Branching processes with competition by pruning of Lévy Trees

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A: Competition by pruning of Levy trees

B: Exceptional times for Generalised Fleming Viot

$\boxed{\mathrm{I}}$ Populations without interaction

Continuous State branching processes (CSBP) :

Markov process $(Z_t)_{t\geq 0}$, in $\mathbb{R}_+$, describe the size of the pop.

No interaction $\Leftrightarrow$ $\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$

Law characterized by branching mechanism $\lambda \to \Psi(\lambda)$

$$\Psi(\lambda) = -\alpha\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\pi(dx)$$

$\Rightarrow$ Laplace exponent of spectrally positive Levy process

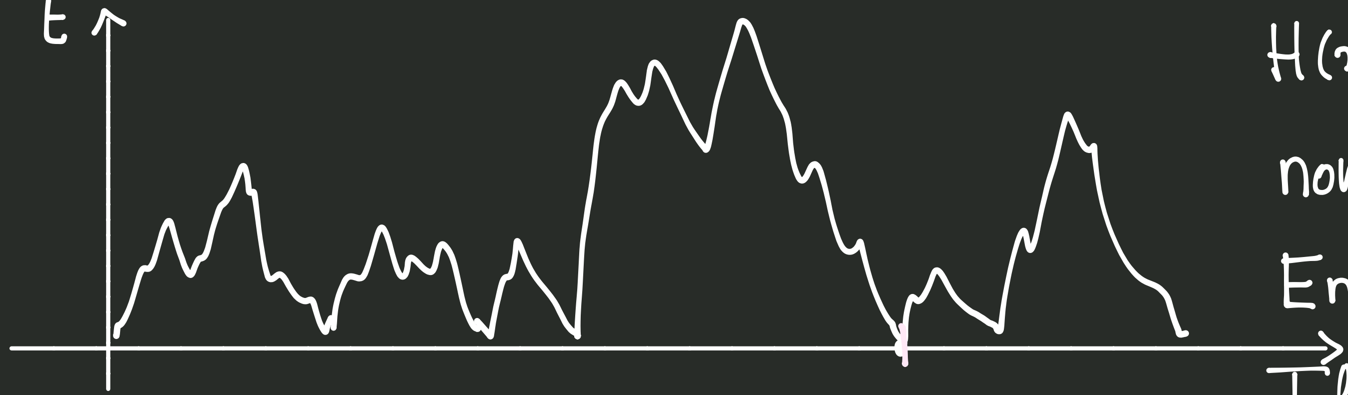
Assume : Conservative ($\mathbb{P}(Z_t < \infty) = 1$)	
(Sub)-critical ($\mathbb{E}(Z_t)$ is $\searrow$)	= regular.
$\mathbb{P}$(extinct. finite time) = 1	

Flows of CSBP's

$\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$: Can we construct all $(\mathbb{P}_x)_x$ simultaneously?

Two approaches: 1) Ray-Knight theorem 2) Levy driven SDE's.

1) Following Duquesne - Le Gall.



$H(x)$ = height process

non - Markovian.

Encodes Levy trees.

If $\Psi(\lambda) = 2\lambda^2$

$H(x) = |B(x)|$

L^t_x = Local time at level t, left of x.

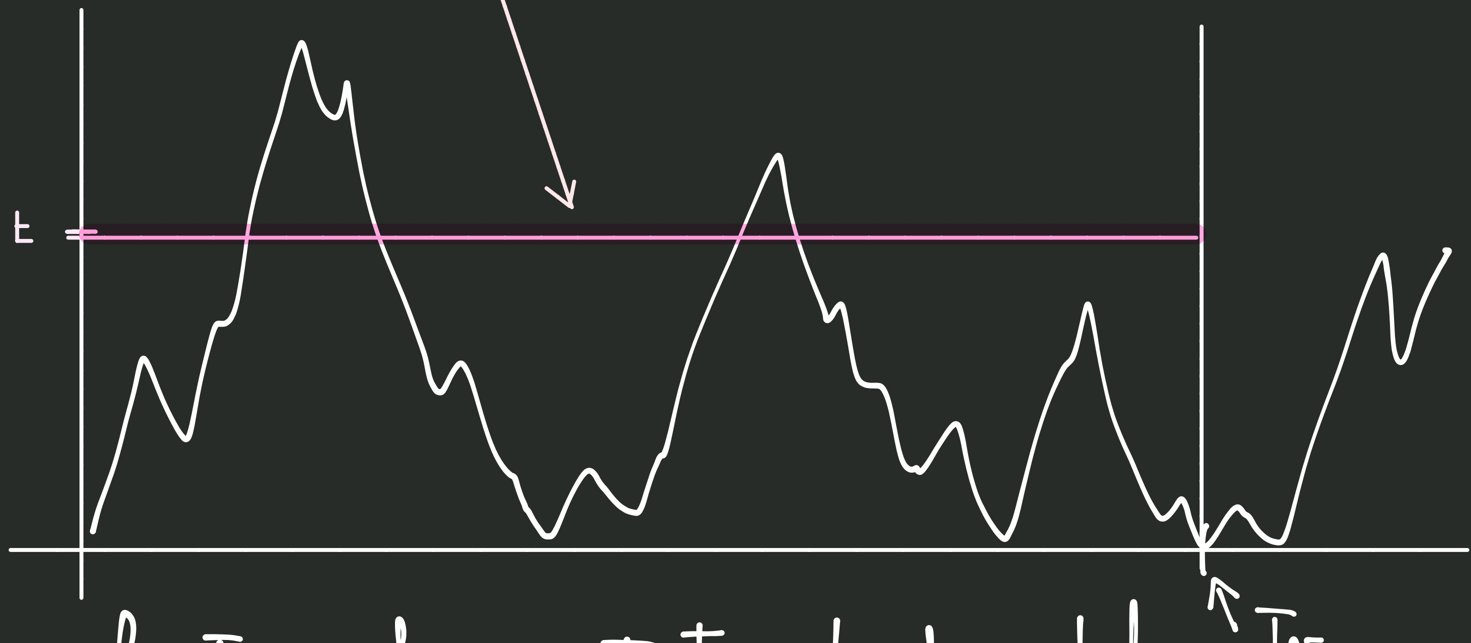
Thm

$\forall v \geq 0$ $\left(L^t_{T_v}, t \geq 0\right)$ has law $\mathbb{P}_v$

$$T_v = \inf\{x : L^0_x \geq v\}$$

Fix $v>0$, $T_v = \inf\{x : L^0_x \geq v\}$

$L^t_{T_v}$ = local Time in this band



$L^{s,t}_v$ = fraction of pop at time t descended from $[0,v]$ at time s

$= L^t_{T^s_v}$ $\quad T^s_v = \inf\{x : L^s_x \geq v\}$

2) Lévy driven SDE's

Following Dawson-Li and Bertoin-Le Gall

$$\begin{cases} Y_t(v) = v - \alpha \int_0^t Y_s(v)\,ds + \sigma \int_0^t \int_0^{Y_{s-}(v)} W(ds,du) \\ \qquad + \int_0^t \int_0^{Y_{s-}(v)} \int_0^\infty r\,\tilde{N}(ds,d\nu,dr) \\ t \geq 0,\ v \geq 0 \end{cases}$$

$\tilde{N}$ = Compensated PPP $ds \times d\nu \times \pi(dr)$

(s_i, ν_i, r_i) at time s_i, if $\nu_i \leq Y_{s_i-}(v)$, jump of size r_i

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Thm (Dawson-Li)

Existence and uniqueness + $(Y_t(v), t \geq 0\ v \geq 0) = (L^t_{T_v}, t \geq 0, v \geq 0)$

II Populations with competition.

Logistic growth equation : $dZ_t = (bZ_t - cZ_t^2)dt$

↳ ecological interpretation

II Populations with competition.

Feller diff w. logistic growth : $dZ_t = (bZ_t - cZ_t^2)dt + \sqrt{\gamma Z_t}\, dB_t$

Scaling limit of GW reprod w. mean $b+1$, var γ, death at rate cn

More general branching mechanism ψ by time change of O.U. process

(Lambert)

II Populations with competition.

Feller diff w. logistic growth : $dZ_t = (bZ_t - cZ_t^2)dt + \sqrt{\gamma Z_t}\, dB_t$

Scaling limit of GW reprod w. mean $b+1$, var γ, death at rate cn

More general branching mechanism Ψ by time change of O.U. process (Lambert)

SDE definition:

$$\begin{cases} Z_t(v) = v - \alpha \int_0^t Z_s(v)\,ds + \sigma \int_0^t \int_0^{Z_{s-}(v)} W(ds, du) \\ \qquad + \int_0^t \int_0^{Z_{s-}(v)} \int_0^{\infty} r\, \tilde{N}(ds, d\nu, dr) - \int_0^t G(Z_s(v))\,ds \\ t \geq 0,\ v \geq 0 \end{cases}$$

$G(z) = \int_0^z g(t)\,dt$ where $g\ \mathbb{R}_+ \to \mathbb{R}_+$ locally bounded.

<u>ex</u> $G(z) = z^2 \iff g(x) = 2x$

SDE definition:

$$\begin{cases} Z_t(v) = v - \alpha \int_0^t Z_s(v)\,ds + \sigma \int_0^t \int_0^{Z_{s-}(v)} W(ds, du) \\ \qquad + \int_0^t \int_0^{Z_{s-}(v)} \int_0^\infty r\, \tilde{N}(ds, d\nu, dr) - \int_0^t G(Z_s(v))\,ds \\ t \geq 0,\ v \geq 0 \end{cases}$$

v fixed. At time t: negative drift $G(Z_t(v))$ = total rate of killing

Indiv. $\eta \in [0, Z_t(v))$ killed at rate

$g(\eta) = g$(pop to its <u>left</u>)

Proposition: SDE has a unique strong solution.

