Binary trees, exploration processes, and an extented Ray–Knight Theorem

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Abstract

We study the bijection betwen binary Galton Watson trees in continuous time and their exploration process, both in the sub- and in the supercritical cases. We then take the limit over renormalized quantities, as the size of the population tends to infinity. We thus deduce Delmas' generalization of a well–known Ray–Knight theorem.

Introduction

There are various forms of bijection between an exploration (or height) process and a random binary tree. Here we describe such a bijection, and prove that a certain law on the exploration paths is in bijection with the law of a continuous time subcritical binary Galton–Watson random tree. The result in the critical case has been first established by Le Gall [5], and in the subcritical case by Pitman, Winkel [7], see also Geiger, Kersting [3], Lambert [6], where the exploration processes are jump processes, while ours is continuous. We believe that our proof is simpler than the above proofs. It uses only a rather elementary property of the Poisson process (see Lemma 2.2). Moreover we consider the supercritical case, which is new. Inspired by

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the work of Delmas [1], we note that in the super critical case, the random tree killed t time $a > 0$ is in bijection with the exploration process reflected below a. One can define a unique local time process, which describes the local times of all the reflected exploration processes, and has the same law as the supercritical Galton Watson process.

We next renormalize our Galton Watson tree and height process, and take the weak limit, thus giving a rigourous proof of Delmas' extension of a well–known Ray–Knight theorem. The classical version of this theorem establishes the identity in law between the local time of reflected Brownian motion at the time when the local time at 0 reaches x, and all levels, and a Feller critical branching diffusion. The same result holds in the subcritical (resp. supercritical) case, Brownian motion being replaced by Brownian motion with drift (in the supercritical case, reflection below an arbitrary level, as above, is requested).

Note that associating an exploration process to a continuous branching process allows us to describe genealogies of the population sampled at times $0, h, 2h, \dots$ from the embeded excursions above levels $0, h, 2h, \dots$ of the exploration process, while those genealogies cannot be read from the branching diffusion itself, see figure 5 at the end of the paper. The exploration process contains the information of the tree, not only of the process of population sizes.

The paper is organized as follows. Section 1 is devoted to the description of the bijection between height curves and binary trees (without introducing probabilities). Section 2 presents the equivalence of laws of height processes and Galton Watson trees in the subritical case. Section 3 describes analogous results in the supercritical case. Section 4 presents the results of convergence of both the population process and the height process, in the limit of large populations. Finally section 5 deduces the generalized Ray–Knight theorem from our convergences and the results at the discrete level.

1 Preliminaries

We denote by $\mathcal{H}_{p,m}$ the set of piecewise linear functions $H : s \mapsto H(s)$ from $[0, T_m]$ into $\mathbb{R}_+$ with alternating derivatives p and $-p$, which starts from (0,0) with slope p, is reflected whenever it hits zero, and is stopped at the time T_m of its m–th return to zero, which is assumed to be finite. We shall write $\mathcal{H}_p$ for $\mathcal{H}_{p,1}$.

We also denote by $\mathcal{T}$ the set of finite rooted binary trees which are defined as follows: an ancestor is born at time 0. Until he eventually dies, he gives birth successively to an arbitrary number of offsprings. The same happens to each of his offsprings, and the offsprings of his offsprings, etc... until eventually the population dies out. We denote by $\mathcal{F}_m$ the set of forests which are the union m elements of $\mathcal{T}$.

There is a well known bijection between binary trees and exploration processes. Under the curve representing an element $H \in \mathcal{H}_p$, we can draw a tree as follows. The height h_{lfmax} of the most left local maximum[1] of H is the lifetime of the ancestor and the heigh h_{lowmin} of the lowest non zero local minimum is the time of the birth of the first offspring of the ancestor. If there is no such local minimum, the ancestor dies before giving birth to any offspring. ' We draw a horizontal line at level h_{lowmin}. H has two excursions above h_{lowmin}. The left one is used to represent the fate of the ancestor and of the rest of his progeny. The right one is used to represent the fate of his first offspring and of the progeny of the later. Continuing until there is no further local minimum to explore, we define clearly a bijection Φ_p from $\mathcal{H}_p$ into $\mathcal{T}$. Repeating the same procedure, we define a bijection between $\mathcal{H}_{p,m}$ and $\mathcal{T}_m$.

We now define probability measures on $\mathcal{H}_p$ (resp. $\mathcal{H}_{p,m}$) and $\mathcal{T}$ (resp. $\mathcal{T}_m$).

Let $0 < \mu \le \lambda$ be two parameters. We define a stochastic process whose trajectories belong to $\mathcal{H}_p$ as follows. Let $\{U_k,\ k \ge 1\}$ and $\{V_k,\ k \ge 1\}$ be two mutually independent sequences of i.i.d exponential random variables with parameter λ and μ respectively. We define $Z_k = U_k - V_k$, $k \ge 1$. $\mathbb{P}_{\lambda,\mu}$ is the law of the random element of $\mathcal{H}_p$, which is such that the height of the first local maximum is U_k, that of the first local minimum is $(Z_1)^+$. If $Z_1 = 0$, the process is stopped. Otherwise, the height of the second local maximum is $Z_1 + U_2$, the height of the second local minimum is $(Z_1 + Z_2)^+$, etc. Because $\mu < \lambda$, the process returns to zero a.s in finite time. The random trajectory which we have constructed is an excursion above zero. We define similarly a law on $\mathcal{H}_{p,m}$ as the concatenation of m i. i. d. such excursions.

To the same pair of parameters (λ, μ), we associate the subcritical binary Galton Watson tree (i. e. random element of $\mathcal{T}$) as follows. The lifetime of each individual is exponential with parameter λ, and during its lifetime, independently of it, each individual gives birth to offsprings according to a rate μ Poisson process. The behaviours of the various individuals are i. i. d. This defines a probability measure $\mathbb{Q}_{\lambda,\mu}$ on $\mathcal{T}$. We use the same notation to denote the law on $\mathcal{F}_m$ of m i. i d. random trees with $\mathbb{Q}_{\lambda,\mu}$ as their common law.

[1]We assume that this minimum is unique, as will be a.s the case in the probabilistic model below

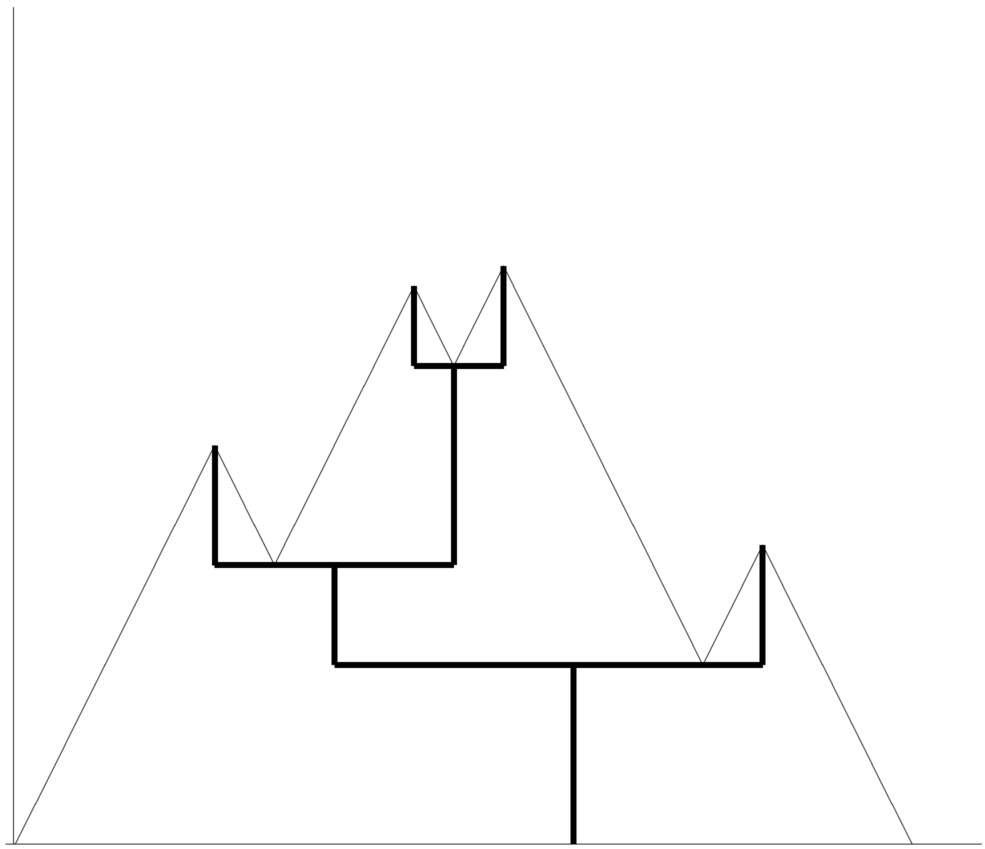

Figure 1: Bijection between binary trees and exploration processes

2 Subcritical case

The aim of this section is to prove that $\mathbb{P}_{\lambda,\mu}\Phi_p^{-1} = \mathbb{Q}_{\lambda,\mu}$.

