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L^p solutions of backward stochastic differential equations

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Abstract

In this paper, we are interested in solving backward stochastic differential equations (BSDEs for short) under weak assumptions on the data. The first part of the paper is devoted to the development of some new technical aspects of stochastic calculus related to BSDEs. Then we derive a priori estimates and prove existence and uniqueness of solutions in L^p $p>1$, extending the results of El Karoui et al. (Math. Finance 7(1) (1997) 1) to the case where the monotonicity conditions of Pardoux (Nonlinear Analysis; Differential Equations and Control (Montreal, QC, 1998), Kluwer Academic Publishers, Dordrecht, pp. 503–549) are satisfied. We consider both a fixed and a random time interval. In the last section, we obtain, under an additional assumption, an existence and uniqueness result for BSDEs on a fixed time interval, when the data are only in L^1.

Keywords: Backward stochastic differential equation; Monotone generator; p-integrable data

1. Introduction

In this paper, we are concerned with backward stochastic differential equations (BSDEs for short in the remaining); a BSDE is an equation of the following type:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}B_r, \qquad 0 \leqslant t \leqslant T, \tag{1}$$

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where B is a standard Brownian motion and ξ is a random variable measurable with respect to the past of B up to time T. ξ is the terminal condition and f the coefficient (also called the generator). The unknowns are the processes $\{Y_t\}_{t\in[0,T]}$ and $\{Z_t\}_{t\in[0,T]}$, which are required to be adapted with respect to the filtration of the Brownian motion: this is a crucial point.

Such equations, in the nonlinear case, have been introduced by Pardoux and Peng (1990). They proved an existence and uniqueness result under the following assumption: f is Lipschitz continuous in both variables y and z and the data, ξ and the process $\{f(t,0,0)\}_{t\in[0,T]}$, are square integrable.

Since this first existence and uniqueness result, many papers have been devoted to existence and/or uniqueness results under weaker assumptions. Among these papers, we can distinguish two different classes: scalar BSDEs and multidimensional BSDEs. In the first case, one can take advantage of the comparison theorem: we refer to El Karoui et al. (1997) for this result. In this spirit, let us mention the contributions of Kobylanski (1997) and Lepeltier and San Martin (1998) which dealt with quadratic growth generators in z. For multidimensional BSDEs, there is no comparison theorem and to overcome this difficulty a monotonicity assumption on the generator f in the variable y is used. This condition is essential in the study of BSDEs with random terminal time and appears for the first time in this context in a paper by Peng (1991). When the terminal time is deterministic, this condition allows to get rid of the growth condition in the variable y: see the work of Briand and Carmona (2000) for a study of polynomial growth in L^p with $p \geqslant 2$ and the work of Pardoux (1999) for an arbitrary growth.

Let us mention also that when the generator is Lipschitz continuous, a result of El Karoui et al. (1997), provides the existence of a solution when the data ξ and $\{f(t,0,0)\}_{t\in[0,T]}$ are in L^p even for $p\in(1,2)$. The first part of this paper is devoted to the generalization of this result to the case of a monotone generator, both for equations on a fixed and on a random time interval.

Let us briefly comment the main issue of our study. Peng (1997) introduced the notion of g-martingales which can be viewed, in some sense, as nonlinear martingales. g-martingales are solutions to BSDEs. It is not so surprising to consider solutions to BSDEs as "martingales" since in the simplest case, namely when the generator is 0, the solution to the BSDE is the martingale $\mathbb{E}(\xi \mid \mathscr{F}_t)$. Since the classical theory of martingales is carried in the space L^1, the question of solving a BSDE when the data are only integrable comes up naturally. Peng gives an answer for real BSDEs only in the case where $f(t,y,z) = f_1(t,z) + f_2(t,y)$ is Lipschitz in (y,z) with $f_1(t,0) = 0$, $f_2(t,0) \geqslant 0$ and for $\xi \geqslant 0$. One of the objectives of this paper is to prove an existence and uniqueness result for BSDEs in $\mathbb{R}^d$ when ξ and the process $\{f(t,0,0)\}_{t\in[0,T]}$ are integrable with f only monotone in the variable y.

The paper is organized as follows: the next section contains all the notations and some basic identities, while Section 3 contains essential estimates. Section 4 is devoted to the case where the data are in L^p with $p\in(1,2)$ on a fixed time interval, Section 5 with the same problem, but for a BSDE on random time interval, and finally the last section studies the case $p=1$, where an additional assumption on the coefficient is required.

2. Preliminaries

2.1. Notations and definition

First of all, $B=\{B_t\}_{t\geqslant 0}$ is a standard Brownian motion with values in $\mathbb{R}^d$ defined on some complete probability space $(\Omega,\mathscr{F},\mathbb{P})$. $\{\mathscr{F}_t\}_{t\geqslant 0}$ is the augmented natural filtration of B which satisfies the usual conditions. In this paper, we will always use this filtration.

In most of this work, the stochastic processes will be defined for $t\in[0,T]$, where T is a positive real number, and will take their values in $\mathbb{R}^n$ for some positive integer n. If $X=\{X_t\}_{t\in[0,T]}$ is such a process, we will simply write X_* or $\sup_t |X_t|$ instead of $\sup_{t\in[0,T]}|X_t|$ where $|x|$ denotes the Euclidean norm of $x\in\mathbb{R}^n$.

For any real $p>0$, $\mathscr{S}^p(\mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued, adapted and càdlàg processes $\{X_t\}_{t\in[0,T]}$ such that

$$\|X\|_{\mathscr{S}^p}=\mathbb{E}\left[\sup_t |X_t|^p\right]^{1\wedge 1/p}<+\infty.$$

If $p\geqslant 1$, $\|\cdot\|_{\mathscr{S}^p}$ is a norm on $\mathscr{S}^p(\mathbb{R}^n)$ and if $p\in(0,1)$, $(X,X')\mapsto \|X-X'\|_{\mathscr{S}^p}$ defines a distance on $\mathscr{S}^p$. Under this metric, $\mathscr{S}^p(\mathbb{R}^n)$ is complete.

$M^p(\mathbb{R}^n)$ denotes the set of (equivalent classes of) predictable processes $\{X_t\}_{t\in[0,T]}$ with values in $\mathbb{R}^n$ such that

$$\|X\|_{M^p}=\mathbb{E}\left[\left(\int_0^T |X_r|^2\,\mathrm{d}r\right)^{p/2}\right]^{1\wedge 1/p}<+\infty.$$

For $p\geqslant 1$, $M^p(\mathbb{R}^n)$ is a Banach space endowed with this norm and for $p\in(0,1)$ M^p is a complete metric space with the resulting distance.

Let us consider a random function $f:[0,T]\times\Omega\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$ measurable with respect to $\mathrm{Prog}\times\mathscr{B}(\mathbb{R}^k)\times\mathscr{B}(\mathbb{R}^{k\times d})$ where Prog denotes the sigma-field of progressive subsets of $[0,T]\times\Omega$, and an $\mathbb{R}^k$-valued $\mathscr{F}_T$-measurable random vector ξ.

$\mathbb{R}^{k\times d}$ is identified with the space of real matrices with k rows and d columns. If $z\in\mathbb{R}^{k\times d}$, we have $|z|^2=\mathrm{trace}(zz^*)$.

Let us recall what we mean by a solution to the BSDE (1).

Definition 2.1. A solution to the BSDE (1) is a pair of progressively measurable processes (Y,Z) with values in $\mathbb{R}^k\times\mathbb{R}^{k\times d}$ such that: $\mathbb{P}$-a.s., $t\mapsto Z_t$ belongs to $L^2(0,T)$, $t\mapsto f(t,Y_t,Z_t)$ belongs to $L^1(0,T)$ $\mathbb{P}$-a.s., and

$$Y_t=\xi+\int_t^T f(r,Y_r,Z_r)\,\mathrm{d}r-\int_t^T Z_r\,\mathrm{d}B_r,\quad 0\leqslant t\leqslant T.$$

2.2. A basic identity

As explained in the introduction, we want to deal with BSDEs with data in L^p with $p<2$ and we would like to use Itô's formula applied to the function $x\mapsto|x|^p$ which is not smooth enough. That is why we start by a generalization to the multidimensional case of the Tanaka formula. Let us now introduce the notation $\hat{x}=|x|^{-1}x\mathbf{1}_{x\neq 0}$. The following lemma will be our basic tool in the treatment of L^p-solutions. It is very likely that this result has already appeared somewhere, but we have not seen it, so we provide a proof.

