \quad x, y \geq 0.$$

Note that the Hypothesis A implies that for all $x \geq 0$,

$$f(x) \leq \beta x.$$

It follows from Theorem 2.1 in [3] that equation (0.2) has a unique strong solution. Indeed, if we let $f_1(x) = \beta x$ and $f_2(x) = \beta x - f(x)$, we have the decomposition $f = f_1 - f_2$ with f_1 Lipschitz continuous and f_2 continuous and nondecreasing. Hence the assumptions of the quoted theorem are satisfied, as a consequence of Hypothesis A. We note that since $f(0) = 0$ (see Remark 1.1 below for discussion of this assumption), the unique solution starting from the initial condition $x = 0$ is $Z_t^0 \equiv 0$. Consequently, from the comparison Theorem 2.2 in [3], we deduce that $Z_t^x \geq 0$ for all $x > 0$, $t \geq 0$.

An equivalent way to write (0.2) is the following.

$$Z_t^x = x + \int_0^t f\left(Z_s^x\right)\mathrm{d}s + 2\int_0^t \sqrt{Z_s^x}\,\mathrm{d}W_s^x, \tag{0.3}$$

where W^x is a standard Brownian motion. However, if we use the formulation (0.3), the joint evolution of the various population sizes $\{Z_t^x, t \geq 0\}$ corresponding to different initial population sizes x has a complicated description, whereas the formulation (0.2) due to Dawson and Li [3] with one unique space–time white noise W, describes exactly the joint evolution of $\{Z_t^x, t \geq 0, x \geq 0\}$ which we have in mind. We call this diffusion the generalized Feller diffusion. In order to motivate this continuous model, we first define a discrete model. For defining jointly the discrete model for all initial population sizes, we need as in [9] to impose a nonsymmetric competition rule between the individuals, which we will describe in Section 1 below. We do a suitable renormalization of the parameters of the discrete model in order to obtain in Section 2 a large population limit of our model which is a generalized Feller diffusion. In Section 3 we establish a Ray–Knight representation for such a generalized Feller diffusion. We now state this representation, which is the main result of this paper. For that sake, let us first introduce some notions. We shall say that the process $\{Z_t^x, t \geq 0\}$ is (sub)critical if this process goes extinct in finite time a.s. This notation is meant as a generalization of the two distinct cases of critical and subcritical branching processes. A necessary and sufficient condition on f will be given at the end of Section 1 for Z^x to be (sub)critical for all $x > 0$. Let us now assume that f is of class C^1, and introduce the "contour process" of the forest of genealogical trees of the population described by the process Z_t^x, which is the solution of the SDE

$$H_s = B_s + \frac{1}{2}\int_0^s f'\left(L_r(H_r)\right)\mathrm{d}r + \frac{1}{2}L_s(0),$$

where for any $s, t \geq 0$, $L_s(t)$ is the local time accumulated by the process H at level t up to time s. Note that the last term in the above SDE means that the solution is reflected above 0. The intuition behind the drift term will be given below. Note for the moment that when $f' > 0$, the interaction with the other individuals in the population favors longer trees, while when $f' < 0$, the interaction is of the type of a competition, which tends to reduce the size of the trees. We moreover define the inverse of the local time of H at level 0:

$$S_x := \inf\{s > 0, L_s(0) > x\}, \quad x > 0.$$

Theorem 0.1. *Suppose that f is of class C^1 and satisfies Hypothesis* A, *and is such that the population process is (sub)critical. Then the two random fields $\{L_{S_x}(t), t \geq 0, x \geq 0\}$ and $\{Z_t^x, t \geq 0, x \geq 0\}$ have the same law.*

1. Discrete model with a general interaction

In this section we set up a discrete mass continuous-time approximation of the generalized Feller diffusion. We consider a discrete model of population dynamics with interaction, in which each individual, independently of the others, gives birth naturally at rate λ and dies naturally at rate μ. Moreover, we suppose that each individual gives birth and dies because of interaction with others at rates which depend upon the current population size. Moreover, we exclude multiple births at any given time and we define the interaction rule through a function f which satisfies Hypothesis A.

In order to define our model jointly for all initial sizes, we need to introduce a nonsymmetric description of the effect of the interaction as in [7], but here we allow the interaction to be favorable to the individuals.

1.1. *The model*

We consider a continuous-time $\mathbb{Z}_+$-valued population process $\{X_t^m, t \geq 0\}$, which starts at time zero from m ancestors who are arranged from left to right, and evolves in continuous-time. The left/right order is passed on to their offsprings: the daughters are placed on the right of their mothers and if at a time t the individual i is located at the left of individual j, then all the daughters of i after time t will be placed on the left of all the daughters of j. Those rules apply inside each genealogical tree, and distinct branches of the trees never cross. Since we have excluded multiple births at any given time, this means that the forest of genealogical trees of the population is a plane forest of trees, where the ancestor of the population X_t^1 is placed on the far left, the ancestor of $X_t^2 - X_t^1$ immediately on his right, etc… This defines in a non-ambiguous way an order from left to right within the population alive at each time t. See Figure 1. We decree that each individual feels the interaction with the others placed on her left but not with those on her right. Precisely, at any time t, the individual i has an interaction death rate equal to $(f(\mathcal{L}_i(t) + 1) - f(\mathcal{L}_i(t)))^-$ or an interaction birth rate equal to $(f(\mathcal{L}_i(t) + 1) - f(\mathcal{L}_i(t)))^+$, where $\mathcal{L}_i(t)$ denotes the number of individuals alive at time t who are located on the left of i in the above planar picture. This means that the individual i is under attack by the others located at her left if $f(\mathcal{L}_i(t) + 1) - f(\mathcal{L}_i(t)) < 0$ while the interaction improves her fertility if $f(\mathcal{L}_i(t) + 1) - f(\mathcal{L}_i(t)) > 0$. Of course, conditionally upon $\mathcal{L}_i(\cdot)$, the ocurence of a "competition death event" or an "interaction birth event" for individual i is independent of the other birth/death events and of what happens to the other individuals. In order to simplify our formulas, we suppose moreover that the first individual in the left/right order has a birth rate equal to $\lambda + f^+(1)$ and a death rate equal to $\mu + f^-(1)$.

The resulting total interaction death and birth rates endured by the population X_t^m at time t is then

$$\sum_{k=1}^{X_t^m} \big[\big(f(k) - f(k-1)\big)^+ - \big(f(k) - f(k-1)\big)^-\big] = \sum_{k=1}^{X_t^m} \big(f(k) - f(k-1)\big) = f\big(X_t^m\big).$$

As a result, $\{X_t^m, t \geq 0\}$ is a discrete-mass $\mathbb{Z}_+$-valued Markov process, which evolves as follows. $X_0^m = m$. If $X_t^m = 0$, then $X_s^m = 0$ for all $s \geq t$. While at state $k \geq 1$, the process

$$X_t^m \text{ jumps to } \begin{cases} k+1, & \text{at rate } \lambda k + \sum_{\ell=1}^k (f(\ell) - f(\ell-1))^+; \\ k-1, & \text{at rate } \mu k + \sum_{k=1}^k (f(\ell) - f(\ell-1))^-. \end{cases}$$

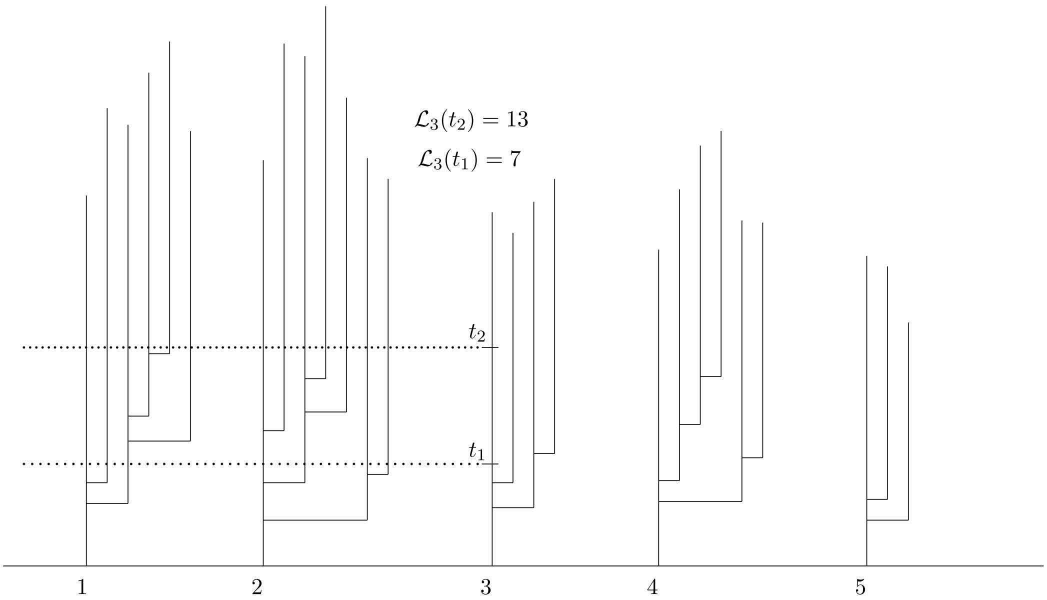

Fig. 1. Plane forest with five ancestors.

1.2. *Coupling over ancestral population size*

The above description specifies the joint evolution of all $\{X_t^m, t \ge 0\}_{m\ge 1}$, or in other words of the two-parameter process $\{X_t^m, t \ge 0, m \ge 1\}$. In the case of a linear function f, for each fixed $t > 0$, $\{X_t^m, m \ge 1\}$ is an independent increments process. In the case of a nonlinear function f, we believe that for t fixed $\{X_t^m, m \ge 1\}$ is not a Markov chain. That is to say, the conditional law of X_t^{n+1} given X_t^n differs from its conditional law given $(X_t^1, X_t^2, \dots, X_t^n)$. The intuitive reason for that is that the additional information carried by $(X_t^1, X_t^2, \dots, X_t^{n-1})$ gives us a clue as to the fertility or the level of competition that the progeny of the $(n+1)$st ancestor had to beneficit or to suffer from, between time 0 and time t.

However, $\{X_\cdot^m, m \ge 1\}$ is a Markov chain with values in the space $D([0,\infty); \mathbb{Z}_+)$ of càdlàg functions from $[0,\infty)$ into $\mathbb{Z}_+$, which starts from 0 at $m = 0$. Consequently, in order to describe the law of the whole process, that is of the two-parameter process $\{X_t^m, t \ge 0, m \ge 1\}$, it suffices to describe the conditional law of $X_\cdot^n$, given $\{X_\cdot^{n-1}\}$. We now describe that conditional law for arbitrary $1 \le m < n$. Let $V_t^{m,n} := X_t^n - X_t^m$, $t \ge 0$. Conditionally upon $\{X_\cdot^\ell, \ell \le m\}$, and given that $X_t^m = x(t)$, $t \ge 0$, $\{V_t^{m,n}, t \ge 0\}$ is a $\mathbb{Z}_+$-valued time inhomogeneous Markov process starting from $V_0^{m,n} = n - m$, whose time-dependent infinitesimal generator $\{Q_{k,\ell}(t), k, \ell \in \mathbb{Z}_+\}$ is such that its off-diagonal terms are given by

$$Q_{0,\ell}(t) = 0, \quad \forall \ell \ge 1, \quad \text{and for any } k \ge 1,$$

$$Q_{k,k+1}(t) = \mu k + \sum_{\ell=1}^{k} \bigl(f\bigl(x(t)+\ell\bigr) - f\bigl(x(t)+\ell-1\bigr)\bigr)^+,$$

$$Q_{k,k-1}(t) = \lambda k + \sum_{\ell=1}^{k} \bigl(f\bigl(x(t)+\ell\bigr) - f\bigl(x(t)+\ell-1\bigr)\bigr)^-,$$

$$Q_{k,\ell}(t) = 0, \quad \forall \ell \notin \{k-1, k, k+1\}.$$

The reader can easily be convinced that this description of the conditional law of $\{X_t^n - X_t^m, t \ge 0\}$, given $X_\cdot^m$ is prescribed by what we have said above, and that $\{X_\cdot^m, m \ge 1\}$ is indeed a Markov chain.