Let us establish some preliminary results which will be useful in the sequel.

2.1 Preliminary results

Let $(T_k)_{k\geq 0}$ be a Poisson point process on $\mathbb{R}_+$ with intensity μ. For any $t > 0$, we define the random variable

$$R_t = \sup_{k\geq 0}\{T_k; T_k \leq t\}.$$

Lemma 2.1 *Let M be a non negative random variable independent of $\{T_k,\ k \geq 0\}$ and define*

$$R_M = \sup_{k\geq 0}\left\{T_k; T_k \leq M\right\}.$$

Then $M - R_M \overset{(d)}{=} V \wedge M$ where V and M are independent, V has an exponential distribution with parameter μ.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ continuous and bounded. It is well knowed that for any $t > 0$, the law of $t - R_t$ is the same that of $V \wedge t$, where $V \backsimeq \mathcal{E}(\mu)$.

$$\begin{aligned}
\mathbb{E}\left(f(M - R_M)\right) &= \int \mathbb{E}\left(f(M - R_M)|M = t\right) d\mathbb{P}_M(dt) \\
&= \int \mathbb{E}\left(f(t - R_t)\right)\mathbb{P}_M(dt) \\
&= \int \mathbb{E}\left(f(V \wedge t)\right)\mathbb{P}_M(dt) \\
&= \int\int \left(f(v \wedge t)\right)\mathbb{P}_V(dv)\mathbb{P}_M(dt) \\
&= \mathbb{E}\left(f(V \wedge M)\right),
\end{aligned}$$

with V whith is independent of M, since it is a function of $(T_k)_{k\geq 0}$. ■

Lemma 2.2 *Let $(T_k)_{k\geq 0}$ be a Poisson point process on $\mathbb{R}_+$ with intensity μ. M a positive random variable which is independent of $(T_k)_{k\geq 0}$. Consider the integer valued random variable K such that $T_K = R_M$ and a second Poisson point process $(T'_k)_{k\geq 0}$ with intensity μ, which is jointly independent of the first and of M. Then $\left(\overline{T}_k\right)_{k\geq 0}$ defined by:*

$$\overline{T}_k = \begin{cases} T_k & \text{if } k < K \\ T_K + T'_{k-K+1} & \text{if } k \geq K \end{cases}$$

is a Poisson point process on $\mathbb{R}_+$ *with intensity* μ *which is independent of* R_M.

PROOF: Let $(N_t, t \geq 0)$, $\big(\overline{N}_t, t \geq 0\big)$ and $(N'_t, t \geq 0))$ be the counting processes associated to T, $\overline{T}$ and T'.
For any $n \geq 1,\ 0 < t_1 < \cdots < t_n$ and $k_1, \ldots, k_n \in \mathbb{N}^*$, we intend to evaluate

$$\xi_t = \mathbb{P}\Big(\overline{N}_{t_1} = k_1, \ldots, \overline{N}_{t_n} = k_n | R_M = t\Big).$$

It suffices to treat the case: $\exists\ i$ such that $2 \leq i \leq n$ and $t_{i-1} < t < t_i$.

$$\xi_t = \mathbb{P}\left(N_{t_1} = k_1, N_{t_2} = k_2, \cdots, N_{t_{i-1}} = k_{i-1}, N_t + N'_{t_i - t} = k_i, \cdots, N_t + N'_{t_n - t} = k_n\right)$$

$$= \mathbb{P}\Bigg(N_{t_1} = k_1, N_{t_2} - N_{t_1} = k_2 - k_1, \cdots, N_{t_{i-1}} - N_{t_{i-2}} = k_{i-1} - k_{i-2}, N_t - N_{t_{i-1}}$$

$$+ N'_{t_i - t} = k_i - k_{i-1}, N'_{t_{i+1}-t} - N'_{t_i - t} = k_{i+1} - k_i, \cdots, N'_{t_n - t} - N'_{t_{n-1}-t} = k_n - k_{n-1}\Bigg)$$

$$= e^{-\mu t_n} \prod_{i=1}^{n} \frac{(\mu(t_i - t_{i-1}))^{k_i - k_{i-1}}}{(k_i - k_{i-1})!}$$

Therefore $\overline{T}$ is a Poisson process with intensity μ and it is independent of R_M since ξ_t does not depend upon t. ■

We are now in a position to prove our main result.

2.2 Main result

Theorem 2.3

$$\mathbb{Q}_{\lambda,\mu} = \mathbb{P}_{\lambda,\mu} \Phi_p^{-1}.$$

The theorem says that the tree associated to the exploration process $\{H_s,\ s \geq 0\}$ is a $\mathbb{N}$-valued continuous binary tree with death rate λ and birth rate μ, and vice versa.
PROOF: A binary tree with birth rate μ and death rate λ can be described as follows: Each individual gives birth to offsprings according to a Poisson process of intensity μ, and dies at an exponential time of parameter λ, independent of the birth times. The individuals will be numbered: $\ell = 1, 2, \ldots$. 1 is the ancestor of the whole family. The subsequent individuals will be

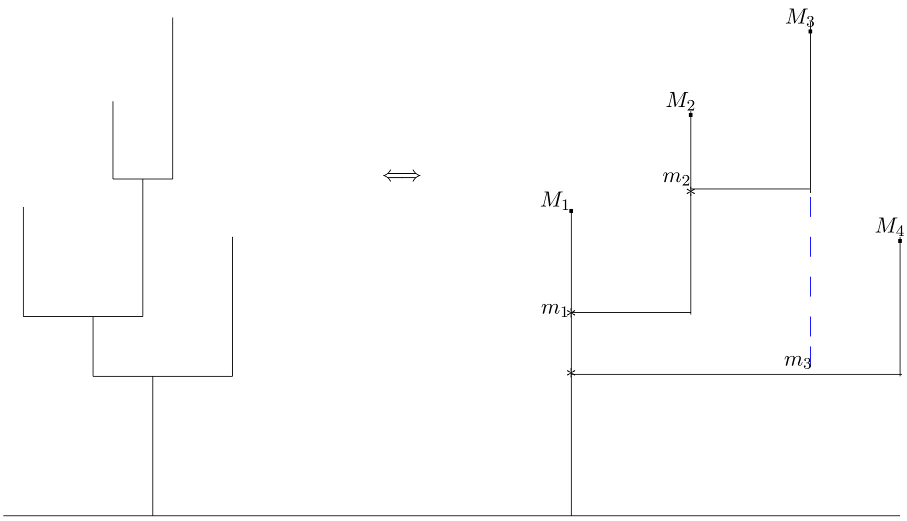

Figure 2: Another view at binary Galton Watson trees

identified below. We will show that there is way to explore this tree with an exploration process whose law is pricisely $\mathbb{P}_{\lambda,\mu}$. We introduce the family $(T_k^\ell, k \geq 0, \ell \geq 1)$ of mutually independent of Poisson processes . For any $\ell \geq 1$, the process T_k^ℓ describes the times of birth of the offsprings of the individual ℓ. We define U_ℓ to be the lifetime of individual ℓ.

