LOCALLY PERIODIC HOMOGENIZATION

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ABSTRACT. *In this paper, two linear second PDEs are homogenized. The coefficients are supposed to be locally periodic, Lipschitz and bounded. Compared to our previous work* [1], *we provide a new and simpler proof and weaken the hypotheses of the main theorem. We use both probabilistic and analytic arguments.*

1. INTRODUCTION

In this paper we deal with the problem of the homogenization property of two singular second order PDE's with locally periodic coefficients, namely first the elliptic PDE in the bounded smooth domain $D \subset \mathbf{R}^d$ with a Dirichlet boundary condition :

$$\begin{aligned} L^\varepsilon u^\varepsilon(x) + f\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x) = 0,\ x \in D,& \\ u^\varepsilon(x) = g(x),\ x \in \partial D,& \end{aligned} \tag{1.1}$$

where f is bounded from above (see (6.2)), f and g are continuous, and second the following parabolic equation with a Cauchy type initial condition

$$\begin{aligned} \partial_t u^\varepsilon(t,x) &= L^\varepsilon u^\varepsilon(t,x) + \lambda^\varepsilon\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(t,x), \\ u^\varepsilon(0,x) &= g(x), \end{aligned} \tag{1.2}$$

where $\lambda^\varepsilon(x, \frac{x}{\varepsilon}) = \varepsilon^{-1} e(x, \frac{x}{\varepsilon}) + f(x, \frac{x}{\varepsilon})$, and g is continuous with at most polynomial growth at infinity. The operator acting on x

$$L^\varepsilon = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x, \frac{x}{\varepsilon}) \partial^2_{x_i x_j} + \sum_{i=1}^d (\frac{1}{\varepsilon} b_i(x, \frac{x}{\varepsilon}) + c_i(x, \frac{x}{\varepsilon})) \partial_{x_i}, \tag{1.3}$$

with the symmetric matrix $a(x,y) = [a_{ij}(x,y)]$, for x and y in $\mathbf{R}^d$, is supposed to be uniformly elliptic. That is $\exists \beta$ strictly positive s.t. for

Date: November 26, 2003.

1991 *Mathematics Subject Classification.* Primary 35B27, 60H30; Secondary 60J60, 60J35.

Key words and phrases. Homogenization, Parabolic PDE, Poisson equation, Diffusion approximation.

all x, y and ξ in $\mathbf{R}^d$

$$\beta \|\xi\|^2 \leq (a(x,y)\xi,\xi). \tag{1.4}$$

All the coefficients are periodic with respect to the second variable with period one in each direction of $\mathbf{R}^d$.

We intend here to improve upon our earlier paper [1] by slightly relaxing our assumptions, and giving a simpler proof, based on a more efficient approximation of our coefficients.

1.1. **Probabilistic approach.** After the pioneer work of Freidlin [4], the probabilistic approach to homogenization has been developed by several authors, see in particular Bensoussan and al. [2], Olla [6], Pardoux [7]. We refer to the introduction in [1] for an outline of Freidlin's approach. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, B_t, \mathbf{P})$ a filtered probability space, where $\{B_t\}$ is under $\mathbf{P}$ an $\mathcal{F}_t$–Brownian motion starting at zero. We define the $\mathbf{R}^d$–valued diffusion X_t^ε as the solution of the SDE

$$X_t^\varepsilon = x + \int_0^t (\frac{1}{\varepsilon} b(X_s^\varepsilon, \frac{X_s^\varepsilon}{\varepsilon}) + c(X_s^\varepsilon, \frac{X_s^\varepsilon}{\varepsilon}))ds + \int_0^t \sigma(X_s^\varepsilon, \frac{X_s^\varepsilon}{\varepsilon})dB_s, \tag{1.5}$$

where the matrix σ satisfies $\sigma\sigma^*(x,y) = a(x,y)$. The following operator will play an important role in our derivation :

$$L_{x,y} = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,y)\partial^2_{y_i y_j} + \sum_{i=1}^d b_i(x,y)\partial_{y_i}. \tag{1.6}$$

$L_{x,y}$ is the infinitesimal generator of the following diffusion process indexed by x, with transition densities $p_x(t,y,y')$

$$Y_t^x = y + \int_0^t b(x, Y_s^x)ds + \int_0^t \sigma(x, Y_s^x)dB_s,$$

which may rather be considered as a diffusion on the torus $\mathbf{T}^d$, i.e.

$$\dot{Y}_t^x = y + \int_0^t b(x, \dot{Y}_s^x)ds + \int_0^t \sigma(x, \dot{Y}_s^x)dB_s,$$

with transition density $\dot{p}_x(t,y,y') = \sum_{k\in Z^d} p_x(t,y,y' + \sum_{i=1}^d k_i e_i)$, $\{e_i\}$ being the canonical basis of $\mathbf{R}^d$. In what follows we shall drop the dots when no ambiguity arises. For each $x \in \mathbf{R}^d$, this diffusion possesses a unique invariant probability measure $\mu(x,dy)$, with density $p_x(\infty, y)$. We shall need the following crucial centering assumption on b :

$$\int_{\mathbf{T}^d} b(x,y)\mu(x,dy) = 0, \quad \text{for each } x \in \mathbf{R}^d, \tag{1.7}$$

under which we can solve the Poisson equation

$$L_{x,y}\widehat{b}(x,y) = -b(x,y), \tag{1.8}$$

where the solution $\widehat{b}$ belongs to the space $W^{2,p}(\mathbf{T}^d)$ and also satisfies $\int_{\mathbf{T}^d} \widehat{b}(x,y)\mu(x,dy) = 0$, $x \in \mathbf{R}^d$.

We now prove a Lemma which will be used several times below, and is a variant of Lemma 9.17 from Gilbarg, Trudinger [5].

Lemma 1. *Let $p \in (1,\infty)$. There exists a constant c such that for all x in $\mathbf{R}^d$ and all $h(x,\cdot) \in W^{2,p}(\mathbf{T}^d)$, satisfying*

$$\int_{\mathbf{T}^d} h(x,y)\mu(x,dy) = 0, \quad \textit{for each } x \in \mathbf{R}^d.$$

Then

$$\|h(x,\cdot)\|_{W^{2,p}(\mathbf{T}^d)} \leq c\|L_{x,\cdot}h(x,\cdot)\|_{L^p(\mathbf{T}^d)}.$$

Proof. We drop the parameter x for notational simplicity. Let us suppose that the result is not true. Then there would exist a sequence h_n s.t. $\|h_n\|_{L^p(\mathbf{T}^d)} = 1$ and $\|Lh_n\|_{L^p(\mathbf{T}^d)} \to 0$. On the other hand, upon decomposing the torus into two domains, we are able to apply Theorem 9.11 in Gilbarg, Trudinger [5], and we can find a constant c obviously independent of x, s.t. for all n

$$\|h_n\|_{W^{2,p}(\mathbf{T}^d)} \leq c(\|h_n\|_{L^p(\mathbf{T}^d)} + \|Lh_n\|_{L^p(\mathbf{T}^d)}).$$

