ON THE LARGE TIME BEHAVIOUR OF THE SOLUTION OF AN SDE DRIVEN BY A POISSON POINT PROCESS

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Abstract

We study a stochastic differential equation driven by a Poisson point process, which models continuous change in a population's environment, as well as the stochastic fixation of beneficial mutations that might compensate for this change. The fixation probability of a given mutation increases as the phenotypic lag X_t between the population and the optimum grows larger, and successful mutations are assumed to fix instantaneously (leading to an adaptive jump). Our main result is that the process is transient (i.e., converges to $-\infty$, so that continued adaptation is impossible) if the rate of environmental change v exceeds a parameter m, which can be interpreted as the rate of adaptation in case every beneficial mutation gets fixed with probability 1. If $v < m$, the process is Harris recurrent and possesses a unique invariant probability measure, while in the limiting case $m = v$, Harris recurrence with an infinite invariant mesure or transience depends upon additional technical conditions. We show how our results can be extended to a class of time varying rates of environmental change.

Keywords: Stochastic Differential Equations with jumps, Large time behaviour, Transience, Recurrence

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1. Introduction

When faced with environmental change, biological populations can increase their fitness by adaptive Darwinian evolution. While adaptation over short timescales often relies on frequency shifts of pre-existing standing genetic variation, long-term adaptation ultimately depends on the establishment and (frequently) fixation of new beneficial mutations, i.e. the entire population is composed of the progeny of the original mutant, which themselves arise in a stochastic manner (e.g. due to copying errors) independent of the current "needs" of the population. Several distinct mathematical frameworks have been developed to model adaptive evolution under various sets of simplifying assumptions. One such framework focuses on cases where fixations events are rare and clearly separated from one another, allowing adaptation to be modeled as a Markovian jump process, which has been called "adaptive walk" see [6], [8] or "trait substitution sequence" (TSS). For the links between TSS models and individual-based birth-death models (primarily for asexual populations) see [5].

Many traits of organisms are thought to be under stabilizing selection, such that fitness (i.e., the expected number of offspring of an individual with a certain trait value or phenotype) is maximal for intermediate values and declines monotonically with the distance to this optimum (a classical example is birth weight in humans, where infant mortality is increased for both very low and very high weights). Environmental change (e.g., climate change) can then be viewed as altering the value of the optimum, making a previously well-adapted population suffer a reduction in mean fitness. This so-called moving-optimum model has been widely used in the theoretical biology literature to investigate the effects of various kinds of environmental change (e.g., sudden, directional or fluctuating) on phenomena such as population extinction risk [4], the maintenance of genetic variation [3], or the fixation probability [17, 15], and fixation time [11, 17] of beneficial mutations. Recently, Kopp and Hermisson [12] developed an adaptive-walk approximation for a model with a linearly moving optimum and used it to study the size-distribution of adaptive jumps by means of which the population phenotype follows the optimum. This model is the starting point for the present paper. The innovative idea here is to model the adaptive walk by means of a stochastic differential

equation, allowing us to obtain results about the large-time behaviour of the solution, which have a precise meaning in biological terms. Our SDE is of the form

$$X_t = X_0 - \mathcal{V}_t + \int_{[0,t] \times \mathbb{R} \times [0,1]} \alpha \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi),$$

where M is an $\mathcal{F}_t$–Poisson point process on $\mathbb{R}_+ \times \mathbb{R} \times [0,1]$ with mean measure $ds \, \nu(d\alpha) \, d\xi$ and $\varphi(x, \alpha, \xi) = \mathbf{1}_{\{\xi \leq g(x,\alpha)\}}$, and the process $\mathcal{V}_t$ is right continuous with left limits at every point and $\mathcal{F}_t$–progressively measurable, satisfying $\mathcal{V}_0 = 0$. X describes the evolution of the phenotypic lag and takes value in $\mathbb{R}$ depending whether the population lags above or below the optimal value 0. $ds \, \nu(d\alpha)$ represents the rate of appearance of new mutations of different size, while $g(x, \alpha)$ is the probability that a mutation of size α, which is proposed while the population's phenotypic lag is x, gets fixed. We assume that $g(x, \alpha) \to 1$ when $x \to \pm\infty$ provided that $x\alpha < 0$.

We start with the simple case $\mathcal{V}_t = vt$, with $v > 0$ a real number. With the notation $m = \int_0^\infty \alpha \nu(d\alpha)$ – in other words, m is the mean movement to the right per time unit produced by the positive mutations if all of them get fixed – our first result says that under some additional assumption, the Markov process X_t is Harris recurrent if $m > v$, transient if $m < v$, with a speed of escape to infinity equal to $v - m$. The most difficult case is the limit situation $m = v$. We show that, depending upon the speed at which $m(x) = \int_0^\infty \alpha g(x, \alpha) \nu(d\alpha)$ converges to m as $x \to -\infty$, the process can be either Harris recurrent or else transient with zero speed. We then generalize our results to the case where $\mathcal{V}_t$ is a more general (and even possibly random) function of time.

Note that [9] has studied similar questions in discrete time. The same results for an SDE driven by Brownian motion with coefficients that do not depend upon the time variable would be easy to obtain. Here we use stochastic calculus and several ad hoc Lyapounov functions. Note that the Itô formula for processes with jumps leads to less explicit computations than in the Brownian case. To circumvent this difficulty, for the treatment of the delicate case $m = v$, we establish a stochastic inequality for C^2 functions whose second derivative is either increasing or decreasing, exploiting the fact that all jumps have the same sign, see Lemma 4.3 below.

The paper is organized as follows. We define our model in detail in section 2, referring

to models already studied in the biological literature. We establish existence and uniqueness of a solution to our equation in section 3 (the result is not immediate, since we do not assume that the measure ν is finite). Section 4 is devoted to the large time behaviour of X_t when $\mathcal{V}_t = vt$, successively with $m < v$, $m > v$, and $m = v$. Finally section 5 is devoted to the large time behaviour of X_t when $\mathcal{V}_t$ takes a more general form, but $\overline{v} = \lim_{t \to \infty} t^{-1}\mathcal{V}_t$ exists and is deterministic.

2. The model

Our starting point is the model by Kopp and Hermisson [12], of which we subsequently relax several assumptions. Kopp and Hermisson modeled a population of constant size N that is subject to Gaussian stabilizing selection with a moving optimum that increases linearly at rate v. Therefore, an individual with phenotype z has a phenotypic lag $x = z - vt$ and fitness $\mathcal{W}(x) = e^{-\sigma^{-2}x^2}$, where σ^{-2} determines the strength of selection. The population is assumed to be monomorphic at all times (i.e., its state is completely characterized by x). Mutations appear at rate $\Theta/2 = N\mu$ (where μ is the *per-capita* mutation rate and $\Theta = 2N\mu$ is a standard population-genetic parameter), and their phenotypic effects α are drawn from a distribution with density $p(\alpha)$. In other words, mutations arise according to a Poisson point process with intensity $ds\ \nu(d\alpha)$, where

$$\nu(d\alpha) = \frac{\Theta}{2}p(\alpha)d\alpha. \tag{1}$$

A mutation that appears while the lag is x has selection coefficient

$$s(x,\alpha) = \left(\frac{\mathcal{W}(x+\alpha)}{\mathcal{W}(x)} - 1\right) \times \mathbf{1}_{\{x\alpha<0\}} \approx \sigma^{-2}[|\alpha|(2|x| - |\alpha|)]^+ \times \mathbf{1}_{\{x\alpha<0\}}, \tag{2}$$

where the approximation is valid as long as $\sigma^{-2}[|\alpha|(2|x| - |\alpha|)]^+$ is small. Even beneficial mutations have a considerable risk of being lost due to genetic drift (i.e., due to stochastic fluctuations while their frequency is low). The probability that a mutation escapes drift loss and instead goes to fixation is

$$g(x,\alpha) = \begin{cases} 1 - \exp(-2s(x,\alpha)) & \text{if } s(x,\alpha) > 0, \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

This equation neglects the probability of fixation of deleterious mutations (with $s < 0$), and otherwise is a good approximation for the fixation probability derived under a diffusion approximation [13, 10], which is valid when the population size N is large enough. Note that [12] used the even simpler approximation $g(x, \alpha) \approx 2s(x, \alpha)$ ([7]; for more exact approximations for the fixation probability in changing environments, see [17, 15]). Once a mutation gets fixed, it is assumed to do so instantaneously, and the phenotypic lag x of the population is updated accordingly.