- **Step 1**: We start from the initial time $t = 0$ and climb up to the level M_1 of height U_1, where U_1 follows $\mathcal{E}(\lambda)$. We go down from M_1 until we find the most recent point of the Poisson process (T_k^1) which gives the moments of birth of the offsprings of individual 1. So we have descended from a height of $V_1 \wedge M_1$, where V_1 follows $\mathcal{E}(\mu)$ and is independent of M_1(see 2.1). This moment corresponds to a level $m_1 = M_1 - V_1 \wedge M_1$.
  if $m_1 = 0$, we stop, else

- **Step 2**: We give the label 2 to this last offspring of the individual 1, born at the time m_1. Let us define $(\overline{T}_k^2)$ by:

$$\overline{T}_k^2 = \begin{cases} T_k^1 & \text{if } k < K_1 \\ T_{K_1}^1 + T_{k-K_1+1}^2 & \text{otherwise} \end{cases}$$

  where K_1 is such that $T_{K_1}^1 = m_1$.
  Thanks to Lemma 2, $(\overline{T}_k^2)$ is a Poisson process with intensity μ on $\mathbb{R}_+$, which is independent of m_1 and in fact also of (U_1, V_1).

  Starting form m_1, the exploration process climb up to level $M_2 = m_1 + U_2$, where U_2 follows $\mathcal{E}(\lambda)$ and is independent of (U_1, V_1). Starting from level M_2, we go down a height $M_2 \wedge V_2$ where V_2 follows $\mathcal{E}(\mu)$ and independent of (U_2, U_1, V_1), to find the most recent point of the Poisson process $(\overline{T}_k^2)$. At this moment we are at the level $m_2 = M_2 - V_2 \wedge M_2$.
  If $m_2 = 0$ we stop. Otherwise we continue.
  Suppose we have made $\ell - 1$ steps and $m_{\ell-1} > 0$, $\ell \geq 3$.

- **Step** ℓ We start from $m_{\ell-1}$ found in $\overline{T}^{\ell-1}$, which is the birthtime of individual ℓ . More precisely $m_{\ell-1}$=$\overline{T}_{K_{\ell-1}}^{\ell-1} = \sup\{\overline{T}_k^{\ell-1}; \overline{T}_k^{\ell-1} \leq M_{\ell-1}\}$. We set $M_\ell = m_{\ell-1} + U_\ell$, which corresponds to the time of death of individual ℓ, where $U_\ell \sim \mathcal{E}(\lambda)$ and is independent of $U_i, V_i, i = 1, ...k-1$. We now define

$$\overline{T}_k^\ell = \begin{cases} \overline{T}_k^{\ell-1} & \text{if } k < K_{\ell-1} \\ \overline{T}_{K_{\ell-1}}^{\ell-1} + T_{k-K_{\ell-1}+1}^\ell & \text{otherwise} \end{cases}$$

Then $(\overline{T}^{\ell}_k)$ is a Poisson point process with intensity μ on $\mathbb{R}_+$ and is independent of $(m_1, M_1, \cdots, M_{\ell-1}, m_{\ell-1})$. Coming down from level M_ℓ, we wait a time $V_\ell \wedge M_\ell$ to find the most recent point of the Poisson process $(\overline{T}^{\ell}_j)$. Consequently the next level is $m_\ell = M_\ell - V_k \wedge M_\ell$, where $V_\ell \sim \mathcal{E}(\mu)$ and is independent of $U_i, i = 1...\ell$ and $V_i, i = 1...\ell - 1$.

Since we are in the (sub)critical case, zero is reached $a.s$ after a finite number of iterations. It is clear that the random variables M_i and m_i determine fully the law $\mathbb{Q}_{\lambda,\mu}$ of the binary tree and they have both the same joint distribution as the levels of the successive local minimas and maximas of the process $\{H_s,\ s \geq 0\}$ under $\mathbb{P}_{\lambda,\mu}$, and the joint distribution of the nodes (including the leaves) of the binary (λ, μ) tree. ■

2.3 A discrete Ray–Knight theorem

Consider the exploration process $\{H_t,\ t \geq 0\}$ defined above which is reflected at zero and stopped at the first moment it reaches zero for the m-th time. To this process we can associate a forest of m binary trees of birth rate μ and death rate λ which all start with a single individual at the initial time $t = 0$. Consider the branching process in continuous time $(Z^m_t)_{t\geq 0}$ giving the number of offsprings alive at time t of the m ancestors born at time 0. Every individual in this population, independently of the others, lives during a exponential time with parameter λ and gives birth to offsprings according to a Poisson process of intensity μ.

We now choose the slopes of the piecewise linear process H to be ± 2 (i. e. $p = 2$). We define the local time accumulated by H at level t up to time s:

$$L_s(t) = \frac{4}{\sigma^2} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t \leq H_r < t+\varepsilon\}} dr. \tag{2.1}$$

$L_s(t)$ equals to $4/\sigma^2$ times the number of pairs of branches of H which cross the level t between times 0 and s. Note that a local minimum at level t counts for two crossings, while a local maximum at level t counts for none. We have the "occupation time formula": for any integrable function g,

$$\frac{4}{\sigma^2} \int_0^s g(H_r) dr = \int_0^\infty g(r) L_s(r) dr.$$

Let

$$\tau_m = \inf \left\{ s > 0 : L_s(0) > \frac{4m}{\sigma^2} \right\}.$$

$(\sigma^2/4) L_{\tau_m}(t)$ count the number of individuals of the forest of m trees which are alive up to time t, since a pair of branches of the exploration process corresponds to one individual in the tree. Then we have:

Lemma 2.4

$$\{L_{\tau_m}(t),\ t\geq 0,\ m\geq 1\}\equiv\left\{\frac{4}{\sigma^2}Z_t^m,\ t\geq 0,\ m\geq 1\right\}.$$

2.4 Renormalization

In this section we make a renormalization. We choose $m=[Nx]$ for some $x>0$. Let $\left(Z_t^N\right)_{t\geq 0}$ the branching process which describes the population size of the $[Nx]$ binary trees with birth rate $\mu_N=\frac{\sigma^2}{2}N+\alpha$ and death rate $\lambda_N=\frac{\sigma^2}{2}N+\delta$, where $0<\alpha<\delta$. We set

$$X_t^{N,x}=\frac{Z_t^N}{N}.$$

In particular we have that $X_0^{N,x}=\frac{[Nx]}{N}\longrightarrow x$ when $N\longrightarrow+\infty$.

Let H^N be the exploration process associated to Z^N defined in the same way as previously, but with slopes $\pm 2N$, and λ, μ are remplaced by λ_N and μ_N. We define also $L_s^N(t)$, the local time accumulated by H^N at level t up to time s, as

$$L_s^N(t)=\frac{4}{\sigma^2}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^s\mathbf{1}_{\{t\leq H_r^N<t+\varepsilon\}}dr$$

$L_s^N(t)$ equals to $4/N\sigma^2$ times the number of pairs of branches of H^N which cross the level t between times 0 and s. Let

$$\tau_x^N=\inf\left\{s>0:L_s^N(0)>\frac{4}{\sigma^2}\frac{[Nx]}{N}\right\}. \tag{2.2}$$

We have again :

Lemma 2.5

$$\left\{L_{\tau_x^N}^N(t),\ t\geq 0,\ x>0\right\}\equiv\left\{\frac{4}{\sigma^2}X_t^{N,x},\ t\geq 0,\ x>0\right\}.$$

3 Supercritical case

3.1 A reflected exploration process

We consider again the process $\{H_t,\ t\geq 0\}$ defined in section 1, but this time in the case $\mu>\lambda$. This process does not come back to zero a.s. For each level $a>0$, we consider the height process $\{H_t^a,\ t\geq 0\}$ reflected at level a

by adding the rule that whenever the process reaches the level a, it stops and starts immediately going down with slope $-p$. Again the process stops when first going back to zero. The reflected process comes back to zero almost surely. Indeed we have

$$\mathbb{P}\big(\text{reaching zero during an arbitrary descent}\big) \geq \mathbb{P}\big(V_i > a\big) = \exp(-\mu a).$$

Note A_n the event "does not reach zero during the n first descents". We have clearly

$$\mathbb{P}(A_n) \leq (1 - \exp(-\mu a))^n,$$

which goes to zero as $n \longrightarrow \infty$. Hence the result.