Hence the sequence h_n is bounded in $W^{2,p}(\mathbf{T}^d)$, and we can extract a subsequence, still noted h_n, which converges weakly to some h in $W^{2,p}(\mathbf{T}^d)$. Note that h is also centered. By the Sobolev embedding theorems, $\|h_n\|_{L^p(\mathbf{T}^d)} \to \|h\|_{L^p(\mathbf{T}^d)} = 1$. On the other hand, $Lh = 0$ by weak convergence. Since h is centered, this entails $h = 0$, which is a contradiction. □

1.2. **Notations.** Let ξ denote any coefficient a, b, c, e, f. For the sake of simplifying the notations, the process $\xi(X_s^\varepsilon, X_s^\varepsilon/\varepsilon)$ will be denoted by $\xi(s,s)$. There should be no confusion with $\xi(x,y)$, since we use different letters for space and time variables. Unimportant constants will invariably be designated by c, the value of which may vary from line to line while proofs are in process but when there are many constants within a string of relations, we will use c, c', ...

1.3. **Assumptions on the coefficients.** Our standing assumptions are, next to (1.4), in which β is fixed, and (1.7), (7.2) below,

Condition 1. *The coefficients are continuous and bounded, in particular there exists a constant k s.t. for any $\xi = a$, b, c, e and f,*

$$\|\xi(x,y)\| \leq k,\ \forall x \in \mathbf{R}^d,\ y \in \mathbf{T}^d.$$

Condition 2. *Global Lipschitz condition : there exists a constant k_L s.t. for any $\xi = a$, b and e*

$$\|\xi(x,y) - \xi(x',y')\| \leq k_L(\|x-x'\| + \|y-y'\|),\ \forall x, x' \in \mathbf{R}^d,\ y, y' \in \mathbf{T}^d.$$

Moreover $\partial_x \xi$ exists and is continuous in x, uniformly with respect to $y \in \mathbf{T}^d$, and the same is true for $\xi = \tilde{b}$, where $\tilde{b}_i = b_i - \frac{1}{2}\sum_j \partial_{y_j} a_{ij}$ is uniformly Lipchitz with respect to x, $1 \leq i \leq d$.

Note that our conditions imply the existence and uniqueness of a global weak solution of equation (1.5), and that the operator $L_{x,y}$ can be rewritten as

$$L_{x,y} = \frac{1}{2}\mathrm{div}_y\,(a\nabla_y \cdot) + \tilde{b}\nabla_y \cdot \tag{1.9}$$

2. A regularization procedure

The key idea, which allows us to maintain minimal regularity assumptions, is to use a regularization procedure, which we now describe.

Let φ be a standard mollifier, which satisfies :

$$\int_{\mathbf{R}^d} x_i \varphi(x) dx = 0, \quad i = 1, \ldots, d. \tag{2.1}$$

For any measurable and locally bounded $h \ : \mathbf{R}^d \times \mathbf{T}^d \to \mathbf{R}$, we define

$$h^\varepsilon(x,y) = \varepsilon^{-d} \int_{\mathbf{R}^d} h(x-x',y)\varphi\left(\frac{x'}{\varepsilon}\right) dx'. \tag{2.2}$$

We also define the regularized second order operator

$$L^\varepsilon_{x,y} = \frac{1}{2}\sum_{i,j=1}^d a^\varepsilon_{ij}(x,y)\partial^2_{y_i y_j} + \sum_{i=1}^d b^\varepsilon_i(x,y)\partial_{y_i}, \tag{2.3}$$

where a^ε and b^ε are defined as in (2.2). It generates a diffusion $\{Y^{x,\varepsilon}_t,\ t \geq 0\}$ on the torus $\mathbf{T}^d$, whose associated x–dependent invariant probability measure on $\mathbf{T}^d$ will be denoted $\mu^\varepsilon(x,dy)$.

We can define moreover the solution $\widehat{h^\varepsilon}$ of the regularized Poisson equation

$$L^\varepsilon_{x,y}\widehat{h^\varepsilon}(x,y) + \check{h}^\varepsilon(x,y) = 0, \tag{2.4}$$

where

$$\check{h}^\varepsilon(x,y) = h^\varepsilon(x,y) - \bar{h^\varepsilon}(x), \tag{2.5}$$

and

$$\bar{h}^{\varepsilon}(x) = \int_{\mathbf{T}^d} h^{\varepsilon}(x,y)\mu^{\varepsilon}(x,dy). \tag{2.6}$$

We also define

$$h_{\varepsilon}(x,y) = h(x,y) - h^{\varepsilon}(x,y), \tag{2.7}$$

and when $h(x,\cdot)$ is centered with respect to $\mu(x,\cdot)$ for all $x \in \mathbf{R}^d$,

$$\widehat{h}_{\varepsilon}(x,y) = \widehat{h}(x,y) - \widehat{h^{\varepsilon}}(x,y).$$

Lemma 2. *Let $h : \mathbf{R}^d \times \mathbf{T}^d \to \mathbf{R}$ be jointly continuous, and such that for some $p > d$ and all $x \in \mathbf{R}^d$, $h(x,\cdot)$ is in $W^{1,p}(\mathbf{T}^d)$ and the $W^{1,p}(\mathbf{T}^d)$ norm of $h(x,\cdot)$ is bounded uniformly with respect to x. Then the following identity holds*

(2.8)

$$\begin{aligned}
\int_0^t h^{\varepsilon}(s,s)ds &= \varepsilon \int_0^t [\partial_x \widehat{h^{\varepsilon}} b + \partial_y \widehat{h^{\varepsilon}} c + Tr(\partial^2_{xy}\widehat{h^{\varepsilon}} a)](s,s)ds \\
&+ \varepsilon \int_0^t (\partial_y \widehat{h^{\varepsilon}}\sigma)(s,s)dB_s \\
&+ \int_0^t (\bar{h}^{\varepsilon}(s) + (L_{s,s} - L^{\varepsilon}_{s,s})\widehat{h^{\varepsilon}}(s,s))ds \\
&+ \varepsilon^2 \int_0^t [\partial_x \widehat{h^{\varepsilon}} c + \frac{1}{2} Tr(\partial^2_x \widehat{h^{\varepsilon}} a)](s,s)ds + \varepsilon^2 \int_0^t (\partial_x \widehat{h^{\varepsilon}}\sigma)(s,s)dB_s \\
&+ \varepsilon^2(\widehat{h^{\varepsilon}}(0,0) - \widehat{h^{\varepsilon}}(t,t)).
\end{aligned}$$