$\forall v : t \to Z_t(v)$ is LBP (Ψ, G) càdlàg

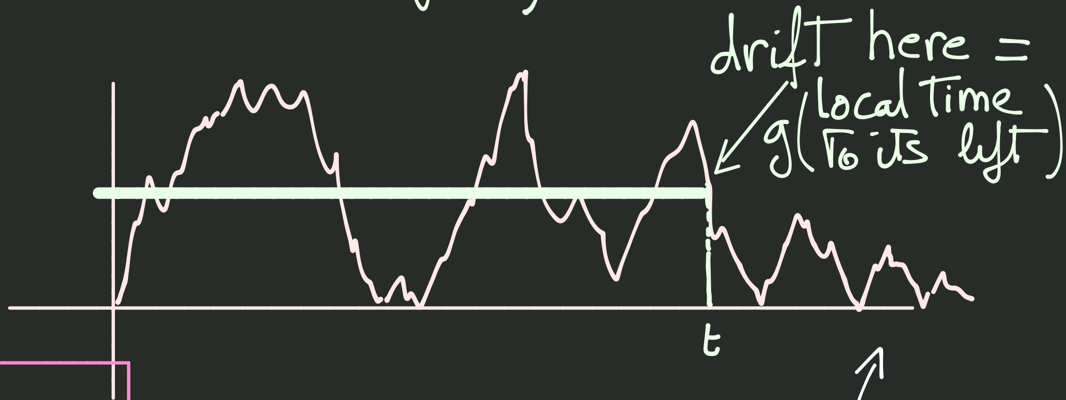
$u < v < w$, $t \to [Z_t(w) - Z_t(v)]$ is indep of $(Z_t(u))_t$ cond on $(Z_t(v))_t$

Le, Pardoux & Wakolbinger

Construct $H^{\searrow}$ = reflected BM + negative drift
such thad drift $\propto g$(local time "to the left")

Thm: (Le, Pardoux, Wakolbinger) + Ba, Pardoux

$\left(L^t_{T_v}(H^{\searrow}), t \geq 0, v \geq 0\right)$
is a LBP flow with quadratic branching and compet. g

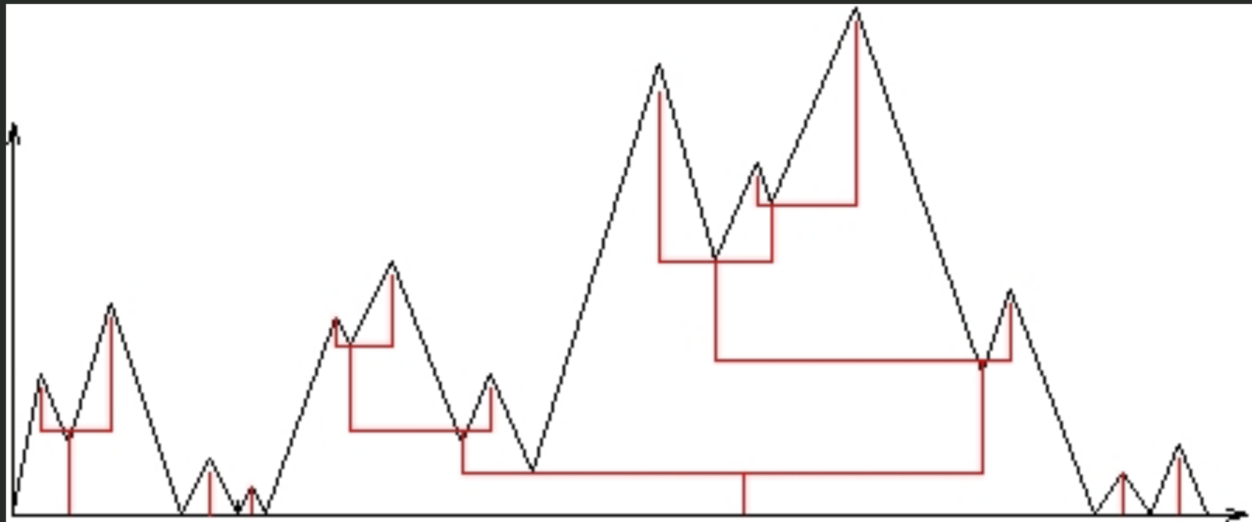


trees under attack from the left.

Sol of
$$\begin{cases} Z_t(v) = v - \alpha \int_0^t Z_s(v)\,ds + \sigma \int_0^t \int_0^{Z_{s-}(v)} W(ds,du) - \int_0^t G(Z_s(v))\,ds \\ Z_0(v) = v \end{cases}$$

No Poissonian term!

Excursions, Trees and Pruning



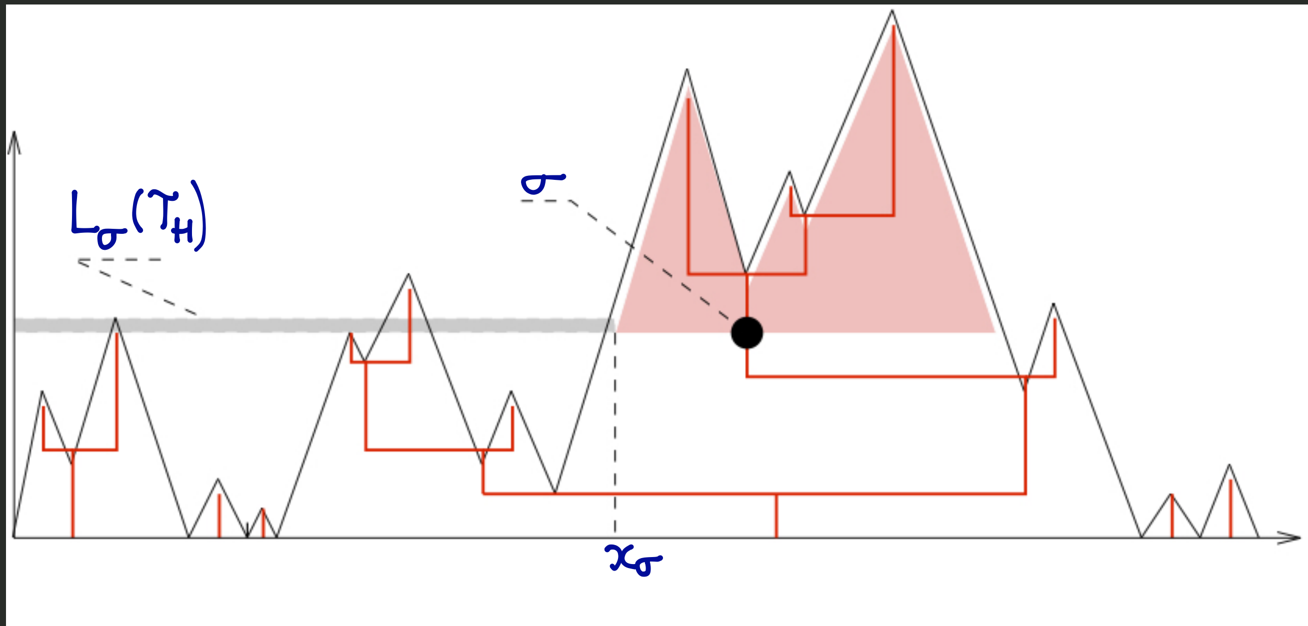
H encodes a real tree (or forrest) $\mathcal{T}$ with

1) $d\sigma$ = Lebesgue measure / skeleton

2) if $\sigma, \sigma' \in \mathcal{T}$ write $\sigma < \sigma'$ if σ is "left of" σ' (or an ancestor)

(the tree comes with a planar embedding)

Excursions, Trees and Pruning



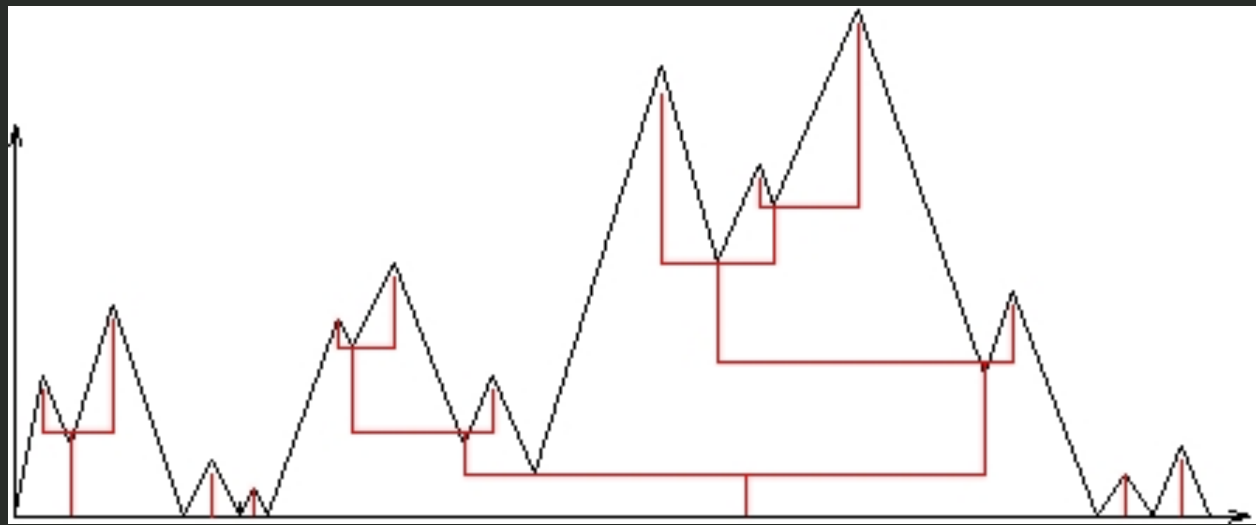
H encodes a real tree (or forrest) $\mathcal{T}$ with

1) $d\sigma$ = Lebesgue measure / skeleton

2) if $\sigma, \sigma' \in \mathcal{T}$ write $\sigma < \sigma'$ if σ is "left of" σ' (or an ancestor)
   (the tree comes with a planar embedding)

3) $\forall \sigma \in \mathcal{T} \Leftrightarrow L_\sigma(\mathcal{T}) = L^{|\sigma|}_{x_\sigma}(H)$ = local time left of σ.