For each $a > 0$, and any pair (λ, μ) of positive numbers, denote by $\mathbb{P}_{\lambda,\mu,a}$ the law of the process H^a. Define $\mathbb{Q}_{\lambda,\mu,a}$ to be the law of the (λ, μ) Galton–Watson tree, killed at time $t = a$ (i. e. all individuals alive at time a^- are killed at time a). An easy adaptation of the proof of Theorem 2.3 yields

Proposition 3.1 *For any $a, \lambda, \mu > 0$,*

$$\mathbb{Q}_{\lambda,\mu,a} = \mathbb{P}_{\lambda,\mu,a}\Phi_p^{-1}.$$

3.2 A Ray–Knight representation

For any $b > a > 0$, we now define the application $\Pi^{a,b}$ which maps trajectories from $\mathcal{H}_p$ with values in $[0, b]$ into trajectories with values in $[0, a]$.

Let $\rho : \mathbb{R} \mapsto \mathbb{R}$ be a function such that:

$$\rho(0) = 0; \qquad \frac{d\rho}{ds} = \mathbf{1}_{\{H^b > a\}}(s)$$

We define

$$\Pi^{a,b}(H^b)(s) = H^b_{s-\rho(s)}$$

Lemma 3.2

$$\Pi^{a,b}(H^b) \stackrel{(d)}{=} H^a$$

Proof: It is in fact sufficient to show that the conditinal law of the level of the first local minimum of H^b after crossing the level a downwards, given the past of H^b, is the same as the conditional law of the level of the first local minimum of H^a after a reflexion at level a, given the past of H^a. This identity follows readily from the "lack of memory" of the exponential law. ■

We now consider the case $p = 2$. For each $a > 0$, $m \geq 1$, define the stopping time

$$\tau_m^a = \inf\{s > 0;\ L_s^a(0) > \frac{4}{\sigma^2} m\},$$

where $L_\cdot^a(0)$ is the local time of the process H^a at level 0. τ_m^b is defined similary in term of $L_\cdot^b(0)$. Thanks to Lemma 3.2 we have for any $b > a > 0$, $m \geq 0$:

$$(L^b_{\tau^b_m}(t), 0 \leq t \leq a) \overset{(d)}{=} (L^a_{\tau^a_m}(t), 0 \leq t \leq a),$$

and $L_s^a(t)$ are (resp. $L_s^b(t)$) the local time associed to H^a (resp. H^b) at level t up to time s.

It follows that, for each $m \geq 1$, we can define the process $\{\mathcal{L}_m(t), t \geq 0\}$, which is such that for each $a > 0$,

$$\{\mathcal{L}_m(t), 0 \leq t \leq a\} \overset{(d)}{=} \{L^a_{\tau^a_m}(t), 0 \leq t \leq a\}.$$

We have the following Ray–Knight type statement

Lemma 3.3

$$\{\mathcal{L}_m(t), t \geq 0, m \geq 1\} \overset{(d)}{=} \left\{ \frac{4}{\sigma^2} Z_t^m,\ t \geq 0, m \geq 0 \right\}.$$

PROOF: It suffice to show that for any $a \geq 0$, $m \geq 1$

$$\{\mathcal{L}_m(t), 0 \leq t \leq a\} \overset{(d)}{=} \{Z_t^m, 0 \leq t \leq a\}.$$

But

$$\{\mathcal{L}_m(t), 0 \leq t \leq a\} \overset{(d)}{=} \{L^a_{\tau^a_m}(t), 0 \leq t \leq a\} = \{Z_t^{m,a}, 0 \leq t \leq a\},$$

and

$$\{Z_t^m, 0 \leq t \leq a\} \overset{(d)}{=} \{Z_t^{m,a}, 0 \leq t \leq a\}.$$

Hence the result. ∎

3.3 Renormalization

Following the same approach as in the subcritical case, we consider the exploration process $\{H_t^{N,a},\ t \geq 0\}$ defined above which is reflected in the interval $[0, a]$ and stopped at the first moment it reaches zero for the m-th time, where $m = [Nx]$, $N \in \mathbb{N}^*$ and some $x > 0$, and remplace respectively λ and μ by λ_N and μ_N. Like the process H^N, the process $H^{N,a}$ has slopes $\pm 2N$.

To this exploration process $H^{N,a}$, we associate the forest of m trees killed at time a. We consider the branching process $\{Z_s^N \mathbf{1}_{\{s\leq a\}}, s \geq 0\}$ which gives the population's size of these m binary trees, killed at time a. We consider also the local time $L_s^{N,a}(t)$ of the process $H^{N,a}$ defined as above. For each $a > 0$, define the stopping time

$$\tau_x^{N,a} = \inf\{s > 0;\ L_s^{N,a}(0) > \frac{4}{\sigma^2}\frac{[Nx]}{N}\}. \tag{3.1}$$

Again we can define a process $\{\mathcal{L}_x^N(t), t \geq 0\}$ which is such that for each $a > 0$,

$$\{\mathcal{L}_x^N(t), 0 \leq t \leq a\} \overset{(d)}{=} \{L^{N,a}_{\tau_x^{N,a}}(t), 0 \leq t \leq a\}.$$

We have the

Lemma 3.4

$$\{\mathcal{L}_x^N(t), t \geq 0, x \geq 0\} \overset{(d)}{=} \left\{\frac{4}{\sigma^2} X_t^{N,x}, t \geq 0, x \geq 0\right\}.$$

4 Weak convergence

4.1 Tightness criteria in D

Consider a sequence $\{X_t^n,\ t \geq 0\}_{n\geq 1}$ of one–dimensional semimartingales, which is such that for each $n \geq 1$,

$$X_t^n = X_0^n + \int_0^t \varphi_n(X_s^n)ds + M_t^n,\ 0 \leq t \leq T;$$
$$\langle M^n \rangle_t = \int_0^t \psi_n(X_s^n)ds,\ t \geq 0;$$

where for eah $n \geq 1$, $M_\cdot^n$ is a locally square–integrable martingale, φ_n and ψ_n are Borel measurable functions from $\mathbb{R}$ into $\mathbb{R}$ and $\mathbb{R}_+$ respectively. Most of the next statement follows readily from Corollary 2.3.3 in Joffe, Métivier [4], while the last part follows from Theorem III.10.2 in Ethier, Kurtz [2].

Proposition 4.1 *A sufficient condition for the sequence $\{X_t^n,\ t \geq 0\}_{n\geq 1}$ to be tight in $D([0,\infty))$ is that both*

$$\text{the sequence of r.v.'s } \{X_0^n,\ n \geq 1\} \text{ is tight;} \tag{4.1}$$

and for each $T > 0$, some $p > 1$,

$$\text{the sequence of r. v.'s } \{\int_0^T [|\varphi_n(X_t^n)| + \psi_n(X_t^n)]^p dt,\ n \geq 1\} \text{ is tight.} \tag{4.2}$$

If moreover, for any $T > 0$, as $n \to \infty$,

$$\sup_{0 \le t \le T} |M_t^n - M_{t-}^n| \to 0 \quad \textit{in probability,}$$

then any limit X of a converging subsequence of the original sequence $\{X^n\}_{n \ge 1}$ is a. s. continuous.