Proof. Since h^{ε} is of class C^{∞} in x, with x–derivatives uniformly bounded on $\mathbf{R}^d \times \mathbf{T}^d$, we can show that $\widehat{h^{\varepsilon}} \in C^2(\mathbf{R}^d \times \mathbf{T}^d)$ in a similar way as in the proof of Lemma 3 below. It then suffices to compute $\widehat{h^{\varepsilon}}(t,t) - \widehat{h^{\varepsilon}}(0,0)$ with the help of Itô's formula. □

We deduce easily from Lemma 2 the

Corollary 1. *Suppose in addition to the hypotheses of Lemma 2 that h is centered, i.e. satisfies*

$$\int_{\mathbf{T}^d} h(x,y)\mu(x,dy) = 0, \quad \textit{for each } x \in \mathbf{R}^d.$$

Then we have

$$\varepsilon^{-1}\int_0^t h(s,s)ds = \int_0^t F_h(s,s)ds + \int_0^t G_h(s,s)dB_s + R^{h,\varepsilon}_t,$$

where

$$F_h(x,y) = (\partial_x \widehat{h} b + \partial_y \widehat{h} c + Tr(\partial^2_{xy} \widehat{h} a))(x,y)$$
$$G_h(x,y) = \left(\partial_y \widehat{h} \sigma\right)(x,y)$$

and

$$R_t^{h,\varepsilon} = R_t^{h,1,\varepsilon} + R_t^{h,2,\varepsilon} + R_t^{h,3,\varepsilon}$$

in which

$$R_t^{h,1,\varepsilon} = -\int_0^t [\partial_x \widehat{h}_\varepsilon b + \partial_y \widehat{h}_\varepsilon c + Tr(\partial^2_{xy} \widehat{h}_\varepsilon a)](s,s)ds - \int_0^t \partial_y \widehat{h}_\varepsilon \sigma(s,s)dB_s,$$
$$R_t^{h,2,\varepsilon} = \varepsilon^{-1} \int_0^t (\overline{h}^\varepsilon(s) + [h_\varepsilon + (L_{s,s} - L^\varepsilon_{s,s})\widehat{h^\varepsilon})](s,s))ds$$

and

$$R_t^{h,3,\varepsilon} = \varepsilon \left(\int_0^t [\partial_x \widehat{h^\varepsilon} c + \frac{1}{2} Tr(\partial^2_x \widehat{h^\varepsilon} a)](s,s)ds + \int_0^t \left(\partial_x \widehat{h^\varepsilon} \sigma\right)(s,s)dB_s + \widehat{h^\varepsilon}(0,0) - \widehat{h^\varepsilon}(t,t) \right).$$

3. Rewriting our SDE

It follows from Corollary 1 applied to each component of b that

$$X_t^\varepsilon = \overline{X}_t^\varepsilon + R_t^\varepsilon, \tag{3.1}$$

where

$$\overline{X}_t^\varepsilon = x + \int_0^t F(s,s)ds + \int_0^t G(s,s)dB_s, \tag{3.2}$$

with

$$F = \left(\partial_x \widehat{b} b + (I + \partial_y \widehat{b})c + Tr \partial^2_{xy} \widehat{b} a\right)(x,y), G \ = ((I + \partial_y \widehat{b})\sigma)(x,y),$$

and $R_t^\varepsilon = R_t^{b,\varepsilon}$. The next section establishes all the technical results needed in order to prove that $R_t^\varepsilon \to 0$ in probability as $\varepsilon \to 0$, uniformly with respect to $t \leq T$.

4. Technical facts

4.1. **Regularity of the solution of the Poisson equation.**

Lemma 3. *Under the above conditions, there exists a constant $c > 0$ s.t. for all x in $\mathbf{R}^d$ and y in $\mathbf{T}^d$*

$$\left\|\widehat{b}(x,y)\right\| + \left\|\partial_x\widehat{b}(x,y)\right\| + \left\|\partial_y\widehat{b}(x,y)\right\| + \left\|\partial_y^2\widehat{b}(x,y)\right\| + \left\|\partial_{xy}^2\widehat{b}(x,y)\right\| \le c.$$

Moreover $\partial_x\widehat{b}$, $\partial_y\widehat{b}$ and $\partial_{xy}^2\widehat{b}$ are continuous in x, uniformly with respect to $y \in \mathbf{T}^d$.

Proof. Let x be an arbitrary point in $\mathbf{R}^d$. It follows from the Sobolev embedding theorem that the first part of the Lemma is a consequence of the existence of a constant c such that

$$\|\widehat{b}(x,\cdot)\|_{W^{3,p}(\mathbf{T}^d)} + \|\partial_x\widehat{b}(x,\cdot)\|_{W^{2,p}(\mathbf{T}^d)} \le c,\ \forall x \in \mathbf{R}^d, \tag{4.1}$$

for some $p > d$. We note that from (1.8),

$$L\partial_x\widehat{b} = -(\partial_x L)\widehat{b} - \partial_x b,$$

and

$$L\partial_y\widehat{b} = -(\partial_y L)\widehat{b} - \partial_y b.$$

Hence (4.1) follows from

$$\|(\partial_x L)\widehat{b}(x,\cdot) + \partial_x b(x,\cdot)\|_{L^p(\mathbf{T}^d)} + \|(\partial_y L)\widehat{b}(x,\cdot) + \partial_y b(x,\cdot)\|_{L^p(\mathbf{T}^d)} \le c,$$

and Lemma 1. The continuity in the last assertion of the Lemma is easily established by varying the coefficient x in the corresponding Poisson equation. □

A similar proof yields the

Lemma 4. *There exists a constant $c > 0$ such that for all x in $\mathbf{R}^d$, y in $\mathbf{T}^d$ and $\varepsilon \in (0,1]$,*

$$\left\|\widehat{b}^\varepsilon(x,y)\right\| + \left\|\partial_x\widehat{b}^\varepsilon(x,y)\right\| + \left\|\partial_y\widehat{b}^\varepsilon(x,y)\right\| + \left\|\partial_y^2\widehat{b}^\varepsilon(x,y)\right\| + \left\|\partial_{xy}^2\widehat{b}^\varepsilon(x,y)\right\| \le c.$$

4.2. **An estimate of the invariant density.**

The following estimate will be needed below. It follows from Proposition 2.1 in [7] and the proof of Lemma 18 in [1] that the invariant measure $\mu^\varepsilon(x,dy)$ has a density $p_x^\varepsilon(\infty,y)$ which satisfies

Lemma 5. *Under the above conditions, there is a constant $c > 0$ s.t. for any $(\varepsilon,x,y) \in (0,1] \times \mathbf{R}^d \times \mathbf{T}^d$*

$$|p_x^\varepsilon(\infty,y)| + \|\partial_y p_x^\varepsilon(\infty,y)\| \le c.$$

Moreover $p_x^\varepsilon(\infty,y)$ is also bounded away from zero, uniformly in (ε,x,y).