Lemma 2.2. *Let $\{K_t\}_{t\in[0,T]}$ and $\{H_t\}_{t\in[0,T]}$ be two progressively measurable processes with values respectively in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$ such that $\mathbb{P}$-a.s.,*

$$\int_0^T (|K_t| + |H_t|^2)\,\mathrm{d}t < +\infty.$$

We consider the $\mathbb{R}^k$-valued semimartingale $\{X_t\}_{t\in[0,T]}$ defined by

$$X_t = X_0 + \int_0^t K_s\,\mathrm{d}s + \int_0^t H_s\,\mathrm{d}B_s, \qquad 0 \leqslant t \leqslant T.$$

Then, for any $p \geqslant 1$, we have

$$\begin{aligned} |X_t|^p - \mathbf{1}_{p=1} L_t = |X_0|^p &+ p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, K_s\rangle\,\mathrm{d}s + p\int_0^t |X_s|^{p-1}\langle \hat{X}_s, H_s\,\mathrm{d}B_s\rangle \\ &+ \frac{p}{2}\int_0^t |X_s|^{p-2}\mathbf{1}_{X_s\neq 0}\{(2-p)(|H_s|^2 - \langle \hat{X}_s, H_s H_s^* \hat{X}_s\rangle) \\ &+ (p-1)|H_s|^2\}\,\mathrm{d}s, \end{aligned}$$

where $\{L_t\}_{t\in[0,T]}$ is a continuous, increasing process with $L_0 = 0$, which increases only on the boundary of the random set $\{t\in[0,T],\ X_t = 0\}$.

Proof. Since the function $x \mapsto |x|^p$ is not smooth enough (for $p\in[1,2)$) to apply Itô's formula we use an approximation. Let $\varepsilon > 0$ and let us consider the function $u_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{1/2}$. It is a smooth function and we have, denoting I the identity matrix of $\mathbb{R}^k$,

$$\nabla u_\varepsilon^p(x) = p u_\varepsilon^{p-2}(x) x, \qquad \mathrm{D}^2 u_\varepsilon^p(x) = p u_\varepsilon^{p-2}(x) I + p(p-2) u_\varepsilon^{p-4}(x)\,(x\otimes x).$$

Itô's formula leads to the equality

$$\begin{aligned} u_\varepsilon^p(X_t) = u_\varepsilon^p(X_0) &+ p\int_0^t u_\varepsilon^{p-2}(X_s)\,\langle X_s, K_s\rangle\,\mathrm{d}s + p\int_0^t u_\varepsilon^{p-2}(X_s)\,\langle X_s, H_s\,\mathrm{d}B_s\rangle \\ &+ \frac{1}{2}\int_0^t \mathrm{trace}(\mathrm{D}^2 u_\varepsilon^p(X_s) H_s H_s^*)\,\mathrm{d}s. \end{aligned} \tag{2}$$

It remains essentially to pass to the limit when $\varepsilon\to 0$ in this identity. To do this, let us first remark that

$$\int_0^t u_\varepsilon^{p-2}(X_s)\,\langle X_s, K_s\rangle\,\mathrm{d}s \to \int_0^t |X_s|^{p-1}\langle \hat{X}_s, K_s\rangle\,\mathrm{d}s,$$

$\mathbb{P}$-a.s., and that, at least uniformly on $[0,T]$ in $\mathbb{P}$-probability, we have

$$\int_0^t u_\varepsilon^{p-2}(X_s)\,\langle X_s, H_s\,\mathrm{d}B_s\rangle \to \int_0^t |X_s|^{p-1}\langle \hat{X}_s, H_s\,\mathrm{d}B_s\rangle;$$

this convergence of stochastic integrals follows from the following convergence:

$$\int_0^T |X_r|^2 \mathbf{1}_{X_r\neq 0}|H_r|^2(|X_r|^{p-2} - u_\varepsilon^{p-2}(X_r))^2\,\mathrm{d}r \to 0,$$

which is clear from the dominated convergence theorem.

It remains to study the convergence of the term including the second derivative of u_ε. Let us write

$$\begin{aligned}
&\operatorname{trace}(D^2u_\varepsilon^p(X_s)H_sH_s^*)\\
&\quad = p(2-p)(|X_s|u_\varepsilon^{-1}(X_s))^{4-p}|X_s|^{p-2}\mathbf{1}_{X_s\neq 0}(|H_s|^2 - \langle \hat{X}_s, H_sH_s^*\hat{X}_s\rangle)\\
&\qquad + p(p-1)(|X_s|u_\varepsilon^{-1}(X_s))^{4-p}|X_s|^{p-2}\mathbf{1}_{X_s\neq 0}|H_s|^2 + C_s^\varepsilon(p),
\end{aligned}$$

where $C_s^\varepsilon(p) = p\varepsilon^2|H_s|^2u_\varepsilon^{p-4}(X_s)$.

One has

$$|H_s|^2 \geqslant \langle \hat{X}_s, H_sH_s^*\hat{X}_s\rangle. \tag{3}$$

Moreover,

$$\frac{|X_s|}{u_\varepsilon(X_s)} \nearrow \mathbf{1}_{\{X_s\neq 0\}}$$

as $\varepsilon\to 0$. Hence by monotone convergence, as $\varepsilon \to 0$,

$$\int_0^t (|X_s|u_\varepsilon^{-1}(X_s))^{p-4}|X_s|^{p-2}\mathbf{1}_{X_s\neq 0}\{(2-p)(|H_s|^2 - \langle \hat{X}_s, H_sH_s^*\hat{X}_s\rangle) + (p-1)|H_s|^2\}\,\mathrm{d}s$$

converges to

$$\int_0^t |X_s|^{p-2}\mathbf{1}_{X_s\neq 0}\{(2-p)(|H_s|^2 - \langle \hat{X}_s, H_sH_s^*\hat{X}_s\rangle) + (p-1)|H_s|^2\}\,\mathrm{d}s$$

$\mathbb{P}$-a.s., for all $0 \leqslant t \leqslant T$.

It now follows from (2) that $\{L_t^\varepsilon(p) := \int_0^t C_s^\varepsilon(p)\,\mathrm{d}s\}_{t\in[0,T]}$ converges as $\varepsilon \to 0$ to a continuous increasing process $\{L_t(p)\}_{t\in[0,T]}$, and the result follows.

For $p \geqslant 4$, $L(p) \equiv 0$ since $C^\varepsilon(p)$ converges to 0 in $L^1(0,T)$. Now, if $p\in(1,4)$, we write

$$C_s^\varepsilon(p) = p(\varepsilon^2|H_s|^2u_\varepsilon^{-3}(X_s))^\theta(\varepsilon^2|H_s|^2)^{1-\theta},$$

where $\theta = (4-p)/3 \in (0,1)$, and then, we get, using Hölder's inequality,

$$L_T^\varepsilon(p) \leqslant pL_T^\varepsilon(1)^\theta\left(\int_0^T \varepsilon^2|H_s|^2\,\mathrm{d}s\right)^{1-\theta},$$

which tends to 0 as $\varepsilon \to 0$ so that $L(p) \equiv 0$.

Let us denote by L the process $L(1)$ and let us set $A = \{t\in[0,T],\ X_t = 0\}$. If t is in the interior of A, then there exists $\delta > 0$ such that $X_s = 0$ whenever $|t-s| \leqslant \delta$; the quadratic variation of X is constant on the interval $[t-\delta, t+\delta]$ and then $H_s = 0$ almost everywhere on this interval. If t is in the complement of the set A, there exists $\delta > 0$

such that $X_s \neq 0$ if $|t-s| \leqslant \delta$. In both cases, $C^\varepsilon(1)$ converges to 0 in $L^1(t-\delta, t+\delta)$ and

$$L_{t+\delta} - L_{t-\delta} = \lim_{\varepsilon \to 0} \int_{t-\delta}^{t+\delta} C_s^\varepsilon(1)\,\mathrm{d}s = 0.$$

This concludes the proof of the lemma. □

Corollary 2.3. *If* (Y,Z) *is a solution of the BSDE* (1), $p \geqslant 1$, $c(p) = p[(p-1)\wedge 1]/2$ *and* $0 \leqslant t \leqslant u \leqslant T$, *then*

$$|Y_t|^p + c(p)\int_t^u |Y_s|^{p-2}\mathbf{1}_{Y_s\neq 0}|Z_s|^2\,\mathrm{d}s \leqslant |Y_u|^p + p\int_t^u |Y_s|^{p-1}\langle \hat{Y}_s, f(s,Y_s,Z_s)\rangle\,\mathrm{d}s$$

$$- p\int_t^u |Y_s|^{p-1}\langle \hat{Y}_s, Z_s\,\mathrm{d}B_s\rangle. \tag{4}$$

Proof. The proof follows from the following consequence of Lemma 2.2, for $0 \leqslant t \leqslant u \leqslant T$ and $c(p) = p[(p-1)\wedge 1]/2$,

$$|X_u|^p \geqslant |X_t|^p + p\int_t^u |X_s|^{p-1}\langle \hat{X}_s, K_s\rangle\,\mathrm{d}s + p\int_t^u |X_s|^{p-1}\langle \hat{X}_s, H_s\,\mathrm{d}B_s\rangle$$

$$+ c(p)\int_t^u |X_s|^{p-2}\mathbf{1}_{X_s\neq 0}|H_s|^2\,\mathrm{d}s. \qquad \square$$