In the present work, we relax these assumptions in three respects: First, we consider a more general model of environmental change, such that, in the absence of evolution, the lag increases due to a random function $\mathcal{V}_t$. Second, we only assume that mutations arise according to a Poisson point process with intensity $ds \, \nu(d\alpha)$ (which subsumes the mutation rate $\Theta/2$), but we do not impose that ν has a density, nor that it is a finite measure. This allows both for heavy-tailed mutational distributions, enabling very large jumps, and for an accumulation of infinitely many small jumps. Note however that some of our results will require additional assumptions about the tail of ν. Third, we only make the following assumptions about the fixation probability: A mutation with effect α that arises in a population with phenotypic lag x has a probability of fixation $g(x, \alpha)$ that satisfies

1. $0 \leq g(x, \alpha) \leq \mathbf{1}_{\{\alpha x < 0\}} \times \mathbf{1}_{\{|\alpha| \leq 2|x|\}}$,

2. For all $\alpha \in \mathbb{R}$, $g(x, \alpha) \uparrow \mathbf{1}_{\{\alpha x < 0\}}$, as $|x| \to \infty$,

3. There exists a compact set $K \subset \mathbb{R}$ and $c_K > 0$ such that $\nu(K^c) < \infty$ and for all $x, y \in \mathbb{R}$

$$\int_K |\alpha| \times |g(x, \alpha) - g(y, \alpha)| \nu(d\alpha) \leq c_K |x - y|. \tag{4}$$

These items represent the basic mathematical assumptions, which will be assumed to hold throughout this paper. Condition 1 assures that only the beneficial mutations can be fixed, which is a reasonable slightly simplifying biological assumption. In reality, there is a small probability that a slightly deleterious mutation gets fixed through chance (i.e. genetic drift), but we neglect this possibility. Condition 2 means that when the phenotypic lag goes to infinity, the probability of fixation of any beneficial

mutation tends to 1. This is quite reasonable: when the fitness of a population is very low, one may expect that any possible improvement will be taken by the population. Condition 3 is imposed for mathematical convenience but is required only in the case $\nu(\mathbb{R}) = \infty$.

The evolution of the phenotypic lag of the population can then be described by the following SDE already given in the introduction :

$$X_t = X_0 - \mathcal{V}_t + \int_{[0,t] \times \mathbb{R} \times [0,1]} \alpha \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi). \tag{5}$$

Here, M is a Poisson point process over $\mathbb{R}_+ \times \mathbb{R} \times [0,1]$ with intensity $ds \, \nu(d\alpha) \, d\xi$. $\nu(d\alpha)$ is a σ-finite measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} |\alpha| \wedge 1 \, \nu(d\alpha) < \infty, \tag{6}$$

and

$$\varphi(x, \alpha, \xi) = \mathbf{1}_{\{\xi \leq g(x,\alpha)\}},$$

where the fixation probability $g(x, \alpha)$ has been defined above. The points of the Poisson point process M (T_i, A_i, Ξ_i) are such that the (T_i, A_i) form a Poisson point process over $\mathbb{R}_+ \times \mathbb{R}$ of the proposed mutations with intensity $ds\nu(d\alpha)$, and the Ξ_i are i.i.d. $\mathcal{U}[0,1]$, globally independent of the Poisson point process of the (T_i, A_i). T_i's are the times when mutations are proposed and A_i's are the effect sizes of those mutations. The Ξ_i are auxiliary variables determining fixation: a mutation gets instantaneously fixed if $\Xi_i \leq g(X_{T_i}, A_i)$, and is lost otherwise.

Finally, we define for all $x \in \mathbb{R}$

$$m(x) = \int_{\mathbb{R}} \alpha g(x, \alpha) \nu(d\alpha),$$

$$m = \int_{\mathbb{R}_+} \alpha \nu(d\alpha), \tag{7}$$

$$V(x) = \int_{\mathbb{R}} \alpha^2 g(x, \alpha) \nu(d\alpha),$$

$$V = \int_{\mathbb{R}_+} \alpha^2 \nu(d\alpha). \tag{8}$$

$m(x)$ is the mean speed towards zero induced by the fixation of random mutations while $X_t = x < 0$. $V(x)$ is related to the second moment of the distribution of these

mutations. m and V are the limits of $m(x)$ and $V(x)$, respectively, as $x \to -\infty$ or in other words, in the case that all mutations with $\alpha > 0$ go to fixation. Note that our assumptions do not exclude cases where $m = \infty$ and/or $V = \infty$, unless stated otherwise. However, since $g(x, \cdot)$ has compact support, for each x, $m(x) < \infty$ and $V(x) < \infty$. The cases $m = \infty$ and $V = \infty$ correspond to a heavy tailed ν. It would be quite acceptable on biological grounds to assume that $m < \infty$ and/or $V < \infty$. However, we refrain from adding unnecessary assumptions.

In the case $\mathcal{V}_t = vt$, X_t is a Markov process, whose generator $\mathcal{L}$ acts on a differentiable function f as

$$\mathcal{L}f(x) = f'(x)(m(x) - v) + \int_{\mathbb{R}} \left(f(x + \alpha) - f(x) - f'(x)\alpha \right) g(x, \alpha)\nu(d\alpha).$$

3. Existence and uniqueness

We rewrite the SDE (5) as follows

$$X_t = X_0 - \mathcal{V}_t + \int_0^t m(X_s)ds + \mathcal{M}_t \tag{9}$$

where the local martingale $\mathcal{M}_t$ is given as

$$\mathcal{M}_t = \int_0^t \int_{\mathbb{R}} \int_0^1 \alpha\varphi(X_{s-}, \alpha, \xi)\bar{M}(ds, d\alpha, d\xi), \tag{10}$$

with $\bar{M}(ds, d\alpha, \xi)$ being the compensated Poisson measure $M(ds, d\alpha, d\xi) - ds\nu(d\alpha)d\xi$.

Proposition 3.1. *Equation* (9) *has a unique solution.*

Proof. If ν is a finite measure, then M has a.s. finitely many points in $[0, t] \times \mathbb{R}$ for any $t > 0$. In that case, the unique solution is constructed explicitly by adding the successive jumps. In the general case, we choose an arbitrary compact set $K = [-k, k]$ (with $k > 0$). Due to (6), there are finitely many jumps (t_i, α_i) of M with $\alpha_i \notin K$. It suffices to prove existence and uniqueness between two such consecutive jumps. In other words, it suffices to prove existence and uniqueness under the assumption $\nu(K^c) = 0$ where K is chosen such that Condition 3 is satisfied, and hence from (6),

we deduce that $\int \left(|\alpha| + \alpha^2 \right) \nu(d\alpha) < \infty$, which we assume from now on. We shall also assume that there exists $C > 0$ such that

$$|\mathcal{V}_t| \leq Ct, \quad \text{for all } t \geq 0. \tag{11}$$

Indeed, if that is not the case, we let $V_t^n = (\mathcal{V}_t \wedge nt) \vee (-nt)$, and define $T_n = \inf\{t > 0, : |\mathcal{V}_t| > nt\}$. Existence and uniqueness under the additional assumption (11) will provide a unique solution X_t^n associated to V_t^n. We will have that $X_t^m = X_t^n$ for $0 \leq t \leq T_n$ if $m > n$, and $T_n \uparrow \infty$ as $n \to \infty$. This clearly implies existence of a unique solution under our standing assumptions. Hence we assume for the rest of this proof that (11) is satisfied.

Define for each $t > 0$

$$\Gamma_t(U) = x - \mathcal{V}_t + \int_{[0,t] \times K \times [0,1]} \alpha \varphi(U_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi). \tag{12}$$

A solution of equation (9) is a fixed point of the mapping Γ a.s. . Hence it suffices to prove that Γ admits a unique fixed point a.s. . For $\lambda > 0$,

$$e^{-\lambda t} \left| \Gamma_t(U) - \Gamma_t(U') \right| = -\lambda \int_0^t e^{-\lambda s} \left| \Gamma_s(U) - \Gamma_s(U') \right| ds + \int_0^t e^{-\lambda s} d \left| \Gamma_s(U) - \Gamma_s(U') \right|$$

$$\leq -\lambda \int_0^t e^{-\lambda s} \left| \Gamma_s(U) - \Gamma_s(U') \right| ds$$

$$+ \int_{[0,t] \times K \times [0,1]} |\alpha| e^{-\lambda s} \left| \varphi(U_{s-}, \alpha, \xi) - \varphi(U'_{s-}, \alpha, \xi) \right| M(ds, d\alpha, d\xi).$$

The above inequality follows readily from the fact that, for all $0 < s < t$,

$$\left| \Gamma_t(U) - \Gamma_t(U') \right| - \left| \Gamma_s(U) - \Gamma_s(U') \right|$$

$$\leq \int_{(s,t) \times K \times [0,1]} |\alpha| \times |\varphi(U_{r-}, \alpha, \xi) - \varphi(U'_{r-}, \alpha, \xi)| M(dr, d\alpha, d\xi).$$

Thus,

$$\lambda \mathbb{E} \int_0^t e^{-\lambda s} \left| \Gamma_s(U) - \Gamma_s(U') \right| ds \leq \mathbb{E} \int_0^t \int_K |\alpha| e^{-\lambda s} \left| g(U_s, \alpha) - g(U'_s, \alpha) \right| \nu(d\alpha) ds$$

$$\leq c_K \mathbb{E} \int_0^t e^{-\lambda s} |U_s - U'_s| ds. \tag{13}$$

The last inequality is due to the assumption (4). Let T be arbitrary. Define for all $\lambda > 0$ the norm on the Banach space $L^1(\Omega \times [0, T])$,

$$\|Z\|_{T,\lambda} = \mathbb{E} \int_0^T e^{-\lambda t} |Z_t| dt.$$

We choose $\lambda_0 > c_K$. We deduce from (13) that

$$\mathbb{E}\|\Gamma(U) - \Gamma(U')\|_{T,\lambda_0} \le \frac{c_K}{\lambda_0}\mathbb{E}\|U - U'\|_{T,\lambda_0}.$$

Since $c_K/\lambda_0 < 1$, Γ has a unique fixed point such that $\Gamma_t(U) = U_t$ a.s. for all $0 \le t \le T$. Since T is arbitrary, the result is proved. $\qquad\square$

We now prove that g given by (3) and (2) satisfies the assumption (4).