Excursions, Trees and Pruning



H = heigh process of ψ-CSBP

π^θ = PPP / Skeleton of $\mathcal{T}$ w. intensity $\theta\, d\sigma$. $\pi = (\sigma_i)$

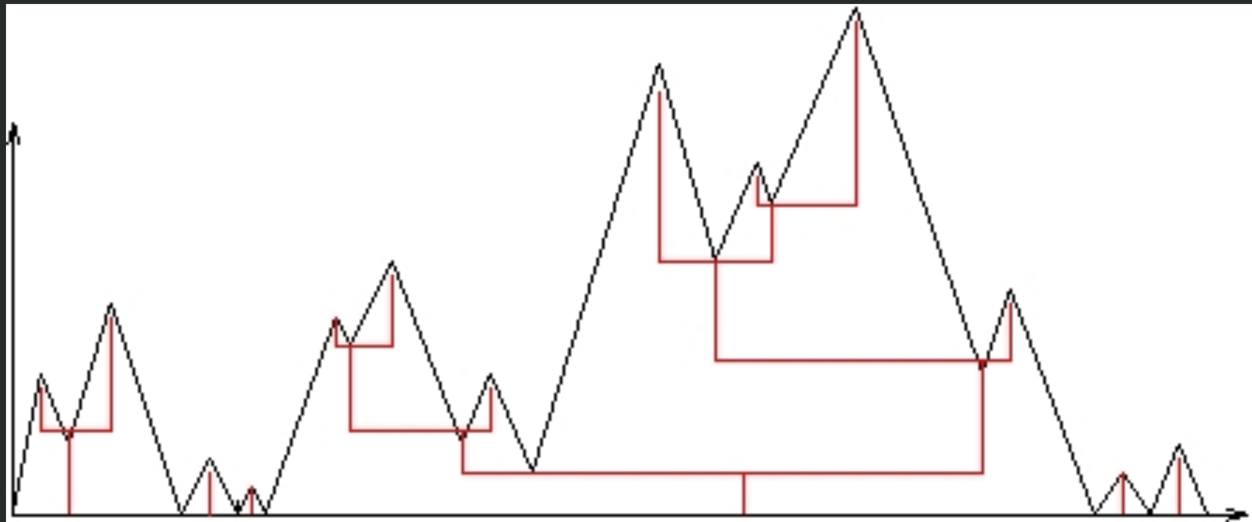
Pruning = cut at the σ_i, keep connected component of root.

Abraham - Delmas

$$\boxed{P(\mathcal{T}_\psi, \pi^\theta) \stackrel{d}{=} \mathcal{T}_{\psi_\theta} \quad \text{where } \psi_\theta(\lambda) = \psi(\lambda) + \theta\lambda}$$

+ embedding and $d\sigma$.

Excursions, Trees and Pruning



Abraham-Delmas

$$\boxed{P(\mathcal{T}_\psi, \pi^\theta) \overset{d}{=} \mathcal{T}_{\psi_\theta} \quad \text{where } \psi_\theta(\lambda) = \psi(\lambda) + \theta\lambda}$$

So $L^t_{\mathcal{T}_v}(P(\mathcal{T}_\psi, \pi^\theta))$ solves

$$\begin{cases} Z_t(v) = v - \alpha \int_0^t Z_s(v)\,ds + \sigma \int_0^t \int_0^{Z_{s-}(v)} W(ds, du) \\ \qquad + \int_0^t \int_0^{Z_{s-}(v)} \int_0^\infty r\, \tilde{N}(ds, d\nu, dr) - \int_0^t \theta Z_s(v)\,ds \\ t \geq 0,\ v \geq 0 \end{cases}$$

Corresponds To $g \equiv \theta$

Excursions, Trees and Pruning

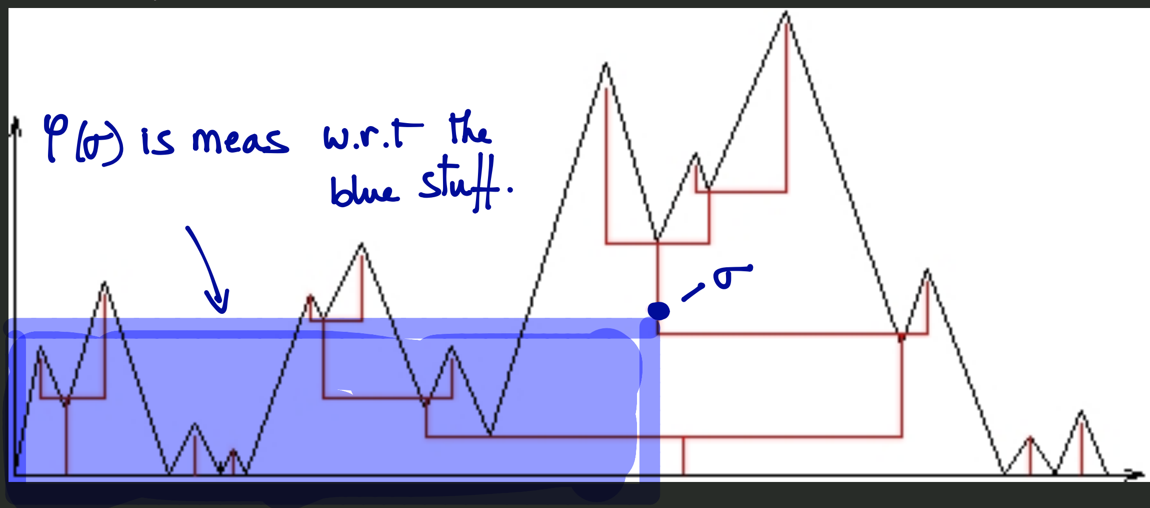
- $\pi = (\sigma_i, U_i)$ = PPP on $\mathcal{T}_\psi \times \mathbb{R}_+$ with intensity $d\sigma \times du$
- Let $\varphi : \mathcal{T}_\psi \to [0,\infty)$ be deterministic

$$\pi^{\varphi} = (\sigma_i \text{ s.t. } U_i \leq \varphi(\sigma_i))$$

- We say we prune with rate φ if we use π^{φ} to cut $\mathcal{T}_\psi$

$$\mathcal{P}(\mathcal{T}_\psi, \varphi) = \text{prune}(\mathcal{T}_\psi, \pi^{\varphi}). \quad \text{Ex: } \varphi \equiv \theta$$

- $\varphi : \mathcal{T}_\psi \to [0,\infty)$ is called an adapted intensity $\varphi(\sigma)$ is measurable w.r.t. $\mathcal{T}_{<\sigma}$ = part of the tree "left of σ" and (σ_i, U_i) s.t. $\sigma_i \in \mathcal{T}_{<\sigma}$



Main results

Thm (B. Fontbona, Fittipaldi '15) Suppose $g \nearrow$

(1) With proba 1, there exists a unique φ^* adapted s.t.

$$\varphi^*(\sigma) = g\left(L_\sigma\left(\mathcal{P}(\mathcal{T},\varphi^*)\right)\right), \quad \forall \sigma \in \mathcal{P}(\mathcal{T},\varphi^*)$$

(2) Furthermore, $\left(L^t_{\mathcal{T}_v}(\mathcal{P}(\mathcal{T},\varphi^*)), t\geq 0, v \geq 0\right)$ is a weak solution of

$$\begin{cases} Z_t(v) = v - \alpha \int_0^t Z_s(v)\,ds + \sigma \int_0^t \int_0^{Z_{s-}(v)} W(ds,du) \\ \qquad + \int_0^t \int_0^{Z_{s-}(v)} \int_0^\infty r\,\tilde{N}(ds,d\nu,dr) - \int_0^t G(Z_s(v))\,ds \\ t \geq 0,\ v \geq 0 \end{cases}$$

Again, $g(x) = x$, $G(z) = z^2/2$ is a typical example.

Proof (1) Picard iterration

Define a sequence $\varphi^{(n)}$. Write $T^{(n)} = P(T, \varphi^{(n)})$

$\varphi^{(0)} \equiv 0$ and $\varphi^{(n+1)}_{(\cdot)} = g(L_\cdot(T^{(n)}))$ (note: $\varphi^* =$ fixed point !)

$\varphi^{(0)} = 0 \Rightarrow T^{(0)} = T$ (whole tree)

$\varphi^{(1)} = L(T^{(0)}) > 0$ so $T^{(1)}$ subtree of T

$\varphi^{(2)} = L(T^{(1)}) < L(T^{(0)})$, so $T^{(1)} < T^{(2)} < T^{(0)}$

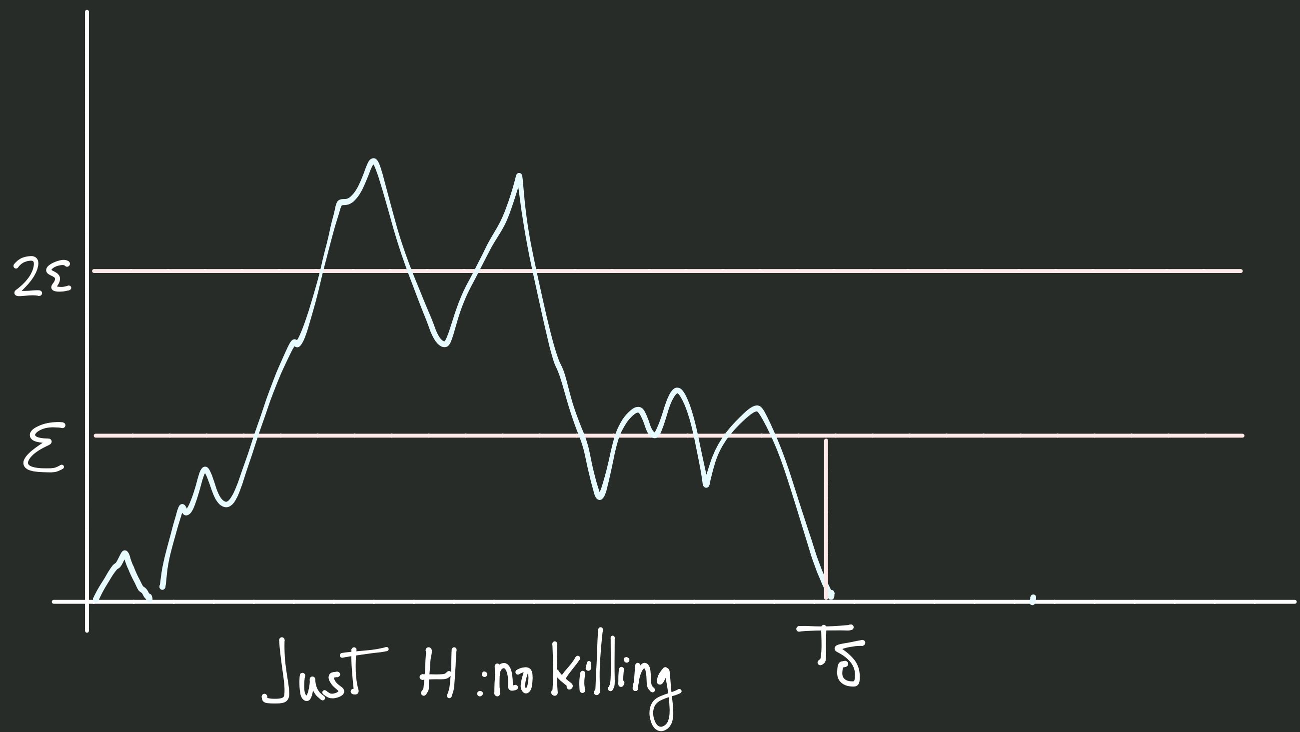
$\varphi^{(3)} = L(T^{(2)}) > \varphi^{(2)}$ so $T^{(1)} < T^{(3)} < T^{(2)} < T^{(0)}$