4.3. **Fine properties of the regularization procedure.** The aim of this subsection is to establish the

Proposition 1. *There exists a mapping θ from $(0,1]\times\mathbf{R}^d$ into $\mathbf{R}_+$ such that*

$$\sup_{(\varepsilon,x)\in(0,1]\times\mathbf{R}^d}\theta(\varepsilon,x)<\infty$$

and

$$\theta(\varepsilon,x)\to 0,\ \text{ as } \varepsilon\to 0,$$

uniformly for x in a compact subset of $\mathbf{R}^d$, and the following holds for each $(\varepsilon,x,y)\in(0,1]\times\mathbf{R}^d\times\mathbf{T}^d$

$$\varepsilon^{-1}\left(\|a_\varepsilon(x,y)\|+\|b_\varepsilon(x,y)\|\right)\le\theta(\varepsilon,x),\tag{4.2}$$

$$\varepsilon^{-1}\|\bar{b^\varepsilon}(x)\|\le\theta(\varepsilon,x),\tag{4.3}$$

$$\|\partial_y\widehat{b_\varepsilon}(x,y)\|\le\theta(\varepsilon,x),\tag{4.4}$$

$$\|\partial_x\widehat{b_\varepsilon}(x,y)\|+\|\partial^2_{xy}\widehat{b_\varepsilon}(x,y)\|+\varepsilon\left\|\partial^2_x\widehat{b^\varepsilon}(x,y)\right\|\le\theta(\varepsilon,x).\tag{4.5}$$

The following Lemma contains the proof of (4.2) and prepares that of the other estimates.

Lemma 6. *For $\xi=a$, b or e, the quantities*

$$\varepsilon^{-1}\xi_\varepsilon(x,y),\quad \partial_x\xi_\varepsilon(x,y),\quad \text{and } \varepsilon\partial^2_x\xi^\varepsilon(x,y)$$

are bounded uniformly with respect to $0<\varepsilon\le 1$, $x\in\mathbf{R}^d$, $y\in\mathbf{T}^d$, and tend to zero as $\varepsilon\to 0$, uniformly for (x,y) in a compact subset of $\mathbf{R}^d\times\mathbf{T}^d$.

Proof. We consider only the term $\varepsilon\partial^2_x\xi^\varepsilon(x,y)$, since it is the most delicate one. We note that for $1\le i,j\le d$, we have

$$\begin{aligned}\left\|\varepsilon\partial^2_{x_ix_j}\xi^\varepsilon(x,y)\right\| &= \varepsilon^{-d}\left\|\int_{\mathbf{R}^d}\partial_{x_i}\xi(x-x',y)\partial_{x_j}\varphi\left(\frac{x'}{\varepsilon}\right)dx'\right\|\\ &= \varepsilon^{-d}\left\|\int_{\mathbf{R}^d}\left(\partial_{x_i}\xi(x-x',y)-\partial_{x_i}\xi(x,y)\right)\partial_{x_j}\varphi\left(\frac{x'}{\varepsilon}\right)dx'\right\|\\ &\le c'\sup_{\|x'\|\le c\varepsilon,y\in\mathbf{T}^d}\|\nabla_x\xi(x-x',y)-\nabla_x\xi(x,y)\|,\end{aligned}$$

where c is the radius of the support of φ and

$$c'=\sup_j\varepsilon^{-d}\int_{\mathbf{R}^d}\left|\partial_{x_j}\varphi\left(\frac{x'}{\varepsilon}\right)\right|dx'=\sup_j\int_{\mathbf{R}^d}\left|\partial_{x_j}\varphi(x)\right|dx.$$

The above quantity is bounded, and tends to zero as $\varepsilon\to 0$ locally uniformly in x, thanks to Condition 2. Note that the proof of each of the two other estimates is similar and uses in an essential way the relation (2.1). □

We can now prove the

Lemma 7. (4.3)*and* (4.4) *hold.*

Proof. Note that

$$\varepsilon^{-1}\bar{b^\varepsilon}(x) = \varepsilon^{-1}\int_{\mathbf{T}^d}(b^\varepsilon - b)(x,y)\mu^\varepsilon(x,dy) + \varepsilon^{-1}\int_{\mathbf{T}^d} b(x,y)\left(\mu^\varepsilon(x,dy) - \mu(x,dy)\right),$$

and consequently (4.3) follows from Lemma 3, Lemma 6 and the following computation

$$\int_{T^d} p_x^\varepsilon(\infty,y)L^\varepsilon_{x,y}\widehat{b}(x,y)dy = \int_{T^d} p_x(\infty,y)L_{x,y}\widehat{b}(x,y)dy$$

$$\begin{aligned}\int_{T^d} p_x^\varepsilon(\infty,y)(L^\varepsilon_{x,y} - L_{x,y})\widehat{b}(x,y)dy &= \int_{T^d} p_x(\infty,y)L_{x,y}\widehat{b}(x,y)dy \\ &\quad - \int_{T^d} p_x^\varepsilon(\infty,y)L_{x,y}\widehat{b}(x,y)dy \\ &= \int_{T^d}(p_x^\varepsilon(\infty,y) - p_x(\infty,y))b(x,y)dy.\end{aligned}$$

Now (4.4) follows from the arguments in the proof of Lemma 3, Lemma 4, (4.2) and (4.3). □

It remains to establish (4.5). The bounds on $\partial_x\widehat{b_\varepsilon}$ and $\partial^2_{xy}\widehat{b_\varepsilon}$ are treated as in the proof of (4.4), i.e. by means of Lemma 1. Next, we need to consider the quantities

$$\partial_x\widehat{b^\varepsilon}(x,y) \text{ and } \partial_x^2\widehat{b^\varepsilon}(x,y).$$

It follows from the equation for $\widehat{b^\varepsilon}$ that for $1 \le i,j \le d$,

$$L^\varepsilon_{x,y}\partial_{x_i}\widehat{b^\varepsilon} + \partial_{x_i}L^\varepsilon_{x,y}\widehat{b^\varepsilon} + \partial_{x_i}\check{b}^\varepsilon = 0,$$

and

$$L^\varepsilon_{x,y}\partial^2_{x_ix_j}\widehat{b^\varepsilon} + \partial_{x_j}L^\varepsilon_{x,y}\partial_{x_i}\widehat{b^\varepsilon} + \partial_{x_i}L^\varepsilon_{x,y}\partial_{x_j}\widehat{b^\varepsilon} + \partial^2_{x_ix_j}L^\varepsilon_{x,y}\widehat{b^\varepsilon} + \partial^2_{x_ix_j}\check{b}^\varepsilon = 0.$$