Lemma 3.1. *For any compact set $K \subset \mathbb{R}$ and for all $u, v \in \mathbb{R}$,*

$$\int_K |\alpha(g(u,\alpha) - g(v,\alpha))|\nu(d\alpha) \le c_K|u - v|,$$

where $c_K = 4\sigma^{-2}\left(\int_K \alpha^2\nu(d\alpha)\right)$.

Proof. For $0 < u < v$ we have that

$$\int_K |\alpha(g(u,\alpha) - g(v,\alpha))|\nu(d\alpha) = \int_{\mathbb{R}_- \cap K} |\alpha\left(e^{-2\sigma^{-2}|\alpha|(2|v|-|\alpha|)^+} - e^{-2\sigma^{-2}|\alpha|(2|u|-|\alpha|)^+}\right)|\nu(d\alpha)$$

$$= \int_{[-2u,0]\cap K} |\alpha\left(e^{-2\sigma^{-2}|\alpha|(2|v|-|\alpha|)} - e^{-2\sigma^{-2}|\alpha|(2|u|-|\alpha|)}\right)|\nu(d\alpha)$$

$$+ \int_{[-2v,-2u]\cap K} |\alpha|\left|e^{-2\sigma^{-2}|\alpha|(2|v|-|\alpha|))} - 1\right|\nu(d\alpha)$$

$$\le 4\sigma^{-2}\left(\int_{[-2u,0]\cap K}\alpha^2\nu(d\alpha)\right)\times|u - v|$$

$$+ 2\sigma^{-2}\int_{[-2v,-2u]\cap K}\alpha^2(2v+\alpha)\nu(d\alpha)$$

$$\le 4\sigma^{-2}\left(\int_K\alpha^2\nu(d\alpha)\right)\times|u - v|.$$

A similar estimate can easily be obtained for $v < u < 0$. For $u < 0 < v$, we have that

$$\int_K |\alpha(g(u,\alpha) - g(v,\alpha))|\nu(d\alpha) = \int_{\mathbb{R}_- \cap K}|\alpha g(v,\alpha)|\nu(d\alpha) + \int_{\mathbb{R}_+ \cap K}|\alpha g(u,\alpha)|\nu(d\alpha)$$

$$\le 2\sigma^{-2}\int_{\mathbb{R}_- \cap K}\alpha^2(2v+\alpha)^+\nu(d\alpha) + 2\sigma^{-2}\int_{\mathbb{R}_+ \cap K}\alpha^2(-2u-\alpha)^+\nu(d\alpha)$$

$$\le 4\sigma^{-2}\int_{\mathbb{R}_- \cap K}\alpha^2|v|\nu(d\alpha) + 4\sigma^{-2}\int_{\mathbb{R}_+ \cap K}\alpha^2|u|\nu(d\alpha)$$

$$\le 4\sigma^{-2}\left(\int_K\alpha^2\nu(d\alpha)\right)\times|u - v|.$$

$$\square$$

We close this section by establishing an a priori estimate on our solution.

Proposition 3.2. *For all $T > 0$, if $\sup\limits_{0 \le t \le T} \mathbb{E}[\,|\mathcal{V}_t|\,] < \infty$, then*

$$\sup_{0 \le t \le T} \mathbb{E}[\,|X_t|\,] < \infty.$$

Proof. It is plain that

$$|X_t| \le |X_0| + |\mathcal{V}_t| + \int_{[0,t] \times \mathbb{R} \times [0,1]} |\alpha| \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi).$$

Furthermore, exploiting the fact that $g(x, \alpha) \le \mathbf{1}_{|\alpha| \le 2|x|}$,

$$
\begin{aligned}
\mathbb{E} \int_{[0,t] \times \mathbb{R} \times [0,1]} |\alpha| \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi) &= \mathbb{E} \int_{[0,t] \times \mathbb{R}} |\alpha| g(X_s, \alpha) \nu(d\alpha) ds \\
&\le \mathbb{E} \int_0^t \int_{-2|X_s|}^{2|X_s|} |\alpha| \nu(d\alpha) ds \\
&\le t \int_{-1}^1 |\alpha| \nu(d\alpha) + \mathbb{E} \int_0^t \int_{1 \le |\alpha| \le 2|X_s|} |\alpha| \nu(d\alpha) \\
&\le t \int_{-1}^1 |\alpha| \nu(d\alpha) + 2\nu([-1,1]^c) \mathbb{E} \int_0^t |X_s| ds.
\end{aligned}
$$

Finally

$$\mathbb{E}[|X_t|] \le |X_0| + \mathbb{E}[|\mathcal{V}_t|] + \left[\int_{-1}^1 |\alpha| \nu(d\alpha) \right] t + 2\nu([-1,1]^c) \int_0^t \mathbb{E}[|X_s|] ds,$$

and the result follows from Gronwall's Lemma. $\square$

It is now easy to deduce from the previous proof the

Corollary 3.1. *If $\sup\limits_{0 \le t \le T} \mathbb{E}\left[|\mathcal{V}_t|\right] < \infty$ for all $T > 0$, then the process $\mathcal{M}_t$ defined by* (10) *is a martingale.*

4. Classification of the large-time behaviour in the case $\mathcal{V}_t = vt$

We now consider the case $\mathcal{V}_t = vt$, $v > 0$.

Proposition 4.1. *If $X_0 > 0$, then X_t becomes negative after a finite time a.s.*

Proof. Let $T_- = \inf(t > 0, X_t < 0)$. We have that

$$0 \le X_{(t \wedge T_-)-} \le X_0 - v(t \wedge T_-).$$

Hence

$$t \wedge T_- \le \frac{X_0 - X_{(t \wedge T_-)-}}{v} < \frac{X_0}{v}, \text{ and } T_- \le \frac{X_0}{v} < \infty.$$

$\square$

Proposition 4.2. *The functions* $x \longmapsto m(x)$ *and* $x \longmapsto V(x)$ *are continuous and decreasing on* $\mathbb{R}_-$ *and*

$$m(x) \xrightarrow[x \to -\infty]{} m,$$

$$V(x) \xrightarrow[x \to -\infty]{} V. \tag{14}$$

Proof. We prove this result for the function $x \longmapsto m(x)$. A similar argument applies to $V(x)$. Let

$$h : \mathbb{R}_- \times \mathbb{R}_+ \to \mathbb{R}_+$$

$$(x, \alpha) \mapsto h(x, \alpha) = \alpha g(x, \alpha).$$

We have that $h(x, \cdot) \in L^1(\nu)$, and $x \mapsto h(x, \alpha)$ is decreasing. For each fixed $\alpha > 0$, $0 \le h(x, \alpha) \uparrow \alpha$, as $x \to -\infty$. By the monotone convergence theorem, it follows that

$$m(x) = \int_{\mathbb{R}} h(x, \alpha)\nu(d\alpha) \xrightarrow[x \to -\infty]{} m.$$

Continuity is proved similarly. $\square$

To determine the large-time behavior of the process, we now consider successively, the three cases $v > m$, $v < m$ and $v = m$.

4.1. The case $v > m$

In particular, here $m = \int_0^\infty \alpha \nu(d\alpha)$ is finite. Let

$$\mathcal{N}_t = \int_{[0,t] \times \mathbb{R} \times [0,1]} \alpha \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi)$$

be the sum of all the jumps on the time interval $[0, t]$. We have that

$$\mathcal{N}_t = \mathcal{N}_t^{(+)} + \mathcal{N}_t^{(-)} \leq \mathcal{N}_t^{(-)},$$

where

$$\mathcal{N}_t^{(+)} = \mathbf{1}_{\{X_{s-}>0\}}d\mathcal{N}_s$$

$$\mathcal{N}_t^{(-)} = \mathbf{1}_{\{X_{s-}<0\}}d\mathcal{N}_s$$

Let $m^{(-)}(x) = \mathbf{1}_{\{x<0\}}m(x)$, hence

$$\mathcal{M}_t^{(-)} = \mathcal{N}_t^{(-)} - \int_0^t m^{(-)}(X_s)ds.$$

Thus,

$$X_t \leq X_0 + \int_0^t (m^{(-)}(X_s) - v)ds + \mathcal{M}_t^{(-)}$$

Lemma 4.1. *If $m < \infty$, then*

$$\frac{\mathcal{M}_t^{(-)}}{t} \xrightarrow[t\to\infty]{} 0 \ a.s. \tag{15}$$

Proof. $\mathcal{M}_t^{(-)}$ is a square-integrable martingale, such that $\mathbb{E}\mathcal{M}_t^{(-)} = 0$. For all $i \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$, define

$$\xi_i = \int_{i-1}^i \int_0^\infty \int_0^1 \alpha\varphi(X_{s-}, \alpha, \xi)M(ds, d\alpha, d\xi),$$

$$\omega_i = \int_{i-1}^i m^{(-)}(X_s)ds,$$

$$\eta_i = \int_{i-1}^i \int_0^\infty \int_0^1 \alpha M(ds, d\alpha, d\xi),$$

$$Y_i = \xi_i - \omega_i,$$

$$\mathcal{M}_n^{(-)} = \sum_{i=1}^n Y_i,$$

Note that for all $i \in \mathbb{N}^*$, $0 \leq \xi_i \leq \eta_i$ and $0 \leq \omega_i \leq m$. We first establish