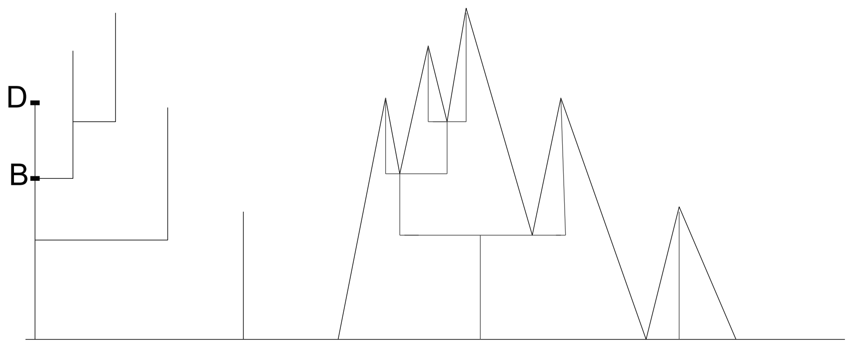

Fig. 2. A forest with two trees and its contour process.

Remark 1.1. *Note that if the function f is increasing on $[0, a]$, $a > 0$ and decreasing on $[a, \infty)$, the interaction improves the rate of fertility in a population whose size is smaller than a but for large size the interaction amounts to competition within the population. This is reasonable because when the population is large, the limitation of resources implies competition within the population. A positive interaction (for moderate population sizes) may be explained by the fact that an increase in the population size allows a more efficient organization of the society, with specalization among its members, thus resulting in better food production, health care, etc... We are mainly interested in the model with interaction defined with functions f such that $\lim_{x\to\infty} f(x) = -\infty$. Note also that we could have generalized our model to the case $f(0) \geq 0$. $f(0) > 0$ would mean an immigration flux. The reader can easily check that results in Section 2 would still be valid in this case. However in Proposition 1.3 and in Section 3 below, assumption $f(0) = 0$ is crucial, since we need the population to become extinct in finite time a.s.*

1.3. *The associated contour process in the discrete model*

The just described reproduction dynamics give rise to a forest $\mathcal{F}^m$ of m trees of descent, drawn into the plane as sketched in Figure 2. Note also that, with the above described construction, the $(\mathcal{F}^m, m \geq 1)$ are coupled: the forest $\mathcal{F}^{m+1}$ has the same law as the forest $\mathcal{F}^m$ to which we add a new tree generated by an ancestor placed at the $(m+1)$st position. If the function f tends to $-\infty$ and m is large enough, the trees further to the right of the forest $\mathcal{F}^m$ have a tendency to stay smaller because of the competition: they are "under attack" from the trees to their left. From $\mathcal{F}^m$ we read off a continuous and piecewise linear $\mathbb{R}_+$-valued path $H^m = (H^m_s)$ (called the contour process of $\mathcal{F}^m$) which is described as follows.

Starting from 0 at the initial time $s = 0$, the process H^m rises at speed p until it hits the top of the first ancestor branch (this is the leaf marked with D in Figure 2). There it turns and goes downwards, now at speed $-p$, until arriving at the next branch point (which is B in Figure 2). From there it goes upwards into the (yet unexplored) next branch, and proceeds in a similar fashion until being back at height 0, which means that the exploration of the leftmost tree is completed. Then explore the next tree, and so on. See Figure 2.

We define the local time $L^m_s(t)$ accumulated by the process H^m at level t up to time s by:

$$L^m_s(t) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^s 1_{t \leq H^m_r < t+\epsilon}\, \mathrm{d}r.$$

The process H^m is piecewise linear, continuous with derivative $\pm p$: at any time $s \geq 0$, the rate of appearance of minima (giving rise to new branches) is equal to

$$p\mu + \left[f\left(\left\lfloor \frac{p}{2} L^m_s(H^m_s) \right\rfloor + 1 \right) - f\left(\left\lfloor \frac{p}{2} L^m_s(H^m_s) \right\rfloor \right) \right]^+, \tag{1.1}$$

and the rate of appearance of maxima (describing deaths of branches) is equal to

$$p\lambda + \left[f\left(\left\lfloor \frac{p}{2} L^m_s(H^m_s) \right\rfloor + 1 \right) - f\left(\left\lfloor \frac{p}{2} L^m_s(H^m_s) \right\rfloor \right) \right]^-. \tag{1.2}$$

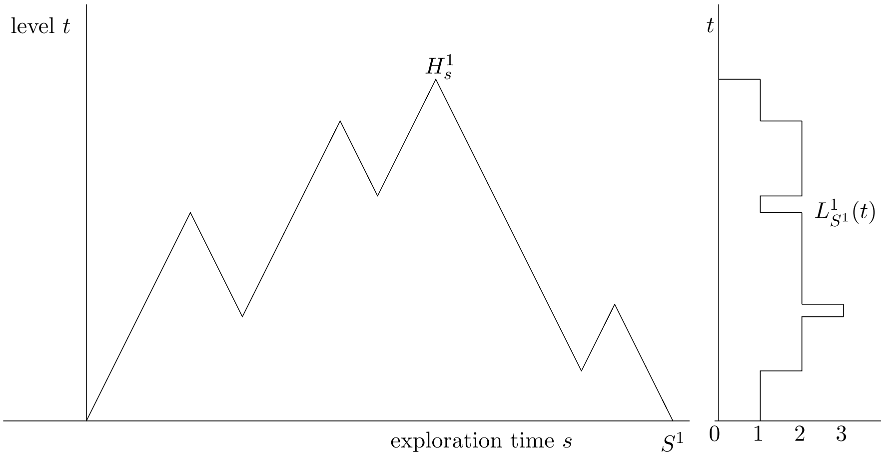

Fig. 3. Discrete Ray–Knight representation.

Let S^m be the time needed in order to explore the forest $\mathcal{F}^m$. We have

$$S^m = \inf\left\{s > 0;\ \frac{p}{2} L_s^m(0) \ge m\right\}.$$

Under the assumption that $S^m < \infty$ a.s. for all $m \ge 1$, we have the following discrete Ray–Knight representation (see Figure 3).

$$\left(X_t^m, t \ge 0, m \ge 1\right) \equiv \left(\frac{p}{2} L^m_{S^m}(t), t \ge 0, m \ge 1\right).$$

Remark 1.2. *We note that the process $\{H_s^m, s \ge 0\}$ is not a Markov process. Its evolution after time s does not depend only upon the present value H_s^m, but also upon its past before s, through the local times accumulated up to time s.*

1.4. *Renormalized discrete model*

Now we proceed to a renormalization of this model. For $x \in \mathbb{R}_+$ and $N \in \mathbb{N}$, we choose $m = \lfloor Nx \rfloor$, $\mu = 2N$, $\lambda = 2N$, we multiply f by N and divide by N the argument of the function f. We affect to each individual in the population a mass equal to $1/N$. Then the total mass process $Z^{N,x}$, which starts from $\frac{\lfloor Nx \rfloor}{N}$ at time $t = 0$, is a Markov process whose evolution can be described as follows.

$$Z^{N,x} \text{ jumps from } \frac{k}{N} \text{ to } \begin{cases} \frac{k+1}{N} \text{ at rate } 2Nk + N\sum_{i=1}^k (f(\frac{i}{N}) - f(\frac{i-1}{N}))^+, \\ \frac{k-1}{N} \text{ at rate } 2Nk + N\sum_{i=1}^k (f(\frac{i}{N}) - f(\frac{i-1}{N}))^-. \end{cases}$$

Clearly there exist two mutually independent standard Poisson processes P_1 and P_2 such that

$$Z_t^{N,x} = \frac{\lfloor Nx \rfloor}{N} + \frac{1}{N} P_1\left(\int_0^t \left(2N^2 Z_r^{N,x} + N \sum_{i=1}^{NZ_r^{N,x}} \left(f\left(\frac{i}{N}\right) - f\left(\frac{i-1}{N}\right)\right)^+\right) \mathrm{d}r\right)$$
$$- \frac{1}{N} P_2\left(\int_0^t \left(2N^2 Z_r^{N,x} + N \sum_{i=1}^{NZ_r^{N,x}} \left(f\left(\frac{i}{N}\right) - f\left(\frac{i-1}{N}\right)\right)^-\right) \mathrm{d}r\right).$$

Consequently there exists a local martingale $M^{N,x}$ such that

$$Z_t^{N,x} = \frac{\lfloor Nx \rfloor}{N} + \int_0^t f\left(Z_r^{N,x}\right) \mathrm{d}r + M_t^{N,x}. \tag{1.3}$$

Since $M^{N,x}$ is a purely discontinuous local martingale, its quadratic variation $[M^{N,x}]$ is given by the sum of the squares of its jumps, i.e.

$$\begin{aligned}\left[M^{N,x}\right]_t = \frac{1}{N^2}\Bigg[& P_1\left(\int_0^t\left(2N^2Z_r^{N,x} + N\sum_{i=1}^{NZ_r^{N,x}}\left(f\left(\frac{i}{N}\right)-f\left(\frac{i-1}{N}\right)\right)^+\right)\mathrm{d}r\right) \\ & + P_2\left(\int_0^t\left(2N^2Z_r^{N,x} + N\sum_{i=1}^{NZ_r^{N,x}}\left(f\left(\frac{i}{N}\right)-f\left(\frac{i-1}{N}\right)\right)^-\right)\mathrm{d}r\right)\Bigg]. \end{aligned} \tag{1.4}$$

We deduce from (1.4) that the predictable quadratic variation $\langle M^{N,x}\rangle$ of $M^{N,x}$ is given by

$$\left\langle M^{N,x}\right\rangle_t = \int_0^t\left\{4Z_r^{N,x} + \frac{1}{N}\|f\|_{N,0,Z_r^{N,x}}\right\}\mathrm{d}r, \tag{1.5}$$

where for any $z=\frac{k}{N}$, $z'=\frac{k'}{N}$, $k\in\mathbb{Z}_+$ such that $k\le k'$,

$$\|f\|_{N,z,z'} = \sum_{i=k+1}^{k'}\left|f\left(\frac{i}{N}\right)-f\left(\frac{i-1}{N}\right)\right|.$$

Now we precise the law of the pair $(Z^{N,x}, Z^{N,y})$, for any $0<x<y$. Consider the pair of process $(Z^{N,x}, V^{N,x,y})$, which starts from $(\frac{\lfloor Nx\rfloor}{N}, \frac{\lfloor Ny\rfloor-\lfloor Nx\rfloor}{N})$ at time $t=0$, and whose dynamic is described by:

$$\left(Z^{N,x}, V^{N,x,y}\right) \text{ jumps from } \left(\frac{i}{N},\frac{j}{N}\right) \text{ to } \begin{cases} (\frac{i+1}{N},\frac{j}{N}) \text{ at rate } 2Ni+\sum_{k=1}^{i}(f(\frac{k}{N})-f(\frac{k-1}{N}))^+, \\ (\frac{i-1}{N},\frac{j}{N}) \text{ at rate } 2Ni+\sum_{k=1}^{i}(f(\frac{k}{N})-f(\frac{k-1}{N}))^-, \\ (\frac{i}{N},\frac{j+1}{N}) \text{ at rate } 2Nj+\sum_{k=1}^{j}(f(\frac{i+k}{N})-f(\frac{i+k-1}{N}))^+, \\ (\frac{i}{N},\frac{j-1}{N}) \text{ at rate } 2Nj+\sum_{k=1}^{j}(f(\frac{i+k}{N})-f(\frac{i+k-1}{N}))^-. \end{cases}$$