4.2 Tightness and convergence of X^{Nx}

The continuous time Galton–Watson process $\{X_t^{N,x},\ t \ge 0\}$ is a Markov process with values in the set $E_N := \{k/N,\ k \ge 1\}$ with generator A^N given by

$$A^N f(x) = Nx\left(N\frac{\sigma^2}{2} + \alpha\right)\left[f\left(x + \frac{1}{N}\right) - f(x)\right] + Nx\left(N\frac{\sigma^2}{2} + \beta\right)\left[f\left(x - \frac{1}{N}\right) - f(x)\right],$$

for any $f : E_N \to \mathbb{R}$, $x \in E_N$. Consequently for any $f \in C(\mathbb{R})$,

$$M_t^{f,N} := f(X_t^{N,x}) - f(X_0^{N,x}) - \int_0^t A^N f(X_s^{N,x}) ds \tag{4.3}$$

is a local martingale. Applying successively the above formula to the cases $f(x) = x$ and $f(x) = x^2$, we get that

$$X_t^{N,x} = X_0^{N,x} + \int_0^t (\alpha - \beta) X_r^{N,x} dr + M_t^{(1),N}, \tag{4.4}$$

$$\left(X_t^{N,x}\right)^2 = \left(X_0^{N,x}\right)^2 + 2(\alpha - \beta)\int_0^t (X_r^{N,x})^2 dr + \left(\sigma^2 + \frac{\alpha + \beta}{N}\right)\int_0^t X_r^{N,x} dr + M_t^{(2),N}, \tag{4.5}$$

where $\{M_t^{(1),N},\ t \ge 0\}$ and $\{M_t^{(2),N},\ t \ge 0\}$ are local martingales. It follows from (4.4) and (4.5) that

$$\langle M^{(1),N}\rangle_t = \left(\sigma^2 + \frac{\alpha + \beta}{N}\right)\int_0^t X_r^{N,x} dr. \tag{4.6}$$

We now prove

Lemma 4.2 *Asume that* $\sup_{N \ge 1} \mathbb{E}\left[(X_0^{N,x})^2\right] < \infty$. *Then for any $T > 0$,*

$$\sup_{N \ge 1} \sup_{0 \le t \le T} \mathbb{E}\left[\left(X_t^{N,x}\right)^2\right] < \infty.$$

An immediate Corollary of this Lemma is that $\{M_t^{(1),N}\}$ and $\{M_t^{(2),N}\}$ are in fact martingales.
PROOF: Let $\tau_n = \inf\{t > 0,\ |M_t^{(2)}| > n\}$. We deduce from (4.5) that with

$$C = \frac{1}{2}\left(\sigma^2 + \alpha + \beta\right),\ C' = 2(\alpha - \beta) + \frac{1}{2}\left(\sigma^2 + \alpha + \beta\right),$$

$$\mathbb{E}\left[(X_{t\wedge\tau_n}^{N,x})^2\right] \le \mathbb{E}\left[(X_0^{N,x})^2\right] + CT + C'\int_0^t (X_{s\wedge\tau_n}^{N,x})^2 ds,$$

hence the result from Gronwall's and Fatou's Lemma. ■

It now follows from Proposition 4.1, (4.4), (4.6), Lemma 4.2 and the fact that that $X_0^{N,x} \to x$ $\{X^{N,x}\}_{N\ge 1}$ is tight in $D([0,+\infty))$.

Standard arguments starting from (4.3) now allow us to deduce

Proposition 4.3 *Since $X_0^{N,x} \to x$ as $N \to \infty$, $X^{N,x} \Rightarrow X^x$ as $N \to \infty$, where X^x is the unique solution of the following Feller diffusion with logistic growth*

$$X_t^x = x + (\alpha - \beta)\int_0^t X_r^x dr + \sigma\int_0^t \sqrt{X_r^x} dB_r,\ t \ge 0.$$

Tightness of H^N(resp $\{H_s^{N,a},\ s \ge 0\}$)

Consider now the exploration process $\{H_s^N,\ s \ge 0\}$(resp $\{H_s^{N,a},\ s \ge 0\}$) of the forest of trees representing the population $\{Z_t^N,\ t \ge 0\}$(resp $\{Z_t^{N,a},\ t \ge 0\}$). Let $\{V_s^N,\ s \ge 0\}$ be the $\{-1,1\}$–valued process which is such that a. e.: In the first case (subcritical)

$$\frac{dH_s^N}{ds} = 2NV_s^N, \tag{4.7}$$

and in the second case

$$\frac{dH_s^{N,a}}{ds} = 2NV_s^{N,a}. \tag{4.8}$$

The $\mathbb{R}_+\times\{-1,1\}$–valued process $\{(H_s^N, V_s^N),\ s \ge 0\}$ (resp $\{(H_s^{N,a}, V_s^{N,a}), s \ge 0\}$) is a Markov process. This process solves a martingale problem, in a sense which we now make precise. In fact we can write the following SDE for this pair in both cases. We have $H_0^N = H_0^{N,a} = 0$, $V_0^N = V_0^{N,a} = 1$, and

$$\frac{dH_s^N}{ds} = 2NV_s^N,$$

$$dV_s^N = 2\mathbf{1}_{\{V_{s^-}^N=-1\}}dP_s^+ - 2\mathbf{1}_{\{V_{s^-}^N=+1\}}dP_s^- + \frac{N\sigma^2}{2}dL_s^N(0);$$

and

$$\frac{dH_s^{N,a}}{ds} = 2NV_s^{N,a},$$
$$dV_s^{N,a} = 2\mathbf{1}_{\{V_{s^-}^{N,a}=-1\}}dP_s^+ - 2\mathbf{1}_{\{V_{s^-}^{N,a}=1\}}dP_s^- + \frac{N\sigma^2}{2}dL_s^{N,a}(0) - \frac{N\sigma^2}{2}dL_s^{N,a}(a^-),$$

where $\{P_s^+,\ s \geq 0\}$ and $\{P_s^-,\ s \geq 0\}$ are two mutually independent Poisson processes, with the intensity resp.

$$\sigma^2 N^2 + 2\alpha N \quad \text{and} \quad \sigma^2 N^2 + 2\beta N.$$

Here $L_s^{N,a}(0)$ and $L_s^{N,a}(a^-)$ denote respectively the number of visits to 0 and a by the process $H^{N,a}$ up to time s, multiplied by $4/N\sigma^2$ (see (2.1)). These two terms in the expression of $V^{N,a}$ stand for the reflection of $H^{N,a}$ above 0 and below a. Note that our definition of $L^{N,a}$ make the mapping $t \longrightarrow L_s^{N,a}(t)$ right continuous. Hence $L_s^{N,a}(t) = 0$ for $t \geq a$, while $L_s^{N,a}(a^-) = \lim_{n\to\infty} L_s^{N,a}(a - \frac{1}{n}) > 0$ if $H^{N,a}$ has reached the level a by time s. We now write a martingale problem satisfied by the process $\{(H_s^N, V_s^N),\ s \geq 0\}$(resp $\{(H_s^{N,a}, V_s^{N,a}),\ s \geq 0\}$). We are not interested in writing it for arbitrary functions of the two variables (h, v), but rather for specific functions, which will be convenient for taking the limit as $N \to \infty$. Note that the process $\{V_s^N,\ s \geq 0\}$ oscillates faster and faster as N grows, and that in the limit some averaging takes place. We thus implement the so called "perturbed test function" method used in stochastic averaging, see e. g. Ethier, Kurtz [2].

For $f \in C^2(\mathbb{R})$, let

$$f^N(h,v) = f(h) + \frac{v}{N\sigma^2}f'(h),$$
$$A^N f^N(h,v) = \frac{2}{\sigma^2}f''(h) + \mathbf{1}_{\{v=-1\}}\frac{4\alpha}{\sigma^2}f'(h) - \mathbf{1}_{\{v=+1\}}\frac{4\beta}{\sigma^2}f'(h).$$

It is easily seen that whenever $f \in C^2(\mathbb{R})$,

$$M_s^{f,N} := f^N(H_s^N, V_s^N) - f^N(0,1) - \int_0^s A^N f^N(H_r^N, V_r^N)dr - \frac{f'(0)}{2}L_s^N(0)$$

$$\left(\text{resp. } M_s^{f,N,a} := f^N(H_s^{N,a}, V_s^{N,a}) - f^N(0,1) - \int_0^s A^N f^N(H_r^{N,a}, V_r^{N,a})dr \right.$$
$$\left. - \frac{f'(0)}{2}L_s^N(0) + \frac{f'(a)}{2}L_s^N(a^-)\right)$$

is a local martingale.