Hence from Lemma 1, there exists a constant c independent of x such that

$$\left\|\partial_{x_i}\widehat{b^\varepsilon}(x,\cdot)\right\|_{W^{2,p}(\mathbf{T}^d)} \le c\left(\left\|\partial_{x_i}L^\varepsilon_{x,y}\widehat{b^\varepsilon}(x,\cdot)\right\|_{L^p(\mathbf{T}^d)} + \left\|\partial_{x_i}\check{b}^\varepsilon(x,\cdot)\right\|_{L^p(\mathbf{T}^d)}\right),$$

and

$$\left\|\partial^2_{x_ix_j}\widehat{b^\varepsilon}(x,\cdot)\right\|_{W^{2,p}(\mathbf{T}^d)} \le c\left(1+\left\|\partial^2_{x_ix_j}L^\varepsilon_{x,y}\widehat{b^\varepsilon}(x,\cdot)\right\|_{L^p(\mathbf{T}^d)}\right.$$
$$\left.+\left\|\partial^2_{x_ix_j}\check{b}^\varepsilon(x,\cdot)\right\|_{L^p(\mathbf{T}^d)}\right).$$

Clearly, only the last term of the right hand side of the two above inequalities needs some care. In fact, we need only to study

$$\bar{b^\varepsilon}(x) = \int_{\mathbf{T}^d} b^\varepsilon(x,y)\mu^\varepsilon(x,dy).$$

The result now follows from

Lemma 8. *For any $1 \le i,j \le d$ the quantities*

$$\|\partial_{x_i}p^\varepsilon_x(\infty,\cdot)\|_{L^2(\mathbf{T}^d)} \text{ and } \varepsilon\|\partial^2_{x_ix_j}p^\varepsilon_x(\infty,\cdot)\|_{L^2(\mathbf{T}^d)}$$

are bounded uniformly in $(\varepsilon,x)\in(0,1]\times\mathbf{R}^d$ and the second one tends to zero as $\varepsilon\to 0$, locally uniformly in x.

Proof. In order to keep as few terms as possible, inspired by Pardoux [7] Proposition 2.2, we will work with $\mu^\varepsilon(x,dy)$ as a reference measure. The scalar product of functions relative to this measure will be denoted by $(\cdot,\cdot)_\varepsilon$, while both the scalar product relative to Lebesgue measure and the ordinary scalar product in $\mathbf{R}^d$ are denoted, when no ambiguity arises, by $(\cdot,\cdot)$. It is easy to check that (see Pardoux [7], page 501)

Lemma 9. *For any x in $\mathbf{R}^d$ and ψ in $H^1(\mathbf{T}^d)$ we have*

$$(L^\varepsilon_{x,y}\psi,\psi)_\varepsilon = -\frac{1}{2}\int_{\mathbf{T}^d}(a^\varepsilon\nabla_y\psi,\nabla_y\psi)\mu^\varepsilon(x,dy).$$

Let $p^\varepsilon_x(t,\nu,y)$ stand for the density of the law of $Y^{x,\varepsilon}_t$ when the process starts with the distribution ν , which we drop when no ambiguity arises. For $1\le i,j\le d$ the derivatives $\partial_{x_i}p^\varepsilon_x(t,y)$ and $\partial_{x_ix_j}p^\varepsilon_x(t,y)$ are denoted respectively by $q^{i,\varepsilon}_x(t,y)$ and $r^{i,j,\varepsilon}_x(t,y)$. We have the PDE with Cauchy initial value $p^\varepsilon_x(0)=\nu$

$$\partial_t p^\varepsilon_x(t,y) = (L^\varepsilon_{x,y})^* p^\varepsilon_x(t,y) \tag{4.6}$$

For fixed $1\le i,j\le d$, the ratios of $p^\varepsilon_x(t,y)$, $q^{i,\varepsilon}_x(t,y)$ and $r^{i,j,\varepsilon}_x(t,y)$ to $p^\varepsilon_x(\infty,y)$ are denoted respectively by $u^\varepsilon_x(t,y)$, $v^\varepsilon_x(t,y)$ and $w^\varepsilon_x(t,y)$. Our bounds are derived by an iterative scheme. Applying Lemma 9 with $\psi = u^\varepsilon_x(t,y)$, we deduce from (4.6) that

$$\frac{d}{dt}\|u^\varepsilon_x(t)\|^2_{2,\varepsilon} + \int_{\mathbf{T}^d}(a^\varepsilon\nabla_y u^\varepsilon_x(t),\nabla_y u^\varepsilon_x(t))\mu^\varepsilon(x,dy) = 0.$$

After choosing a suitable ν (e.g. with constant density 1, which we assume from now on) we obtain thanks to ellipticity that there exists a constant $c > 0$ s.t. for all (ε, x) in $(0,1] \times \mathbf{R}^d$,

$$\sup_{0\leq t} \|u_x^\varepsilon(t)\|_{2,\varepsilon}^2 + \int_0^\infty \|\nabla_y u_x^\varepsilon(s)\|_{2,\varepsilon}^2 \, ds \leq c. \tag{4.7}$$

Let us now turn to the first derivative. We obviously have for $1 \leq i \leq d$ and Cauchy initial value $q_x^{i,\varepsilon}(0) = 0$

$$\partial_t q_x^{i,\varepsilon}(t,y) = (L_{x,y}^\varepsilon)^* q_x^{i,\varepsilon}(t,y) + (\partial_{x_i} L_{x,y}^\varepsilon)^* p_x^\varepsilon(t,y) \ .$$

The same procedure as above gives

$$\frac{d}{dt}\|v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + \int_{\mathbf{T}^d} (a^\varepsilon \nabla_y v_x^\varepsilon(t), \nabla_y v_x^\varepsilon(t)) \mu^\varepsilon(x,dy)$$
$$= -(\partial_{x_i} a^\varepsilon \nabla_y p_x^\varepsilon(t), \nabla_y v_x^\varepsilon(t)) + 2(\partial_{x_i} \widetilde{b}^\varepsilon p_x^\varepsilon(t), \nabla_y v_x^\varepsilon(t)).$$

It is then easy to see that there exist three positive constants c, c' and c'', c' sufficiently small thanks to an interpolated Young's inequality, s.t.

$$\frac{d}{dt}\|v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + c\|\nabla_y v_x^\varepsilon(t)\|_{2,\varepsilon}^2 \leq c'\|\nabla_y v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + c''(\|p_x^\varepsilon(t)\|_2^2 + \|\nabla_y p_x^\varepsilon(t)\|_2^2).$$

Now clearly

$$\int_{\mathbf{T}^d} v_x^\varepsilon(t,y)\mu^\varepsilon(x,dy) = 0.$$

It then follows from the Poincaré inequality that there exist three positive constants c, c' and c'', s.t. for all (ε, x) in $(0,1] \times \mathbf{R}^d$

$$\frac{d}{dt}\|v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + c\|v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + c'\|\nabla_y v_x^\varepsilon(t)\|_{2,\varepsilon}^2 \leq c''(\|p_x^\varepsilon(t)\|_2^2 + \|\nabla_y p_x^\varepsilon(t)\|_2^2). \tag{4.8}$$