The process $V^{N,x,y}$ can be expressed as follows.

$$\begin{aligned} V_t^{N,x,y} = {} & \frac{\lfloor Ny\rfloor-\lfloor Nx\rfloor}{N} + \frac{1}{N}P^1\left(N\int_0^t\sum_{k=1}^{NV_s^{N,x,y}}\left(f\left(Z_s^{N,x}+\frac{k}{N}\right)-f\left(Z_s^{N,x}+\frac{k-1}{N}\right)\right)^+\mathrm{d}s\right) \\ & - \frac{1}{N}P^2\left(N\int_0^t\sum_{k=1}^{NV_s^{N,x,y}}\left(f\left(Z_s^{N,x}+\frac{k}{N}\right)-f\left(Z_s^{N,x}+\frac{k-1}{N}\right)\right)^-\mathrm{d}s\right) \\ & + \frac{1}{N}P^3\left(2N^2\int_0^t V_s^{N,x,y}\,\mathrm{d}s\right) - \frac{1}{N}P^4\left(2N^2\int_0^t V_s^{N,x,y}\,\mathrm{d}s\right), \end{aligned} \tag{1.6}$$

where P^1, P^2, P^3 and P^4 are mutually independent standard Poisson processes which are all independent of $\{Z_\cdot^{N,x'}, x'\le x\}$. Consequently

$$V_t^{N,x,y} = \frac{\lfloor Ny\rfloor-\lfloor Nx\rfloor}{N} + \int_0^t\left[f\left(Z_r^{N,x}+V_r^{N,x,y}\right)-f\left(Z_r^{N,x}\right)\right]\mathrm{d}r + M_t^{N,x,y}, \tag{1.7}$$

where $M^{N,x,y}$ is a local martingale whose predictable quadratic variation $\langle M^{N,x,y}\rangle$ is given by

$$\left\langle M^{N,x,y}\right\rangle_t = \int_0^t\left\{4V_r^{N,x,y} + \frac{1}{N}\|f\|_{N,Z_r^{N,x},V^{N,x,y}+Z_r^{N,x}}\right\}\mathrm{d}r. \tag{1.8}$$

Since $Z^{N,x}$ and $V^{N,x,y}$ never jump at the same time,

$$[M^{N,x}, M^{N,x,y}] = 0, \quad \text{hence } \langle M^{N,x}, M^{N,x,y}\rangle = 0, \tag{1.9}$$

which implies that the martingales $M^{N,x}$ and $M^{N,x,y}$ are orthogonal.

Consequently, $Z^{N,x} + V^{N,x,y}$ solves the SDE

$$Z_t^{N,x} + V_t^{N,x,y} = \frac{\lfloor Ny\rfloor}{N} + \int_0^t f\big(Z_r^{N,x} + V_r^{N,x,y}\big)\,\mathrm{d}r + \tilde{M}_t^{N,x,y},$$

where $\tilde{M}^{N,x,y}$ is a local martingale with $\langle \tilde{M}^{N,x,y}\rangle$ given by

$$\langle \tilde{M}^{N,x,y}\rangle_t = \langle M^{N,x}\rangle_t + \langle M^{N,x,y}\rangle_t = \langle M^{N,x+y}\rangle_t, \quad \forall t \ge 0.$$

We then deduce that for any $x, y \in \mathbb{R}_+$ such $x \le y$,

$$Z^{N,x} + V^{N,x,y} \stackrel{(d)}{=} Z^{N,y}.$$

It follows from (1.6) that conditionally upon $\{Z^{N,x'}, x' \le x\}$, $M^{N,x,y}$ is a local martingale.

1.5. *Continuous model with a general interaction*

Given a space–time white noise $W(\mathrm{d}s, \mathrm{d}u)$, consider an $\mathbb{R}_+$-valued two-parameter stochastic process $\{Z_t^x, t \ge 0, x \ge 0\}$ which is such that for each fixed $x > 0$, $\{Z_t^x, t \ge 0\}$ is a continuous process, solution of the SDE (0.2). Now fix $0 < x < y$, and let $\{V_t^{x,y}, t \ge 0\}$ denote the solution of the SDE

$$V_t^{x,y} = y - x + \int_0^t \big[f\big(Z_s^x + V_s^{x,y}\big) - f\big(Z_s^x\big)\big]\,\mathrm{d}s + 2\int_0^t \int_{Z_s^x}^{Z_s^x + V_s^{x,y}} W(\mathrm{d}s, \mathrm{d}u). \tag{1.10}$$

The process $V^{x,y}$ is again nonnegative almost surely, from Theorem 2.2 in [3]. We note that

$$\int_0^t \int_0^{Z_s^x} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_{Z_s^x}^{Z_s^x + V_s^{x,y}} W(\mathrm{d}s, \mathrm{d}u) = \int_0^t \int_0^{Z_s^x + V_s^{x,y}} W(\mathrm{d}s, \mathrm{d}u) \quad \text{a.s.}$$

This implies that $Z^y = Z^x + V^{x,y}$ a.s. It follows that, for each $t \ge 0$, the process $\{Z_t^x, x \ge 0\}$ is almost surely non decreasing and for $0 \le x < y$, the conditional law of $Z_\cdot^y$, given $\{Z_t^{x'}, x' \le x, t \ge 0\}$ and $Z_t^x = z(t)$, $t \ge 0$, is the law of the sum of z plus the solution of (1.10) with Z_t^x replaced by $z(t)$. When $Z_\cdot^x$ is replaced by a deterministic trajectory z, the solution of (1.10) is independent of $\{Z_\cdot^{x'}, x' < x\}$. Hence the process $\{Z_\cdot^x, x \ge 0\}$ is a Markov process with values in $C([0,\infty), \mathbb{R}_+)$, the space of continuous functions from $[0,\infty)$ into $\mathbb{R}_+$, starting from 0 at $x = 0$. When f is a linear function, the increments of the mapping $x \to Z_t^x$ are independent, for each $t > 0$.

For $x \ge 0$, define T_0^x the extinction time of the process Z^x by:

$$T_0^x = \inf\{t > 0; Z_t^x = 0\}.$$

For any $x \ge 0$, we call the process Z^x (sub)critical (this strange notation means either critical or subcritical) if it becomes extinct almost surely in finite time i.e if T_0^x is finite almost surely. Hypothesis A implies that $\frac{f(x)}{x}$ is bounded from above. Let us introduce the notation

$$\Lambda(f) := \int_1^\infty \exp\left(-\frac{1}{2}\int_1^u \frac{f(r)}{r}\,\mathrm{d}r\right)\mathrm{d}u. \tag{1.11}$$

Proposition 1.3. *Suppose that f satisfies Hypothesis* A. *For any $x \ge 0$, Z^x is* (*sub*)*critical if and only if $\Lambda(f) = \infty$. In particular we have*:

(i) *A sufficient condition for* $\mathbb{P}(T_0^x < \infty) = 1$ *is*: *there exists* $z_0 \geq 1$ *such that* $f(z) \leq 2, \forall z \geq z_0$,
(ii) *A sufficient condition for* $\mathbb{P}(T_0^x = \infty) > 0$ *is*: *there exists* $z_0 > 1$ *and* $\delta > 0$ *such that* $f(z) \geq 2 + \delta, \forall\, z \geq z_0$.

Proof. The function

$$S(z) = \int_1^z \exp\left(-\frac{1}{2}\int_1^u \frac{f(r)}{r}\,\mathrm{d}r\right)\mathrm{d}u$$

is a scale function of the diffusion Z^x. Let us denote by T_y^x the random time at which Z^x hits y for the first time. We have for any $0 \leq a < x < b$

$$\mathbb{P}\big(T_a^x < T_b^x\big) = \frac{S(b) - S(x)}{S(b) - S(a)}, \quad \text{and} \quad \mathbb{P}\big(T_a^x < \infty\big) = \lim_{b \to \infty} \mathbb{P}\big(T_a^x < T_b^x\big).$$

If the function $S(z)$ tends to infinity as z goes to infinity, then $\mathbb{P}(T_a^x < \infty) = 1$. Otherwise $0 < \mathbb{P}(T_a^x < \infty) < 1$. From this we deduce that Z^x goes extinct almost surely in finite time if and only if $\lim_{z \to \infty} S(z) = \infty$, i.e. if and only if $\Lambda(f) = \infty$. The rest of the proposition is immediate. □

2. Convergence as $N \to \infty$

The aim of this section is to prove the convergence in law as $N \to \infty$ of the two-parameter process $\{Z_t^{N,x}, t \geq 0, x \geq 0\}$ defined in Section 1.4 towards the process $\{Z_t^x, t \geq 0, x \geq 0\}$ defined in Section 1.5. We need to make precise the topology for which this convergence will hold. We note that the process $Z_t^{N,x}$ (resp. Z_t^x) is a Markov process indexed by x, with values in the space of càdlàg (resp. continuous) functions of t $D([0,\infty); \mathbb{R}_+)$ (resp. $C([0,\infty); \mathbb{R}_+)$). So it will be natural to consider a topology of functions of x, with values in functions of t.

For each fixed x, the process $t \to Z_t^{N,x}$ is càdlàg, constant between its jumps, with jumps of size $\pm N^{-1}$, while the limit process $t \to Z_t^x$ is continuous. On the other hand, both $Z_t^{N,x}$ and Z_t^x are discontinuous as functions of x. The mapping $x \to Z_\cdot^x$ has countably many jumps on any compact interval, but the mapping $x \to \{Z_t^x, t \geq \epsilon\}$, where $\epsilon > 0$ is arbitrary, has finitely many jumps on any compact interval, and it is constant between its jumps. This fact is well-known in the case where f is linear. The general case follows via a coupling argument, see [10]. Recall that $D([0,\infty); \mathbb{R}_+)$ equipped with the distance d_∞^0 defined by (16.4) in [2] is separable and complete, see Theorem 16.3 in [2]. We have the following statement.

Theorem 2.1. *Suppose that the Hypothesis* A *is satisfied. Then as* $N \to \infty$,

$$\big\{Z_t^{N,x}, t \geq 0, x \geq 0\big\} \Rightarrow \big\{Z_t^x, t \geq 0, x \geq 0\big\}$$

in $D([0,\infty); D([0,\infty); \mathbb{R}_+))$, *equipped with the Skohorod topology of the space of càdlàg functions of* x, *with values in the Polish space* $D([0,\infty); \mathbb{R}_+)$ *equipped with the metric* d_∞^0.

To prove the theorem, we first show that for fixed $x \geq 0$ the sequence $\{Z^{N,x}, N \geq 0\}$ is tight in $D([0,\infty); \mathbb{R}_+)$.

2.1. *Tightness of* $Z^{N,x}$

To this end, we first establish a few lemmas.