If we assume moreover that $f'(0) \geq 0$ (resp. $f'(0) \geq 0$ and $f'(a) \leq 0$), then

$$\tilde{M}_s^{f,N} := f^N(H_s^N, V_s^N) - f^N(0,1) - \int_0^s A^N f^N(H_r^N, V_r^N) dr$$
$$\left(\text{resp. } \tilde{M}_s^{f,N,a} := f^N(H_s^{N,a}, V_s^N) - f^N(0,1) - \int_0^s A^N f^N(H_r^N, V_r^N) dr\right)$$
is a local sub–martingale.
$$\tag{4.9}$$

If we choose successively $f(h) = h$ and $f(h) = h^2$, we deduce that there exist two local martingales $\{M_s^{1,N},\ s \geq 0\}$ and $\{M_s^{2,N},\ s \geq 0\}$(resp. $\{M_s^{1,N,a},\ s \geq 0\}$ and $\{M_s^{2,N,a},\ s \geq 0\}$) such that

$$H_s^N + \frac{V_s^N}{N\sigma^2} = \frac{1}{N\sigma^2} + \frac{4\alpha}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^N=-1\}} dr - \frac{4\beta}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^N=+1\}} dr + \frac{1}{2}\left[L_s^N(0) - L_{0^+}^N(0)\right] + M_s^{1,N} \tag{4.10}$$

$$\left(H_s^N\right)^2 + \frac{2}{N\sigma^2} H_s^N V_s^N = \frac{4}{\sigma^2} s + \frac{8\alpha}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^N=-1\}} H_r^N dr - \frac{8\beta}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^N=+1\}} H_r^N dr + M_s^{2,N}. \tag{4.11}$$

resp.

$$H_s^{N,a} + \frac{V_s^{N,a}}{N\sigma^2} = \frac{1}{N\sigma^2} + \frac{4\alpha}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^{N,a}=-1\}} dr - \frac{4\beta}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^{N,a}=+1\}} dr + \frac{1}{2}\left[L_s^{N,a}(0) - L_{0^+}^{N,a}(0)\right] - \frac{1}{2} L_s^{N,a}(a^-) + M_s^{1,N,a}$$

$$(H_s^{N,a})^2 + \frac{2}{N\sigma^2} H_s^{N,a} V_s^{N,a} = \frac{4}{\sigma^2} s + \frac{8\alpha}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^{N,a}=-1\}} H_r^{N,a} dr - \frac{8\beta}{\sigma^2}\int_0^s \mathbf{1}_{\{V_r^{N,a}=+1\}} H_r^{N,a} dr - aL_s^{N,a}(a^-) + M_s^{2,N,a}.$$

We deduce from the above computations in both cases that

$$[M^{1,N}]_s = \frac{4}{N^2\sigma^4}\sum_{r\le s}\left(\mathbf{1}_{\{V^N_{r^-}=-1\}}\Delta P^+_r + \mathbf{1}_{\{V^N_{r^-}=1\}}\Delta P^-_r\right),$$

$$\langle M^{1,N}\rangle_s = \frac{4}{\sigma^2}s + \frac{8\alpha}{N\sigma^4}\int_0^s \mathbf{1}_{\{V^N_r=-1\}}dr + \frac{8\beta}{N\sigma^4}\int_0^s \mathbf{1}_{\{V^N_r=1\}}dr$$

$$\langle M^{1,N,a}\rangle_s = \frac{4}{\sigma^2}s + \frac{8\alpha}{N\sigma^4}\int_0^s \mathbf{1}_{\{V^{N,a}_r=-1\}}dr + \frac{8\beta}{N\sigma^4}\int_0^s \mathbf{1}_{\{V^{N,a}_r=1\}}dr$$

It follows immediately from this formula that $\{M^{1,N}_s,\ s\ge 0\}$ (resp.$\{M^{1,N,a}_s,\ s\ge 0\}$) is in fact a martingale. One difficulty which we want to get rid of is the local time term(s) in the expression for $H^N_s + \frac{V^N_s}{N\sigma^2}$ (resp. $H^{N,a}_s + \frac{V^{N,a}_s}{N\sigma^2}$), which introduces some additional complication for checking tightness. For that sake, we consider a new pair of processes (G^N, W^N) (resp. $(G^{N,a}, W^{N,a})$), which is $\mathbb{R}\times\{-1,1\}$–valued and satisfies:
in the **subcritical case**

$$G^N_s = 2N\int_0^s W^N_r dr,$$

$$W^N_s = 1 + 2\int_0^s \mathrm{sign}(G^N_r)\mathbf{1}_{\{W^N_{r^-}=-\mathrm{sign}(G^N_r)\}}dP^+_r - 2\int_0^s \mathrm{sign}(G^N_r)\mathbf{1}_{\{W^N_{r^-}=\mathrm{sign}(G^N_r)\}}dP^-_r;$$

in the **supercritical case** (with the additional reflection at level a)

$$G^{N,a}_s = 2N\int_0^s W^{N,a}_r dr,$$

$$W^{N,a}_s = 1 + \sum_{i\in\mathbb{Z}}\left\{2\int_0^s \mathbf{1}_{\{ai\le G^{N,a}_r\le(i+1)a\}}(-1)^i\mathbf{1}_{\{W^{N,a}_{r^-}=-(-1)^i\}}dP^+_r \right.$$
$$\left. - 2\int_0^s \mathbf{1}_{\{ai\le G^{N,a}_r\le(i+1)a\}}(-1)^i\mathbf{1}_{\{W^{N,a}_{r^-}=-(-1)^i\}}dP^-_r\right\}$$

with the same P^+ and P^- as above. We claim that in the subcritical case

$$(H^N, V^N) \equiv (|G^N|, \mathrm{sign}(G^N)W^N)$$

and in the supercritical case

$$H^{N,a} = \lim_{k\to\infty}\Phi_k(G^N)$$

and

$$V^N = \sum_{i\in\mathbb{Z}}(-1)^i\mathbf{1}_{\{ai\le G^N\le(i+1)a\}}W^N,$$

where

$$\Phi_k = \psi_k \circ \cdots \circ \psi_1$$

and for every j, the mapping ψ_j from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

$$\psi_j(x) = \begin{cases} |x|, & \text{if j is odd;} \\ a - |x-a|, & \text{if j is even.} \end{cases}$$

Clearly tightness of $\{G^N\}$ (resp. $\{G^{N,a}\}$ will imply that of $\{H^N\}$ (resp $\{H^{N,a}\}$), since

$$\forall s,t \quad |H_s^N - H_t^N| \le |G_s^N - G_t^N|$$
$$\text{resp.}\forall s,t \quad |H_s^{N,a} - H_t^{N,a}| \le |G_s^{N,a} - G_t^{N,a}|.$$

Now we have in the first case

$$G_s^N + \frac{W_s^N}{N\sigma^2} = \frac{1}{N\sigma^2} + \frac{4\alpha}{\sigma^2}\int_0^s \text{sign}(G_r^N)\mathbf{1}_{\{W_r^N = -\text{sign}(G_r^N)\}}dr$$
$$- \frac{4\beta}{\sigma^2}\int_0^s \text{sign}(G_r^N)\mathbf{1}_{\{W_r^N = \text{sign}(G_r^N)\}}dr + \tilde{M}_s^{1,N} \tag{4.12}$$

$$\langle \tilde{M}^{1,N}\rangle_s = \frac{4}{\sigma^2}s + \frac{8\alpha}{N\sigma^4}\int_0^s \mathbf{1}_{\{W_r^N = -\text{sign}(G_r^N)\}}dr + \frac{8\beta}{N\sigma^4}\int_0^s \mathbf{1}_{\{W_r^N = \text{sign}(G_r^N)\}}dr. \tag{4.13}$$

Similarly we have in the supercritical case with the reflection at level a:

$$G_s^{N,a} + \frac{W_s^{N,a}}{N\sigma^2} = \frac{1}{N\sigma^2} + \frac{4\alpha}{\sigma^2}\sum_{i\in\mathbb{Z}}\int_0^s \mathbf{1}_{\{ai \le G_r^{N,a} \le (i+1)a\}}(-1)^i\mathbf{1}_{\{W_{r^-}^{N,a} = -(-1)^i\}}dr$$
$$- \frac{4\beta}{\sigma^2}\sum_{i\in\mathbb{Z}}\int_0^s \mathbf{1}_{\{ai \le G_r^{N,a} \le (i+1)a\}}(-1)^i\mathbf{1}_{\{W_{r^-}^{N,a} = (-1)^i\}}dr + \tilde{M}_s^{1,N} \tag{4.14}$$

$$\langle \tilde{M}^{1,N}\rangle_s = \frac{4}{\sigma^2}s + \frac{8\alpha}{N\sigma^4}\sum_{i\in\mathbb{Z}}\int_0^s \mathbf{1}_{\{ai \le G_r^{N,a} \le (i+1)a\}}\mathbf{1}_{\{W_{r^-}^{N,a} = -(-1)^i\}}dr$$
$$+ \frac{8\beta}{N\sigma^4}\sum_{i\in\mathbb{Z}}\int_0^s \mathbf{1}_{\{ai \le G_r^{N,a} \le (i+1)a\}}\mathbf{1}_{\{W_{r^-}^{N,a} = (-1)^i\}}dr. \tag{4.15}$$

Tightness of $\{G^N\}_{N\ge1}$ (resp. $\{G^{N,a}\}_{N\ge1}$) in $D([0,\infty))$, hence in $C([0,\infty))$, follows immediately from (4.12), (4.13) (resp (4.14), (4.15)) and Proposition 4.1. We deduce the

Lemma 4.4 *The sequence $\{H^N,\ s \ge 0\}_{N\ge1}$ is tight in $C([0,\infty))$.*

4.3 Weak convergence of H^N

Let us state our convergence results.