Lemma 4.2. *The event*

$$\left\{ \frac{\sum_{i=1}^n Y_i}{n} \to 0 \ \ as \ n \to \infty \right\}$$

entails the event

$$\left\{ \frac{\mathcal{M}_t^{(-)}}{t} \to 0 \quad as \ t \to \infty \right\}.$$

Proof.

$$\frac{\mathcal{M}_t^{(-)}}{t} = \frac{\mathcal{M}_{\lfloor t \rfloor}^{(-)}}{\lfloor t \rfloor} \times \frac{\lfloor t \rfloor}{t} + \frac{\tilde{\mathcal{M}}_t^{(-)}}{t},$$

where

$$\frac{\tilde{\mathcal{M}}_t^{(-)}}{t} = \frac{1}{t} \left(\int_{\lfloor t \rfloor}^t \int \int \alpha \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi) - \int_{\lfloor t \rfloor}^t m^{(-)}(X_s) ds \right)$$

$$\leq \frac{1}{t} \left(\int_{\lfloor t \rfloor}^{\lceil t \rceil} \int \int \alpha \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi) + \int_{\lfloor t \rfloor}^{\lceil t \rceil} m^{(-)}(X_s) ds \right)$$

$$= \frac{1}{t} \left(\xi_{\lceil t \rceil} + \omega_{\lceil t \rceil} \right) = \frac{\lceil t \rceil}{t} \times \frac{1}{\lceil t \rceil} \left(Y_{\lceil t \rceil} + 2\omega_{\lceil t \rceil} \right) \xrightarrow[t \to \infty]{} 0,$$

since for all $n > 0$,

$$\frac{Y_{n+1}}{n+1} = \frac{\sum_{i=1}^{n+1} Y_i}{n+1} - \frac{\sum_{i=1}^n Y_i}{n} \times \frac{n}{n+1} \xrightarrow[n \to \infty]{} 0$$

and

$$0 \leq \frac{\omega_n}{n} \leq \frac{m}{n},$$

hence

$$\frac{\omega_n}{n} \xrightarrow[n \to \infty]{} 0.$$

$\square$

Back to the proof of Lemma 4.1. We now define

$$A_i = \{\eta_i > i\},$$

$$\tilde{Y}_i = Y_i \mathbf{1}_{\{\eta_i \leq i\}}.$$

Since the $(\eta_i, i \in \mathbb{N}^*)$ are i.i.d, integrable and

$$\sum_{i \geq 1} \mathbb{P}(\eta_i > i) = \sum_{i \geq 1} \mathbb{P}(\eta_1 > i) \leq \mathbb{E}\eta_1 < \infty,$$

it follows from Borel Cantelli's Lemma that $\mathbb{P}(\limsup A_i) = 0$. Hence, a.s. there exists $N(\alpha)$ such that for all $n > N(\alpha)$, we have $\tilde{Y}_n = Y_n$. But since $\mathbb{E}(\tilde{Y}_n) \to \mathbb{E}(Y_1) = 0$ due to the dominated convergence theorem, it is sufficient to prove that

$$\frac{\sum_{i=1}^n \left(\tilde{Y}_i - \mathbb{E}(\tilde{Y}_i) \right)}{n} \xrightarrow[n]{} 0.$$

Due to corollary 3.22 in [2]†, it is again sufficient to prove that

$$\sum_{i=1}^{\infty} \frac{\mathbb{E}(\tilde{Y}_i^{\,2})}{i^2} < \infty.$$

Indeed, we have that

$$\sum_{i=1}^{\infty} \frac{\mathbb{E}(\tilde{Y}_i^{\,2})}{i^2} \leq 2m.$$

The underlying calculation can be found in the proof of theorem 3.30 in [2]. $\square$

Remark 4.1. In the case $m < \infty$ and $X_t \to -\infty$, we have that

$$\frac{1}{t}\mathcal{M}_t^{(+)} \to 0 \text{ a.s. as } t \to \infty,$$

since eventually X_t remains negative. Furthermore, if we assume that $\int_{-\infty}^{0} \alpha\nu(d\alpha) > -\infty$ then the previous Lemma implies that

$$\frac{\mathcal{M}_t}{t} \to 0 \text{ a.s. as } t \to \infty,$$

whether $X_t \to -\infty$ or not. But we refrain from adding any unnecessary assumption on ν.

Theorem 4.1. *In the case $v > m$, $X_t \to -\infty$ with speed $v - m$ in the sense that*

$$\frac{X_t}{t} \xrightarrow[t \to \infty]{} m - v \text{ a.s.}$$

Proof. We have that

$$X_t = X_0 - vt + \int_{[0,t]\times\mathbb{R}\times[0,1]} \alpha\varphi(X_{s-},\alpha,\xi)M(ds,d\alpha,d\xi)$$

$$\leq X_0 - vt + \int_0^t m^{(-)}(X_s)ds + \mathcal{M}_t^{(-)},$$

since we have deleted negative jumps (recall the notation defined before Lemma 4.1). Hence

$$\frac{X_t}{t} = \frac{X_0}{t} - v + \frac{1}{t}\int_0^t m(X_s)ds + \frac{\mathcal{M}_t}{t} \leq \frac{X_0}{t} - v + m + \frac{\mathcal{M}_t^{(-)}}{t}.$$

---

†In the proof of this Corollary, we replace Kolmogorov's inequality by Doob's inequality for martingales, and the result holds in our case.

It follows from Lemma 4.1 that

$$\limsup_{t\to\infty} \frac{X_t}{t} \le -v + m \text{ a.s.}$$

In particular, $X_t \to -\infty$ a.s. as $t \to \infty$, and combined with (14), we deduce that

$$\forall \epsilon > 0 \quad \exists t_\epsilon \text{ such that } \forall s \ge t_\epsilon, \ m(X_s) > m - \epsilon.$$

Then, $\forall \epsilon > 0$ and $t > t_\epsilon$

$$\frac{X_t}{t} \ge \frac{X_{t_\epsilon}}{t} + (m - \epsilon - v) \times \frac{t - t_\epsilon}{t} + \frac{\mathcal{M}_t - \mathcal{M}_{t_\epsilon}}{t}.$$

Hence,

$$\liminf_{t\to\infty} \frac{X_t}{t} \ge -v + m \text{ a.s.}$$

We conclude that $X_t \to -\infty$ a.s. with speed $v - m$. $\qquad\square$

4.2. The case $v < m$

Define for all $x \in \mathbb{R}$

$$\psi(x) = m(x) - v. \tag{16}$$

In this section, we assume that either $\nu(\mathbb{R}) < \infty$, or else the following mild growth condition on $g(x, \alpha)$ for small $x > 0$, which is satisfied in case g is given by (3) and (2) : for some $\delta > 0$, $c > 0$,

$$|g(x, \alpha)| \le c|x|, \quad \text{for all } 0 < x \le \delta. \tag{17}$$

Theorem 4.2. *In the case $v < m$, if (17) is satisfied, then X_t is recurrent in the sense of Harris. Moreover, X_t possesses a unique invariant probability measure.*

Proof. **Step 1:** *X_t returns to $[-K, K]$ in finite time*

We have

$$|X_t| = |X_0| + \int_0^t \text{sign}(X_s)\psi(X_s)ds + \int_0^t \text{sign}(X_{s-})d\mathcal{M}_s$$

$$+ \sum_{s \leq t} (|X_{s-} + \Delta X_s| - |X_{s-}| - \text{sign}(X_{s-}\Delta X_s))$$

$$= |X_0| + \int_0^t \text{sign}(X_s)\psi(X_s)ds + \int_0^t \text{sign}(X_{s-})d\mathcal{M}_s$$

$$+ 2\sum_{s \leq t} \mathbf{1}_{\{X_{s-}(X_{s-} + \Delta X_s) < 0\}}|X_{s-} + \Delta X_s|$$

$$= |X_0| + \int_0^t \left(\text{sign}(X_s)\psi(X_s) + \Phi(X_s)\right)ds + M_t^\star,$$

where

$$M_t^\star = \int_0^t \text{sign}(X_{s-})d\mathcal{M}_s + 2\int_{[0,t]\times\mathbb{R}\times[0,1]} |X_{s-}+\alpha|\mathbf{1}_{\{X_{s-}(X_{s-}+\alpha)<0\}}\varphi(X_{s-},\alpha,\xi)\bar{M}(ds,d\alpha,d\xi),$$

and

$$\Phi(x) = \begin{cases} -2\int_{-2x}^{-x}(x+\alpha)g(x,\alpha)\nu(d\alpha), & \text{if } x > 0; \\ 2\int_{-x}^{-2x}(x+\alpha)g(x,\alpha)\nu(d\alpha), & \text{if } x < 0. \end{cases}$$