Lemma 2.2. *For all* $T > 0$, $x \geq 0$, *there exist a constant* $C_0 > 0$ *such that for all* $N \geq 1$,

$$\sup_{0 \leq t \leq T} \mathbb{E}\big(Z_t^{N,x}\big) \leq C_0.$$

Moreover, for all $t \geq 0$, $N \geq 1$,

$$\mathbb{E}\left(-\int_0^t f\big(Z_r^{N,x}\big)\,\mathrm{d}r\right) \leq x.$$

Proof. Let $(\tau_n, n \geq 0)$ be a sequence of stopping times such that τ_n tends to infinity as n goes to infinity and for any n, $(M^{N,x}_{t\wedge\tau_n}, t \geq 0)$ is a martingale and $Z^{N,x}_{t\wedge\tau_n} \leq n$. Taking the expectation on both sides of equation (1.3) at time $t \wedge \tau_n$, we obtain

$$\mathbb{E}\big(Z^{N,x}_{t\wedge\tau_n}\big) = \frac{\lfloor Nx \rfloor}{N} + \mathbb{E}\bigg(\int_0^{t\wedge\tau_n} f\big(Z^{N,x}_r\big)\,\mathrm{d}r\bigg). \tag{2.1}$$

It follows from the Hypothesis A on f that

$$\mathbb{E}\big(Z^{N,x}_{t\wedge\tau_n}\big) \leq \frac{\lfloor Nx \rfloor}{N} + \beta\int_0^t \mathbb{E}\big(Z^{N,x}_{r\wedge\tau_n}\big)\,\mathrm{d}r.$$

From Gronwall and Fatou Lemmas, we deduce that there exists a constant $C_0 > 0$ which depends only upon x and T such that

$$\sup_{N\geq 1}\sup_{0\leq t\leq T} \mathbb{E}\big(Z^{N,x}_t\big) \leq C_0.$$

From (2.1), we deduce that

$$-\mathbb{E}\bigg(\int_0^{t\wedge\tau_n} f\big(Z^{N,x}_r\big)\,\mathrm{d}r\bigg) \leq \frac{\lfloor Nx \rfloor}{N}.$$

Since $-f(Z^{N,x}_r) \geq -\beta Z^{N,x}_r$, the second statement follows using Fatou's Lemma and the first statement. $\square$

We now have the following lemma.

Lemma 2.3. *For all $T > 0$, $x \geq 0$, there exists a constant $C_1 > 0$ such that*

$$\sup_{N\geq 1}\mathbb{E}\big(\langle M^{N,x}\rangle_T\big) \leq C_1.$$

Proof. For any $N \geq 1$ and $k, k' \in \mathbb{Z}_+$ such that $k \leq k'$, we set $z = \frac{k}{N}$ and $z' = \frac{k'}{N}$. By definition,

$$\begin{aligned}
\|f\|_{N,z,z'} &= \sum_{i=k+1}^{k'}\bigg\{\bigg(f\Big(\frac{i}{N}\Big) - f\Big(\frac{i-1}{N}\Big)\bigg)^+ + \bigg(f\Big(\frac{i}{N}\Big) - f\Big(\frac{i-1}{N}\Big)\bigg)^-\bigg\} \\
&= \sum_{i=k+1}^{k'}\bigg\{2\bigg(f\Big(\frac{i}{N}\Big) - f\Big(\frac{i-1}{N}\Big)\bigg)^+ - \bigg(f\Big(\frac{i}{N}\Big) - f\Big(\frac{i-1}{N}\Big)\bigg)\bigg\}.
\end{aligned}$$

Consequently we obtain from Hypothesis A on f that

$$\|f\|_{N,z,z'} \leq 2\beta(z'-z) + f(z) - f(z'). \tag{2.2}$$

We deduce from (2.2), (1.5) and Lemma 2.2 that

$$\begin{aligned}
\mathbb{E}\big(\langle M^{N,x}\rangle_T\big) &\leq \int_0^T \bigg\{\bigg(4 + \frac{2\beta}{N}\bigg)\mathbb{E}\big(Z^{N,x}_r\big) - \frac{1}{N}\mathbb{E}\big(f\big(Z^{N,x}_r\big)\big)\bigg\}\,\mathrm{d}r \\
&\leq \bigg(4 + \frac{2\beta}{N}\bigg)C_0 T + \frac{x}{N}.
\end{aligned}$$

Hence the lemma. $\square$

It follows from this that $M^{N,x}$ is in fact a square integrable martingale. We also have

Lemma 2.4. *For all $T>0$, $x\geq 0$, there exist two constants $C_2, C_3>0$ such that*:

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\big[\big(Z_t^{N,x}\big)^2\big]\leq C_2,$$

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\bigg(-\int_0^t Z_r^{N,x} f\big(Z_r^{N,x}\big)\,\mathrm{d}r\bigg)\leq C_3.$$

Proof. We deduce from (1.3) and Itô's formula that

$$\big(Z_t^{N,x}\big)^2=\bigg(\frac{\lfloor Nx\rfloor}{N}\bigg)^2+2\int_0^t Z_r^{N,x} f\big(Z_r^{N,x}\big)\,\mathrm{d}r+\big\langle M^{N,x}\big\rangle_t+M_t^{N,x,(2)}, \tag{2.3}$$

where $M^{N,x,(2)}$ is a local martingale. Let $(\sigma_n, n\geq 1)$ be a sequence of stopping times such that $\lim_{n\to\infty}\sigma_n=+\infty$ a.s. and for each $n\geq 1$, $(M_{t\wedge\sigma_n}^{N,x,(2)}, t\geq 0)$ is a martingale. Taking the expectation on the both sides of (2.3) at time $t\wedge\sigma_n$ and using Hypothesis A, Lemma 2.3, the Gronwall and Fatou Lemmas we obtain that for all $T>0$, there exists a constant $C_2>0$ such that:

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\big(Z_t^{N,x}\big)^2\,\mathrm{d}r\leq C_2.$$

We also have that

$$2\mathbb{E}\bigg(-\int_0^{t\wedge\sigma_n} Z_r^{N,x} f\big(Z_r^{N,x}\big)\,\mathrm{d}r\bigg)\leq\bigg(\frac{\lfloor Nx\rfloor}{N}\bigg)^2+C_1.$$

From Hypothesis A, we have $-Z_r^{N,x} f(Z_r^{N,x})\geq -\beta(Z_r^{N,x})^2$. The result now follows from Fatou's Lemma. □

We want to check tightness of the sequence $\{Z^{N,x}, N\geq 0\}$ using Aldous's criterion. Let $\{\tau_N, N\geq 1\}$ be a sequence of stopping times in $[0,T]$. We deduce from Lemma 2.4

Proposition 2.5. *For any $T>0$ and $\eta, \epsilon>0$, there exists $\delta>0$ such that*

$$\sup_{N\geq 1}\sup_{0\leq\theta\leq\delta}\mathbb{P}\bigg(\bigg|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f\big(Z_r^{N,x}\big)\,\mathrm{d}r\bigg|\geq\eta\bigg)\leq\epsilon.$$

Proof. Let c be a non negative constant. Provided $0\leq\theta\leq\delta$, we have

$$\bigg|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f\big(Z_r^{N,x}\big)\,\mathrm{d}r\bigg|\leq\sup_{0\leq r\leq c}\big|f(r)\big|\delta+\int_{\tau_N}^{(\tau_N+\theta)\wedge T}\mathbf{1}_{\{Z_r^{N,x}>c\}}\big|f\big(Z_r^{N,x}\big)\big|\,\mathrm{d}r.$$

But

$$\begin{aligned}\int_{\tau_N}^{(\tau_N+\theta)\wedge T}\mathbf{1}_{\{Z_r^{N,x}>c\}}\big|f\big(Z_r^{N,x}\big)\big|\,\mathrm{d}r&\leq c^{-1}\int_0^T Z_r^{N,x}\big(f^+\big(Z_r^{N,x}\big)+f^-\big(Z_r^{N,x}\big)\big)\,\mathrm{d}r\\&\leq c^{-1}\int_0^T\big(2Z_r^{N,x} f^+\big(Z_r^{N,x}\big)-Z_r^{N,x} f\big(Z_r^{N,x}\big)\big)\,\mathrm{d}r\\&\leq c^{-1}\int_0^T\big(2\beta\big(Z_r^{N,x}\big)^2-Z_r^{N,x} f\big(Z_r^{N,x}\big)\big)\,\mathrm{d}r.\end{aligned}$$

From this and Lemma 2.4, we deduce that $\forall\ N \ge 1$, again with $\theta \le \delta$,

$$\sup_{0\le\theta\le\delta} \mathbb{P}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f\left(Z_r^{N,x}\right)\mathrm{d}r\right| \ge \eta\right) \le \eta^{-1}\mathbb{E}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f\left(Z_r^{N,x}\right)\mathrm{d}r\right|\right)$$
$$\le \sup_{0\le r\le c} \frac{|f(r)|\delta}{\eta} + \frac{A}{c\eta},$$

with $A = 2\beta C_2 T + C_3$. The result follows by choosing $c = 2A/\epsilon\eta$, and then $\delta = \epsilon\eta/2\sup_{0\le r\le c}|f(z)|$. □

From Proposition 2.5, the Lebesgue integral term in the right hand side of (1.3) satisfies Aldous's condition $[A]$, see [1]. The same proposition, Lemma 2.2, (1.5) and (2.2) imply that $\langle M^{N,x}\rangle$ satisfies the same condition, hence so does $M^{N,x}$, according to Rebolledo's Theorem, see [4]. Since all jumps are of size $\frac{1}{N}$, tightness follows. We have proved

Proposition 2.6. *For any fixed $x \ge 0$, the sequence of processes $\{Z^{N,x}, N \ge 1\}$ is tight in $D([0,\infty);\mathbb{R}_+)$.*

We deduce from Proposition 2.6 the following corollary.

Corollary 2.7. *For any $0 \le x < y$ the sequence of processes $\{V^{N,x,y}, N \ge 1\}$ is tight in $D([0,\infty);\mathbb{R}_+)$.*

Proof. For any x fixed the process $Z^{N,x}$ has jumps equal to $\pm\frac{1}{N}$ which tends to zero as $N \to \infty$. It follows from that and equation (1.3) that any weak limit of a converging subsequence of $Z^{N,x}$ is continuous and is the unique weak solution of equation (0.2). We deduce that for any $x, y \ge 0$, the sequence $\{Z^{N,y} - Z^{N,x}, N \ge 1\}$ is tight since $\{Z^{N,x}, N \ge 1\}$ and$\{Z^{N,y}, N \ge 1\}$ are tight and both have a continuous limit as $N \to \infty$. □

2.2. *Proof of Theorem* 2.1

From Theorem 13.5 in [2], Theorem 2.1 follows from the two next propositions.

Proposition 2.8. *For any $n \in \mathbb{N}$, $0 \le x_1 < x_2 < \cdots < x_n$,*

$$\left(Z^{N,x_1}, Z^{N,x_2}, \ldots, Z^{N,x_n}\right) \Rightarrow \left(Z^{x_1}, Z^{x_2}, \ldots, Z^{x_n}\right)$$

as $N \to \infty$, for the topology of locally uniform convergence in t.