$\vdots$

$\varphi^{(0)} < \varphi^{(2)} < \cdots < \varphi^{(5)} < \varphi^{(3)} < \varphi^{(1)}$

Claim: $\varphi^{(n)} \longrightarrow \varphi^*$ uniformly on compacts. (Gronwall)

Proof (2) Discretisation $\varepsilon > 0$, $\delta > 0$ fixed



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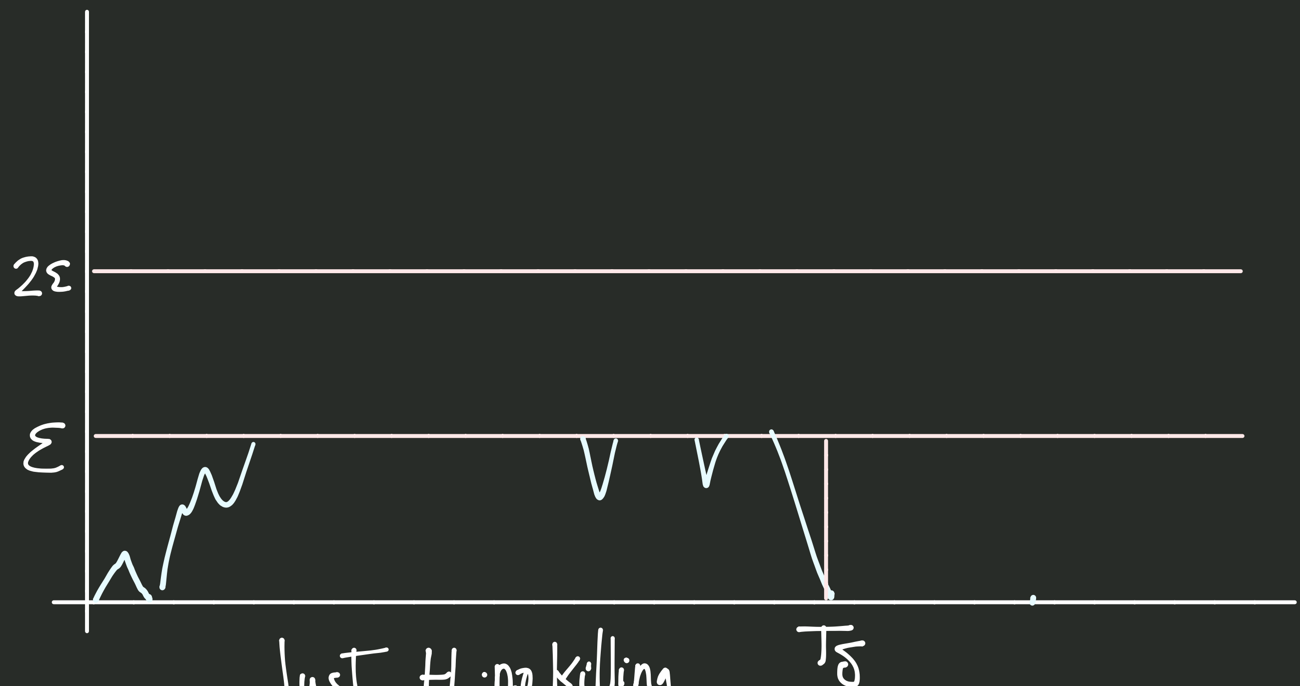
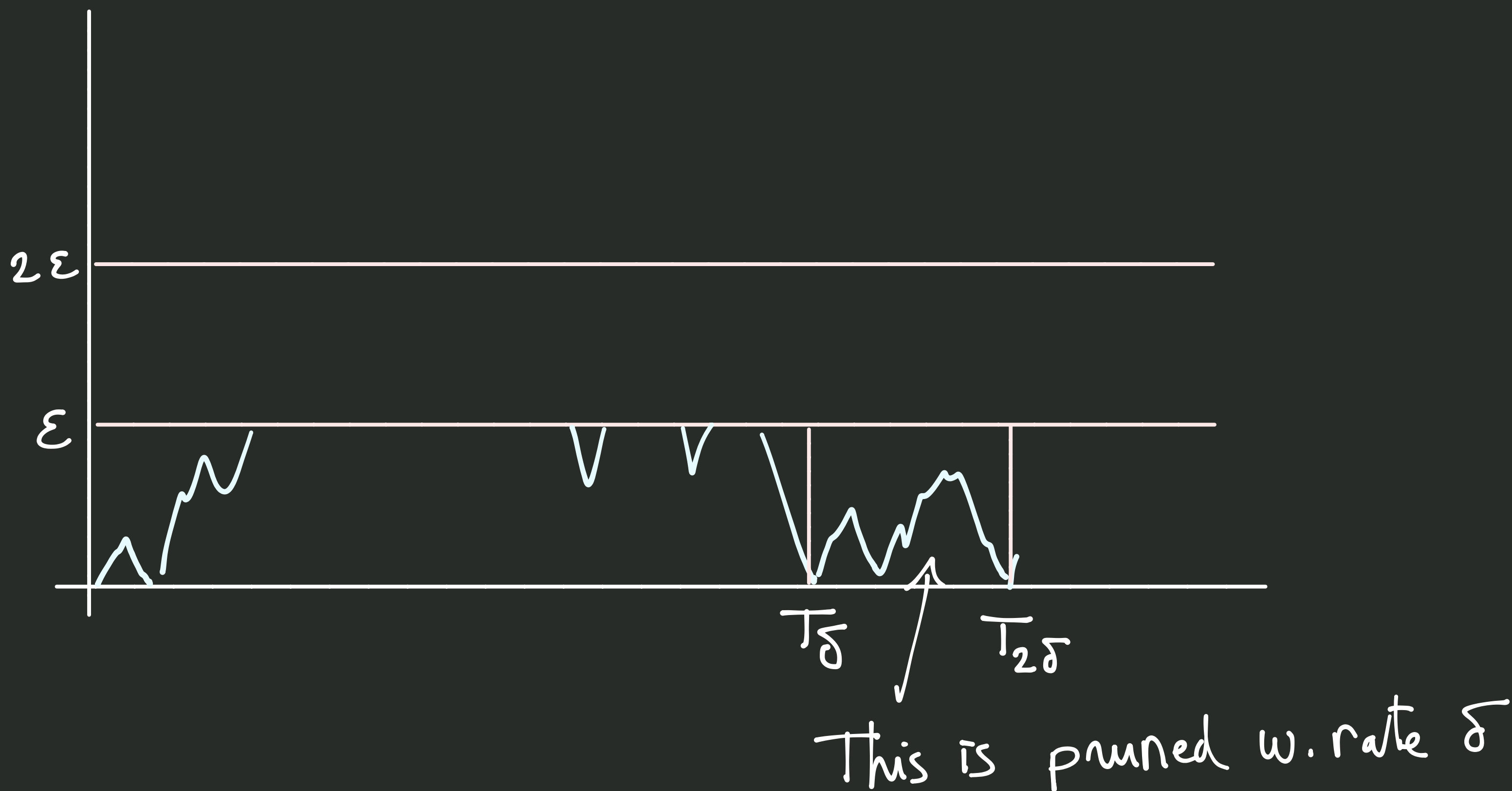


Just H: no killing

Proof (2) Discretisation $\varepsilon > 0$, $\delta > 0$ fixed



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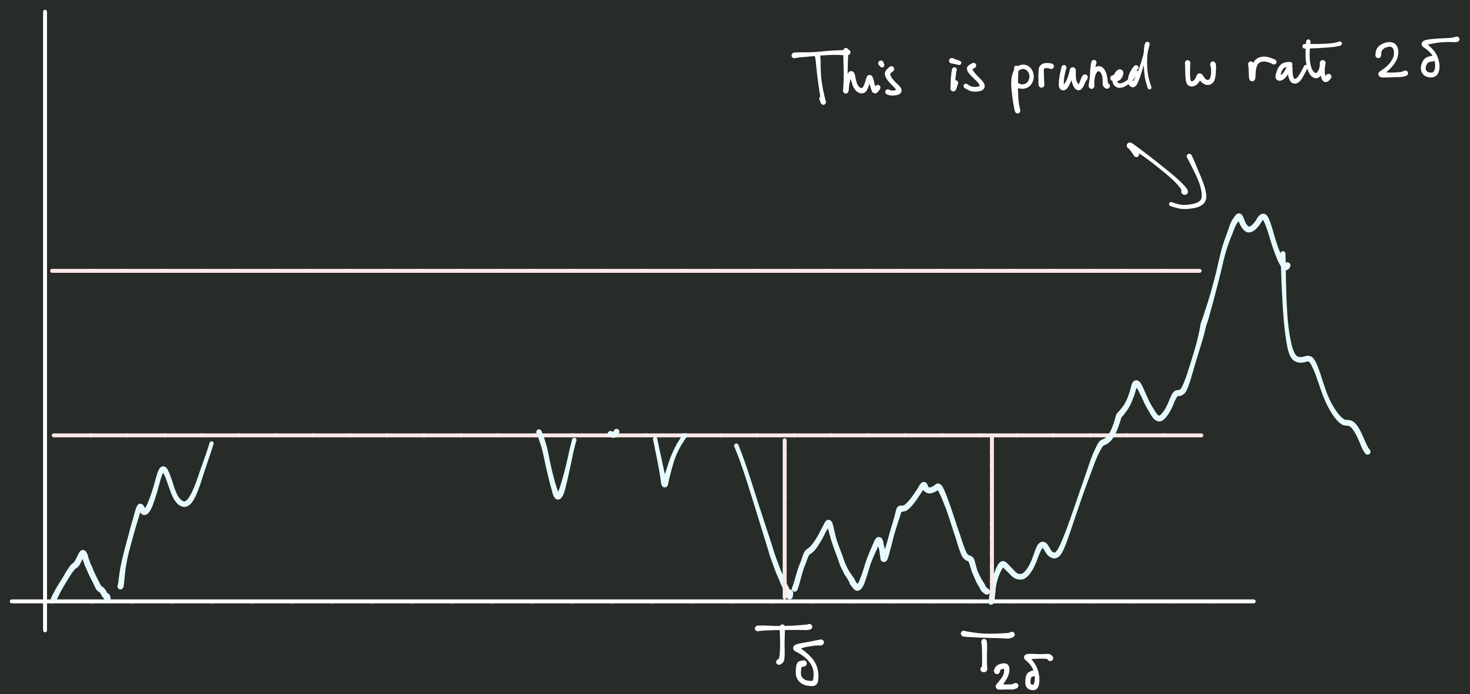