We will need the following variant of the Gronwall-Bellman Lemma

Lemma 10. *Let f be a real function in $L^1_{loc}([0,\infty))$ and $\alpha \in \mathbf{R}$. If the following differential inequality holds*

$$\psi'(t) + \alpha\psi(t) \leq f(t),$$

then

$$\psi(t) \leq \exp(-\alpha t)\psi(0) + \int_0^t \exp(-\alpha(t-s)) f(s) ds.$$

Proof. It suffices to write $\varphi(t) = \exp(\alpha t)\psi(t)$ and to notice that $\varphi'(t) \leq \exp(\alpha t) f(t)$. □

It follows from equation (4.8) and Lemma 10 that there are two positive constants c and α s.t.

$$\begin{aligned}
&\|v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + c\int_0^t e^{-\alpha(t-s)}\,\|\nabla_y v_x^\varepsilon(s)\|_{2,\varepsilon}^2\,ds \\
&\le c'\int_0^t e^{-\alpha(t-s)}(\|p_x^\varepsilon(s)\|_2^2 + \|\nabla_y p_x^\varepsilon(s)\|_2^2)ds \\
&\le c'\int_0^t e^{-\alpha(t-s)}(\|u_x^\varepsilon(s)\|_2^2\|p_x^\varepsilon(\infty)\|_{W^{1,\infty}(\mathbf{T}^d)}^2 + \|\nabla_y u_x^\varepsilon(s)\|_2^2\|p_x^\varepsilon(\infty)\|_\infty^2)ds.
\end{aligned}$$

Now from Lemma 5 and equation (4.7), we conclude that there exists a constant c such that for all $x \in \mathbf{R}^d$, $\varepsilon > 0$,

$$\sup_{0\le t}\left(\|v_x^\varepsilon(t)\|_{2,\varepsilon}^2 + \int_0^t \exp(-\alpha(t-s))\,\|\nabla_y v_x^\varepsilon(s)\|_{2,\varepsilon}^2\,ds\right) \le c. \tag{4.9}$$

Considering the second derivative, again we have for $1 \le i,j \le d$ and Cauchy initial value $r_x^{i,j,\varepsilon}(0) = 0$

$$\begin{aligned}
\partial_t r_x^{i,j,\varepsilon}(t,y) &= (L_{x,y}^\varepsilon)^* r_x^{i,j,\varepsilon}(t,y) + (\partial_{x_i}L_{x,y}^\varepsilon)^* q_x^{j,\varepsilon}(t,y) + (\partial_{x_j}L_{x,y}^\varepsilon)^* q_x^{i,\varepsilon}(t,y) \\
&\quad + (\partial_{x_ix_j}L_{x,y}^\varepsilon)^* p_x^\varepsilon(t,y),
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{d}{dt}\,\|w_x^\varepsilon(t)\|_{2,\varepsilon}^2 + c\,\|\nabla_y w_x^\varepsilon(t)\|_{2,\varepsilon}^2 \\
&\le c'(\left\|q_x^{i,\varepsilon}(t)\right\|_2^2 + \left\|\nabla_y q_x^{i,\varepsilon}(t)\right\|_2^2 + \left\|q_x^{j,\varepsilon}(t)\right\|_2^2 + \left\|\nabla_y q_x^{j,\varepsilon}(t)\right\|_2^2 \\
&+ \left\|\partial_{x_ix_j}a_{i,j}^\varepsilon\right\|_\infty^2\,\|\nabla_y p_x^\varepsilon(t)\|_2^2 + \left\|\partial_{x_ix_j}\widetilde{b}^\varepsilon\right\|_\infty^2\,\|p_x^\varepsilon(t)\|_2^2),
\end{aligned}$$

therefore we deduce from similar arguments as above, using (4.7) and (4.9),

$$\begin{aligned}
\|w_x^\varepsilon(t)\|_{2,\varepsilon}^2 &\le c\int_0^t \exp(-\alpha(t-s))(\|q_x^\varepsilon(s)\|_2^2 + \|\nabla_y q_x^\varepsilon(s)\|_2^2)ds \\
+c(\left\|\partial_x^2 a^\varepsilon\right\|_\infty^2 &+ \left\|\partial_x^2\widetilde{b}^\varepsilon\right\|_\infty^2)\int_0^t \exp(-\alpha(t-s))(\|p_x^\varepsilon(s)\|_2^2 + \|\nabla_y p_x^\varepsilon(s)\|_2^2)ds \\
&\le c(1 + \left\|\partial_x^2 a^\varepsilon\right\|_\infty^2 + \left\|\partial_x^2\widetilde{b}^\varepsilon\right\|_\infty^2),
\end{aligned}$$

where c is a constant independent of t, x and ε. This concludes the proof of Lemma 8.

5. Convergence in law

Define again

$$F(x,y) = \left(\partial_x\widehat{b}b + (I + \partial_y\widehat{b})c + Tr\partial^2_{xy}\widehat{b}a\right)(x,y) \tag{5.1}$$

and

$$G(x,y) = ((I + \partial_y\widehat{b})\sigma)(x,y). \tag{5.2}$$

Note that F and G are continuous and bounded. We can now state the following Theorem, whose proof is carried out as in [1].

Theorem 1. *The sequence of processes* $\{X^\varepsilon_t,\ 0 \le t \le T;\ 0 < \varepsilon \le 1\}$ *is tight in* $\mathcal{C}([0,T];\mathbf{R}^d)$*, and*

$$\sup_{0\le t\le T} \|R^\varepsilon_t\| \to 0 \textit{ in probability}, \qquad \textit{as } \varepsilon \to 0.$$

In order to identify the limit, we need in addition the following result, namely a locally periodic ergodic theorem, which is an improvement over Theorem 3 in [1]. We first define $\Lambda(x,y) = GG^*(x,y)$,

$$\overline{F}(x) = \int_{\mathbf{T}^d} F(x,y)\mu(x,dy) \tag{5.3}$$

and

$$\overline{\Lambda}(x) = \int_{\mathbf{T}^d} \Lambda(x,y)\mu(x,dy). \tag{5.4}$$

Observe that the continuity of $\overline{F}$ and $\overline{G}$ has already been proved in [1].

Theorem 2. *Let* φ *be a smooth function with compact support from* $\mathbf{R}^d$ *into* $\mathbf{R}$*, and* $h(x,y)$ *be a bounded continuous function from* $\mathbf{R}^d \times \mathbf{T}^d$ *into* $\mathbf{R}$ *such that for all* $x \in \mathbf{R}^d$

$$\int_{\mathbf{T}^d} h(x,y)\mu(x,dy) = 0.$$

Then

$$H^\varepsilon(t) = \int_0^t \varphi(\overline{X}^\varepsilon_s)h(X^\varepsilon_s, \frac{X^\varepsilon_s}{\varepsilon})ds$$

converges to zero in $L^1(\Omega)$ *for any* $t > 0$.