3. Apriori estimates

First of all, we state some estimates concerning solutions to the BSDE (1). In what follows, we assume that $p>1$, ξ is an $\mathbb{R}^k$-valued, $\mathscr{F}_T$-measurable random vector and f is a random function from $[0,T]\times\Omega\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$ into $\mathbb{R}^k$, which is measurable with respect to $\mathrm{Prog}\times\mathscr{B}(\mathbb{R}^k)\times\mathscr{B}(\mathbb{R}^{k\times d})$. We will make use of the following assumption: $\mathbb{P}$-a.s.,

$$\forall (t,y,z)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}, \qquad \langle \hat{y}, f(t,y,z)\rangle \leqslant f_t + \mu|y| + \lambda|z|, \tag{A}$$

where $\mu\in\mathbb{R}$, $\lambda \geqslant 0$ and $\{f_t\}_{t\in[0,T]}$ is a non-negative progressively measurable process. Let us set $F = \int_0^T f_r\,\mathrm{d}r$.

Here, we want to obtain estimates for solutions to a BSDE in L^p in the spirit of the work (El Karoui et al., 1997) which shows that these estimates are very useful for the study of existence and uniqueness of solutions. The difficulty here comes from two facts: firstly, the function f is not supposed to be Lipschitz continuous and secondly, we want to obtain L^p-estimates for $p\in(1,2)$.

We start by showing how to control the process Z in terms of the data and Y.

Lemma 3.1. *Let assumption* (A) *hold and let* (Y,Z) *be a solution to BSDE* (1). *Let us assume moreover that, for some* $p>0$, F^p *is integrable.*

If $Y \in \mathscr{S}^p$ then Z belongs to M^p and there exists a constant C_p depending only on p such that for any $a \geqslant \mu + \lambda^2$,

$$\mathbb{E}\left[\left(\int_0^T e^{2at}|Z_r|^2\,dr\right)^{p/2}\right] \leqslant C_p\,\mathbb{E}\left[\sup_t e^{apt}|Y_t|^p + \left(\int_0^T e^{ar} f_r\,dr\right)^p\right].$$

Proof. Let us fix $a \geqslant \mu + \lambda^2$ and define $\tilde{Y}_t = e^{at}Y_t$, $\tilde{Z}_t = e^{at}Z_t$. $(\tilde{Y}, \tilde{Z})$ solves the BSDE

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{f}(r, \tilde{Y}_r, \tilde{Z}_r)\,dr - \int_t^T \tilde{Z}_r\,dB_r, \quad 0 \leqslant t \leqslant T,$$

where $\tilde{\xi} = e^{aT}\xi$ and $\tilde{f}(t,y,z) = e^{at} f(t, e^{-at}y, e^{-at}z) - ay$ which satisfies assumption (A) with $\tilde{f}_t = e^{at} f_t$, $\tilde{\lambda} = \lambda$ and $\tilde{\mu} = \mu - a$. Since we are working on a compact time interval, the integrability conditions are equivalent with or without the superscript $\sim$. Thus, with this change of variable we reduce to the case $a = 0$ and $\mu + \lambda^2 \leqslant 0$. We forget the superscript $\sim$ for notational convenience.

For each integer $n \geqslant 1$, let us introduce the stopping time

$$\tau_n = \inf\left\{t \in [0,T],\ \int_0^t |Z_r|^2\,dr \geqslant n\right\} \wedge T.$$

Itô's formula gives us

$$|Y_0|^2 + \int_0^{\tau_n} |Z_r|^2\,dr = |Y_{\tau_n}|^2 + 2\int_0^{\tau_n} \langle Y_r, f(r, Y_r, Z_r)\rangle\,dr - 2\int_0^{\tau_n} \langle Y_r, Z_r\,dB_r\rangle.$$

But, from the assumption on f, we have, since $\mu + \lambda^2 \leqslant 0$,

$$2\langle y, f(r,y,z)\rangle \leqslant 2|y| f_r + 2\mu|y|^2 + 2\lambda^2|y|^2 + |z|^2/2 \leqslant 2|y| f_r + |z|^2/2.$$

Thus, since $\tau_n \leqslant T$, we deduce that

$$\frac{1}{2}\int_0^{\tau_n} |Z_r|^2\,dr \leqslant Y_*^2 + 2Y_* \int_0^T f_r\,dr + 2\left|\int_0^{\tau_n} \langle Y_r, Z_r\,dB_r\rangle\right|.$$

It follows that

$$\int_0^{\tau_n} |Z_r|^2\,dr \leqslant 4\left(Y_*^2 + \left(\int_0^T f_r\,dr\right)^2 + \left|\int_0^{\tau_n} \langle Y_r, Z_r\,dB_r\rangle\right|\right)$$

and thus that

$$\left(\int_0^{\tau_n} |Z_r|^2\,dr\right)^{p/2} \leqslant c_p\left(Y_*^p + \left(\int_0^T f_r\,dr\right)^p + \left|\int_0^{\tau_n} \langle Y_r, Z_r\,dB_r\rangle\right|^{p/2}\right). \tag{5}$$

But by the BDG inequality, we get

$$\begin{aligned} c_p\,\mathbb{E}\left[\left|\int_0^{\tau_n} \langle Y_r, Z_r\,dB_r\rangle\right|^{p/2}\right] &\leqslant d_p\,\mathbb{E}\left[\left(\int_0^{\tau_n} |Y_r|^2\,|Z_r|^2\,dr\right)^{p/4}\right] \\ &\leqslant d_p\,\mathbb{E}\left[Y_*^{p/2}\left(\int_0^{\tau_n} |Z_r|^2\,dr\right)^{p/4}\right] \end{aligned}$$

and thus

$$c_p\,\mathbb{E}\left[\left|\int_0^{\tau_n}\langle Y_r, Z_r\,\mathrm{d}B_r\rangle\right|^{p/2}\right] \leqslant \frac{d_p^2}{2}\,\mathbb{E}[Y_*^p] + \frac{1}{2}\,\mathbb{E}\left[\left(\int_0^{\tau_n}|Z_r|^2\,\mathrm{d}r\right)^{p/2}\right].$$

Coming back to estimate (5), we get, for each $n \geqslant 1$,

$$\mathbb{E}\left[\left(\int_0^{\tau_n}|Z_r|^2\,\mathrm{d}r\right)^{p/2}\right] \leqslant C_p\,\mathbb{E}\left[Y_*^p + \left(\int_0^T f_r\,\mathrm{d}r\right)^p\right]$$

and, Fatou's lemma implies that

$$\mathbb{E}\left[\left(\int_0^{T}|Z_r|^2\,\mathrm{d}r\right)^{p/2}\right] \leqslant C_p\,\mathbb{E}\left[Y_*^p + \left(\int_0^T f_r\,\mathrm{d}r\right)^p\right].$$

The result follows. □

We keep on this study by stating the standard estimate in our context. The difficulty comes from the fact that f is not Lipschitz in y and also from the fact that the function $y \mapsto |y|^p$ is not $\mathscr{C}^2$ since we will work with $p \in (1,2)$.