Proof. We prove the statement in the case $n = 2$ only. The general statement can be proved in a very similar way. For $0 \le x_1 < x_2$, we consider the process $(Z^{N,x_1}, V^{N,x_1,x_2})$, using the notations from Section 1. The argument preceding the statement of Proposition 2.6 implies that the sequences of martingales M^{N,x_1} and M^{N,x_1,x_2} are tight. Hence $(Z^{N,x_1}, V^{N,x_1,x_2}, M^{N,x_1}, M^{N,x_1,x_2})$ is tight. Thanks to (1.3), (1.5), (1.7), (1.8) and (1.9), any converging subsequence of $\{Z^{N,x_1}, V^{N,x_1,x_2}, M^{N,x_1}, M^{N,x_1,x_2}, N \ge 1\}$ has a weak limit $(Z^{x_1}, V^{x_1,x_2}, M^{x_1}, M^{x_1,x_2})$ which satisfies

$$Z_t^{x_1} = x_1 + \int_0^t f\left(Z_s^{x_1}\right)\mathrm{d}s + M_t^{x_1},$$
$$V_t^{x_1,x_2} = x_2 - x_1 + \int_0^t \left[f\left(Z_s^{x_1} + V_s^{x_1,x_2}\right) - f\left(Z_s^{x_1}\right)\right]\mathrm{d}s + M_t^{x_1,x_2},$$

where the continuous martingales M^{x_1} and M^{x_1,x_2} satisfy

$$\left\langle M^{x}\right\rangle_t = 4\int_0^t Z_s^{x_1}\,\mathrm{d}s, \qquad \left\langle M^{x_1,x_2}\right\rangle_t = 4\int_0^t V_s^{x_1,x_2}\,\mathrm{d}s, \qquad \left\langle M^{x_1}, M^{x_1,x_2}\right\rangle_t = 0.$$

This implies that the pair (Z^{x_1}, V^{x_1,x_2}) is a weak solution of the system of SDEs (0.2) and (1.10), driven by the same space–time white noise. The result follows from the uniqueness of the system, see again Theorem 2.1 in [3] and recall from the Introduction that Hypothesis A implies that the hypotheses of that theorem are satisfied. □

Proposition 2.9. *There exists a constant C, which depends only upon θ and T, such that for any $0 \le x < y < z$, which are such that $y - x \le 1$, $z - y \le 1$,*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|Z_t^{N,y} - Z_t^{N,x}\big|^2 \times \sup_{0\le t\le T}\big|Z_t^{N,z} - Z_t^{N,y}\big|^2\Big] \le C|z-x|^2.$$

We first prove the

Lemma 2.10. *For any $0 \le x < y$, we have*

$$\sup_{0\le t\le T}\mathbb{E}\big(Z_t^{N,y} - Z_t^{N,x}\big) = \sup_{0\le t\le T}\mathbb{E}\big(V_t^{N,x,y}\big) \le \left(\frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right)\mathrm{e}^{\beta T}.$$

Proof. Let $(\sigma_n, n \ge 0)$ be a sequence of stopping times such that $\lim_{n\to\infty}\sigma_n = +\infty$ and for each $n \ge 1$, $(M_{t\wedge\sigma_n}^{N,x,y}, t \ge 0)$ is a martingale. Taking the expectation on both sides of (1.7) at time $t \wedge \sigma_n$ we obtain that

$$\mathbb{E}\big(V_{t\wedge\sigma_n}^{N,x,y}\big) \le \left(\frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right) + \beta\int_0^t \mathbb{E}\big(V_{r\wedge\sigma_n}^{N,x,y}\big)\,\mathrm{d}r. \tag{2.4}$$

Using Gronwall's and Fatou's Lemmas, we obtain that

$$\sup_{0\le t\le T}\mathbb{E}\big(V_t^{N,x,y}\big) \le \left(\frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right)\mathrm{e}^{\beta T}.$$

□

Proof of Proposition 2.9. Using equation (1.7), a stopping time argument as above, Lemma 2.10 and Fatou's Lemma, where we take advantage of the inequality $f(Z_r^{N,x}) - f(Z_r^{N,x} + V_r^{N,x,y}) \ge -\beta V_r^{N,x,y}$, we deduce that

$$\mathbb{E}\left(\int_0^t \big[f\big(Z_r^{N,x}\big) - f\big(Z_r^{N,x} + V_r^{N,x,y}\big)\big]\,\mathrm{d}r\right) \le \frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}. \tag{2.5}$$

We now deduce from (1.8), Lemma 2.10, inequalities (2.5) and (2.2) that for each $t > 0$, there exists a constant $C(t) > 0$ such that

$$\mathbb{E}\big(\langle M^{N,x,y}\rangle_t\big) \le C(t)\left(\frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right). \tag{2.6}$$

This implies that $M^{N,x,y}$ is in fact a square integrable martingale. For any $0 \le x < y < z$, we have $Z_t^{N,z} - Z_t^{N,y} = V_t^{N,y,z}$ and $Z_t^{N,y} - Z_t^{N,x} = V_t^{N,x,y}$ for any $t \ge 0$. On the other hand we deduce from (1.7) and Hypothesis A that

$$\begin{aligned}\sup_{0\le t\le T}\big(V_t^{N,x,y}\big)^2 &\le 3\left(\frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right)^2 + 3\beta^2 T\int_0^T \sup_{0\le s\le r}\big(V_s^{N,x,y}\big)^2\,\mathrm{d}r \\ &\quad + 3\sup_{0\le t\le T}\big(M_t^{N,x,y}\big)^2\end{aligned}$$

and

$$\begin{aligned}\sup_{0\le t\le T}\big(V_t^{N,y,z}\big)^2 &\le 3\left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}\right)^2 + 3\beta^2 T\int_0^t \sup_{0\le s\le r}\big(V_s^{N,y,z}\big)^2\,\mathrm{d}r \\ &\quad + 3\sup_{0\le t\le T}\big(M_t^{N,y,z}\big)^2.\end{aligned}$$

Now let $\mathcal{G}^{x,y} := \sigma(Z_t^{N,x}, Z_t^{N,y}, t \geq 0)$ be the filtration generated by $Z^{N,x}$ and $Z^{N,y}$. It is clear that for any t, $V_t^{N,x,y}$ is measurable with respect to $\mathcal{G}^{x,y}$. We then have

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|V_t^{N,x,y}\big|^2 \times \sup_{0\le t\le T}\big|V_t^{N,y,z}\big|^2\Big] = \mathbb{E}\Big[\sup_{0\le t\le T}\big|V_t^{N,x,y}\big|^2 \mathbb{E}\Big(\sup_{0\le t\le T}\big|V_t^{N,y,z}\big|^2\big|\mathcal{G}^{x,y}\Big)\Big].$$

Conditionally upon $Z^{N,x}$ and $Z^{N,y} = u(\cdot)$, $V^{N,y,z}$ solves the following SDE

$$V_t^{N,y,z} = \frac{\lfloor Nz\rfloor - \lfloor Ny\rfloor}{N} + \int_0^t \big[f\big(V_r^{N,y,z} + u(r)\big) - f\big(u(r)\big)\big]\,\mathrm{d}r + M_t^{N,y,z},$$

where $M^{N,y,z}$ is a martingale conditionally upon $\mathcal{G}^{x,y}$, hence the arguments used in Lemma 2.10 lead to

$$\sup_{0\le t\le T} \mathbb{E}\big(V_t^{N,y,z}|\mathcal{G}^{x,y}\big) \le \left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}\right)\mathrm{e}^{\beta T},$$

and those used to prove (2.5) yield

$$\mathbb{E}\left(\int_0^t f\big(Z_r^{N,y}\big) - f\big(Z_r^{N,y} + V_r^{N,y,z}\big)\,\mathrm{d}r\big|\mathcal{G}^{x,y}\right) \le \frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}.$$

From this we deduce (see the proof of (2.6)) that

$$\mathbb{E}\big(\langle M^{N,y,z}\rangle_t|\mathcal{G}^{x,y}\big) \le C(t)\left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}\right).$$

From Doob's inequality we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}\big|M_t^{N,y,z}\big|^2\big|\mathcal{G}^{x,y}\Big) &\le 4\mathbb{E}\big(\langle M^{N,y,z}\rangle_T|\mathcal{G}^{x,y}\big)\\ &\le C(T)\left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}\right).\end{aligned}$$

Since $0 < z - y < 1$, we deduce that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}\big|V_t^{N,y,z}\big|^2\big|\mathcal{G}^{x,y}\Big) &\le 3\big(1 + C(T)\big)\left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}\right)\\ &\quad + 3\beta^2 T\int_0^T \mathbb{E}\Big(\sup_{0\le s\le r}\big(V_s^{N,y,z}\big)^2\big|\mathcal{G}^{x,y}\Big)\,\mathrm{d}r.\end{aligned}$$

From this and Gronwall's Lemma we deduce that there exists a constant $K_1 > 0$ such that

$$\mathbb{E}\Big(\sup_{0\le t\le T}\big|V_t^{N,y,z}\big|^2\big|\mathcal{G}^{x,y}\Big) \le K_1\left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Ny\rfloor}{N}\right). \tag{2.7}$$

Similary we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big(V_s^{N,x,y}\big)^2\Big] \le K_1\left(\frac{\lfloor Ny\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right).$$

Since $0 \le y - x < z - x$ and $0 \le z - y < z - x$, we deduce that

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|V_t^{N,x,y}\big|^2 \times \sup_{0\le t\le T}\big|V_t^{N,y,z}\big|^2\Big] \le K_1^2\left(\frac{\lfloor Nz\rfloor}{N} - \frac{\lfloor Nx\rfloor}{N}\right)^2,$$

hence the result. $\square$

Proof of Theorem 2.1. We now show that for any $T > 0$,

$$\{Z_t^{N,x}, 0 \le t \le T, x \ge 0\} \Rightarrow \{Z_t^x, 0 \le t \le T, x \ge 0\}$$

in $D([0,\infty); D([0,T],\mathbb{R}_+))$. From Theorems 13.1 and 16.8 in [2], since from Proposition 2.8, for all $n \ge 1$, $0 < x_1 < \cdots < x_n$,

$$\left(Z_\cdot^{N,x_1}, \dots, Z_\cdot^{N,x_n}\right) \Rightarrow \left(Z_\cdot^{x_1}, \dots, Z_\cdot^{x_n}\right)$$

in $D([0,T];\mathbb{R}^n)$, it suffices to show that for all $\bar{x} > 0$, $\epsilon, \eta > 0$, there exists $N_0 \ge 1$ and $\delta > 0$ such that for all $N \ge N_0$,

$$\mathbb{P}\left(w_{\bar{x},\delta}\left(Z^N\right) \ge \epsilon\right) \le \eta, \tag{2.8}$$

where for a function $(x,t) \to z(x,t)$

$$w_{\bar{x},\delta}(z) = \sup_{0\le x_1\le x\le x_2\le\bar{x}, x_2-x_1\le\delta} \inf\left\{\left\|z(x,\cdot) - z(x_1,\cdot)\right\|, \left\|z(x_2,\cdot) - z(x,\cdot)\right\|\right\},$$

with the notation $\|z(x,\cdot)\| = \sup_{0\le t\le T}|z(x,t)|$. But from the proof of Theorem 13.5 in [2], (2.8) for Z^N follows from Proposition 2.9. $\square$

3. Ray–Knight representation of a general Feller diffusion

In this section we establish a Ray–Knight representation of Feller's branching diffusion solution of (0.2), in terms of the local time of a reflected Brownian motion H with a drift that depends upon the local time accumulated by H at its current level, through the function f' where f is a function which is from now on assumed to satisfy the following hypothesis.

Hypothesis B. $f \in C^1(\mathbb{R}_+;\mathbb{R}_+)$, $f(0) = 0$ *and there exist a constant* $\beta > 0$ *such that*

$$f'(x) \le \beta, \quad \textit{for all } x \ge 0.$$

Note that Hypothesis B follows from Hypothesis A if we assume that f is continuously differentiable.

Our proof relies upon the result in [9], which itself is inspired by previous work of Norris, Rogers and Williams [8]. Our process H solves the SDE

$$H_s = B_s + \frac{1}{2}\int_0^s f'\left(L_r(H_r)\right)\mathrm{d}r + \frac{1}{2}L_s(0).$$

One way to understand the form of the drift is to see (H_s) as the limit of the contour process H^N of the forest of random trees associated to $Z^{N,x}$. The drift of the process H^N appears in (1.1) + (1.2).