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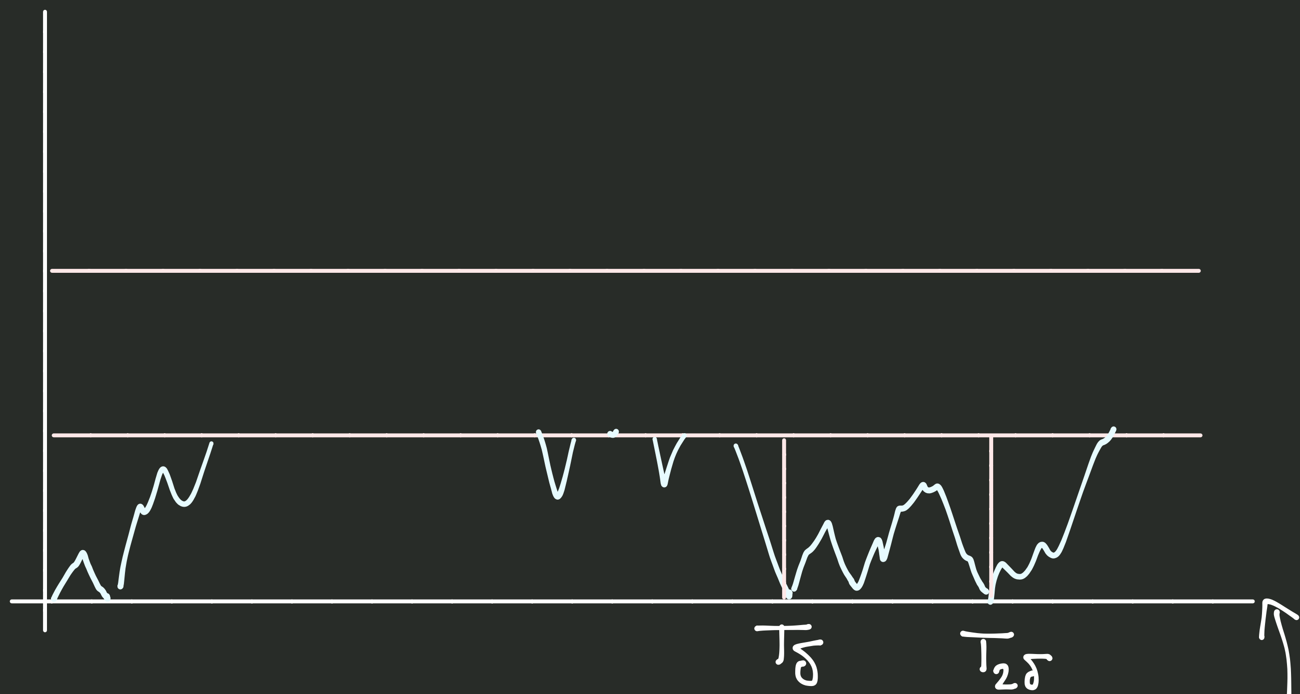
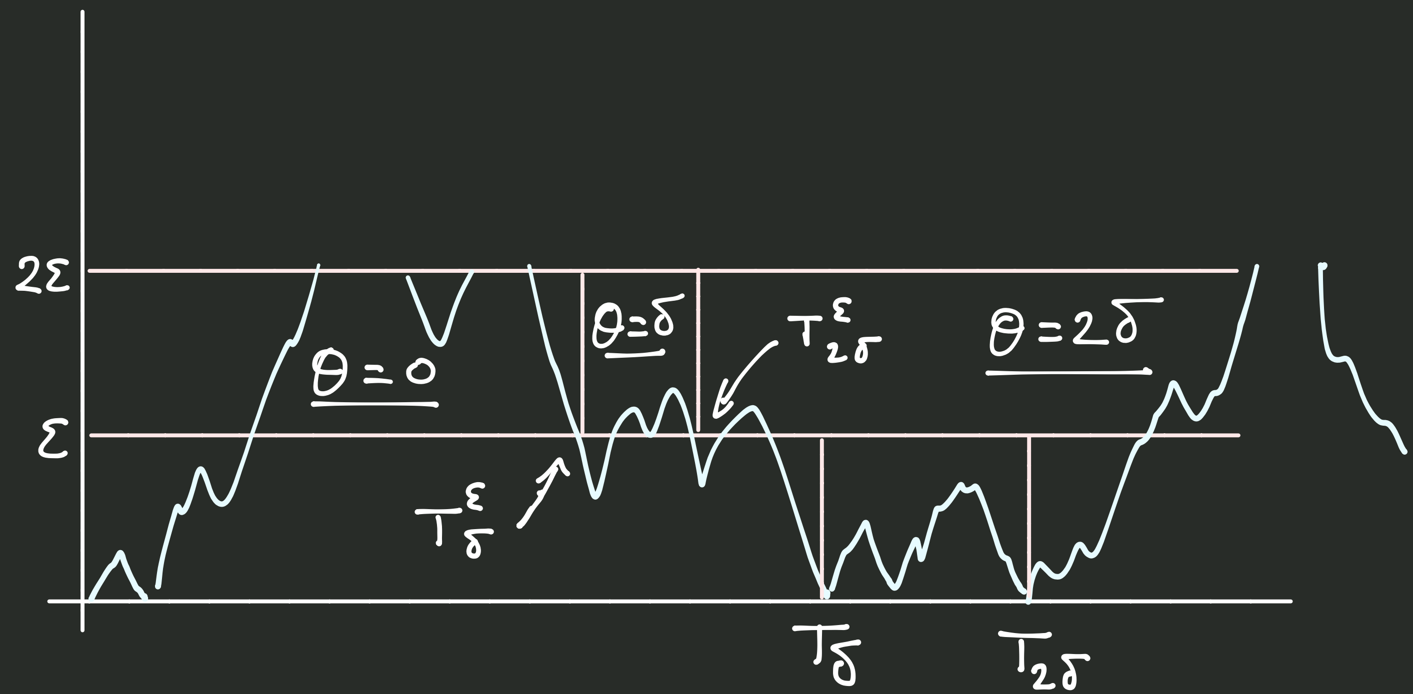


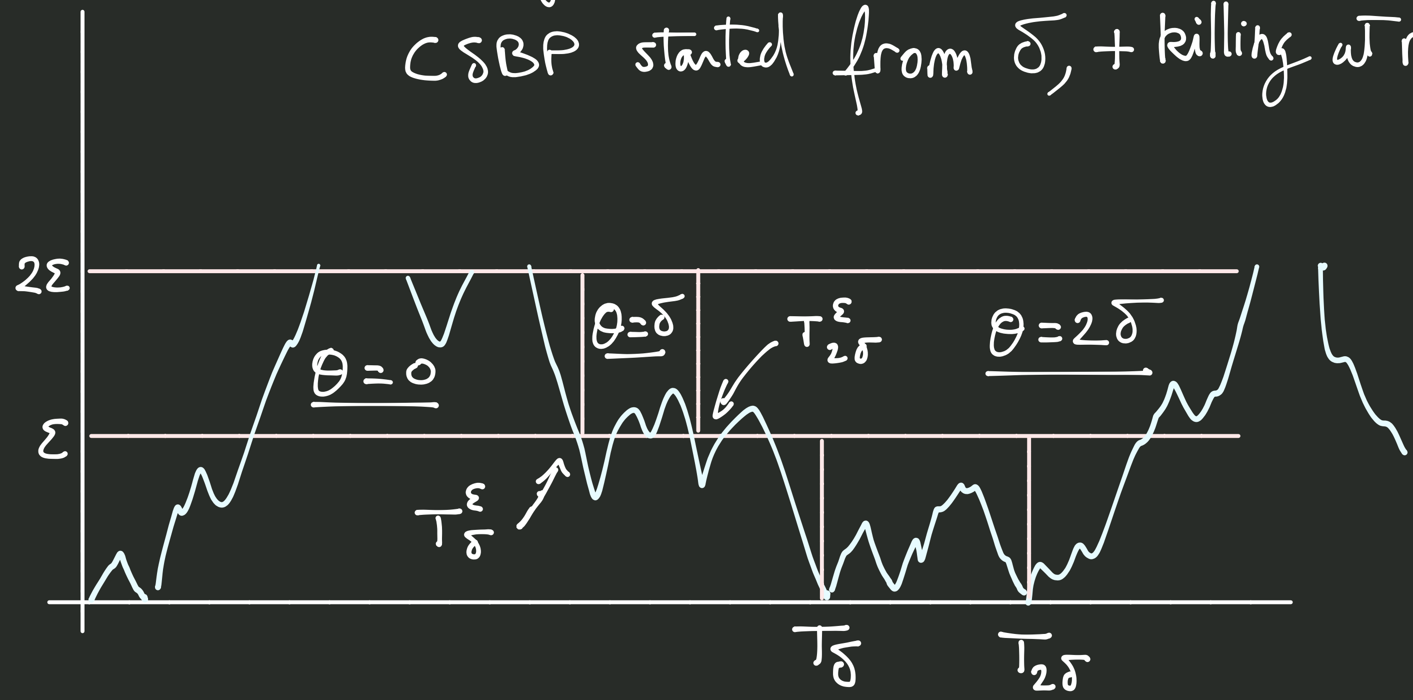
This is pruned w rate 2δ

Proof (2) Discretisation $\varepsilon>0$, $\delta>0$ fixed



Proof (2) Discretisation $\varepsilon > 0$, $\delta > 0$ fixed

In each rectangle $\underset{k\delta}{\boxed{}}^{(i+1)\varepsilon}_{i\varepsilon}$ I have an indep CSBP started from δ, + killing at rate $k\delta$