Proposition 3.2. *Let assumption* (A) *hold and let us assume that, for some* $p > 1$, F *belongs to* L^p. *Let* (Y,Z) *be a solution to BSDE* (1) *where* Y *belongs to* $\mathscr{S}^p$. *Then, there exists a constant* C_p, *depending only on* p, *such that for any* $a \geqslant \mu + \lambda^2/[1 \wedge (p-1)]$,

$$\mathbb{E}\left[\sup_t \mathrm{e}^{apt}|Y_t|^p + \left(\int_0^T \mathrm{e}^{2ar}|Z_r|^2\,\mathrm{d}r\right)^{p/2}\right]$$
$$\leqslant C_p\,\mathbb{E}\left[\mathrm{e}^{apT}|\xi|^p + \left(\int_0^T \mathrm{e}^{ar} f_r\,\mathrm{d}r\right)^p\right].$$

Proof. Let us fix $a \geqslant \mu + \lambda^2/[1 \wedge (p-1)]$. As in the proof of the previous lemma, we make the change of variables $\tilde{Y}_t = \mathrm{e}^{at}Y_t$, $\tilde{Z}_t = \mathrm{e}^{at}Z_t$. This reduces the proof to the case $a = 0$ and $\mu + \lambda^2/[1 \wedge (p-1)] \leqslant 0$; omitting the superscript $\sim$, we have to prove that

$$\mathbb{E}\left[Y_*^p + \left(\int_0^T |Z_r|^2\,\mathrm{d}r\right)^{p/2}\right] \leqslant C_p\,\mathbb{E}\left[|\xi|^p + \left(\int_0^T f_r\,\mathrm{d}r\right)^p\right].$$

From Corollary 2.3, we get the following inequality:

$$|Y_t|^p + c(p)\int_t^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r$$
$$\leqslant |\xi|^p + p\int_t^T |Y_r|^{p-1}\langle \hat{Y}_r, f(r,Y_r,Z_r)\rangle\,\mathrm{d}r - p\int_t^T |Y_r|^{p-1}\langle \hat{Y}_r, Z_r\,\mathrm{d}B_r\rangle.$$

The assumption on f yields the inequality

$$\langle \hat{y}, f(r,y,z)\rangle \leqslant f_r + \mu|y| + \lambda|z|,$$

from which we deduce that, with probability one, for all $t\in[0,T]$,

$$|Y_t|^p + c(p)\int_t^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r$$
$$\leqslant |\xi|^p + p\int_t^T (|Y_r|^{p-1}f_r + \mu|Y_r|^p)\,\mathrm{d}r + p\lambda\int_t^T |Y_r|^{p-1}|Z_r|\,\mathrm{d}r$$
$$-p\int_t^T |Y_r|^{p-1}\langle \hat{Y}_r, Z_r\,\mathrm{d}B_r\rangle.$$

First of all we deduce from the previous inequality that, $\mathbb{P}$-a.s.,

$$\int_0^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r < +\infty.$$

Moreover, we have

$$p\lambda|Y_r|^{p-1}|Z_r| \leqslant \frac{p\lambda^2}{1\wedge(p-1)}|Y_r|^p + \frac{c(p)}{2}|Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2$$

and thus, since $\mu + \lambda^2/[1\wedge(p-1)] \leqslant 0$, we get the inequality

$$|Y_t|^p + \frac{c(p)}{2}\int_t^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r$$
$$\leqslant |\xi|^p + p\int_t^T |Y_r|^{p-1}f_r\,\mathrm{d}r - p\int_t^T |Y_r|^{p-1}\langle \hat{Y}_r, Z_r\,\mathrm{d}B_r\rangle.$$

Let us set $X = |\xi|^p + p\int_0^T |Y_r|^{p-1}f_r\,\mathrm{d}r$; then, we have, a.s., for each $t\in[0,T]$,

$$|Y_t|^p + \frac{c(p)}{2}\int_t^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r \leqslant X - p\int_t^T |Y_r|^{p-1}\langle \hat{Y}_r, Z_r\,\mathrm{d}B_r\rangle. \tag{6}$$

It follows from the BDG inequality that $\{M_t := \int_0^t |Y_r|^{p-1}\langle \hat{Y}_r, Z_r\,\mathrm{d}B_r\rangle\}_{0\leqslant t\leqslant T}$ is a uniformly integrable martingale. Indeed, we have, by Young's inequality

$$\mathbb{E}[\langle M,M\rangle_T^{1/2}] \leqslant \mathbb{E}\left[Y_*^{p-1}\left(\int_0^T |Z_r|^2\,\mathrm{d}r\right)^{1/2}\right]$$
$$\leqslant \frac{(p-1)}{p}\mathbb{E}[Y_*^p] + \frac{1}{p}\mathbb{E}\left[\left(\int_0^T |Z_r|^2\,\mathrm{d}r\right)^{p/2}\right],$$

the last term being finite since Y belongs to $\mathscr{S}^p$ and then Z belongs to M^p by Lemma 3.1.

Coming back to inequality (6), and taking the expectation for $t=0$, we get both

$$\frac{c(p)}{2}\mathbb{E}\left[\int_0^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r\right] \leqslant \mathbb{E}[X], \tag{7}$$

and,

$$\mathbb{E}[Y_*^p] \leqslant \mathbb{E}[X] + k_p \mathbb{E}[\langle M, M\rangle_T^{1/2}]. \tag{8}$$

On the other hand, we have also,

$$k_p \mathbb{E}[\langle M, M\rangle_T^{1/2}] \leqslant k_p \mathbb{E}\left[Y_*^{p/2}\left(\int_0^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}|Z_r|^2\,\mathrm{d}r\right)^{1/2}\right]$$

$$\leqslant \frac{1}{2}\,\mathbb{E}[Y_*^p] + \frac{k_p^2}{2}\,\mathbb{E}\left[\int_0^T |Y_r|^{p-2}\mathbf{1}_{Y_r\neq 0}\,|Z_r|^2\,\mathrm{d}r\right].$$

Coming back to inequalities (7) and (8), we obtain

$$\mathbb{E}[Y_*^p] \leqslant d_p\,\mathbb{E}[X].$$

Applying once again Young's inequality, we get

$$pd_p\,\mathbb{E}\left[\int_0^T |Y_r|^{p-1} f_r\,\mathrm{d}r\right] \leqslant pd_p\,\mathbb{E}\left[Y_*^{p-1}\int_0^T f_r\,\mathrm{d}r\right]$$

$$\leqslant \frac{1}{2}\,\mathbb{E}[Y_*^p] + d'_p\mathbb{E}\left[\left(\int_0^T f_r\,\mathrm{d}r\right)^p\right],$$

from which we deduce, coming back to the definition of X, that

$$\mathbb{E}[Y_*^p] \leqslant C_p\mathbb{E}\left[|\xi|^p + \left(\int_0^T f_r\,\mathrm{d}r\right)^p\right].$$

The result follows from Lemma 3.1. □

4. Existence and uniqueness of a solution

With the help of the above a priori estimates, we can obtain an existence and uniqueness result.

As before, let us consider an $\mathbb{R}^k$-valued $\mathscr{F}_T$-measurable random vector ξ and a random function $f:[0,T]\times\Omega\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$ which is $\mathrm{Prog}\times\mathscr{B}(\mathbb{R}^k)\times\mathscr{B}(\mathbb{R}^{k\times d})$-measurable.

We will work under the following assumptions: for some $p>1$,

$$\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(s,0,0)|\,\mathrm{d}s\right)^p\right] < +\infty, \tag{H1}$$

there exist constants $\lambda \geqslant 0$ and $\mu\in\mathbb{R}$ such that, $\mathbb{P}$-a.s., for each $(t,y,y',z,z')\in[0,T]\times\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^{k\times d}$:

$$|f(t,y,z) - f(t,y,z')| \leqslant \lambda|z-z'|, \tag{H2}$$

$$\langle y-y', f(t,y,z) - f(t,y',z)\rangle \leqslant \mu|y-y'|^2. \tag{H3}$$

We assume also that,

$$\mathbb{P}\text{-a.s.},\quad \forall (t,z)\in[0,T]\times\mathbb{R}^{k\times d},\quad y\mapsto f(t,y,z) \text{ is continuous} \tag{H4}$$

and finally that

$$\forall r>0,\quad \psi_r(t):=\sup_{|y|\leqslant r}|f(t,y,0)-f(t,0,0)|\in L^1([0,T]\times\Omega, m\otimes\mathbb{P}). \tag{H5}$$

We want to obtain an existence and uniqueness result for BSDE (1) under the previous assumptions for all $p>1$.

Firstly, let us recall the result of Pardoux (1999, Theorem 2.2). For this, let us introduce the following assumption:

$$\mathbb{P}\text{-a.s.},\quad \forall (t,y)\in[0,T]\times\mathbb{R}^{k},\quad |f(t,y,0)|\leqslant|f(t,0,0)|+\varphi(|y|), \tag{H5'}$$

where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a deterministic continuous increasing function.

Theorem 4.1. *Let* $p=2$. *Under assumptions* (H1)–(H4) and (H5′), *BSDE* (1) *has a unique solution in* $\mathcal{S}^2\times M^2$.

We now prove our existence and uniqueness result.

Theorem 4.2. *Under assumptions* (H1)–(H5), *BSDE* (1) *has a unique solution in* $\mathcal{S}^p\times\mathcal{M}^p$.

Proof. Let us start by studying the uniqueness part. Let us consider (Y,Z) and (Y',Z') two solutions of our BSDE in the appropriate space. We denote by (U,V) the process $(Y-Y',Z-Z')$; this process is solution to the following BSDE:

$$U_t=\int_t^T g(s,U_s,V_s)\,\mathrm{d}s-\int_t^T V_s\,\mathrm{d}B_s,\quad 0\leqslant t\leqslant T,$$

where g stands for the random function

$$g(t,y,z)=f(t,y+Y'_t,z+Z'_t)-f(t,Y'_t,Z'_t).$$

Thanks to assumptions (H2) and (H3), the function g satisfies assumption (A) with $f_t\equiv 0$. By Proposition 3.2, we get immediately that $(U,V)=(0,0)$.