Since we want to identify the law of $\{Z_t^x, t \ge 0, x \ge 0\}$, we will need to identify the law of $\{V_t^{x,y}, t \ge 0\}$ for all $x, y > 0$. This will force us to consider a more general SDE for the process H. Fix $z \in C(\mathbb{R}_+;\mathbb{R}_+)$, the set of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ and consider the stochastic differential equation

$$H_s = B_s + \frac{1}{2}\int_0^s f'\left(z(H_r) + L_r(H_r)\right)\mathrm{d}r + \frac{1}{2}L_s(0), \tag{3.1}$$

where B is a standard Brownian motion, and for $s, t \ge 0$ $L_s(t)$ is the local time accumulated by the solution H at level t up to time s. For $x > 0$ define

$$S_x = \inf\left\{r > 0\colon\ L_s(0) > x\right\} \quad \text{and} \quad S = \sup_{x>0} S_x.$$

Going back one moment to the case $z \equiv 0$, recall from Theorem 0.1 that we want to identify the law of $\{Z_t^x, t \ge 0, x \ge 0\}$ with that of $\{L_{S_x}(t), t \ge 0, x \ge 0\}$. For that sake, we need that $S_x < \infty$ a.s. for all $x > 0$. This is not obvious from the form of the SDE for H, since in particular we do not impose any condition on the sign of f'. Once we have the Ray–Knight representation, it will be a consequence of the (sub)criticality of Z^x. Hence we shall first prove the Ray–Knight representation for H reflected in the interval $[0, K]$, $K > 0$ arbitrary.

This motivates the study of the SDE

$$H_s^K = B_s + \frac{1}{2}\int_0^s f'\big(z\big(H_r^K\big) + L_r^K\big(H_r^K\big)\big)\,\mathrm{d}r - \frac{1}{2}L_s^K\big(K^-\big) + \frac{1}{2}L_s^K(0). \tag{3.2}$$

Note that the reason for evaluating the local time at K^- instead of K is the fact that we assume as usual that the local time is right-continuous as a function of the level, so that in particular $L_s^{K,z}(K) = 0$. This explains the disymetry between the notations at 0 and at K. We define

$$S_x^K = \inf\big\{s > 0, L_s^K(0) > x\big\} \quad \text{and} \quad S^K = \sup_{x>0} S_x^K.$$

We also introduce the process $Z^{x,z}$ which is a solution, for fixed $x > 0$, of the SDE

$$Z_t^{x,z} = x + \int_0^t \big[f\big(z(r) + Z_r^{x,z}\big) - f\big(z(r)\big)\big]\,\mathrm{d}r + 2\int_0^r \sqrt{Z_r^{x,z}}\,\mathrm{d}W_r. \tag{3.3}$$

3.1. *Equation* (3.2) *and a first Ray–Knight representation*

We recall that f satisfies Hypothesis B, as in all of this section. We have

Proposition 3.1. *For any $K > 0$, $z \in C(\mathbb{R}_+; \mathbb{R}_+)$, equation* (3.2) *has a unique weak solution on the random interval $[0, S^K)$, which is such for each $x > 0$, that the law of $\{L_{S_x^K}^{K,z}(t), 0 \le t < K\}$ coincides with that of $\{Z_t^{x,z}, 0 \le t < K\}$, where $Z^{x,z}$ is the solution of* (3.3).

Proof. 1. *Weak existence and uniqueness of* (3.2). Consider the Brownian motion H^K reflected in the interval $[0, K]$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$

$$H_s^K = B_s + \frac{1}{2}L_s^K(0) - \frac{1}{2}L_s^K\big(K^-\big).$$

Let

$$M_s^K = \frac{1}{2}\int_0^s f'\big(z\big(H_r^K\big) + L_r^K\big(H_r^K\big)\big)\,\mathrm{d}B_r,$$

$$\big\langle M^K\big\rangle_s = \frac{1}{4}\int_0^s \big[f'\big]^2\big(z\big(H_r^K\big) + L_r^K\big(H_r^K\big)\big)\,\mathrm{d}r,$$

$$G_s^K = \exp\left(M_s^K - \frac{1}{2}\big\langle M^K\big\rangle_s\right).$$

Let $N = \sup_{0\le t\le K} z(t)$. For each $n > N$, we define the stopping time

$$T_n = \inf\Big\{s > 0; \sup_{0\le t<K} L_s^K(t) > n - N\Big\},$$

so that on $[0, T_n]$, $z(t) + L_s^K(t) \le n$, for all $0 \le t \le K$. We can define a probability measure $\tilde{\mathbb{P}}^{K,z}$ on $\bigcup_n \mathcal{F}_{T_n}$, such that on each $\mathcal{F}_{T_n}$, $\tilde{\mathbb{P}}^{K,z} \ll \mathbb{P}$ and

$$\frac{\mathrm{d}\tilde{\mathbb{P}}^{K,z}|_{\mathcal{F}_{T_n}}}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_{T_n}}} = G_{T_n}^K.$$

Consider the process

$$\tilde{B}_s^K = B_s - \frac{1}{2}\int_0^s f'\big(z\big(H_r^K\big) + L_r^K\big(H_r^K\big)\big)\,\mathrm{d}r.$$

From Girsanov's Theorem, for each $n > N$, $\{\tilde{B}_s^K, 0 \le s \le T_n\}$ is a $\tilde{\mathbb{P}}^{K,z}$ Brownian motion. We have thus constructed a solution of (3.2) on $\bigcup_n [0, T_n]$, which by the same argument is clearly unique. Assume for a moment

Lemma 3.2. *For any $x > 0$, $\mathbb{P}(T_n < S_x^K) \to 0$, and $\tilde{\mathbb{P}}^{K,z}(T_n < S_k^K) \to 0$, as $n \to \infty$.*

Let now $A \in \mathcal{F}_{S_x^K}$ be arbitrary. Since $A \cap \{S_x^K \le T_n\} \in \mathcal{F}_{T_n} \cap \mathcal{F}_{T_n \wedge S_x^K}$,

$$\begin{aligned}\tilde{\mathbb{P}}^{K,z}\big(A \cap \big\{S_x^K \le T_n\big\}\big) &= \int_{A\cap\{S_x^K \le T_n\}} G_{T_n \wedge S_x^K}^K \,\mathrm{d}\mathbb{P} \\ &= \int_{A\cap\{S_x^K \le T_n\}} G_{S_x^K}\,\mathrm{d}\mathbb{P}.\end{aligned}$$

Taking the limit in this identity as $n \to \infty$, we deduce with the help of Lemma 3.2

$$\tilde{\mathbb{P}}^{K,z}(A) = \int_A G_{S_x^K}^K \,\mathrm{d}\mathbb{P}.$$

Consequently $\tilde{\mathbb{P}}^{K,z} \ll \mathbb{P}$ on $\mathcal{F}_{S_x^K}$, and

$$\frac{\mathrm{d}\tilde{\mathbb{P}}^{K,z}|_{\mathcal{F}_{S_x^K}}}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_{S_x^K}}} = G_{S_x^K}^K.$$

We have that $\tilde{B}^K$ is a $\tilde{\mathbb{P}}^{K,z}$-Brownian motion on $[0, S_x^K]$ for all $x > 0$, hence on $[0, S^K)$, and

$$H_s^K = \tilde{B}_s^K + \frac{1}{2}\int_0^s f'\big(z\big(H_r^K\big) + L_r^K\big(H_r^K\big)\big)\,\mathrm{d}r - \frac{1}{2}L_s^K\big(K^-\big) + \frac{1}{2}L_s^K(0), \quad 0 \le s < S^K.$$

2. *Step* 1 *of the proof of the Ray–Knight representation*. For each $n > N$, let $f_n \in C^1(\mathbb{R}_+; \mathbb{R})$ be such that

(i) $f_n(u) = f(u)$, for all $0 \le u \le n$.
(ii) f_n' is uniformly bounded.

Let $H^{K,n}$ and $Z^{x,z,n}$ denote respectively the solutions of

$$H_s^{K,n} = B_s + \frac{1}{2}\int_0^s f_n'\big(z\big(H_r^{K,n}\big) + L_r^{K,n}\big(H_r^{K,n}\big)\big)\,\mathrm{d}r - \frac{1}{2}L_s^{K,n}\big(K^-\big) + \frac{1}{2}L_s^{K,n}(0), \tag{3.4}$$

and

$$Z_t^{x,z,n} = x + \int_0^t \big[f_n\big(z(r) + Z_r^{x,z,n}\big) - f_n\big(z(r)\big)\big]\,\mathrm{d}r + 2\int_0^r \sqrt{Z_r^{x,z,n}}\,\mathrm{d}W_r. \tag{3.5}$$

We can apply Proposition 4 from [9] (strictly speaking, this result is proved in case f_n is a quadratic function, and we detail in Section A.1 below the minor modifications which have to be done in our case), which says that for each fixed z, K, x and n, the laws of $\{L_{S_x^{K,n}}^{K,n}(t), 0 \le t < K\}$ and $\{Z_t^{x,z,n}, 0 \le t < K\}$ coincide. But this means that the laws of $\{L_{S_x^K}^K(t), 0 \le t < K\}$ and $\{Z_t^{x,z}, 0 \le t < K\}$ coincide on the event $\{S_x^K \le T_n\}$.

3. *Proof of Lemma* 3.2. It is plain that $\{T_n < S_x^K\} = \{\sup_{0\le t\le K} L^K_{S_x^K}(t) > n - N\}$. From the second Ray–Knight Theorem (see e.g. Theorem XI.2.3 in [11]) under the probability $\mathbb{P}$, the process $\{L^K_{S_x^K}(t), 0 \le t < K\}$ solves the Feller SDE $Y_t = x + 2\int_0^t \sqrt{Y_r}\, \mathrm{d}W_r$, $0 \le t < K$. But it is clear that

$$\mathbb{P}\Big(\sup_{0\le t\le K} Y_t > n - N\Big) \to 0, \quad \text{as } n \to \infty,$$

hence the first result. Now from the last statement of the previous step of the proof,

$$\begin{aligned}\tilde{\mathbb{P}}^{K,z}\Big(\sup_{0\le t\le K} L^K_{S_x^K}(t) > n - N\Big) &= \mathbb{P}\Big(\sup_{0\le t\le K} Z_t^{x,z} > n - N\Big)\\ &\to 0, \quad \text{as } n \to \infty,\end{aligned}$$

hence the second result.

4. *Step* 2 *of the proof of the Ray–Knight representation*. Let A denote a Borel subset of $C([0, K]; \mathbb{R}_+)$. Combining the last two steps of the proof, we deduce

$$\begin{aligned}\tilde{\mathbb{P}}^{K,z}\big(L^K_{S_x^K} \in A\big) &= \lim_{n\to\infty} \tilde{\mathbb{P}}^{K,z}\big(L^K_{S_x^K} \in A \cap \big\{S_x^K \le T_n\big\}\big)\\ &= \lim_{n\to\infty} \tilde{\mathbb{P}}^{K,z}\big(Z^{x,z} \in A \cap \big\{S_x^K \le T_n\big\}\big)\\ &= \tilde{\mathbb{P}}^{K,z}\big(Z^{x,z} \in A\big).\end{aligned}$$

□

3.2. *Ray–Knight Theorem in the* (*sub*)*critical case*

We first prove the following proposition (recall the definition (1.11) of $\Lambda(f)$).