Furthermore, if $\Psi(x) = \text{sign}(x)\psi(x) + \Phi(x)$, we have for $x > 0$

$$\Psi(x) = -v + \int_{-x}^0 \alpha g(x,\alpha)\nu(d\alpha) + \int_{-2x}^{-x}(|\alpha| - 2x)g(x,\alpha)\nu(d\alpha)$$

$$\leq -v + \int_{-x}^0 \alpha g(x,\alpha)\nu(d\alpha),$$

and for $x < 0$,

$$\Psi(x) = v - \int_0^{|x|} \alpha g(x,\alpha)\nu(d\alpha) + \int_{|x|}^{2|x|}(\alpha - 2|x|)g(x,\alpha)\nu(d\alpha)$$

$$\leq v - \int_0^{|x|} \alpha g(x,\alpha)\nu(d\alpha).$$

For $x > 0$, $\Psi(x) \leq -v$, and $\limsup_{x\to-\infty} \Psi(x) \leq v - m$. Consequently if we let

$$c_1 = \begin{cases} \inf\left(v, \frac{m-v}{2}\right), & \text{if } m < \infty, \\ v, & \text{if } m = \infty, \end{cases}$$

there exists $K > 0$ sufficiently large such that for all $|x| > K$,

$$\Psi(x) = \text{sign}(x)\psi(x) + \Phi(x) \leq -c_1.$$

We now show that for some $C > 0$,

$$0 \le \Phi(x) \le C(1 + |x|), \quad \text{for all } x \in \mathbb{R}. \tag{18}$$

Indeed, for all (α, x) such that $\alpha(x + \alpha) < 0$ and $g(x, \alpha) > 0$, $|x + \alpha| \le |x| \le |\alpha|$. Now if $|x| \le 1$,

$$\Phi(x) \le 2 \int_{-2}^{2} |\alpha| \nu(d\alpha) < \infty,$$

and if $|x| > 1$,

$$\Phi(x) \le 2|x| \int_{[-1,1]^c} \nu(d\alpha).$$

(18) follows from the last two inequalities. We can deduce from Proposition 3.2, Corollary 3.1 and (18) that $M_t^\star$ is a martingale. Let us now assume that $|X_0| > K$ and define

$$T_K = \inf\{t > 0, |X_t| \le K\}$$

From Doob's optional sampling theorem,

$$\mathbb{E}|X_{t \wedge T_K}| \le |X_0| - c_1 \, \mathbb{E}(t \wedge T_K),$$

yielding

$$\mathbb{E}T_K < \frac{|X_0|}{c_1} < \infty.$$

Step 2: X_t *visits* $[0, K + vT]$ *infinitely often*

Here $K' = K + vT$ where $T > 0$ and p is the lower bound of the probability that, starting from any given point $x \in [-K, 0)$ at time t_0, X hits $[0, K']$ before time $t_0 + T$. Clearly we can choose T such that $p > 0$. We now define a random variable β as follows:

If $X_{T_K} \in [0, K]$, then $\beta = 0$. Otherwise, we restart our process X at time $t_0 = T_K$ from $x_0 \in [-K, 0)$. If X hits $[0, K']$ before time T, then $\beta = 1$. If not, we look at the position X_T of X at time T. Two cases are possible:

- If $|X_T| > K$, we wait until X enters $[-K, K]$. Since $|X_T| \le K'$, the time α_2 needed to do so satisfies (see Step 1)

$$\mathbb{E}(\alpha_2) \le \frac{K'}{c_1}.$$

- If $-K \leq X_T < 0$, we start afresh from there, since the probability to reach $[0, K']$ in a time less than T is greater than or equal to p.

So either at time T or at time $T + \alpha_2$, we start again from a position belonging to the interval $[-K, K]$. If $[0, K']$ is reached during the next time interval of length T, then $\beta = 2$. If not, we repeat the procedure. A.s. one of the mutually independent trials is successful. We have that

$$T_{K'}^+ \leq T_K + \sum_{i=1}^{\beta} (T + \alpha_i),$$

where $T_{K'}^+ = \inf \{t > 0; \ X_t \in [0, K']\}$ and β is a stopping time associated with the sequence $(\alpha_i)_{i \geq 1}$. It follows from the martingale version of Wald's formula that

$$\mathbb{E}T_{K'}^+ < \mathbb{E}T_K + \frac{1}{p}\left(T + \frac{K'}{c_1}\right),$$

since $\mathbb{P}(\beta > k) \leq (1 - p)^k$, hence $\mathbb{E}\beta < 1/p$.

Step 3: X_t *hits zero infinitely often*

If $\nu(\mathbb{R}_-) < \infty$, starting from any point in $(0, K']$ at time 0, there is a no jump of X_t before it hits 0 with probability

$$\exp\left(-\nu(\mathbb{R}_-)\frac{K'}{v}\right).$$

If $\nu(\mathbb{R}_-) = \infty$, we choose $0 < \delta < K'$ such that (17) is satisfied and

$$q = \frac{c\delta}{v}\int_{-\delta}^{0} |\alpha|\nu(d\alpha) < 1.$$

Let A_δ be the event that there is no jump of size $< -\delta$ before X_t hits $\mathbb{R}_-$.

$$\mathbb{P}(A_\delta) \leq \exp\left(-\nu(-\infty, -\delta)\frac{K'}{v}\right).$$

Moreover for $0 < x < \delta$

$$
\begin{aligned}
\mathbb{P}_x(X_t \text{ jumps over } 0 \mid A_\delta) &\leq \mathbb{E}\int_{[0, \frac{\delta}{v}]\times[-\delta, -X_{s-}]\times[0,1]} \mathbf{1}_{X_{s-}<0}\varphi(X_{s-}, \alpha, \xi)M(ds, d\alpha, d\xi) \\
&= \mathbb{E}\int_0^{\frac{\delta}{v}} \mathbf{1}_{X_s<0}\int_{-\delta}^{-X_s} g(X_s, \alpha)\nu(d\alpha)ds \\
&\leq c\int_0^{\frac{\delta}{v}}\int_{-\delta}^{0} |\alpha|\nu(d\alpha)ds = q.
\end{aligned}
$$

If $p = \mathbb{P}(A_\delta^c)$ then, starting from any point in $(0, K']$,

$$\mathbb{P}(X_t \text{ hits } \mathbb{R}_- \text{ without visiting } 0) \le p + (1 - p)q < 1.$$

Since X_t visits $(0, K']$ infinitely often, and hits 0 with positive probability after any such visit, the same argument as above yields that for any $x \in \mathbb{R}$, $\mathbb{E}_x T_0 < \infty$, where $T_0 = \inf\{t > 0;\ X_t = 0\}$. In particular, the process is Harris recurrent, since it satisfies condition (b) on page 490 of Meyn and Tweedie [14] with $\mu = \delta_0$.

Step 4: *Existence of an invariant probability measure*

Let X_t start from $X_0 = 0$ and define

$$T_+ = \inf\{t > 0;\ X_t \ge 0\}.$$

We want to show that $T_+ > 0$ a.s. Let $\xi_t = \mathbf{1}_{\{T_+ \le t\}}$. We have

$$\xi_t = \int_{[0,t] \times \mathcal{R}_+ \times [0,1]} \mathbf{1}_{\{X_{s-} \le 0\}} \mathbf{1}_{\{\alpha \ge -X_{s-}\}} \varphi(X_{s-}, \alpha, \xi) M(ds, d\alpha, d\xi).$$

Hence

$$
\begin{aligned}
\mathbb{E}\xi_t &= \mathbb{E} \int_0^t \mathbf{1}_{\{X_{s-} \le 0\}} \int_{-X_s}^{-2X_s} g(X_s, \alpha)\, \nu(d\alpha)\, ds \\
&\le c\, \mathbb{E} \int_0^t \mathbf{1}_{\{X_{s-} \le 0\}} \int_{-X_s}^{-2X_s} |X_s|\, \nu(d\alpha)\, ds \\
&\le c\, t \int_0^{2vt} \alpha\, \nu(d\alpha),
\end{aligned}
$$

where c is the constant defined by condition (17). Thus

$$\mathbb{P}(T_+ \le t) = \mathbb{P}(\xi_t = 1) = \mathbb{E}\xi_t \le c\, t \int_0^{2vt} \alpha\, \nu(d\alpha),$$

yielding $\mathbb{P}(T_+ > t) \to 1$ as $t \downarrow 0$. Since $T_0 = \inf\{t > 0;\ X_t = 0\} \ge T_+$, $T_0 > 0$ $\mathbb{P}_0$ a.s.