Let us turn to the existence part. In order to simplify the calculations, we will always assume that condition (H3) is satisfied with $\mu=0$. If it is not true, the change of variables $\tilde{Y}_t=\mathrm{e}^{\mu t}Y_t$, $\tilde{Z}_t=\mathrm{e}^{\mu t}Z_t$ reduces to this case. We set $f_t^0=f(t,0,0)$.

The proof will be split into two steps.

First step: We assume that ξ and $\sup_t|f_t^0|$ are bounded random variables. Let r be a positive real such that

$$\sqrt{\mathrm{e}^{(1+\lambda^2)T}}(\|\xi\|_\infty+T\|f^0\|_\infty)<r.$$

Let θ_r be a smooth function such that $0\leqslant\theta_r\leqslant 1$, $\theta_r(y)=1$ for $|y|\leqslant r$ and $\theta_r(y)=0$ as soon as $|y|\geqslant r+1$. For each $n\in\mathbb{N}^*$, we denote $q_n(z)=zn/(|z|\vee n)$ and set

$$h_n(t,y,z)=\theta_r(y)(f(t,y,q_n(z))-f_t^0)\frac{n}{\psi_{r+1}(t)\vee n}+f_t^0.$$

This function still satisfies quadratic condition (H3) but with a positive constant. Indeed, let us pick y and y' in $\mathbb{R}^k$. If $|y| > r+1$ and $|y'| > r+1$, the inequality is trivially satisfied and thus we reduce to the case where $|y'| \leqslant r+1$. We write

$$\begin{aligned}
&\langle y - y', h_n(t,y,z) - h_n(t,y',z)\rangle \\
&\quad = \theta_r(y)\,\frac{n}{n \vee \psi_{r+1}(t)}\,\langle y - y', f(t,y,q_n(z)) - f(t,y',q_n(z))\rangle \\
&\qquad + \frac{n}{n \vee \psi_{r+1}(t)}\,(\theta_r(y) - \theta_r(y'))\langle y - y', [f(t,y',q_n(z)) - f_t^0]\rangle.
\end{aligned}$$

The first term of the right-hand side of the previous equality is negative since condition (H3) is in force for f with $\mu = 0$. For the second term, one can use the fact that θ_r is $C(r)$-Lipschitz, to get, since $|y'| \leqslant r+1$,

$$\begin{aligned}
&(\theta_r(y) - \theta_r(y'))\langle y - y', [f(t,y',q_n(z)) - f_t^0]\rangle \\
&\quad \leqslant C(r)\,|y - y'|^2 |f(t,y',q_n(z)) - f_t^0| \leqslant C(r)(\lambda n + \psi_{r+1}(t))\,|y - y'|^2
\end{aligned}$$

and thus

$$\frac{n}{n \vee \psi_{r+1}(t)}(\theta_r(y) - \theta_r(y'))\langle y - y', [f(t,y',q_n(z)) - f_t^0]\rangle \leqslant C(r)(\lambda + 1)n|y - y'|^2.$$

Then the pair (ξ, h_n) satisfies the assumptions of Theorem 4.1. Hence, for each $n \in \mathbb{N}^*$, the BSDE associated to (ξ, h_n) has a unique solution (Y^n, Z^n) in the space $\mathcal{S}^2 \times M^2$.

Since

$$\langle y, h_n(t,y,z)\rangle \leqslant |y|\,\|f^0\|_\infty + \lambda|y|\,|z|$$

and ξ is bounded, Lemma 2.2 in Briand and Carmona (2000) shows that the process Y^n satisfies the inequality $\|Y^n\|_\infty \leqslant r$. In addition, from Proposition 3.2,

$$\|Z^n\|_{M^2} \leqslant r', \tag{9}$$

where r' is another constant. As a byproduct (Y^n, Z^n) is a solution to the BSDE associated to (ξ, f_n) where

$$f_n(t,y,z) = (f(t,y,q_n(z)) - f_t^0)\,\frac{n}{\psi_{r+1}(t) \vee n} + f_t^0$$

for this function (H3) is satisfied with "$\mu = 0$".

We now have, for $i \in \mathbb{N}$, setting $U = Y^{n+i} - Y^n$, $V = Z^{n+i} - Z^n$, using assumptions (H2) and (H3) on f_{n+i}

$$\begin{aligned}
&e^{2\lambda^2 t}|U_t|^2 + \frac{1}{2}\int_t^T e^{2\lambda^2 s}|V_s|^2\,ds \\
&\quad \leqslant 2\int_t^T e^{2\lambda^2 s}\langle U_s, f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)\rangle\,ds - 2\int_t^T e^{2\lambda^2 s}\langle U_s, V_s\,dB_s\rangle.
\end{aligned}$$

But $\|U\|_\infty \leqslant 2r$ so that

$$e^{2\lambda^2 t}|U_t|^2 + \frac{1}{2}\int_t^T e^{2\lambda^2 s}|V_s|^2\,ds$$
$$\leqslant 4r\int_0^T e^{2\lambda^2 s}|f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)|\,ds - 2\int_t^T e^{2\lambda^2 s}\langle U_s, V_s\,dB_s\rangle$$

and using the BDG inequality, we get, for a constant C depending only on λ and T,

$$\mathbb{E}\left[\sup_t |U_t|^2 + \int_0^T |V_s|^2\,ds\right] \leqslant Cr\,\mathbb{E}\left[\int_0^T |f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)|\,ds\right].$$

On the other hand, since $\|Y^n\|_\infty \leqslant r$, we have

$$|f_{n+i}(s,Y_s^n,Z_s^n) - f_n(s,Y_s^n,Z_s^n)|$$
$$\leqslant 2\lambda|Z_s^n|\mathbf{1}_{|Z_s^n|>n} + 2\lambda|Z_s^n|\mathbf{1}_{\psi_{r+1}(s)>n} + 2\psi_{r+1}(s)\mathbf{1}_{\psi_{r+1}(s)>n},$$

from which we deduce, with the help of inequality (9) and assumption (H5), that (Y^n,Z^n) is a Cauchy sequence in $\mathscr{S}^2\times M^2$. It is easy to pass to the limit in the approximating equation, yielding a solution to BSDE (1).

Second step: We now treat the general case. For each $n\in\mathbb{N}^*$, let us define

$$\xi_n = q_n(\xi), \qquad f_n(t,y,z) = f(t,y,z) - f_t^0 + q_n(f_t^0).$$

For each pair (ξ_n, f_n), BSDE (1) has a unique solution (Y^n,Z^n) in L^2 thanks to the first step of this proof, but in fact also in all L^p, $p>1$ according to Lemma 3.1. Now from Proposition 3.2, for $(i,n)\in\mathbb{N}\times\mathbb{N}^*$,

$$\mathbb{E}\left[\sup_t |Y_t^{n+i} - Y_t^n|^p + \left(\int_0^T |Z_s^{n+i} - Z_s^n|^2\,ds\right)^{p/2}\right]$$
$$\leqslant C\,\mathbb{E}\left[|\xi_{n+i} - \xi_n|^p + \left(\int_0^T |q_{n+i}(f_t^0) - q_n(f_t^0)|\,dt\right)^p\right],$$

where C depends on T and λ.

The right-hand side of the last inequality clearly tends to 0, as $n\to\infty$, uniformly in i, so we have again a Cauchy sequence and the limit is a solution to BSDE (1). □

Remark 4.3. In the case $k=1$, Theorem 4.2 remains valid if we replace (H5) by the weaker condition

$$\psi_r \in L^1(0,T), \quad \text{a.s.} \quad \forall r>0.$$

The additional estimate in this case which allows that generalization is the following:

$$\mathbb{E}\left[\left(\int_0^T |f(s,Y_s,Z_s)|\,ds\right)^p\right] \leqslant c\,\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f_t^0|\,dt\right)^p\right],$$

for a certain constant c depending only upon T, μ and λ. Indeed, it follows from (4), that

$$\begin{aligned} &\mathrm{e}^{\mu t}|Y_t| + \int_t^T \mathrm{e}^{\mu s}|f(s,Y_s,0) - f_s^0 - \mu Y_s|\,\mathrm{d}s \\ &\quad\leqslant \mathrm{e}^{\mu T}|\xi| + \int_t^T \mathrm{e}^{\mu s}|f_s^0|\,\mathrm{d}s + \lambda\int_t^T \mathrm{e}^{\mu s}|Z_s|\,\mathrm{d}s - \int_t^T \mathrm{e}^{\mu s}\,\mathrm{sgn}(Y_s)Z_s\,\mathrm{d}B_s \end{aligned}$$

and it remains to combine this last inequality with Proposition 3.2.