Proposition 3.3. *Suppose that f satisfies Hypothesis* B *and $\Lambda(f) = \infty$, and that $z \in C(\mathbb{R}_+; \mathbb{R}_+)$ has compact support*. *Then equation* (3.1) *admits a unique weak solution on* $[0, S)$, *and $S_x < \infty$ a.s. for all $x > 0$.*

Proof. For $x > 0$ and $K > 0$, let

$$\Omega^{K,x} = \Big\{\sup_{s\in[0,S_x^{K'}]} H_s^{K'} \le K, \forall K' > K\Big\}.$$

For any $x \ge 0$ and $z \in C(\mathbb{R}_+; \mathbb{R}_+)$ with compact support, since we are in the (sub)critical case, there exists $T_{x,z} < \infty$ a.s. such that $Z_t^{x,z} = 0$, $\forall\, t \ge T_{x,z}$. We note here that the compact support property of z clearly implies that $\Lambda(f) = \infty$ is a sufficient condition for $Z^{x,z}$ to be (sub)critical. We deduce from this and Proposition 3.1 that for any $x > 0$,

$$\tilde{\mathbb{P}}^{K,z}\big(\big(\Omega^{K,x}\big)^c\big) = \mathbb{P}(T_{x,z} > K) \to 0,$$

as $K \to \infty$. We now define H_s, S_x and S as follows. For $\omega \notin \bigcap_{x>0}\bigcup_K \Omega^{K,x}$, we let $H_s(\omega) \equiv 0$, $S_x(\omega) = S(\omega) = +\infty$. For $\omega \in \bigcap_{x>0}\bigcup_K \Omega^{K,x}$, and each $x > 0$, we choose an arbitrary $K(\omega, x) > 0$ such that $\omega \in \Omega^{K(\omega,x),x}$ and define unambiguously $H_s(\omega) = H_s^{K(\omega,x)}(\omega)$, for any $0 \le s \le S_x(\omega) = S_x^{K(\omega,x)}(\omega)$. This being done for all $x > 0$, we define $S(\omega) = \lim_{x\to\infty} S_x(\omega)$, and H_s is defined for all $0 \le s < S$. We now define $\tilde{\mathbb{P}}^z : \bigcup_{x>0} \mathcal{F}_{S_x} \to [0, 1]$ as follows. For any $x > 0$, $A \in \mathcal{F}_{S_x}$,

$$\tilde{\mathbb{P}}^z(A) = \lim_{K\to\infty} \tilde{\mathbb{P}}^{K,z}\big(A \cap \Omega^{K,x}\big).$$

We note that $\tilde{\mathbb{P}}^z((\bigcup_{K>0} \Omega^{K,x})^c) = 0$ and if $A \subset \Omega^{K,x}$, then $\tilde{\mathbb{P}}^z(A) = \tilde{\mathbb{P}}^{K,z}(A)$. This defines a unique additive functional on $\bigcup_{x>0} \mathcal{F}_{S_x}$, which is a probability measure on each $\mathcal{F}_{S_x}$. Indeed, finite additivity is plain, and whenever $A_n \downarrow \varnothing$, $A_n \in \mathcal{F}_{S_x}$ for all n,

$$\tilde{\mathbb{P}}^z(A_n) \le \tilde{\mathbb{P}}^{K,z}\big(A_n \cap \Omega^{K,z}\big) + \mathbb{P}(T_{x,z} > K).$$

and the second term of the right-hand side can be made arbitrarily small by a proper choice of K, while for fixed $K > 0$, the first term tends to 0 as $n \to \infty$, hence as well as $\tilde{\mathbb{P}}^z(A_n)$. Note also that $S_x < \infty$ $\tilde{\mathbb{P}}^z$ a.s., as a consequence of (sub)criticality.

Now under $\tilde{\mathbb{P}}^z$, the process

$$\tilde{B}_s^z = H_s - \frac{1}{2}\int_0^s f'\big(z(r) + L_r(H_r)\big)\,\mathrm{d}r - \frac{1}{2}L_s(0)$$

is a standard Brownian motion on the random interval $[0, S)$. This proves that (3.1) has a weak solution, whose uniqueness follows from the uniqueness result in Proposition 3.1. □

From now on, z is always assumed to have compact support. We will next show that one can extend $\tilde{\mathbb{P}}^z$ as a probability measure on $\mathcal{F}_{S^K} = \sigma(\bigcup_x \mathcal{F}_{S_x^K})$. Let us first establish

Proposition 3.4. *On the set* $\{S < \infty\}$, $H_s \to 0$ $\tilde{\mathbb{P}}^z$ *a.s., as* $s \uparrow S$.

Proof. Let $S_0 = 0$, and for each $n \geq 1$, S_n is defined as above. We claim that $0 \leq H_s \leq X_s$ for $0 \leq s < S$, where for any $n \geq 0$, $S_n \leq s < S_{n+1}$,

$$X_s := \frac{\beta}{2}(s - S_n) + \tilde{B}_s^z - \tilde{B}_{S_n}^z + \frac{1}{2}\big[L_s^X(0) - L_{S_n}^X(0)\big].$$

Since $H_{S_n} = 0$ and $f' \leq \beta$, this fact follows from Proposition A.1 below used on each interval $[S_n, S_{n+1})$. It remains to show that on the event $\{S = \lim_n S_n < \infty\}$, $X_s \to 0$, as $s \uparrow S$.

Let $Y_s := \theta s/2 + \tilde{B}_s^z$. It is plain that for $S_n \leq s < S_{n+1}$,

$$\begin{aligned} 0 \leq X_s &= Y_s - Y_{S_n} - \inf_{S_n \leq r \leq s}(Y_r - Y_{S_n}) \\ &\leq 2 \sup_{S_n \leq r < S_{n+1}} |Y_r - Y_{S_n}|. \end{aligned}$$

Consequently

$$\sup_{S_n \leq s < S} |X_s| \leq 2 \sup_{S_n \leq r < s < S} |Y_s - Y_r|,$$

and the right hand side of the last inequality tends to 0 a.s. as $n \to \infty$ on the event $\{S < \infty\}$, from the continuity of Y. □

Without loss of generality, we may and do assume from now on that the measure $\tilde{\mathbb{P}}^z$ has been constructed on the space $\Omega = C([0, +\infty))$, equipped with its Borel field, and the usual filtration. We can now prove

Proposition 3.5. *There exists a probability measure* $\tilde{\mathbb{P}}^z$ *on* $\mathcal{F}_S = \sigma(\bigcup_{x>0}\mathcal{F}_{S_x})$, *which coincides with the probability which we have constructed in Proposition* 3.3 *on each* $\mathcal{F}_{S_x}$.

Proof. In the case $S = +\infty$ $\tilde{\mathbb{P}}^z$ a.s. i.e. $\tilde{\mathbb{P}}^z(S_x < s) \to 0$ as $x \to 0$, the result is a direct consequence of Theorem 1.3.5 from Stroock–Varadhan [12], with $\tau_n = S_n$. The general case can be treated following the proof of Theorem 1.1.9 from [12], thanks to Proposition 3.4, which implies that condition (1.1.9) from [12] is satisfied in this case. A complete proof is given in Section A.2 below. □

In particular, in the case $\tilde{\mathbb{P}}^z(S < +\infty) = 1$, the SDE (3.2) has a unique weak solution on the closed interval $[0, S]$. The reader may be interested by the following result from [6].

Proposition 3.6. *Assume that* $g(x) = f(x)/x$ *satisfies Hypothesis* A *on* $[1, +\infty)$, *and that* $f(x) \neq 0$ *on* $[K, +\infty)$ *for some* $K > 0$. *Then*

1. *if* $\int_K^\infty \frac{x}{|f(x)|}\,\mathrm{d}x = \infty$, $S = \infty$ $\tilde{\mathbb{P}}^z$ *a.s.*;
2. *if* $\int_K^\infty \frac{x}{|f(x)|}\,\mathrm{d}x < \infty$, $S < \infty$ $\tilde{\mathbb{P}}^z$ *a.s.*

For $z \equiv 0$, we write $\tilde{\mathbb{P}}$ for $\tilde{\mathbb{P}}^0$. Let us reformulate our main result, Theorem 0.1, which we will now prove.

Theorem 3.7. *Suppose that* f *satisfies Hypothesis* B *and* $\Lambda(f) = \infty$. *Then the law of the random field* $\{L_{S_x}(t), t \geq 0, x \geq 0\}$ *under the probability* $\tilde{\mathbb{P}}$ *is the same as the law of* $\{Z_t^x, t \geq 0, x \geq 0\}$.

We first establish the following proposition.

Proposition 3.8. *Assume that the two assumptions of Theorem* 3.7 *hold. Then for any* x *and* $z \in C(\mathbb{R}_+;\mathbb{R}_+)$ *with compact support fixed, the law of* $\{L_{S_x}(t), t \geq 0\}$ *under* $\tilde{\mathbb{P}}^z$ *coincides with the law of* $\{Z_t^{x,z}, t \geq 0\}$.

Proof. It follows from Proposition 3.1 that for any $K > 0$, $z \in C(\mathbb{R}_+;\mathbb{R}_+)$ with compact support, under $\tilde{\mathbb{P}}^{K,z}$, $(L^K_{S^K_x}(t), 0 \leq t < K)$ has the same law as $(Z_t^{x,z}, 0 \leq t < K)$. A consequence of this is that for any $0 < K < K'$,

$$\bigl\{L^K_{S^K_x}(t), 0 \leq t < K\bigr\} \stackrel{(d)}{=} \bigl\{L^{K'}_{S^{K'}_x}(t), 0 \leq t < K\bigr\}. \tag{3.6}$$

It now follows that for any K, under $\tilde{\mathbb{P}}^z$, $(L_{S_x}(t), 0 \leq t < K)$ has the same law as $(L^K_{S^K_x}(t), 0 \leq t < K)$ under $\tilde{\mathbb{P}}^{K,z,x}$. We then obtain that for any $K > 0$

$$\bigl(L_{S_x}(t), 0 \leq t < K\bigr) \stackrel{(d)}{=} \bigl(Z_t^{x,z}, 0 \leq t < K\bigr).$$

Our result follows by letting $K \to \infty$. □

In particular, for x fixed, the law of $\{L_{S_x}(t), t \geq 0\}$ under $\tilde{\mathbb{P}}$ is the same as the law of $\{Z_t^x, t \geq 0\}$. From this follows the following remarkable identity

Corollary 3.9. *If* f *satisfies Hypothesis* B *and* $\Lambda(f) = \infty$, *then* S_x *is the total mass of the process* $\{Z_t^x, t \geq 0\}$, *in the sense that*

$$S_x \text{ and } \int_0^\infty Z_t^x\,\mathrm{d}t \text{ have the same law.}$$

Proof. Let $g(h) = 1$, for any $h > 0$. By the occupation times formula, we have

$$\begin{aligned} S_x &= \int_0^{S_x} g(H_r)\,\mathrm{d}r \\ &= \int_0^\infty L_{S_x}(t)\,\mathrm{d}t \\ &\stackrel{(d)}{=} \int_0^\infty Z_r^x\,\mathrm{d}r. \end{aligned}$$

Clearly the upper limit in the last integral can be replaced by the extinction time of the process Z^x. □

Proof of Theorem 3.7. Recall that $(Z^x_\cdot, x \geq 0)$ is a Markov process with values in the space of continuous paths from $\mathbb{R}_+$ into $\mathbb{R}_+$ with compact support. From Proposition 3.8 with $z \equiv 0$, its marginal laws coincide with those of $L_{S_x}(\cdot)$. We now check that $(L_{S_x}(\cdot), x \geq 0)$ is a Markov process. This follows readily from the fact that for any $0 \leq x < y$, conditionnaly upon $(L_{S_{x'}}(\cdot), x' \leq x)$ and given $L_{S_x}(\cdot) = z(\cdot)$, on $[0, S_y]$ the process $H_s^x := H_{S_x+s}$ solves the SDE

$$H_s^x = \bar{B}_s + \frac{1}{2}\int_0^s \bigl(f'\bigl(z\bigl(H_r^x\bigr) + L_r^z\bigl(H_r^x\bigr)\bigr)\bigr)\,\mathrm{d}r + \frac{1}{2}L_s^z(0),$$

where $\bar{B}$ is a Brownian motion independent of $(L_{S_{x'}}(t), x' \le x, 0 \le t \le S_x)$ and L^z denotes the local time of H^x, which is also the additional local time accumulated by H after time S_x. To complete the proof of the theorem it now suffices to prove that for any $x, y \ge 0$ the conditional law of $(L_{S_{x+y}}(t), t \ge 0)$ given $(L_{S_x}(t), t \ge 0)$ is the same as the conditional law of $(Z_t^{x+y}, t \ge)$ given $(Z_t^x, t \ge 0)$. Conditioned upon $L_{S_x}(\cdot) = z(\cdot)$, $L_{S_{x+y}}(\cdot) - L_{S_x}(\cdot)$ is the collection of local times accumulated by H^x up to time S_y, and it has the same law as $L^z_{S_y}(\cdot)$ while conditionally upon $Z^x = z(\cdot)$, $Z^{x+y} - Z^x$ has the same law as $Z^{y,z}$. The identity of those two laws has been established in Proposition 3.8. □