Define the measure μ on $(\mathbb{R}, \mathcal{B})$ by

$$\mu(A) = \mathbb{E}_0 \int_0^{T_0} \mathbf{1}_A(X_s) ds.$$

It follows from Step 3 that $\mu(\mathbb{R}) = \mathbb{E}_0(T_0) < \infty$, hence μ is a finite measure and we define the probability measure

$$\bar\mu(A) = \frac{\mu(A)}{\mu(\mathbb{R})}.$$

$\bar{\mu}$ is invariant under the semi group P_t associated to the Markov process X_t. Indeed for any t,

$$\mathbb{E}_0 \int_0^t \mathbf{1}_A(X_s)ds = \mathbb{E}_0 \int_{T_0}^{T_0+t} \mathbf{1}_A(X_s)ds.$$

Hence,

$$\begin{aligned}
\mathbb{E}_0 \int_0^{T_0} \mathbf{1}_A(X_s)ds &= \mathbb{E}_0 \int_t^{T_0+t} \mathbf{1}_A(X_s)ds \\
&= \mathbb{E}_0 \int_0^{T_0} \mathbf{1}_A(X_{t+s})ds \\
&= \int_0^\infty \mathbb{P}_0(X_{t+s} \in A, \ s < T_0)ds \\
&= \int_0^\infty \int_{\mathbb{R}} \mathbb{P}_0(X_s \in dz, \ s < T_0)\mathbb{P}_z(X_t \in A)ds \\
&= \int_{\mathbb{R}} \mathbb{P}_z(X_t \in A)\mu(dz).
\end{aligned}$$

Step 5: *Uniqueness of the invariant probability measure*

Since an invariant probability measure exists, there exists an invariant ergodic probability measure which we again denote by $\bar{\mu}$. From the ergodic theorem, if f is continuous and bounded, as $t \to \infty$

$$\frac{1}{t} \int_0^t f(X_s)ds \to \int f(x)\bar{\mu}(dx) \ \mathbb{P}_{\bar{\mu}} \text{ a.s.}$$

Now, if $t > T_0$

$$\frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{t} \int_0^{T_0} f(X_s)ds + \frac{1}{t} \int_{T_0}^t f(X_s)ds.$$

Consequently, as $t \to \infty$

$$\frac{1}{t} \int_{T_0}^t f(X_s)ds \to \int f(x)\bar{\mu}(dx).$$

Thus, if $\bar{\mu}'$ is another ergodic invariant probability measure,

$$\int f(x)\bar{\mu}(dx) = \int f(x)\bar{\mu}'(dx),$$

for all f continuous and bounded. Uniqueness of the ergodic invariant probability measure and hence of the invariant probability measure follows.

$\square$

Remark 4.2. Note that condition (17) is rather weak. It is hard to find an example of a g that would satisfy (4), but not (17). This new condition is not really necessary. It simplifies our proof, since it allows us to prove that 0 is visited infinitely many times. We have decided to adopt it, since it is not a serious restriction. At any rate, Steps 1 and 2 of our proof do not necessitate this assumption.

4.3. The case $v = m$

We first state a lemma that we will apply several times in this section.

Lemma 4.3. *Let X_t be a finite variation càdlàg process.*

1. *If $\Phi \in C^1$, then*

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \Phi'(X_{s-})dX_s + \sum_{s \leq t, \Delta X_s \neq 0} \Phi(X_{s-} + \Delta X_s) - \Phi(X_{s-}) - \Phi'(X_{s-})\Delta X_s,$$

   where $\Delta X_s = X_s - X_{s-}$, $\forall s$.

2. *Moreover, if $\Phi \in C^2$ such that Φ'' is an increasing function and $\Delta X_s \geq 0$ for all s, then*

$$\Phi(X_t) - \Phi(X_0) - \int_0^t \Phi'(X_{s-})dX_s \leq \frac{1}{2} \sum_{s \leq t, \Delta X_s \neq 0} \Phi''(X_{s-} + \Delta X_s)(\Delta X_s)^2.$$

   If $\Phi \in C^2$ such that Φ'' is a decreasing function and $\Delta X_s \geq 0$ for all s, then

$$\Phi(X_t) - \Phi(X_0) - \int_0^t \Phi'(X_{s-})dX_s \leq \frac{1}{2} \sum_{s \leq t, \Delta X_s \neq 0} \Phi''(X_{s-})(\Delta X_s)^2.$$

   In particular, choosing $\Phi(x) = x^2$, we deduce that

$$X_t^2 = X_0^2 + 2 \int_0^t X_{s-}dX_s + \sum_{s \leq t} (\Delta X_s)^2. \tag{19}$$

Proof. The first part of this lemma is a well known result (see [16]). We will only prove part 2 of the lemma. If $\Phi \in C^2$ then it follows from Taylor's formula that there exists a random function β taking its values in $[0, 1]$ such that for all s

$$\Phi(X_s) - \Phi(X_{s-}) - \Phi'(X_{s-})\Delta X_s = \frac{1}{2}\Phi''(X_{s-} + \beta_s \Delta X_s)(\Delta X_s)^2.$$

If Φ'' is an increasing function and $y \geq 0$ then

$$\Phi''(x) \leq \Phi''(x + \beta_s y) \leq \Phi''(x + y).$$

If Φ'' is a decreasing function and $y \geq 0$ then

$$\Phi''(x + y) \leq \Phi''(x + \beta_s y) \leq \Phi''(x).$$

$\square$

Note that $V \leq \infty$ and at this stage we do not assume that V is finite. In the case $m = v$, the asymptotic behavior of the process X_t depends on the asymptotic behavior of the mean net rate of adaptation $\psi(x)$ defined in (16) as $x \to -\infty$.

Theorem 4.3. *We assume that $m = v$ and that*

$$supp(\nu) \subset (-\infty, K], \ for \ some \ K > 0. \tag{20}$$

If moreover

$$\limsup_{x \to -\infty} |x\psi(x)| < \frac{V}{2}, \tag{21}$$

then the process X_t is Harris recurrent but the mean return time to a compact is infinite.

Proof. First note that, since $m = v$ implies $\psi(x) \leq 0$ for all $x \leq 0$, condition (21) is equivalent to

$$\liminf_{x \to -\infty} |x|\psi(x) > -\frac{V}{2}.$$

To prove recurrence under condition (21), we recall that

$$X_t = X_0 + \int_0^t \psi(X_s)ds + \mathcal{M}_t. \tag{22}$$

We will apply Lemma 4.3 with $\Phi(x) = \log|x|$, with $x < 0$. Here Φ'' is decreasing.

Hence as long as X_t remains negative,

$$\log|X_t| \leq \log|X_0| + \int_0^t \frac{\psi(X_s)}{X_s}ds + \int_0^t \frac{1}{X_{s-}}d\mathcal{M}_s - \frac{1}{2}\sum_{s \leq t}\frac{(\Delta X_s)^2}{X_{s-}^2}$$

$$= \log|X_0| + \int_0^t \frac{\psi(X_s)}{X_s}ds + \int_0^t \frac{1}{X_{s-}}d\mathcal{M}_s$$

$$- \frac{1}{2}\int_0^t \int_{\mathbb{R}_+}\int_0^1 \frac{\alpha^2\varphi(X_{s-},\alpha,\xi)}{X_{s-}^2}\bar{M}(ds,d\alpha,d\xi) - \frac{1}{2}\int_0^t \frac{V(X_s)}{X_s^2}ds$$

$$= \log|X_0| + \int_0^t \left(\frac{\psi(X_s)}{X_s} - \frac{V(X_s)}{2X_s^2}\right)ds + \hat{\mathcal{M}}_t,$$

where $\hat{\mathcal{M}}$ is a martingale. For all $a < b < 0$, define the stopping time

$$S_{a,b} = \inf(t > 0, X_t \leq a \text{ or } X_t \geq b).$$

It follows from our assumption that there exists $L > 0$ such that

$$\inf_{x \leq -L}\left\{|x|\psi(x) + \frac{V(x)}{2}\right\} > 0. \tag{23}$$

For any $N > L$, from Doob's optional sampling theorem, if $-N < X_0 < -L < -K-1$,

$$\mathbb{E}\log|X_{t\wedge S_{-N,-L}}| \leq \log|X_0| + \mathbb{E}\int_0^{t\wedge S_{-N,-L}}\left(\frac{\psi(X_s)}{X_s} - \frac{V(X_s)}{2X_s^2}\right)ds.$$

Letting t tend to ∞,

$$\mathbb{E}\log|X_{S_{-N,-L}}| \leq \log|X_0|.$$

Define the stopping times

$$T_{-L}^{\uparrow} = \inf\{t > 0, X_t \geq -L\},$$

$$T_{-N}^{\downarrow} = \inf\{t > 0, X_t \leq -N\}.$$

From (20) and the condition on L, $\log|X_{S_{-N,-L}}| > 0$. It follows from the previous estimate that

$$\log N \times \mathbb{P}(T_{-N}^{\downarrow} < T_{-L}^{\uparrow}) < \log|X_0|.$$

By the same argument as in the proof of Theorem 4.2, X_t will visit $[-L, L]$, thus also $(0, L)$ infinitely often, and also 0. Note that the process remains in $(-\infty, L]$ when starting there. Therefore, the process X is Harris recurrent.