5. L^p solution of a BSDE with a random terminal time

We now assume that T is a stopping time for the filtration $\mathscr{F}_t$, which need not be bounded ($T \equiv +\infty$ is an interesting particular case, which we have in mind). Assumptions (H2)–(H4) are still in force. We shall assume in this section that $p > 1$. We shall follow closely the approach in Pardoux (1999), which treats the same problem in the case $p = 2$.

Assumption (H1) will be replaced by the following condition. For some

$$\rho > v := \mu + \frac{\lambda^2}{2(p-1)}$$

(where μ and λ are the constants appearing in conditions (H3) and (H2), respectively),

$$\mathbb{E}\left[\mathrm{e}^{p\rho T}|\xi|^p + \int_0^T \mathrm{e}^{p\rho t}|f(t,0,0)|^p\,\mathrm{d}t\right] < +\infty. \tag{H1$'$}$$

Assumption (H5) is replaced by

$$\psi_r \in L^1((0,n)\times\Omega, m\otimes\mathbb{P}) \quad \forall n\in\mathbb{N}^* \quad \forall r>0 \tag{H5$''$}$$

and we shall need the following additional assumption: ξ is $\mathscr{F}_T$-measurable and

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{p\rho t}|f(t,\mathrm{e}^{-vt}\bar{\xi}_t,\mathrm{e}^{-vt}\bar{\eta}_t)|^p\,\mathrm{d}t\right] < +\infty, \tag{H6}$$

where $\bar{\xi} = \mathrm{e}^{vT}\xi$, $\bar{\xi}_t = \mathbb{E}(\mathrm{e}^{vT}\xi \mid \mathscr{F}_t)$ and $\bar{\eta}$ is predictable and such that

$$\bar{\xi} = \mathbb{E}[\bar{\xi}] + \int_0^\infty \bar{\eta}_t\,\mathrm{d}B_t, \quad \mathbb{E}\left[\left(\int_0^\infty |\bar{\eta}_t|^2\,\mathrm{d}t\right)^{p/2}\right] < +\infty.$$

Definition 5.1. A pair $(Y_t,Z_t)_{t\geqslant 0}$ of progressively measurable processes with values in $\mathbb{R}^k\times\mathbb{R}^{k\times d}$ is a solution to the BSDE with random terminal time T with data (ξ,f) if on the set $\{t\geqslant T\}$ $Y_t=\xi$ and $Z_t=0$, $\mathbb{P}$-a.s., $t\mapsto \mathbf{1}_{t\leqslant T}\,f(t,Y_t,Z_t)$ belongs to $L^1_{\mathrm{loc}}(0,\infty)$, $t\mapsto Z_t$ belongs to $L^2_{\mathrm{loc}}(0,\infty)$ and, $\mathbb{P}$-a.s., for all $0\leqslant t\leqslant u$,

$$Y_{t\wedge T} = Y_{u\wedge T} + \int_{t\wedge T}^{u\wedge T} f(s,Y_s,Z_s)\,\mathrm{d}s - \int_{t\wedge T}^{u\wedge T} Z_s\,\mathrm{d}B_s. \tag{10}$$

A solution is said to be in L^p if we have moreover

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} e^{p\rho t}|Y_t|^p + \int_0^T e^{p\rho t}|Y_t|^p\,dt + \int_0^T e^{p\rho t}|Y_t|^{p-2}|Z_t|^2\,dt\right] < +\infty.$$

We have

Theorem 5.2. *Under assumptions* (H1′), (H2)–(H4), (H5″) *and* (H6), *the BSDE with random terminal time* (10) *has a unique solution satisfying*

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} e^{p\rho t}|Y_t|^p + \int_0^T e^{p\rho t}|Y_t|^{p-2}\{|Y_t|^2 + |Z_t|^2\}\,dt\right]$$

$$\leqslant c\mathbb{E}\left[e^{p\rho T}|\xi|^p + \int_0^T e^{p\rho t}|f(t,0,0)|^p\,dt\right]$$

for some constant c depending upon p, λ, ρ and μ.

Proof. The proof follows the steps of the proof of Pardoux (1999, Theorem 4.1). Firstly, we make the change of variables $\hat{Y}_t = e^{\nu t}Y_t$ to reduce to the terminal condition $\bar{\xi}$ which belongs to L^p. We derive the apriori estimate in L^p with $p\in(1,2)$, which is the only difference with the proof in Pardoux (1999). It follows easily from inequality (4) that, for $0\leqslant t\leqslant u$,

$$e^{p\rho(t\wedge T)}|Y_{t\wedge T}|^p + p\int_{t\wedge T}^{u\wedge T} e^{p\rho s}\left(\frac{p-1}{2}|Y_s|^{p-2}|Z_s|^2 + \rho|Y_s|^p\right)ds$$

$$\leqslant e^{p\rho(u\wedge T)}|Y_{u\wedge T}|^p + p\int_{t\wedge T}^{u\wedge T} e^{p\rho s}|Y_s|^{p-1}\langle\hat{Y}_s, f(s,Y_s,Z_s)\rangle\,ds$$

$$-p\int_{t\wedge T}^{u\wedge T} e^{p\rho s}|Y_s|^{p-1}\langle\hat{Y}_s, Z_s\,dB_s\rangle.$$

The assumptions on f together with Young's inequality leads to the inequality, denoting as before $f_s^0 = f(s,0,0)$, for any $0<\delta<(p-1)/2$,

$$|y|^{p-1}\langle\hat{y}, f(s,y,z)\rangle \leqslant \left(\mu+\delta+\frac{\lambda^2}{2(p-1-2\delta)}\right)|y|^p$$

$$+\left(\frac{p-1}{2}-\delta\right)|y|^{p-2}|z|^2 + \frac{1}{p}|f_s^0|^p\left(\frac{p\delta}{p-1}\right)^{1-p}.$$

We choose $\delta>0$ small enough so that $\mu+2\delta+\lambda^2/(2(p-1-2\delta))\leqslant\rho$ and deduce from the previous inequalities that

$$e^{p\rho(t\wedge T)}|Y_{t\wedge T}|^p + p\delta\int_{t\wedge T}^{u\wedge T} e^{p\rho s}(|Y_s|^p + |Y_s|^{p-2}|Z_s|^2)\,ds$$

$$\leqslant \mathrm{e}^{p\rho(u\wedge T)}|Y_{u\wedge T}|^p + C(p,\delta)\int_{t\wedge T}^{u\wedge T} \mathrm{e}^{p\rho s}|f_s^0|^p\,\mathrm{d}s$$

$$-p\int_{t\wedge T}^{u\wedge T} \mathrm{e}^{p\rho s}|Y_s|^{p-1}\langle \hat{Y}_s, Z_s\,\mathrm{d}B_s\rangle.$$

Taking the expectation and sending u to infinity in the last inequality, we get

$$\mathbb{E}\left[\mathrm{e}^{p\rho(t\wedge T)}|Y_{t\wedge T}|^p + \delta\int_0^T \mathrm{e}^{p\rho s}(|Y_s|^p + |Y_s|^{p-2}|Z_s|^2)\,\mathrm{d}s\right]$$

$$\leqslant C(p,\delta)\,\mathbb{E}\left[\mathrm{e}^{p\rho T}|\xi|^p + \int_0^T \mathrm{e}^{p\rho s}|f_s^0|^p\,\mathrm{d}s\right].$$

Using a standard argument based on the Burkholder–Davis–Gundy inequality, we can moreover include a $\sup_t$ inside the expectation of the left-hand side. □

Remark 5.3. In most interesting applications, in particular to elliptic PDEs, if T is an unbounded stopping time (e.g., $\equiv +\infty$), (H1′) cannot be satisfied unless $\rho < 0$. This implies, in particular that $\mu < 0$, which should be expected, from the relation with elliptic PDEs, see Pardoux (1999).

In the case $p = 2$, the condition $\rho > \mu + (2(p-1))^{-1}\lambda^2$ reduces to $\rho > \mu + \lambda^2/2$, which is the condition in Pardoux (1999). On the other hand, as $p \to 1$, the condition

$$\mu + \frac{\lambda^2}{2(p-1)} < \rho < 0$$

requires μ to be negative, with larger and larger absolute value. No result for the case $p = 1$ can be deduced from the above.