Theorem 4.5 *$H^N \Rightarrow H$ in $C([0,\infty))$ as $N \to \infty$, where $\{H_s,\ s \geq 0\}$ is the unique solution of the submartingale problem:*
$\forall\ f \in C^2(\mathbb{R})$ such that $f'(0) \geq 0$, the process

$$\tilde{M}_t^f := f(H_t) - f(0) + \int_0^t Af(H_r)dr$$

is a local submartingale where

$$Af(h) = \frac{2(\alpha-\beta)}{\sigma^2}f'(h) + \frac{2}{\sigma}f''(h)$$

In other words, H is the unique weak solution of the reflected SDE

$$H_s = \frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0), \tag{4.16}$$

where $L_s(0)$ denotes the local time at level 0 accumulated by the process H up to time s. In other words, the process H equals the process

$$\frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s$$

reflected at 0.

Theorem 4.6 *For any $a > 0$, $H^{N,a} \Rightarrow H^a$ in $C([0,\infty))$ as $N \to \infty$, where $\{H_s^a,\ s \geq 0\}$ is the unique solution of the submartingale problem:*
$\forall\ f \in C^2(\mathbb{R})$ such that $f'(0) \geq 0$ and $f'(a) \leq 0$, the process

$$\tilde{M}_t^f := f(H_t^a) - f(0) + \int_0^t Af(H_r^a)dr$$

is a local submartingale where

$$Af(h) = \frac{2(\alpha-\beta)}{\sigma^2}f'(h) + \frac{2}{\sigma}f''(h)$$

In other words, H is the unique weak solution of the reflected SDE

$$H_s^a = \frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s + \frac{1}{2}L_s^a(0) - \frac{1}{2}L_s^a(a^-),$$

where $L_s(0)$ and $L_s(a^-)$ denote respectively the local time at level 0 and a^- accumulated by the process H^a up to time s. In other words, the process H^a equals the process

$$\frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s$$

reflected in $[0,a]$.

The relation between submartingales problem and reflected SDE and the weak convergence, is exposed in Stroock-Varadhan[9]. Let us prove theorem 4.5. The proof of theorem 4.6 is analogous.

Proof: From Lemma 4.4, we can extract a subsequence, still denoted as an abuse $\{H^N\}$, such that

$$H^N \Rightarrow H \quad \text{in} \;\; D([0,\infty)).$$

From (4.9), we know that for all $N \geq 1$, $f \in C^2(\mathbb{R})$ satisfying $f'(0) \geq 0$,

$$M_s^{f,N} := f(H_s^N) + \frac{1}{N\sigma^2} V_s^N f'(H_s^N) - f(0) - \frac{1}{N\sigma^2} f'(0) - \frac{2}{\sigma^2}\int_0^s f''(H_r^N)dr \\ - \frac{4}{\sigma^2}\int_0^s \left[\alpha \mathbf{1}_{\{V_r^N=-1\}} - \beta \mathbf{1}_{\{V_r^N=1\}}\right] f'(H_r^N)dr$$

is a submartingale. Let us admit for a moment the Lemma :

Lemma 4.7 *If $X^{N,x} \Rightarrow X$ as $N \to \infty$ in $C([0,+\infty))$, then for all $s > 0$,*

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} X_r^{N,x} dr \Rightarrow \frac{1}{2}\int_0^s X_r dr, \qquad \int_0^s \mathbf{1}_{\{V_r^N=-1\}} X_r^{N,x} dr \Rightarrow \frac{1}{2}\int_0^s X_r dr.$$

It then follows that for all $f \in C^2(\mathbb{R})$ satisfying $f'(0) \geq 0$,

$$M_s^f := f(H_s) - f(0) - \frac{2}{\sigma^2}\int_0^s [\alpha - \beta] f'(H_r)dr - \frac{2}{\sigma^2}\int_0^s f''(H_r)dr$$

is a submartingale, which establishes the result, since this submartingale problem has a unique solution, which is the Brownian motion with drift reflected at level 0 described in the statement of the Theorem, see Stroock, Varadhan [9]. ∎

Proof of Lemma 4.7: It is an easy exercise to check that the mapping

$$\Phi : C([0,+\infty)) \times C_\uparrow([0,+\infty)) \to C([0,+\infty))$$

defined by

$$\Phi(x,y)(t) = \int_0^t x(s)dy(s),$$

where $C_\uparrow([0,+\infty))$ denotes the set of increasing continuous functions from $[0,\infty)$ into $\mathbb{R}$, and the three spaces are equipped with the topology of locally uniform convergence, is continuous. Consequently it suffices to prove that locally uniformly in $s > 0$,

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} dr \to \frac{s}{2}$$

in probability, as $N \to \infty$. In fact since both the sequence of processes and the limit are continuous and monotone, it follows from an argument "à la Dini" that it suffices to prove the

Lemma 4.8 *For any $s > 0$,*

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} dr \to \frac{s}{2}$$

in probability, as $N \to \infty$.

PROOF: Let A_s^N (resp. I_s^N) denote the number of local maxima (resp. minima) of the process H^N on the interval $[0, s]$.We have

$$I_s^N \leq A_s^N \leq I_s^N + 1 \tag{4.17}$$

and

$$P^1_{\sigma^2 N^2 s} \leq A_s^N + I_s^N \leq P^1_{\sigma^2 N^2 s} + P^2_{2\alpha N s} + P^3_{2\beta N s} + N L_s^N(0),$$

where $\{P_r^1, P_r^2, P_r^3,\ r \geq 0\}$ are three mutually independent Poisson processes with intensity one. We deduce from this system of inequalities that

$$\frac{A_s^N + I_s^N}{\sigma^2 N^2} \to s \quad \text{in probability, as } N \to \infty.$$

Indeed, we deduce from Proposition 4.1 that both H^N and $M^{1,N}$ are tight, hence from (4.10) $L_s^N(0)$ is tight, and $L_s^N(0)/N \longrightarrow 0$ in probability as $N \longrightarrow \infty$. Hence from (4.17)

$$\frac{A_s^N}{\sigma^2 N^2} \to \frac{s}{2},\ \frac{I_s^N}{\sigma^2 N^2} \to \frac{s}{2} \quad \text{in probability, as } N \to \infty.$$

Now

$$\begin{aligned} A_s^N &= \int_0^s \mathbf{1}_{\{V_{r^-}^N=1\}} dP_r^- \\ &= (\sigma^2 N^2 + 2\beta N) \int_0^s \mathbf{1}_{\{V_r^N=1\}} dr + \int_0^s \mathbf{1}_{\{V_{r^-}^N=1\}} dM_r^-, \end{aligned}$$

where

$$\langle M^- \rangle_s = (\sigma^2 N^2 + 2\beta N)s.$$

Consequently

$$\begin{aligned} \int_0^s \mathbf{1}_{\{V_r^N=1\}} dr &= \frac{A_s^N}{\sigma^2 N^2 + 2\beta N} - (\sigma^2 N^2 + 2\beta N)^{-1} \int_0^s \mathbf{1}_{\{V_{r^-}^N=1\}} dM_r^- \\ &\to \frac{s}{2} \quad \text{in probability, as } n \to \infty. \end{aligned}$$

Indeed

$$\mathbb{P}\left(\left|\int_0^s \mathbf{1}_{\{V_{r^-}^N=1\}} dM_r^-\right| > \varepsilon N^2\right) \leq \frac{\sqrt{\mathbb{E}(|M_s^-|^2)}}{\varepsilon N^2}$$
$$\leq \frac{\sqrt{(\sigma^2 N^2 + 2\beta N)\, s}}{\varepsilon N^2}$$
$$\to 0, \quad \text{as } n \to \infty.$$

Corollary 4.9

$$\left(H^N, M^{1,N}, L^N(0)\right) \Longrightarrow \left(H, \frac{2}{\sigma}B, L(0)\right),$$

where B is a standard Brownian motion and $L(0)$ is the local time of H, and for each $a > 0$,

$$\left(H^{N,a}, M^{1,N,a}, L^{N,a}(0), L^{N,a}(a^-)\right) \Longrightarrow \left(H^a, \frac{2}{\sigma}B, L^a(0), L^a(a^-)\right),$$

where B is as above, $L^a(0)$ [resp. $L^a(a^-)$] denotes the local time of the continuous semimartingale H^a at level 0 [resp. at level a^-; it is the limit of the local time at levels $a - 1/n$ as $n \to \infty$].