B. Exceptional times for stable Fleming-Viot with mutations

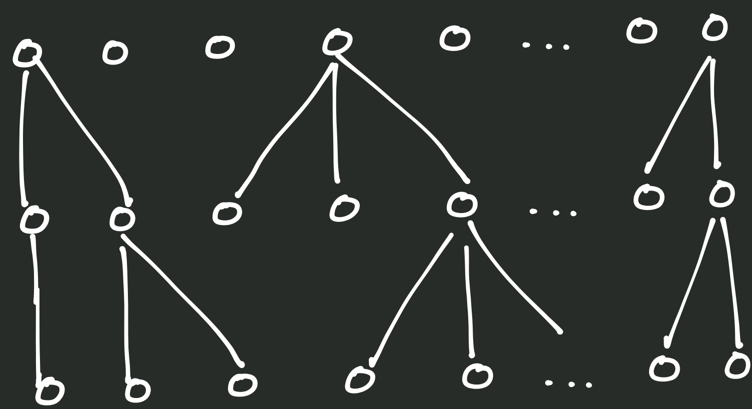
1. Classical Fleming-Viot

Model for evol. of pop. of fixed size

take a Moran model

Each indiv has ν_i offsprings

ν_i are exchangeable



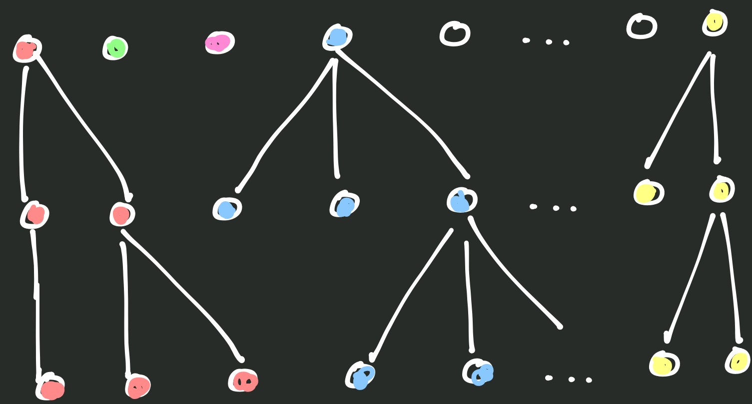
1. Classical Fleming-Viot

Model for evol. of pop. of fixed size

take a Cannings model

Each indiv has ν_i offspring

ν_i are exchangeable, 2^{nd} moment.



Each indiv at gen 0 has type $U_i \sim$ iid $\mathcal{U}_{[0,1]}$

$X_i(n) =$ type of i in gen. n.

$$\mu_n^{(N)}(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(n)}(\cdot)$$

Scaling limit: $\mu_{[Nt]}^{(N)}(\cdot) \longrightarrow \mu_t(\cdot)$ Markov process on $\mathbb{P}/[0,1]$

Called Fleming-Viot process.

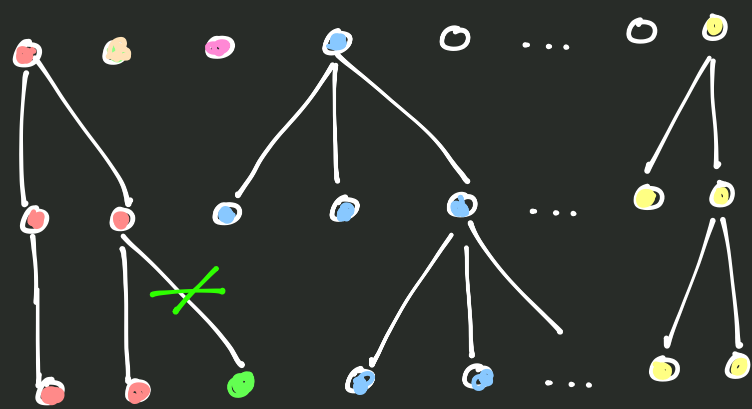
$\mu_0(dy) = dy$ on $[0,1]$ but $\forall t>0$, $\mu_t(\cdot) = \sum_{1}^{K_t} a_i \, \delta_{X_i(t)}(\cdot)$ $\leftarrow$ finite.

Classical Fleming-Viot + mutations FVM(θ)

each indiv gets a mutation w. proba θ/N

Scaling limit $\mu_t^\theta(\cdot)$

Fleming-Viot with mutations



a new mutant type appears

For a fixed $t>0$,

$\#$types $=\infty$ almost surely!

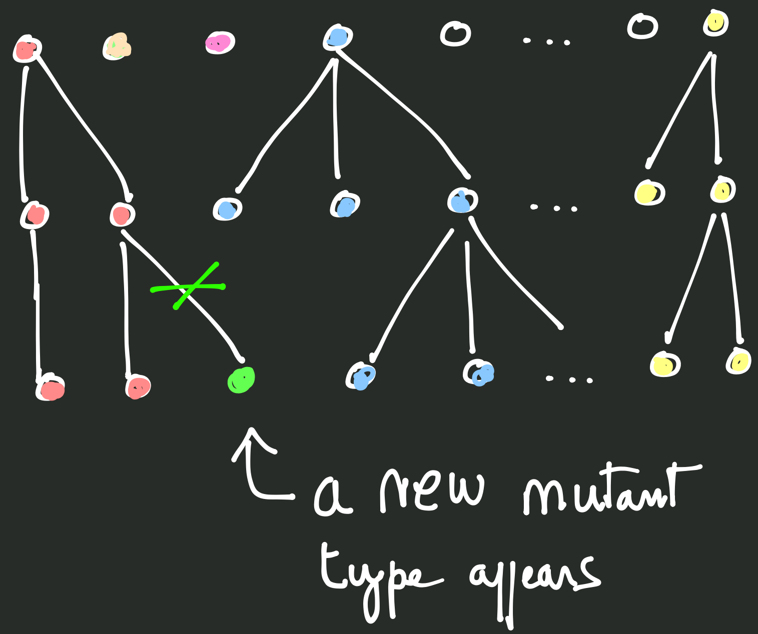
(recall: when no mutation, $\forall t>0$ $\#$types $<\infty$ almost surely).

Classical Fleming-Viot + mutations FVM(θ)

each indiv gets a mutation w. proba θ/N

Scaling limit $\mu_t^{\theta}(\cdot)$

Fleming-Viot with mutations



Thm (Schmuland)

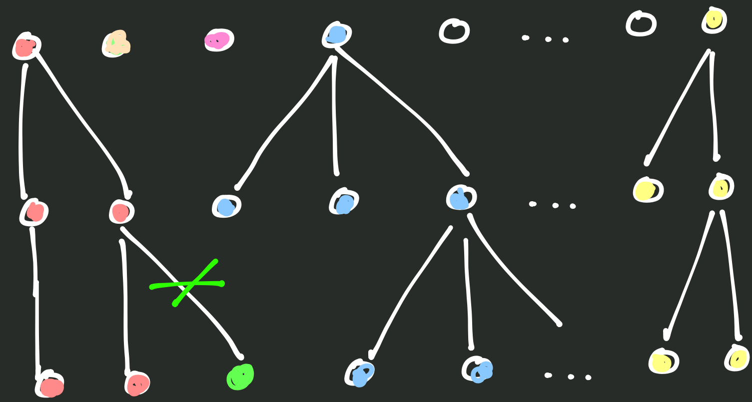
$$\mathbb{P}(\exists t>0 : \#\text{ types} < \infty) = \begin{cases} 1 & \text{if } \theta < 1 \\ 0 & \text{if } \theta \geq 1 \end{cases}$$

Very analytical proof.

Generalized FV $\quad \alpha$ stable FVM (θ)

Take ν_i by generating X_i iid with $\mathbb{P}(X_i > y) \sim y^{-\alpha}$ then sample N out of $\sum_{i=1}^{N} X_i$

then scaling limit is G.F.V α-stable. $(\alpha \in (1,2))$



Add mutations at rate $\theta \cdot N^{1-\alpha} \Rightarrow \forall t > 0$ fixed $\#$Types $= \infty$ a.s.

Thm (B. Doering Mytnik, Zambotti) For α-stable GFVM(θ)

$\forall \theta \quad \mathbb{P}(\exists t : \#\text{types} < \infty) = 0$

FV and CB process $\Psi(\lambda)=\lambda^2$

$\left(Z_t(v),\ t\geq 0,\ v\in[0,1]\right)$

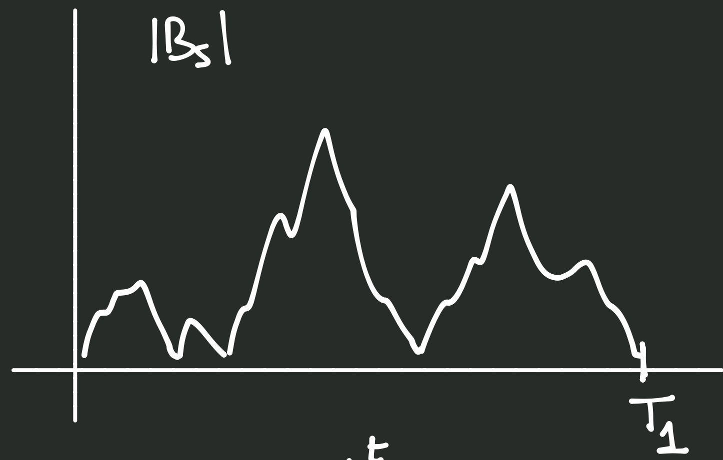
measure valued Feller diffusion

$\Psi(q)=q^2$

Then $\mu_t(\cdot) := \dfrac{Z_t(\cdot)}{Z_t(1)}$,

$\mu_{S^{-1}(t)}(\cdot)$ with $S(t)=\displaystyle\int_0^t \frac{1}{Z_s(1)}\,ds$

is a FV process.