6. Integrable parameters

In this section, we will deal with the case $p = 1$ which appears to be very different from the previous one. We assume here that T is a fixed terminal time. Let us denote Σ_T the set of all stopping times τ such that $\tau \leqslant T$; we recall that, for a process $Y = \{Y_t\}_{0\leqslant t\leqslant T}$, Y belongs to class (D) if the family $\{Y_\tau,\ \tau \in \Sigma_T\}$ is uniformly integrable. For a process Y in class (D), we put

$$\|Y\|_1 = \sup\{\mathbb{E}[|Y_\tau|],\ \tau \in \Sigma_T\}.$$

The space of progressive measurable continuous processes which belong to class (D) is complete under this norm, see Dellacherie and Meyer (1980, p. 90).

We will need a further assumption on the function f: we will assume that there exist two constants $\gamma \geqslant 0$, $\alpha \in (0,1)$ and a non-negative progressively measurable process $\{g_t\}_{t\in[0,T]}$ such that

$$\forall (t,y,z) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d},$$

$$|f(t,y,z) - f(t,y,0)| \leqslant \gamma(g_t + |y| + |z|)^\alpha. \tag{H7}$$

Note that this assumption is trivially satisfied if f does not depend on z.

We will also assume that

$$\mathbb{E}\left[|\xi| + \int_0^T (f_t + g_t)\,\mathrm{d}t\right] < +\infty. \tag{H1''}$$

Firstly, let us recall the following result which can be found in Revuz and Yor (1991) with a different constant but in a more general context.

Lemma 6.1. *Let $\{M_t\}_{t\in[0,T]}$ be a martingale. Then, for all $\beta \in (0,1)$,*

$$\mathbb{E}[M_*^\beta] \leqslant \frac{1}{1-\beta}\,\mathbb{E}[|M_T|]^\beta.$$

Proof. Let us denote $c = \mathbb{E}[|M_T|]$. We have, by Doob's inequality, for each $x > 0$, $x\,\mathbb{P}(M_* > x) \leqslant c$. Then,

$$\mathbb{E}[M_*^\beta] = \mathbb{E}\left[\int_0^{+\infty} \mathbf{1}_{M_*>x}\beta x^{\beta-1}\,\mathrm{d}x\right]$$
$$\leqslant \int_0^{+\infty} \min(1, c/x)\,\beta x^{\beta-1}\,\mathrm{d}x = c^\beta/(1-\beta). \qquad \square$$

Our main results are Theorems 6.2 and 6.3 below.

Theorem 6.2. *Let assumptions* (H1″), (H2)–(H5) *and* (H7) *hold. Then BSDE* (1) *has at most one solution* (Y,Z) *such that* Y *belongs to the class* (D) *and* Z *belongs to the space* $\bigcup_{\beta>\alpha} M^\beta$.

Proof. We can assume without loss of generality that $\mu = 0$.

Let us consider (Y,Z) and (Y',Z') two solutions to (1) with the appropriate conditions. Once again we introduce, for $n \in \mathbb{N}^*$,

$$\tau_n = \inf\left\{t \in [0,T], \int_0^t (|Z_r|^2 + |Z'_r|^2)\,\mathrm{d}r \geqslant n\right\} \wedge T.$$

Setting $y_t = Y_t - Y'_t$, $z_t = Z_t - Z'_t$, formula (4) yields the inequality

$$|y_{t\wedge\tau_n}| \leqslant |y_{\tau_n}| + \int_{t\wedge\tau_n}^{\tau_n} \langle \hat{y}_r, f(r,Y_r,Z_r) - f(r,Y'_r,Z'_r)\rangle\,\mathrm{d}r - \int_{t\wedge\tau_n}^{\tau_n} \langle \hat{y}_r, z_r\,\mathrm{d}B_r\rangle.$$

Thus, using the monotonicity of f in y, we get

$$|y_{t\wedge\tau_n}| \leqslant |y_{\tau_n}| + \int_0^T |f(r,Y_r,Z_r) - f(r,Y_r,Z'_r)|\,\mathrm{d}r - \int_{t\wedge\tau_n}^{\tau_n} \langle \hat{y}_r, z_r\,\mathrm{d}B_r\rangle$$

and conditioning with respect to $\mathscr{F}_t$ we have

$$|y_{t\wedge\tau_n}| \leqslant \mathbb{E}\left(|y_{\tau_n}| + \int_0^T |f(r,Y_r,Z_r) - f(r,Y_r,Z'_r)|\,\mathrm{d}r \,\Big|\, \mathscr{F}_t\right).$$

Of course, we want to send n to infinity. To do this, let us mention that the process y is continuous and belongs to class (D). It follows that, $\mathbb{P}$-a.s., $y_{\tau_n} = y_{T\wedge\tau_n} \to y_T = 0$

and, moreover, this convergence holds in L^1. As a byproduct, we deduce that the continuous martingale $\mathbb{E}(y_{\tau_n} \mid \mathscr{F}_t)$ converges to 0 in ucp. Extracting a subsequence, we obtain, $\mathbb{P}$-a.s.,

$$\forall t \in [0,T], \quad |y_t| \leqslant \mathbb{E}\left(\int_0^T |f(r,Y_r,Z_r) - f(r,Y_r,Z'_r)| \,\mathrm{d}r \mid \mathscr{F}_t \right) \tag{11}$$

and from assumption (H7) we get, $\mathbb{P}$-a.s.,

$$\forall t \in [0,T], \quad |y_t| \leqslant 2\gamma \mathbb{E}\left(\int_0^T (g_r + |Y_r| + |Z_r| + |Z'_r|)^\alpha \,\mathrm{d}r \mid \mathscr{F}_t \right).$$

Since there exists $\beta > \alpha$ such that Z and Z' belongs to M^β and since Y is of class (D), we deduce immediately from the previous inequality and assumption (H3) that y belongs to the space $\mathscr{S}^q$ for some $q > 1$. Lemma 3.1 and Proposition 3.2 imply that $(y,z) = (0,0) \in \mathscr{S}^q \times M^q$. □

We turn now to the existence part of our study. We are going to prove the following result.

Theorem 6.3. *Let assumptions* (H1″), (H2)–(H5) *and* (H7) *hold. Then BSDE* (1) *has a solution* (Y,Z) *such that* Y *belongs to the class* (D).

Moreover, for each $\beta \in (0,1)$, (Y,Z) *belongs to the space* $\mathscr{S}^\beta \times M^\beta$.

Before giving the proof of this result, we study the case where the generator is independent of the variable z.

Proposition 6.4. *Let assumptions* (H1″), (H3)–(H5) *hold and let us suppose that* f *does not depend on* z. *Then, BSDE* (1) *has a solution* (Y,Z) *such that* Y *belongs to the class* (D). *Moreover, for each* $\beta \in (0,1)$, (Y,Z) *belongs to the space* $\mathscr{S}^\beta \times M^\beta$.

Proof. We use a standard truncation method still assuming that $\mu = 0$. Let us introduce, for each integer $n \geqslant 1$, $\xi^n = q_n(\xi)$ and $f^n(t,y) = f(t,y) - f(t,0) + q_n(f(t,0))$ with $q_n(y) = y\,n/(|y| \vee n)$. We know from our previous result (Theorem 4.2) that the BSDE associated to the parameter (ξ^n, f^n) has a unique solution in the space $\mathscr{S}^2 \times M^2$.

We set $\delta Y = Y^{n+i} - Y^n$, $\delta Z = Z^{n+i} - Z^i$. Using the same computation as in the uniqueness part, see (11), we have

$$|\delta Y_t| \leqslant \mathbb{E}\left(|\delta\xi| + \int_0^T |f^{n+i}(r,Y_r^n) - f^n(r,Y_r^n)| \,\mathrm{d}r \mid \mathscr{F}_t \right)$$

from which we derive the inequality

$$|\delta Y_t| \leqslant \mathbb{E}\left(|\xi| \mathbf{1}_{|\xi|>n} + \int_0^T |f(r,0)| \mathbf{1}_{|f(r,0)|>n} \,\mathrm{d}r \mid \mathscr{F}_t \right). \tag{12}$$

We deduce immediately from this inequality that

$$\|\delta Y\|_1 \leqslant \mathbb{E}\left[|\xi| \mathbf{1}_{|\xi|>n} + \int_0^T |f(r,0)| \mathbf{1}_{|f(r,0)|>n} \,\mathrm{d}r \right]$$

and from Lemma 6.1 that, for any $\beta\in(0,1)$,

$$\mathbb{E}\left[\sup_t |\delta Y_t|^\beta\right] \leqslant \frac{1}{1-\beta}\,\mathbb{E}\left[|\xi|\mathbf{1}_{|\xi|>n} + \int_0^T |f(r,0)|\mathbf{1}_{|f(r,0)|>n}\,\mathrm{d}r\right]^\beta.$$

Thus $(Y^n)_{\mathbb{N}}$ is a Cauchy sequence for the $\|\cdot\|_1$ norm and for natural distance on $\mathscr{S}^\beta$ for each $\beta\in(0,1)$. Let Y be the progressive measurable continuous process limit of this sequence: Y belongs to class (D) and to $\mathscr{S}^\beta$ for each $\beta\in(0,1)$.