Let now $X_0 < -(L+1)$. For all $N > L$, multiplying (22) by -1, we have

$$-X_{t\wedge S_{-N,-L}} = |X_0| - \int_0^{t\wedge S_{-N,-L}}\psi(X_s)ds - \int_0^{t\wedge S_{-N,-L}}d\mathcal{M}_s,$$

By Doob's theorem and letting t tend to ∞, since again $\psi(x) \le 0$ for $x \le 0$

$$-\mathbb{E}X_{S_{-N,-L}} = |X_0| - \mathbb{E}\int_0^{S_{-N,-L}} \psi(X_s)ds \ge |X_0|, \text{ hence}$$

$$L\mathbb{P}(T^{\uparrow}_{-L} < T^{\downarrow}_{-N}) + N\mathbb{P}(T^{\downarrow}_{-N} < T^{\uparrow}_{-L}) \ge |X_0|,$$

since $X_{S_{-N,-L}} = -N$ on the event $\left\{T^{\downarrow}_{-N} < T^{\uparrow}_{-L}\right\}$, and $X_{S_{-N,-L}} \ge -L$ on the complementary event. We have

$$\liminf_{N\to\infty} N\mathbb{P}(T^{\downarrow}_{-N} < T^{\uparrow}_{-L}) \ge |X_0| - L > 0. \tag{24}$$

It follows from Lemma 4.3 that

$$X_t^2 = X_0^2 - \int_0^t 2|X_s|\psi(X_s)ds + \int_0^t 2X_{s-}d\mathcal{M}_s + \sum_{s\le t}(\Delta X_s)^2.$$

On the other hand, for $t \le S_{-N,-L}$,

$$\sum_{s\le t}(\Delta X_s)^2 = \int_0^t \int_{\mathbb{R}_+} \int_0^1 \alpha^2 \varphi(X_{s-},\alpha,\xi)\bar{M}(ds,d\alpha,d\xi)$$

$$+ \int_0^t \int_{\mathbb{R}_+} \alpha^2 g(X_{s-},\alpha)\nu(d\alpha)ds.$$

Thus, from (23) and the fact that from (20), $V(x) \le V < \infty$,

$$X_{t\wedge S_{-N,-L}}^2 \le X_0^2 + 2V \times (t \wedge S_{-N,-L}) + \tilde{\mathcal{M}}_{t\wedge S_{-N,-L}},$$

where $\tilde{\mathcal{M}}_{\cdot\wedge S_{-N,-L}}$ is a martingale. Letting t tend to ∞, we have for all $\epsilon > 0$

$$\mathbb{E}X_{S_{-N,-L}}^2 \le X_0^2 + 2V\,\mathbb{E}S_{-N,-L}, \text{ hence}$$

$$\mathbb{E}S_{-N,-L} \ge \frac{N^2\mathbb{P}(T^{\downarrow}_{-N} < T^{\uparrow}_{-L}) - X_0^2}{2V}.$$

It follows by monotone convergence that

$$\mathbb{E}(T^{\uparrow}_{-L}) = \lim_{N\to\infty}\mathbb{E}S_{-N,-L} \ge \liminf_{N\to\infty}\left\{N\mathbb{P}(T^{\downarrow}_{-N} < T^{\uparrow}_{-L}) \times \frac{N}{2V} - \frac{X_0^2}{2V}\right\}.$$

Combining this with (24), we deduce that $\mathbb{E}T^{\uparrow}_{-L} = \infty$. In other words, the return times to compacts have infinite expectation. □

Using similar arguments as in Step 4 of the proof of Theorem 4.2, one should be able to conclude that X_t has an infinite invariant measure, which is unique up to a multiplicative constant.

Remark 4.3. Condition (21) is rather weak. Under the current assumption on the support of ν, it can be shown to hold in particular if ν is finite and g given by (3) and (2). On the other hand, our very strong condition (20) on the support of ν may not be necessary. It might however very well be that in case $m = v$ a stronger condition on the tail of the law of ν than in case $m > v$ is necessary for recurrence. We do not know what is the optimal condition.

We now consider the case $m = v$ and $\liminf_{x \to -\infty} |x\psi(x)| > \frac{V}{2}$, which implies in particular that $V < \infty$.

Theorem 4.4. *Assume that $m = v$ and*

$$\liminf_{x \to -\infty} |x\psi(x)| > \frac{V}{2}. \tag{25}$$

If, moreover, there exist $0 < p_0 < 1$ and $0 < \beta_0 < 1$ such that for all $0 < \beta < \beta_0$

$$|x|^{p_0+2} \int_{-\beta x}^{\infty} \alpha^2 g(x, \alpha)\nu(d\alpha) \xrightarrow[x \to -\infty]{} 0, \tag{26}$$

then X_t is transient, that is $X_t \to -\infty$ a.s., and moreover $\frac{X_t}{t} \to 0$ a.s.

Remark 4.4. The conditions of Theorem 4.4 are satisfied in the case where both ν is infinite and its tail is not too heavy, while g is given by (3) and (2). For example, if

$$\nu(d\alpha) = \left(\frac{1}{\alpha^{1+\delta}} \mathbf{1}_{|\alpha|<1} + \rho(\alpha)\mathbf{1}_{|\alpha|>1} \right) d\alpha,$$

where $\rho(\alpha) \leq C|\alpha|^{-(5+\delta')}$, $|\alpha| > 1$ for some $\delta, \delta' > 0$. Condition (25) follows from the fact that $V < \infty$ while $|x\psi(x)| \to \infty$ as $|x| \to \infty$.

Condition (26) is easy to check.

Proof. First note that condition (25) is equivalent to

$$\limsup_{x \to -\infty} |x|\psi(x) < -\frac{V}{2}.$$

Hence there exist $K > 0$ and $0 < p \leq p_0$ such that

$$\sup_{x \leq -K} \left\{ |x|\psi(x) + (2p+1)\frac{V(x)}{2} \right\} < 0. \tag{27}$$

Let f be the $C^2(\mathbb{R})$-function such that $f(-1) = 1$, $f'(-1) = p$, and

$$f''(x) = \frac{p(p+1)}{|x|^{p+2}}\mathbf{1}_{\{x \leq -1\}} + p(p+1)\mathbf{1}_{\{x \geq -1\}},$$

with p being a real number in $(0, 1)$ for which (27) holds. Then it follows from Lemma 4.3 applied to f, since f'' is an increasing function, as long as $X_s \leq 0$ for $s \leq t$

$$f(X_t) \leq f(X_0) + \int_0^t \psi(X_s)f'(X_s)ds + \frac{1}{2}\int_0^t \int_0^\infty f''(X_s + \alpha)\alpha^2 g(X_s, \alpha)\nu(d\alpha)ds + \mathcal{N}_t,$$

where the martingale $\mathcal{N}$ is defined by

$$\mathcal{N}_t = \frac{1}{2}\int_0^t \int_0^\infty \int_0^1 \left[f'(X_{s-}) + f''(X_{s-} + \alpha)\alpha^2 \right] \varphi(X_{s-}, \alpha, \xi)\bar{M}(ds, d\alpha, d\xi).$$

Let us admit for the moment:

Lemma 4.4. If (26) holds, then

$$\lim_{x \to -\infty} |x|^{p+2} \int_0^\infty f''(x + \alpha)\alpha^2 g(x, \alpha)\nu(d\alpha) = p(p+1)V.$$

This implies that

$$\lim_{x \to -\infty} |x|^{p+2} \int_0^\infty f''(x + \alpha)\alpha^2 g(x, \alpha)\nu(d\alpha) < \lim_{x \to -\infty} p(2p+1)V(x).$$

Hence, there exists $N \geq K$ such that for all $x \leq -N$,

$$\int_0^\infty f''(x + \alpha)\alpha^2 g(x, \alpha)\nu(d\alpha) < p(2p+1)\frac{V(x)}{|x|^{p+2}}.$$

Thus, for all $k > 0$ satisfying $-kN < X_0 < -N$,

$$f(X_{t \wedge S_{-kN, -N}}) \leq f(X_0) + \int_0^{t \wedge S_{-kN, -N}} \frac{p}{|X_s|^{p+1}}\left[\psi(X_s) + (2p+1)\frac{V(X_s)}{2|X_s|} \right] ds$$

$$+ \mathcal{N}_{t \wedge S_{-kN, -N}}.$$

Now if $k \geq 3$, letting $X_0 = -2N$, it follows from (27) that

$$\mathbb{E}(f(X_{t \wedge S_{-kN, -N}})) \leq \frac{1}{(2N)^p}.$$

Thus, if we let t tend to ∞,

$$\frac{1}{N^p}\mathbb{P}(S_{-kN,-N} = T^{\uparrow}_{-N})) \leq \mathbb{E}\frac{1}{|X_{S_{-kN,-N}}|^p} \leq \frac{1}{(2N)^p}.$$

Now letting k tend to ∞,

$$\mathbb{P}(T^{\uparrow}_{-N} < \infty) \leq \frac{1}{2^p}. \tag{28}$$

Let for all $k \geq 1$

$$B_k = \left\{T^{\uparrow}_{-N} \geq T^{\downarrow}_{-kN}\right\}$$

The $(B_k)_{k\geq 1}$ is a decreasing sequence of sets such that

$$\lim_k \mathbb{P}(B_k) = \mathbb{P}(B),$$

where $B = \{X_t \text{ hits } -kN \text{ before } -N \text{ for all } k \geq 1\} \subset \left\{X_t \xrightarrow[t\to\infty]{} -\infty\right\}$. It follows from (28) that

$$\mathbb{P}(X_t \xrightarrow[t\to\infty]{} -\infty) > \mathbb{P}(B) \geq 1 - \frac{1}{2^p}.$$