Appendix

A.1. Adaptation of the proof of Proposition 4 *from* [9]

Let us first explain the few results preceding that Proposition 4, which are necessary for its proof. It is claimed in [9] that for any $\theta, \gamma \ge 0$, $K > 0$, the following reflected SDE has a unique weak solution

$$H_s^K = B_s + \frac{\theta}{2}s - \gamma \int_0^s \big[z\big(H_r^K\big) + L_r^K\big(H_r^K\big)\big]\,\mathrm{d}r + \frac{1}{2}L_s^K(0) - \frac{1}{2}L_s^K\big(K^-\big).$$

The result is proved via Girsanov's Theorem. We note that the same result for our equation (3.4) is simpler to establish, since the drift there is bounded. Next [9] proceeds with proving Lemma 4, which translated in our context says that the expectation of $S_x^{K,n}$ is finite. The proof exploits the fact that the drift in the above SDE is bounded from above by the constant $\theta/2$. The same proof works in our case, since the drift in (3.4) is bounded. Finally Proposition 4 from [9] says, again translated into our notations, that the law of $\{L^{K,n}_{S_x^{K,n}}(t), 0 \le t \le K\}$ is that of the solution of (3.5), killed at time K. The proof translates word to word to our situation. The only minor difference is the computation which uses the generalized occupation times formula (from Exercise 1.15 in Chapter VI of [11]), namely formulas (23) and (24) in [9], which in our case should be replaced by the unique identity

$$\begin{aligned}
&\int_0^{S_x^{K,n}} \mathbf{1}_{\{H_s^{K,n} \le t\}} f_n'\big(z\big(H_s^{K,n}\big) + L_s^{K,n}\big(H_s^{K,n}\big)\big)\,\mathrm{d}s \\
&\quad = \int_0^t \int_0^{S_x^{K,n}} f_n'\big(z(u) + L_s^{K,n}(u)\big)\,\mathrm{d}L_s^{K,n}(u)\,\mathrm{d}u \\
&\quad = \int_0^t \big[f_n\big(z(u) + L^{K,n}_{S_x^{K,n}}(u)\big) - f_n\big(z(u)\big)\big]\,\mathrm{d}u.
\end{aligned}$$

This computation makes the connection with the SDE (3.5).

A.2. Proof of Proposition 3.5

In this proof, we shall write $\tilde{\mathbb{P}}$ instead of $\tilde{\mathbb{P}}^z$. Our probability space Ω will be defined as follows

$$\Omega = \big\{\omega \in C\big([0, +\infty); \mathbb{R}_+\big), \omega(0) = 0 \text{ and whenever } S < \infty, \omega(t) = 0 \text{ for } t \ge S\big\}.$$

Plainly $S(\omega)$ is defined as the limit as $n \to \infty$ of S_n, where S_n is the first time when the local time of ω at level 0 exceeds the value n. Thanks to Proposition 3.4, $\tilde{\mathbb{P}}(\Omega) = 1$. Note that whenever we have a sequence $\{\omega_n, n \ge 1\} \subset \Omega$ which satisfies the property that for all $n \ge 1$, $\omega_{n+1}(t) = \omega_n(t)$, $t \in [0, S_n]$, then there exists $\omega \in \Omega$ which is such that for all $n \ge 1$, $\omega(t) = \omega_n(t)$, $t \in [0, S_n]$. This existence is clear on the event $\{S = +\infty\}$, while on the event $\{S < \infty\}$, it follows from the definition of Ω, by adding to the above requirements $\omega(t) = 0$ for $t \ge S$.

This means that condition (1.1.9) from [12] is satisfied. The proof which we shall now detail is a transcription to our set-up of the proof of Theorem 1.1.9 from the same reference.

In order to simplify slightly the notations, we shall write $\mathcal{G}_n = \mathcal{F}_{S_n}$. All we have to show is that whenever $\{A_n, n \geq 1\}$ is a decreasing sequence of events which is such that $A_n \in \mathcal{G}_n$ for all $n \geq 1$, if $\bigcap_n A_n = \varnothing$, then $\tilde{\mathbb{P}}(A_n) \to 0$. For proving that fact, we will assume that $\tilde{\mathbb{P}}(A_n) \geq \varepsilon$ for all $n \geq 1$, some $\varepsilon > 0$, and deduce that $\bigcap_n A_n \neq \varnothing$. In view of the above remark, it suffices to construct a sequence $\{\omega_n, n \geq 1\} \subset \Omega$ such that $\omega_n \in A_n$ and $\omega_{n+1}(t \wedge S_n) = \omega_n(t \wedge S_n)$ for all $t \geq 0$, $n \geq 1$.

Let us introduce a notation. For all $m, n \geq 1$ let $\pi^{m,n}(\omega, \mathrm{d}\omega')$ denote a regular version of the conditional probability $\tilde{\mathbb{P}}|_{\mathcal{G}_n}(\cdot|\mathcal{G}_m)$. In other words, if $n \leq m$,

$$\pi^{m,n}(\omega, \mathrm{d}\omega') = \delta_{\omega(\cdot \wedge S_n)}(\mathrm{d}\omega'),$$

while if $n > m$,

$$\pi^{m,n}(\omega, \mathrm{d}\omega') = \text{ the law of } \{H_s, 0 \leq s \leq S_n\} \text{ s.t. } H_s \begin{cases} = \omega(s) \text{ on } [0, S_m], \\ \text{solves (3.2) on } [S_m, S_n]. \end{cases}$$

Let us define for each $n \geq 1$

$$F_n^1 = \left\{\omega; \pi^{1,n}(\omega, A_n) \geq \frac{\varepsilon}{2}\right\}.$$

Clearly $F_{n+1}^1 \subset F_n^1$, $F_1^1 = A_1$, and for $n > 1$,

$$\begin{aligned} \varepsilon \leq \tilde{\mathbb{P}}(A_n) &= \int_{(F_n^1)^c} \pi^{1,n}(\omega, A_n)\tilde{\mathbb{P}}(\mathrm{d}\omega) + \int_{F_n^1} \pi^{1,n}(\omega, A_n)\tilde{\mathbb{P}}(\mathrm{d}\omega) \\ &\leq \frac{\varepsilon}{2} + \tilde{\mathbb{P}}(F_n^1). \end{aligned}$$

Consequently $\tilde{\mathbb{P}}(F_n^1) \geq \frac{\varepsilon}{2}$ for all $n \geq 1$, hence also $\tilde{\mathbb{P}}(\bigcap_n F_n^1) \geq \frac{\varepsilon}{2}$, and there exists ω_1 such that $\pi^{1,n}(\omega_1, A_n) \geq \frac{\varepsilon}{2}$, for all $n \geq 1$.

Suppose we have found $\omega_1, \omega_2, \dots, \omega_m$ such that $\omega_k(t \wedge S_{k-1}) = \omega_{k-1}(t \wedge S_{k-1})$ for $2 \leq k \leq m$ and $\pi^{k,n}(\omega_k, A_n) \geq \frac{\varepsilon}{2^k}$, for $1 \leq k \leq m$, $n \geq 1$. Let now

$$F_n^{m+1} = \left\{\omega; \pi^{m+1,n}(\omega, A_n) \geq \frac{\varepsilon}{2^{m+1}}\right\} \in \mathcal{G}_{m+1}.$$

Then $F_{n+1}^{m+1} \subset F_n^{m+1}$ and for $n > m$

$$\frac{\varepsilon}{2^m} \leq \pi^{m,n}(\omega_m, A_n) \leq \frac{\varepsilon}{2^{m+1}} + \pi^{m,m+1}(\omega_m, F_n^{m+1}),$$

and consequently $\pi^{m,m+1}(\omega_m, F_n^{m+1}) \geq \frac{\varepsilon}{2^{m+1}}$, and also $\pi^{m,m+1}(\omega_m, \bigcap_n F_n^{m+1}) \geq \frac{\varepsilon}{2^{m+1}}$. Hence there exists ω_{m+1} such that $\omega_{m+1}(\cdot \wedge S_m) = \omega_m(\cdot \wedge S_m)$, and $\pi^{m+1,n}(\omega_{m+1}, A_n) \geq \frac{\varepsilon}{2^{m+1}}$ for all $n \geq 1$. Our sequence ω_m satisfies in particular the property $\pi^{m,m}(q_m, A_m) > 0$, hence $\omega_m \in A_m$ for all $m \geq 1$. The announced sequence has been constructed.

A.3. *A comparison theorem for reflected SDEs*

We do not claim that the following result is new, but we could not find a reference for it. Let $b \in \mathbb{R}$, $b_2 : \mathbb{R}_+ \to \mathbb{R}$ be measurable, with the properties that

$$b_2(x) \leq b, \quad \text{for all } x \geq 0.$$

Suppose now that $\{X_t^1; t \geq 0\}$ and $\{X_t^2; t \geq 0\}$ are progressively mesurable solutions of the SDEs

$$X_t^1 = x_1 + bt + B_t + K_t^1, \quad t \geq 0,$$
$$X_t^2 = x_2 + \int_0^t b_2(X_s^2)\,\mathrm{d}s + B_t + K_t^2, \quad t \geq 0,$$
$$X_t^1 \geq 0, \qquad X_t^2 \geq 0, \qquad \int_0^t X_s^1\,\mathrm{d}K_s^1 = \int_0^t X_s^2\,\mathrm{d}K_s^2 = 0, \quad t \geq 0,$$

where K^1 and K^2 are continuous and increasing and $x_1 \geq x_2$. We then have

Proposition A.1. *Under the above assumptions*,

$$\mathbb{P}(X_t^1 \geq X_t^2, \forall t \geq 0) = 1.$$

Proof. We note that

$$X_t^2 - X_t^1 = x_2 - x_1 + \int_0^t \big[b_2(X_s^2) - b\big]\,\mathrm{d}s + K_t^2 - K_t^1.$$

Since the mapping $\varphi(x) = (x^+)^2$ is of class C^1, we deduce that.

$$\begin{aligned}\big|(X_t^2 - X_t^1)^+\big|^2 &= 2\int_0^t \mathbf{1}_{\{X_s^2 \geq X_s^1\}}(X_s^2 - X_s^1)\big(b_2(X_s^2) - b\big)\,\mathrm{d}s \\ &\quad + 2\int_0^t \mathbf{1}_{\{X_s^2 \geq X_s^1\}}(X_s^2 - X_s^1)\big(\mathrm{d}K_s^2 - \mathrm{d}K_s^1\big).\end{aligned}$$

The first term on the right hand side is clearly non positive. The same is true for the second one, since $\int_0^t X_s^1\,\mathrm{d}K_s^1 = \int_0^t X_s^2\,\mathrm{d}K_s^2 = 0$. Consequently

$$\big|(X_t^2 - X_t^1)^+\big|^2 \leq 0,$$

for all $t \geq 0$, hence the result. □

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