Now, $(\delta Y,\delta Z)$ solves the following BSDE:

$$\delta Y_t = \xi^{n+i} - \xi^n + \int_t^T F(r,\delta Y_r)\,\mathrm{d}r - \int_t^T \delta Z_r\,\mathrm{d}B_r,$$

where F stands for the random function

$$F(t,y) = f^{n+i}(t,y+Y_t^n) - f^n(t,Y_t^n);$$

since f^{n+i} is monotone, F satisfies the inequality

$$\langle y, F(t,y)\rangle \leqslant |y|\,|f(t,0)|\mathbf{1}_{|f(t,0)|>n}.$$

Thus, using Lemma 3.1, we deduce that, for $\beta\in(0,1)$,

$$\mathbb{E}\left[\left(\int_0^T |\delta Z_r|^2\,\mathrm{d}r\right)^{\beta/2}\right] \leqslant C_\beta\,\mathbb{E}\left[\sup_t |\delta Y_t|^\beta + \left(\int_0^T |f(r,0)|\mathbf{1}_{|f(r,0)|>n}\,\mathrm{d}r\right)^\beta\right].$$

It follows that, for each $\beta\in(0,1)$, $(Z^k)_k$ is a Cauchy sequence in M^β for the natural metric and then converges in this space to some progressively measurable process Z.

Since $\int_0^t Z_r^n\,\mathrm{d}B_r$ converges to $\int_0^t Z_r\,\mathrm{d}B_r$ in ucp and since the map $y\mapsto f(t,y)$ is continuous, we easily check by taking a limit in ucp that (Y,Z) solves the correct BSDE. □

With this proposition in hands we can prove our main existence result.

Proof of Theorem 6.3. Once again, we can assume that $\mu=0$ without loss of generality. We will use some kind of Picard's iterative procedure. Let us set as usual $(Y^0,Z^0)=(0,0)$ and define recursively, in view of the previous proposition, for each $n\geqslant 0$,

$$Y_t^{n+1} = \xi + \int_t^T f(r,Y_r^{n+1},Z_r^n)\,\mathrm{d}r - \int_t^T Z_r^{n+1}\,\mathrm{d}B_r, \quad 0\leqslant t\leqslant T.$$

For each n, Y^n belongs to class (D) and (Y^n,Z^n) belongs to $\mathscr{S}^\beta\times M^\beta$ for all $\beta\in(0,1)$.

For $n\geqslant 1$, arguing as in the proof of uniqueness, we deduce that

$$\forall t\in[0,T],\quad |Y_t^{n+1}-Y_t^n| \leqslant 2\gamma\mathbb{E}\left(\int_0^T (g_r+|Y_r^n|+|Z_r^n|+|Z_r^{n-1}|)^\alpha\,\mathrm{d}r\,\middle|\,\mathscr{F}_t\right).$$

Z^n and Z^{n-1} belongs to M^β for each $\beta \in (0,1)$, Y^n belongs to class (D) and $\{g_t\}_{t\in[0,T]}$ is integrable. Thus the random variable

$$I_n = \int_0^T (g_r + |Y_r^n| + |Z_r^n| + |Z_r^{n-1}|)^\alpha \, dr$$

belongs to the space L^q as soon as $\alpha q < 1$. Let us fix $q \in (1,2)$ such that $\alpha q < 1$. Then, for $n \geqslant 1$, $y^n = Y^{n+1} - Y^n$ belongs to the space $\mathscr{S}^q$. Let us set $z^n = Z^{n+1} - Z^n$. (y^n, z^n) is solution to the following BSDE:

$$y_t^n = \int_t^T f_n(r, y_r^n) \, dr - \int_t^T z_r^n \, dB_r,$$

where

$$f_n(r,y) = f(r, y + Y_r^n, Z_r^n) - f(r, Y_r^n, Z_r^{n-1}).$$

Since f is assumed to satisfy condition (H3) with $\mu = 0$, f_n satisfies assumption (A) and, using (H7), we have the inequality

$$\langle \hat{y}, f_n(r,y) \rangle \leqslant |f(r, Y_r^n, Z_r^n) - f(r, Y_r^n, Z_r^{n-1})| \leqslant 2\gamma (g_r + |Y_r^n| + |Z_r^n| + |Z_r^{n-1}|)^\alpha.$$

Lemma 3.1 shows that z^n is in the space M^q since I_n is in L^q.

Proposition 3.2 implies that there exists a constant C_q depending only on q such that for each $n \geqslant 1$,

$$\|(y^n, z^n)\|^q \leqslant C_q \mathbb{E}\left[\left(\int_0^T |f(r, Y_r^n, Z_r^n) - f(r, Y_r^n, Z_r^{n-1})| \, dr\right)^q\right],$$

where $\|\cdot\|$ stands for the following norm on $\mathscr{S}^q \times M^q$:

$$\|(Y,Z)\| = \left(\mathbb{E}\left[\sup_t |Y_t|^q + \left(\int_0^T |Z_r|^2 \, dr\right)^{q/2}\right]\right)^{1/q}.$$

For $n \geqslant 2$, we use the fact that f is λ-Lipschitz in z to get, using Hölder's inequality,

$$\|(y^n, z^n)\|^q \leqslant c \mathbb{E}\left[\left(\int_0^T |z_r^{n-1}|^2 \, dr\right)^{q/2}\right],$$

where $c = C_q \lambda^q T^{q/2}$. Thus, we have, for $n \geqslant 2$,

$$\|(y^n, z^n)\|^q \leqslant c^{n-1} \|(y^1, z^1)\|^q.$$

Let us assume first that $c = C_q \lambda^q T^{q/2} < 1$. Since $\|(y^1, z^1)\|^q$ is finite, it follows immediately that $(Y^n - Y^1, Z^n - Z^1)$ converges in the space $\mathscr{S}^q \times M^q$ to some (U, V). We deduce that (Y^n, Z^n) converges to $(Y = U + Y^1, Z = V + Z^1)$ in $\mathscr{S}^\beta \times M^\beta$ for each $\beta \in (0,1)$ since (Y^1, Z^1) belongs to it. Moreover Y^n converges to Y for the norm $\|\cdot\|_1$ since Y^1 belongs to class (D) and the convergence in $\mathscr{S}^q$ with $q \geqslant 1$ in stronger than the convergence in $\|\cdot\|_1$-norm.

To conclude the proof in this case, it remains to pass to the limit in the equation satisfied by (Y^n, Z^n) to see that (Y, Z) solves BSDE (1). This is easily done in ucp.

For the general case, we have only to subdivide the time interval $[0, T]$ into a finite number of small intervals. This completes the proof. □

References

Briand, Ph., Carmona, R., 2000. BSDEs with polynomial growth generators. J. Appl. Math. Stochastic Anal. 13 (3), 207–238.

Dellacherie, C., Meyer, P.-A., 1980. Probabilités et Potentiel. Théorie des Martingales. Hermann, Paris (Chapitres V à VIII).

El Karoui, N., Peng, S., Quenez, M.-C., 1997. Backward stochastic differential equations in finance. Math. Finance 7 (1), 1–71.

Kobylanski, M., 1997. Résultats d'existence et d'unicité pour des équations différentielles stochastiques rétrogrades avec des générateurs à croissance quadratique. C. R. Acad. Sci. Paris Sér. I Math. 324 (1), 81–86.

Lepeltier, J.-P., San Martin, J., 1998. Existence for BSDE with superlinear-quadratic coefficient. Stochastics Stochastics Rep. 63 (3,4), 227–240.

Pardoux, E., 1999. BSDEs, weak convergence and homogenization of semilinear PDEs, Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998). Kluwer Academic Publishers, Dordrecht, pp. 503–549.

Pardoux, E., Peng, S., 1990. Adapted solution of a backward stochastic differential equation. Systems Control Lett. 14 (1), 55–61.

Peng, S., 1991. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. Stochastics Stochastics Rep. 37 (1,2), 61–74.

Peng, S., 1997. Backward SDE and related g-expectation. In: El Karoui, N., Mazliak, L. (Eds.), Backward Stochastic Differential Equations, Pitman Research Notes Mathematical Series, Vol. 364. Longman, Harlow, pp. 141–159.

Revuz, D., Yor, M., 1991. Continuous martingales and Brownian motion. In: Grundlehren Math. Wiss., Vol. 293. Springer, Berlin, Heidelberg, New York.