On B^c, X_t enters $(-N, +\infty)$. The arguments from section 4.2 show that sooner or later the process X_t will hit $-2N$ again, and from there the probability of going to $-\infty$ is bounded from below by $1 - 1/2^p$ since this will happen each time the process gets above $-N$, hence

$$X_t \xrightarrow[t\to\infty]{} -\infty \text{ a.s.}$$

And since $m = v < \infty$, it follows from Lemma 4.1 and Remark 4.1 that $\frac{\mathcal{M}_t}{t} \to 0$, hence by the same arguments as in the proof of Therorem 4.1.

$$\frac{X_t}{t} \xrightarrow[t\to\infty]{} 0 \text{ a.s..}$$

$\square$

Proof of Lemma 4.4. For any $0 < \beta < \beta_0 < 1$, if $x < -(1-\beta)^{-1}$,

$$|x|^{p+2}\int_0^\infty f''(x+\alpha)g(x,\alpha)\alpha^2\nu(d\alpha) = |x|^{p+2}\int_0^{-\beta x} f''(x+\alpha)\alpha^2 g(x,\alpha)\nu(d\alpha)$$
$$+ |x|^{p+2}\int_{-\beta x}^\infty f''(x+\alpha)\alpha^2 g(x,\alpha)\nu(d\alpha)$$
$$\leq \int_0^{-\beta x} \frac{p(p+1)}{(1-\beta)^{p+2}}\alpha^2 g(x,\alpha)\nu(d\alpha)$$
$$+ |x|^{p+2}p(p+1)\int_{-\beta x}^\infty \alpha^2 g(x,\alpha)\nu(d\alpha).$$

On the other hand,

$$|x|^{p+2} \int_0^\infty f''(x+\alpha)\alpha^2 g(x,\alpha)\nu(d\alpha) \geq p(p+1) \int_0^{-\beta x} \frac{|x|^{p+2}}{|x+\alpha|^{p+2}} \alpha^2 g(x,\alpha)\nu(d\alpha)$$

$$> p(p+1) \int_0^{-\beta x} \alpha^2 g(x,\alpha)\nu(d\alpha).$$

Letting $x \to -\infty$ in the two above inequalities, we deduce from (26), which holds with p_0 replaced by $p \leq p_0$,

$$p(p+1)V \leq \liminf_{x \to -\infty} |x|^{p+2} \int_0^\infty f''(x+\alpha)\alpha^2 g(x,\alpha)\nu(d\alpha)$$

$$\leq \limsup_{x \to -\infty} |x|^{p+2} \int_0^\infty f''(x+\alpha)\alpha^2 g(x,\alpha)\nu(d\alpha)$$

$$\leq \frac{p(p+1)}{(1-\beta)^{p+2}} V.$$

Thus, letting $\beta \to 0$, it follows that

$$|x|^{p+2} \int_0^\infty f''(x+\alpha)\alpha^2 g(x,\alpha)\nu(d\alpha) \xrightarrow[x \to -\infty]{} p(p+1)V.$$

$\square$

Remark 4.5. We have not been able to precise the large time behavior of the process X_t when the measure ν is of the type

$$\nu(d\alpha) \approx \frac{d\alpha}{\alpha^{2+\delta}} \mathbf{1}_{\{\alpha \geq 1\}}, \quad 0 < \delta \leq \frac{1}{2},$$

which still satisfies $m < \infty$. In this case, $V = \infty$, $|x\psi(x)| \to \infty$ as $|x| \to \infty$, and (26) also fails.

5. Generalization to the case of a time-variable speed

In the following, we treat the case where the speed of environmental change is a random function of time $\mathcal{V}_t$, $\mathcal{V}_t$ being $\mathcal{F}_t$ progressively measurable, where again $\mathcal{F}_t$ is such that our Poisson Point Process M satisfies both that $M|_{[0,t] \times \mathbb{R} \times [0,1]}$ is $\mathcal{F}_t$ measurable and $M|_{(t,+\infty) \times \mathbb{R} \times [0,1]}$ is independent of $\mathcal{F}_t$. The stochastic equation describing the evolution of phenotypic lag becomes

$$X_t = X_0 - \mathcal{V}_t + \int_0^t m(X_s)ds + \mathcal{M}_t. \tag{29}$$

As above, we study three cases:

5.1. The transient case

Here we assume that there exists a constant $\bar{v} \in \mathbb{R}$ such that

$$\lim_{t \to \infty} \frac{\mathcal{V}_t}{t} = \bar{v} \quad \text{a.s.}$$

It is easy to see that results (14) and (15) hold in the new context of equation (29), provided $\bar{v} > m$. Following the steps of the proof in section 4.1, we can see that $X_t \to -\infty$ a.s. with speed $\bar{v} - m$.

5.2. The recurrent case

In this section we assume that

$$\mathcal{V}_t = \int_0^t v_1(s)ds + \mathcal{M}_V(t),$$

where $\mathcal{M}_V(t)$ is a continuous $\mathcal{F}_t$–martingale, $v_1(t) \to \bar{v}$ a.s., as $t \to \infty$, and for all $T > 0$, $\sup_{0 \le t \le T} \mathbb{E}[|\mathcal{V}_t|] < \infty$.

Under the above conditions, together with $\bar{v} < m$ we will prove that the process X_t is Harris recurrent. We define again

$$T_K = \inf\{t > 0, |X_t| \le K\}.$$

Assume that $|X_0| > K$. We rewrite the same inequality from Step 1 of the proof of Theorem 4.2 :

$$|X_t| \le |X_0| + \int_0^t \big\{ \text{sign}(X_s)[m(X_s) - v_1(s)] + \Phi(X_s) \big\} ds + \int_0^t \text{sign}(X_s)d\mathcal{M}_V(s) + M_t^\star,$$

where $M_t^\star$ is a martingale. We prove again that $\mathbb{E}T_K < \infty$. Then we prove that X_t visits $[0, M + (\bar{v} + 1)T]$ infinitely often using the same arguments as Step 2. The rest of the proof remains unchanged. Thus, X_t is Harris recurrent and the return time to $[-K, K]$ has finite expectation. If $\mathcal{V}_t$ is deterministic, then X_t is a Markov process and we again conclude the existence of a unique invariant probability measure.

5.3. The limiting case

Here we assume that

$$\mathcal{V}_t = \int_0^t v_1(s)ds, \quad \text{and } v_1(t) \to \bar{v} \text{ a.s. as } t \to \infty.$$

Define moreover

$$v_{\sup} = \sup_s v_1(s),$$

$$v_{\inf} = \inf_s v_1(s),$$

$$\psi_{\sup}(x) = m(x) - v_{\sup},$$

$$\psi_{\inf}(x) = m(x) - v_{\inf},$$

We formulate two sets of assumptions:

Assumptions A

- $v_{\sup} < \infty,$

- $\liminf\limits_{x \to -\infty} |x|\psi_{\sup}(x) > -\frac{V}{2}.$

Assumptions B

- $v_{\inf} < \infty,$

- $\limsup\limits_{x \to -\infty} |x|\psi_{\inf}(x) < -\frac{V}{2}.$

Under the set of assumptions A and hypothesis (20), we can prove that the process is Harris recurrent. We have, however, not been able to prove that the return time to compacts has infinite expectation.

Ideas of Proof. Apply Lemma 4.3 to the process in equation (29) with $f(x) = \log|x|$, with $x < 0$. Here f'' is decreasing. Hence, as long as X_t remains negative,

$$\log|X_t| \le \log|X_0| + \int_0^t \left(\frac{\psi_{\sup}(X_s)}{X_s} - \frac{V(X_s)}{2X_s^2} \right) ds + \int_0^t \frac{v_{\sup} - v_1(s)}{X_s} ds + \mathcal{M}'_t$$

$$< \log|X_0| + \int_0^t \left(\frac{\psi_{\sup}(X_s)}{X_s} - \frac{V(X_s)}{2X_s^2} \right) ds + \mathcal{M}'_t,$$

where $\mathcal{M}'$ is a martingale. Then we continue the proof as for the case of constant speed. □

Under the set of assumptions B and hypothesis (26), we can prove that

$$X_t \xrightarrow[t\to\infty]{} -\infty \quad \text{and} \quad \frac{X_t}{t} \xrightarrow[t\to\infty]{} 0.$$

Ideas of Proof. We take the same function f we constructed in the case of constant speed. We have $f' > 0$, and

$$f(X_t) \leq f(X_0) + \int_0^t \psi_{\inf}(X_s)f'(X_s)ds + \int_0^t (v_{\inf} - v_1(s))f'(X_s)ds$$
$$+ \frac{1}{2}\int_0^t \int_0^\infty f''(X_s + \alpha)\alpha^2 g(X_s, \alpha)\nu(d\alpha)ds + \mathcal{N}'_t$$
$$\leq f(X_0) + \int_0^t \psi_{\inf}(X_s)f'(X_s)ds + \frac{1}{2}\int_0^t \int_0^\infty f''(X_s + \alpha)\alpha^2 g(X_s, \alpha)\nu(d\alpha)ds + \mathcal{N}'_t,$$

where $\mathcal{N}'$ is a martingale. Then we continue the proof as for the case of constant speed. □

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