Proof. Since $H^\varepsilon(t)$ is bounded, it suffices to prove convergence in probability. We use tightness in an essential way, as in Theorem 3 in [1]. We first need to regularize $h(x,y)$ with respect to both variables, i.e. we define (our notation differs here from the one used above) for $0 < \delta < 1$

$$h^\varepsilon(x,y) = \varepsilon^{-2\delta d} \int_{\mathbf{R}^{2d}} h(x - x', y - y')\psi(\frac{x'}{\varepsilon^\delta}, \frac{y'}{\varepsilon^\delta})dx'dy',$$

where ψ is a standard mollifier. Next, we solve the Poisson equation

$$L^{\varepsilon}_{x,y}\widehat{h^{\varepsilon}}(x,y) + \check{h}^{\varepsilon}(x,y) = 0,$$

where $\check{h}^{\varepsilon}$ is formally defined as in (2.5). However, $L^{\varepsilon}_{x,y}$ stays the same as in Section 2, i.e. a and b are regularized only in x, with a symmetric kernel. We now write the analog of (2.8).

Let φ be an infinitely differentiable function with compact support and define

$$\Phi(\overline{X}^{\varepsilon}_t, X^{\varepsilon}_t) = \varphi(\overline{X}^{\varepsilon}_t)\widehat{h^{\varepsilon}}(X^{\varepsilon}_t, \frac{X^{\varepsilon}_t}{\varepsilon}).$$

We have

$$\begin{aligned}
&\int_0^t \varphi(\overline{X}^{\varepsilon}_s)h(s,s)ds = \int_0^t \varphi(\overline{X}^{\varepsilon}_s)h_{\varepsilon}(s,s)ds + \int_0^t \varphi(\overline{X}^{\varepsilon}_s)\bar{h^{\varepsilon}}(X^{\varepsilon}_s)ds \\
&+ \int_0^t \varphi(\overline{X}^{\varepsilon}_s)((L_{s,s} - L^{\varepsilon}_{s,s})\widehat{h^{\varepsilon}})(s,s)ds \\
&+ \varepsilon(\int_0^t \varphi(\overline{X}^{\varepsilon}_s)(\partial_x\widehat{h^{\varepsilon}}b + \partial_y\widehat{h^{\varepsilon}}c)(s,s)ds \\
&+ \int_0^t Tr\varphi'(\overline{X}^{\varepsilon}_s)(\partial_y\widehat{h^{\varepsilon}}G\sigma)(s,s)ds \\
&+ \int_0^t Tr\varphi(\overline{X}^{\varepsilon}_s)(\partial^2_{xy}\widehat{h^{\varepsilon}}a)(s,s)ds) + \varepsilon\int_0^t \varphi(\overline{X}^{\varepsilon}_s)(\partial_y\widehat{h^{\varepsilon}}\sigma)(s,s)dB_s \\
&+ \varepsilon^2(\int_0^t \varphi'(\overline{X}^{\varepsilon}_s)(\widehat{h^{\varepsilon}}F)(s,s)ds \\
&+ \int_0^t \varphi(\overline{X}^{\varepsilon}_s)(\partial_x\widehat{h^{\varepsilon}}c)(s,s)ds \\
&+ \frac{1}{2}\int_0^t Tr\varphi''(\overline{X}^{\varepsilon}_s)(\widehat{h^{\varepsilon}}\Lambda)(s,s)ds + \int_0^t Tr\varphi'(\overline{X}^{\varepsilon}_s)(\partial_x\widehat{h^{\varepsilon}}G\sigma)(s,s)ds \\
&+ \frac{1}{2}\int_0^t Tr\varphi(\overline{X}^{\varepsilon}_s)(\partial^2_x\widehat{h^{\varepsilon}}a)(s,s)ds) \\
&+ \varepsilon^2(\int_0^t \varphi'(\overline{X}^{\varepsilon}_s)(\widehat{h^{\varepsilon}}G)(s,s)dB_s + \int_0^t \varphi(\overline{X}^{\varepsilon}_s)(\partial_x\widehat{h^{\varepsilon}}\sigma)(s,s)dB_s) \\
&+ \varepsilon^2(\Phi(\overline{X}^{\varepsilon}_0, X^{\varepsilon}_0) - \Phi(\overline{X}^{\varepsilon}_t, X^{\varepsilon}_t)).
\end{aligned}$$

It remains to let ε go to 0. In that respect, the only nontrivial estimates are those of the terms

$$\int_0^t \varphi(\overline{X}^{\varepsilon}_s)[(L_{s,s} - L^{\varepsilon}_{s,s})\widehat{h^{\varepsilon}}](s,s)ds.$$

and

$$\varepsilon^2 \int_0^t Tr\varphi(\overline{X}^\varepsilon_s) \left(\partial_x^2 \widehat{h^\varepsilon} a\right)(s,s)ds.$$

Thanks to Lemma 1,

$$\varepsilon \|\partial_y^2 \widehat{h^\varepsilon}\|_{L^\infty(\mathbf{R}^d\times\mathbf{T}^d)} \le c\varepsilon^{1-\delta},$$

and as $\varepsilon \to 0$,

$$\varepsilon^2 \|\partial_x^2 \widehat{h^\varepsilon}\|_{L^\infty(\mathbf{K}\times\mathbf{T}^d)} = o(\varepsilon^{1-\delta}),$$

where $\mathbf{K}$ is any compact subset of $\mathbf{R}^d$. It remains to use (4.2). □

We are now in the position to state our main result :

Theorem 3. *There is only one limit point as* $\varepsilon \to 0$ *of the family* $\{X^\varepsilon, \varepsilon > 0\}$, *namely* X^0, *the solution of the SDE*

$$X_t^0 = x + \int_0^t \overline{F}(X_s^0)ds + \int_0^t \overline{\Lambda}^{1/2}(X_s^0)dB_s, \tag{5.5}$$

where $\overline{F}(x)$ *and* $\overline{\Lambda}(x)$ *are defined in* (5.3) *(see also* (5.1), (5.2)*).*

Proof. The fact that any limit point of $\{X^\varepsilon\}$ solves the martingale problem associated to the SDE (5.5) follows from Theorem 1 and 2, since both $F(x,y) - \bar{F}(x)$ and $\Lambda(x,y) - \bar{\Lambda}(x)$ satisfy the assumptions of Theorem 2. Next, uniqueness in law of the solution of (5.5) follows from Corollary 2 in [1]. □