$Z_t(v) = L^t_{T_v}$

FVM(θ) and CBI $\quad \psi(\lambda)=\lambda^2$

$$(Z_t(v),\ t\geq 0,\ v\in[0\ 1])$$

$$= \mathrm{CBI}\,(\psi(\lambda)=\lambda^2, \theta)$$

Then $\mu_t(\cdot) := \dfrac{Z_t(\cdot)}{Z_t(1)}$,

$\mu_{S^{-1}(t)}(\cdot)$ with $S(t)=\displaystyle\int_0^t \frac{1}{Z_s(1)}\,ds$

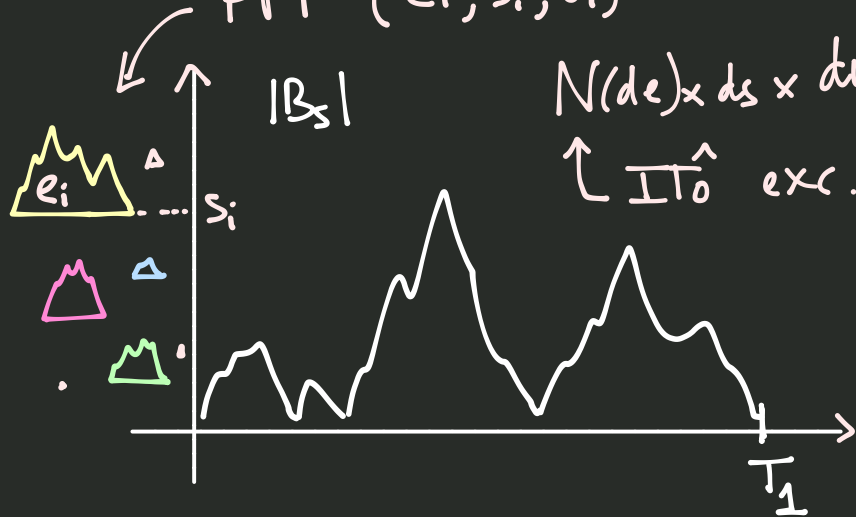
is a FVM(θ) process.

exc $\quad$ time $\quad$ color

PPP (e_i, s_i, u_i)

$N(de)\times ds \times du\, \mathbb{1}_{[0\,1]}$

Itô exc. meas.



$$Z_t(v) = L^t_{T_v} + \sum_{u_i\leq v} L^t(e_i)$$

GFV and CB process

$\left(Z_t(v),\ t \geq 0,\ v \in [0\,1]\right)$

$= CBi\left(\psi(\lambda) = \lambda^2, \theta\right)$

Then $\mu_t(\cdot) := \dfrac{Z_t(\cdot)}{Z_t(1)}$,

$\mu_{S^{-1}(t)}(\cdot)$ with $S(t) = \int_0^t \frac{1}{Z_s(1)}\, ds$

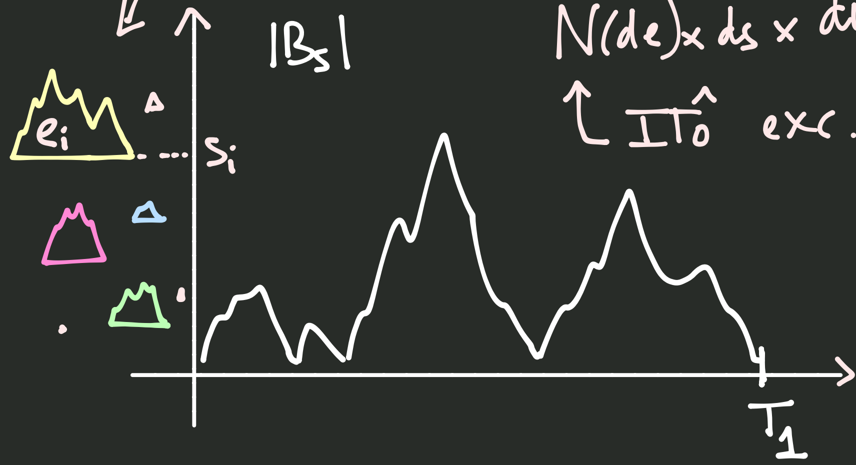
is a $FVM(\theta)$ process.

exc time color

PPP (e_i, s_i, v_i)

$N(de) \times ds \times du\, \mathbb{1}_{[0\,1]}$

ITô exc. meas.



$Z_t(v) = L^t_{T_v} + \sum_{v_i \leq v} L^t(e_i)$

$dZ_t = \sqrt{2Z_t}\, dB_t + \theta\, dt$

$\to$ Pitman Yor

$\to$ $\theta = 1$ critical.

Dawson-Li SDE's for FVM(θ) and CBi $\quad \Psi(\lambda)=\lambda^\alpha$

- $$Z_t(v) = v + \int_0^t \int_0^{Z_{s-}(v)} \int_0^\infty r\,\tilde{N}(ds, dv, dr) + v\int_0^t G(Z_s(1))\,ds$$

CBI = Continuous branching with interactive immigration.

- $$Y_t(v) = v + \int_0^t \int_0^1 \int_0^1 r\left[\mathbb{1}_{v \le Y_{s-}(v)} - Y_{s-}(v)\right] M(ds, dv, dr) + \theta\int_0^t \left[v - Y_s(v)\right] ds$$

M = non comp. Poisson
$ds \otimes dv \otimes r^{-2}\Lambda(dr)$ $\qquad \Lambda(dr) = \mathrm{Beta}(2-\alpha, \alpha)$

$Y_t(\cdot) = \alpha$-stable FVM(θ).

Time change and renormalisation $\quad \Psi(\lambda) = \lambda^\alpha$

Take Z α-stable with immigration $G(z) = c_\alpha z^{2-\alpha}$

- $$Z_t(v) = v + \int_0^t \int_0^{Z_{s-}(v)} \int_0^\infty r \, \tilde{N}(ds, du, dr) + v \int_0^t \theta \, c_\alpha Z_s(1)^{2-\alpha} \, ds$$

$$S(t) = c_\alpha \int_0^t Z_s(1)^{1-\alpha} \, ds$$

Then $\mu_t(\cdot) = \dfrac{Z_{S^{-1}(t)}(\cdot)}{Z_{S^{-1}(t)}(1)}$ is an α stable $FV(\theta)$.

Pitman-Yor representation for Z ?

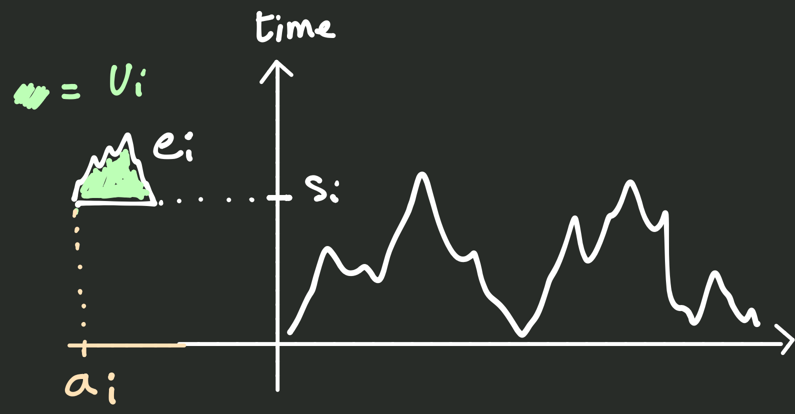
Interactive immigration

$\psi(\lambda) = \lambda^\alpha$

$t \to \varphi(t) \geq 0$ (the rate)

$\text{PPP}\,(e_i, s_i, u_i, a_i)$

e_i: exc, s_i: Time, u_i: color, a_i: filter



$$\text{intensity} = N(de) \times ds\, \mathbb{1}_{s>0} \times du\, \mathbb{1}_{[0,1]}$$
$$\times\, da\, \mathbb{1}_{a>0}$$

Interactive immigration

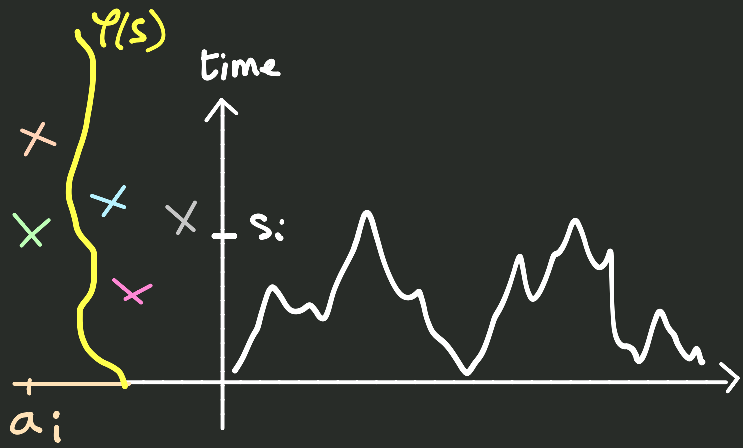
$\psi(\lambda) = \lambda^\alpha$

$t \to \varphi(t) \geq 0$ (the rate)

PPP (e_i, s_i, u_i, a_i) → filter

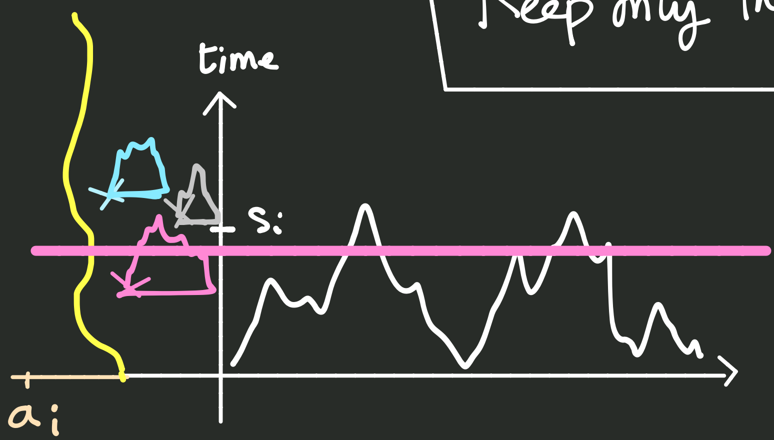
e_i: exc, s_i: Time, u_i: color

intensity $= N(de) \times ds\, \mathbb{1}_{s>0} \times du\, \mathbb{1}_{[0,1]} \times da\, \mathbb{1}_{a>0}$



Keep only the e_i s.t. $a_i \leq \varphi(s_i)$

$L^t(\varphi)$ = local Time at level t of CSBP + immig.