6. Convergence of the solution of the elliptic equation

We need to formulate additional assumptions. Let τ^ε denotes the random time defined as

$$\tau^\varepsilon = \inf\{t \ge 0,\, X_t^\varepsilon \notin \overline{D}\},$$

and $\alpha \ge 0$ be such that for each $x \in D$,

$$\sup_{\varepsilon>0} \mathbf{E}_x \exp(\alpha\tau^\varepsilon) < \infty, \tag{6.1}$$

where $\mathbf{E}_x$ denotes expectation under the law of $\{X_t^\varepsilon\}$ such that $X_0^\varepsilon = x$. We assume that f is jointly continuous and for some $\delta > 0$, all $x \in \mathbf{R}^d$, $y \in \mathbf{T}^d$,

$$f(x,y) \le (\alpha - \delta)^+. \tag{6.2}$$

The solution of equation (1.1) is given by the Feynman-Kac formula

$$u^\varepsilon(x) = \mathbf{E}_x\left[g(X_{\tau^\varepsilon}^\varepsilon)\exp\left(\int_0^{\tau^\varepsilon} f(s,s)ds\right)\right].$$

Since the matrix $\overline{\Lambda}(x)$ is non degenerate,

$$\tau(X^0) = \inf\{t \geq 0,\ X_t^0 \notin \overline{D}\}$$

is a.s. a continuous function of the trajectory X^0. Moreover from Theorem 1 and 2

$$\int_0^t f(s,s)ds \Rightarrow \int_0^t C(X_s^0)ds,$$

where $\Rightarrow$ means "converges in law towards", and

$$C(x) = \int_{\mathbf{T}^d} f(x,y)\mu(x,dy).$$

Consequently, by uniform integrability (see (6.1) and (6.2)), we have the (below $\tau = \inf\{t \geq 0,\ X_t \notin \overline{D}\}$)

Theorem 4. *Under Condition 1, Condition 2,* (6.1) *and* (6.2)*, for all* $x \in D$,

$$u^\varepsilon(x) \to \mathbf{E}_x\left[g(X_\tau^0)\exp\left(\int_0^\tau C(X_s^0)ds\right)\right],$$

which is the solution of the elliptic PDE

$$\begin{aligned} Lu(x) + C(x)u(x) &= 0,\ x \in D, \\ u(x) &= g(x),\ x \in \partial D, \end{aligned}$$

with

$$L = \frac{1}{2}\sum_{i,j=1}^d \overline{\Lambda}_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^d \overline{F}_i(x)\partial_{x_i}.$$

7. Convergence of the solution of the parabolic equation

For us, the solution $u^\varepsilon(t,x)$ to equation (1.2) for all $\varepsilon > 0$, where $g(x)$, it is reminded, is supposed to be continuous with at most polynomial growth at infinity, is given by the Feynman–Kac formula

$$u^\varepsilon(t,x) = \mathbf{E}_x g(X_t^\varepsilon)\exp\int_0^t \lambda^\varepsilon(s,s)ds. \tag{7.1}$$

Clearly, $e(x,y)$ must be centered, i.e. we assume that

$$\int_{\mathbf{T}^d} e(x,y)\mu(x,dy) = 0, \quad \forall x \in \mathbf{R}^d. \tag{7.2}$$

It follows from Condition 1, Condition 2 and (7.2) that Lemma 3, Lemma 4 and Proposition 1 are true with b replaced by e. Hence, if we set

$$Y_t^\varepsilon = \int_0^t \lambda^\varepsilon(s,s)ds,$$

then

$$Y_t^\varepsilon = \overline{Y}_t^\varepsilon + R_t^{e,\varepsilon},$$

where

$$\overline{Y}_t^\varepsilon = \int_0^t (f+F_e)(s,s)ds + \int_0^t G_e(s,s)dB_s,$$

and $R_t^{e,\varepsilon}$ converges to zero in probability (uniformly in $t \le T$), as $\varepsilon \to 0$.

Some care is needed to deal with the stochastic integral in the argument of the exponential in (7.1). The reader is referred to [1] for a complete treatment. Let us define

$$\Lambda_e = G_e G_e^* \quad \text{and} \quad \tilde{f}(x,y) = (f + F_e + \frac{1}{2}\Lambda_e)(x,y).$$

We now state the final result :

Theorem 5. *Under Conditions 1 and 2, for any $t \le T$ and x in $\mathbf{R}^d$,*

$$u^\varepsilon(t,x) \to u(t,x),$$

as $\varepsilon \to 0$, and the limiting PDE is the following equation

$$\partial_t u(t,x) = \frac{1}{2}\sum_{i,j=1}^d \overline{\Lambda}_{ij}(x)\partial^2_{x_i x_j} u(t,x) + \sum_{i=1}^d E_i(x)\partial_{x_i} u(t,x) + C(x)u(t,x),$$

where

$$E(x) = \int_{\mathbf{T}^d} (F + (I + \partial_y \widehat{b})a\partial_y \widehat{e}^*)(x,y)\mu(x,dy),$$

and

$$C(x) = \int_{\mathbf{T}^d} \tilde{f}(x,y)\mu(x,dy).$$

Acknowledgement 1. *The first author is thankful for the hospitality and support of the LATP at University of Provence, Marseille, France, and of CNRS under PICS 444.*

References

[1] BENCHERIF-MADANI, A. and PARDOUX, E. (2003) Homogenization of a diffusion with locally periodic coefficients, in *Séminaire de Probabilités*, Lecture Notes in Mathematics, to appear.

[2] BENSOUSSAN, A., LIONS, J.-L. and PAPANICOLAOU, G. (1978). *Asymptotic analysis of periodic structures*, North-Holland Publ. Comp., Amsterdam.

[3] FRIEDMAN, A. (1964). *Partial differential equations of parabolic type*, Prentice–Hall, Englewood Cliffs, N. J..

[4] FREIDLIN, M. (1964). The Dirichlet problem for an equation with periodic coefficients depeding on a small parameter, *Teor. Primenen*, **9**, 133-139.

[5] GILBARG, D. and TRUDINGER, N. S. (1998). *Elliptic partial differential equations of second order*, Revised third printing, Springer-Verlag, New York.

[6] OLLA, S.(2002). Notes on the Central Limit Theorems for Tagged Particles and Diffusions in Random Fields, in *Milieux aléatoires*, F. Comets, E. Pardoux, eds, Panoramas et Synthèses **12**, SMF.

[7] PARDOUX, E. (1999). Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients : a probabilistic approach, *J. Funct. Anal.*, **167**, 498-520.

[8] PARDOUX, E. and VERETENNIKOV, A.Yu. (2001). On the Poisson equation and diffusion approximation, I, *The Annals of Probab.*, **29**, 3, 1061-1085.

[9] PARDOUX, E. and VERETENNIKOV, A.Yu. (2002). On the Poisson equation and diffusion approximation, II, *The Annals of Probab.*, to appear.

[10] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional diffusion processes*, Springer-Verlag, New York, Berlin.

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