Thm (B., Doering, Mytnik, Zambotti) $\exists! \varphi^*$ such that

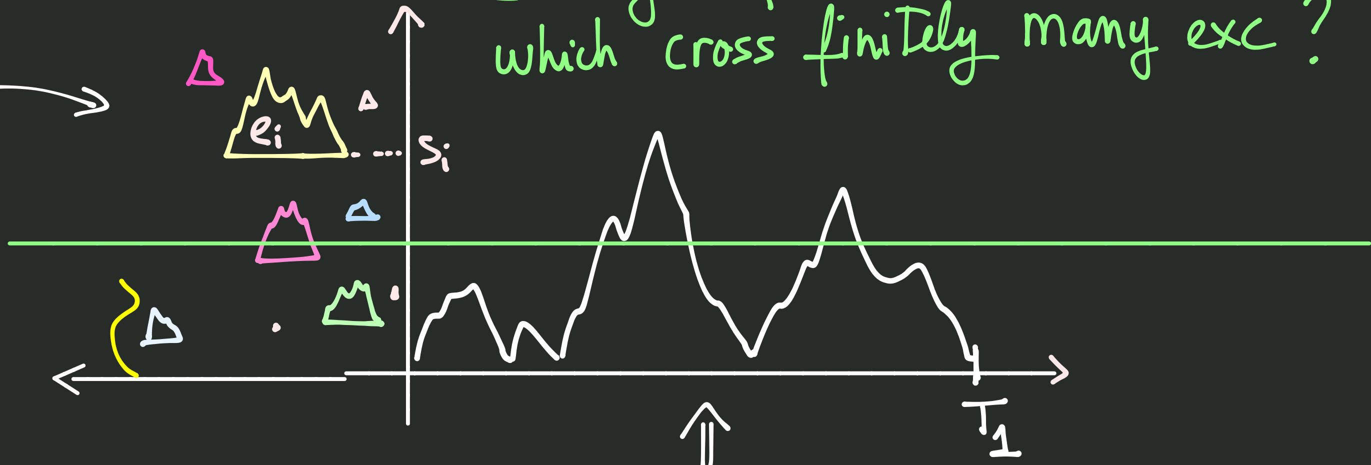
$$\boxed{G(L^t(\varphi^*)) = \varphi^*(t)}$$

Think $G(x) = x^{2-\alpha}$

Exceptional Times

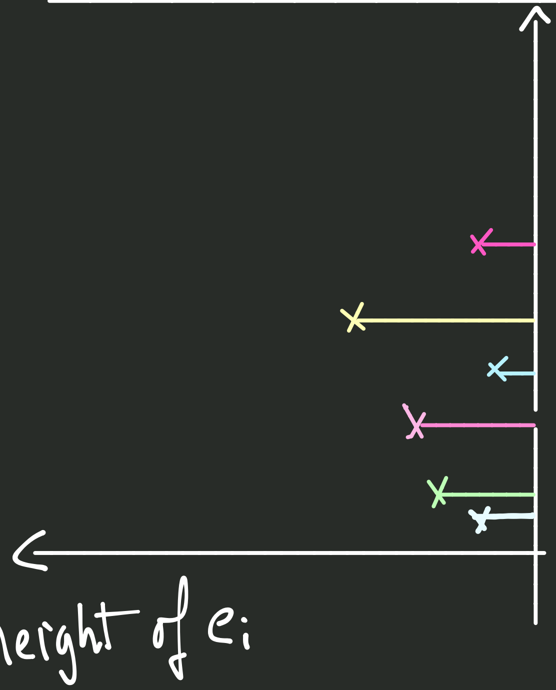
only those
s.t. $a_i \le Z^{2-\alpha}_{s_i}(1)$
i.e right of $\int$

$\Rightarrow$ Can you find a line which cross finitely many exc?

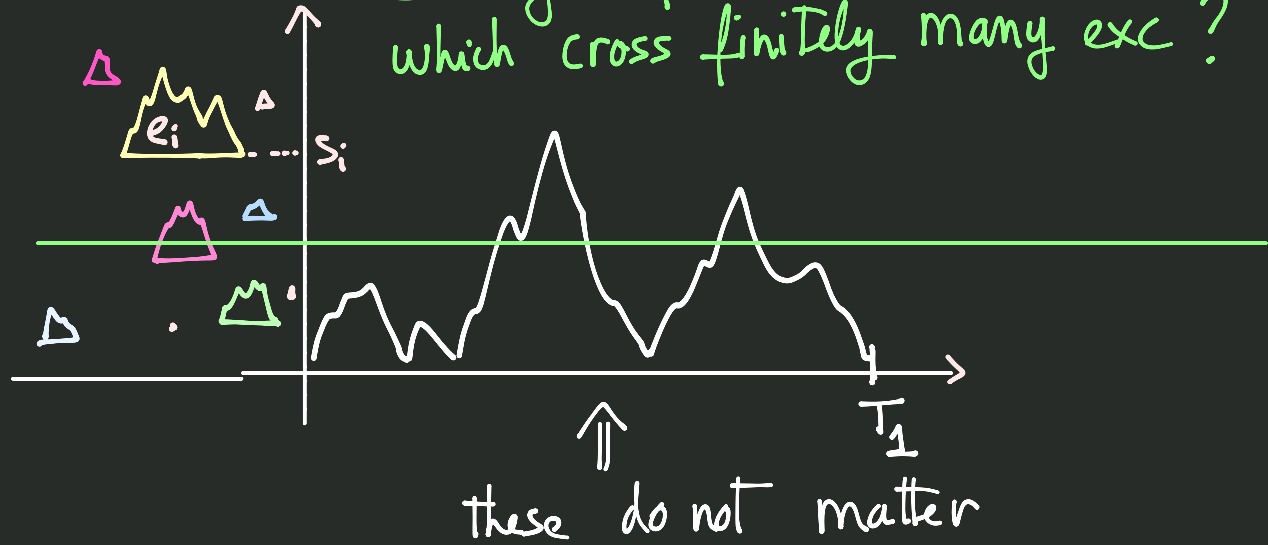


these do not matter

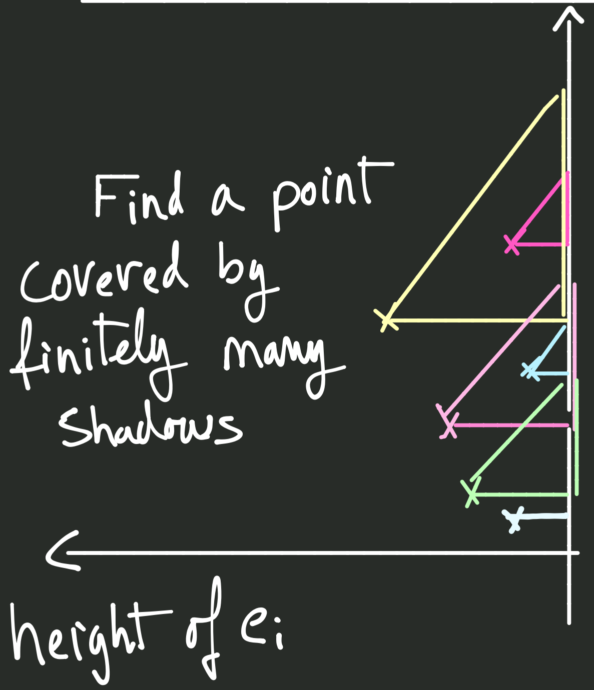
Exceptional Times

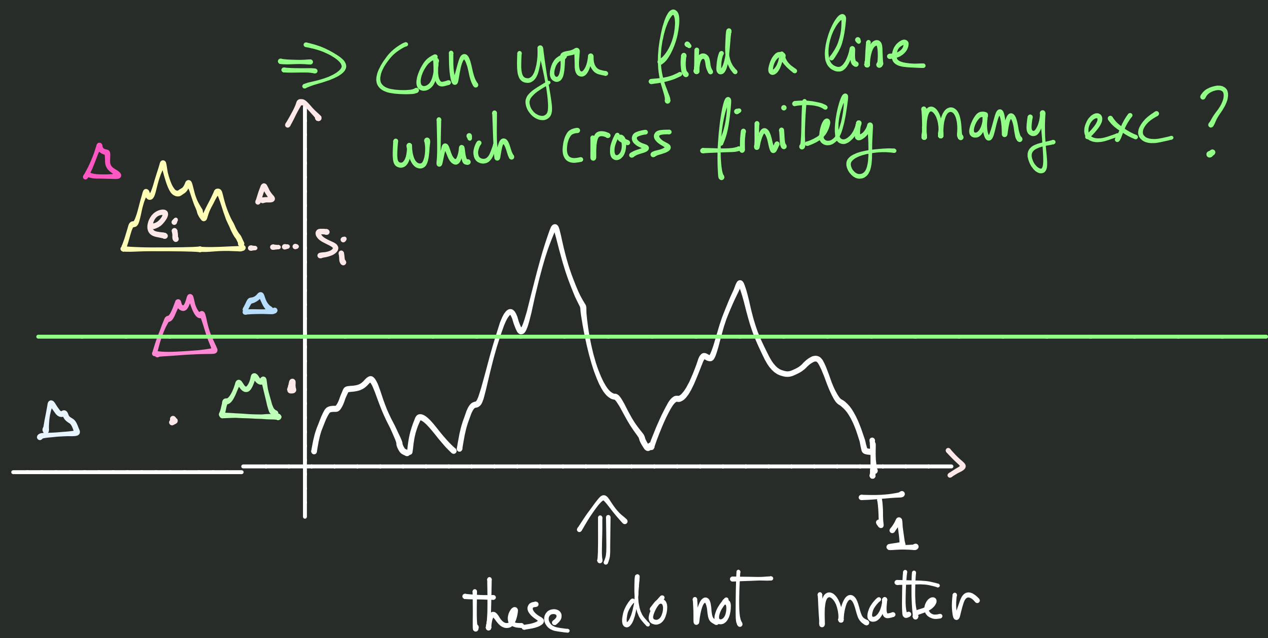


$\Rightarrow$ Can you find a line which cross finitely many exc?



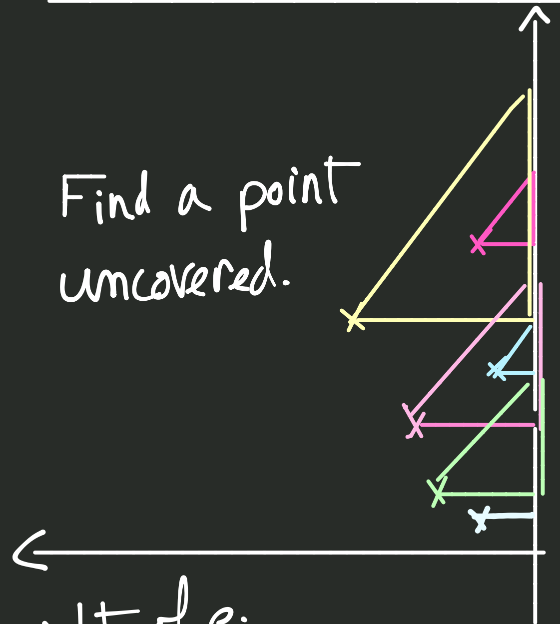
Exceptional Times





Exceptional Times

Find a point uncovered.



height of e_i

$\Rightarrow$ Can you find a line which cross finitely many exc ?

$\Uparrow$ these do not matter

$\Rightarrow$ Shepp's criteria. Only depend on tail behavior of $N(h(exc) \le x)$ when $x \to 0$.