Proof: We prove the first part only. As already noted in the proof of Lemma 4.5, $\left(H^N, M^{1,N}, L_\cdot(0)\right)_{N\leq 1}$ is tight in $C([0,\infty)) \times [D([0,\infty))]^2$.

Morever any weak limit of $M^{1,N}$ along a subsequence equals $\frac{2}{\sigma}B$. We deduce from (4.10) that any weak limit of $\left(H^N, M^{1,N}, L^N(0)\right)$ along a subsequence is a triple of the form $\left(H, \frac{2}{\sigma}B, K\right)$ satisfying

$$H_s = \frac{2(\alpha-\beta)}{\sigma^2}s + \frac{2}{\sigma}B_s + \frac{1}{2}K_s.$$

Comparing with (4.16), we deduce that

$$K = \frac{1}{2}L(0).$$

Now the limit is unique, hence the whole sequence converges. ∎

5 Generalized Ray Knight Theorem

In this section we give an new proof of Delmas' generalization of the Ray Knight Theorem. Define $L_{\cdot}(0)$ the local time of H at level 0, and in the subcritical case $\alpha \leq \beta$

$$\tau_x = \inf\{s > 0; L_s(0) > \frac{4}{\sigma^2}x\}.$$

In the supercritical case, of course the construction is more complex. It follows from Lemma 3.2 and Corollary 4.9 that for any $0 < a < b$,

$$\Pi^{a,b}(H^b) \overset{(d)}{=} H^a, \tag{5.1}$$

where H^a [resp. H^b] is now Brownian motion multiplied by $2/\sigma$, with drift $2(\alpha - \beta)s/\sigma^2$, reflected in the interval $[0, a]$ [resp. $[0, b]$], see Theorem 4.6. Note that (5.1) is Lemma 2.1 in [1].

Now define for each $a, x > 0$,

$$\tau_x^a = \inf\{s > 0,\ L_s^a(0) > \frac{4}{\sigma^2}x\}.$$

It follows from (5.1) that, as in the discrete case, $\forall 0 < a < b$,

$$\{L^b_{\tau_x^b}(t),\ 0 \leq t \leq a\} \overset{(d)}{=} \{L^a_{\tau_x^a}(t),\ 0 \leq t \leq a\}.$$

Consequently for each $x > 0$, we can define a process$\{\mathcal{L}_x(t),\ t \geq 0\}$ which is such that for each $a > 0$,

$$\{\mathcal{L}_x(t),\ 0 \leq t \leq a\} \overset{(d)}{=} \{L^a_{\tau_x^a}(t),\ 0 \leq t \leq a\}.$$

In the subcritical case, we use the same notation $\mathcal{L}_x(t)$ for the quantity $L_{\tau_x}(t)$. We have the (see Theorem 3.1 in Delmas [1])

Theorem 5.1 (Generalized Ray Knight theorem)

$$\{\mathcal{L}_x(t),\ t \geq 0,\ x > 0\} \overset{(d)}{=} \left\{\frac{4}{\sigma^2}X_t^x,\ t \geq 0\ x > 0\right\},$$

where X^x is the Feller branching diffusion process, solution of the SDE

$$X_t^x = x + (\alpha - \beta)\int_0^t X_r^x dr + \sigma \int_0^t \sqrt{X_r^x} dB_r,\ t \geq 0.$$

PROOF: Since both sides have stationary independent increments in x, it suffices to show that for any $x > 0$,

$$\{\mathcal{L}_x(t),\ t \geq 0\} \stackrel{(d)}{=} \left\{\frac{4}{\sigma^2}X_t^x,\ t \geq 0\right\}.$$

We treat the supercritical case only. The subcritical case is similar but simpler. Fix an arbitrary $a > 0$. By applying the "occupation time formula" to $H^{N,a}$, and Lemma 3.4, we have for any $g \in C(\mathbb{R}_+)$ with support in $[0, a]$,

$$\begin{aligned}\frac{4}{\sigma^2}\int_0^{\tau_x^{N,a}} g(H_r^{N,a})dr &= \int_0^\infty g(t)L_{\tau_x^{N,a}}^{N,a}(t)dt \\ &= \frac{4}{\sigma^2}\int_0^\infty g(t)X_t^{N,x}dt \qquad (5.2)\end{aligned}$$

We deduce clearly from Proposition 4.3

$$\int_0^\infty g(t)X_t^{N,x}dt \Longrightarrow \int_0^\infty g(t)X_t^x dt. \qquad (5.3)$$

Let us admit for a moment that as $N \to \infty$

$$\int_0^{\tau_x^{N,a}} g(H_r^{N,a})dr \Longrightarrow \int_0^{\tau_x^a} g(H_r^a)dr \qquad (5.4)$$

From the occupation time formula for the continuous semi-martingale $(H_s^a, s \geq 0)$, we have that

$$\frac{4}{\sigma^2}\int_0^{\tau_x^a} g(H_r^a)dr = \int_0^\infty g(t)L_{\tau_x^a}^a(t)dt. \qquad (5.5)$$

We deduce from (5.2), (5.3), (5.4) and (5.5) that for any $g \in C(\mathbb{R}_+)$ with compact in $[0, a]$,

$$\frac{4}{\sigma^2}\int_0^{\tau_x^a} g(H_r^a)dr = \int_0^\infty g(t)\mathcal{L}_x(t)dt,$$

from which the result follows. ■

It remains to prove (5.4), which clearly is a consequence of (recall the definition (3.1) of $\tau_x^{N,a}$)

Proposition 5.2 *For any $a > 0$, as $N \to \infty$,*

$$(H^{N,a}, \tau_x^{N,a}) \Longrightarrow (H^a, \tau_x^a).$$

PROOF: For the sake of simplifying the notations, we suppress the superscript a. Let us define the function ϕ from $\mathbb{R}_+ \times C_\uparrow([0,+\infty))$ into $\mathbb{R}_+$ by

$$\phi(x,y) = \inf\{s > 0 : y(s) > \frac{4}{\sigma^2}x\}.$$

For any fixed x, the function $\phi(x,.)$ is continuous in the neighborhood of a function y which is strictly increasing at the time when it first reaches the value x. Define

$$\tau_x'^N := \phi(x, L_\cdot^N(0)).$$

We note that for any $x > 0$, $s \longmapsto L_s(0)$ is a.s. strictly increasing at time τ_x. Indeed $\tau_x - \tau_{x^-} > 0$ iff $e_x \neq \delta$ (with the notation of Definition XII.2.1 in [8]), where $e = \{e_x,\ x \geq 0\}$ is the excursion process of the reflected Brownian motion with drift H. The claimed property follows from the fact that e is a Poisson process, hence for each $x > 0$, $\mathbb{P}(e_x \neq \delta) = 0$. Consequently τ_x is a. s. a continuous function of the trajectory $L_\cdot(0)$, and from Corollary 4.9,

$$(H^N, \tau_x'^N) \Longrightarrow (H, \tau_x).$$

It remains to prove that $\tau_x'^N - \tau_s^N \longrightarrow 0$ in probability. For any $y < x$, for N large enough

$$0 \leq \tau_x'^N - \tau_x^N \leq \tau_x'^N - \tau_y'^N.$$

Clearly $\tau_x'^N - \tau_y'^N \Longrightarrow \tau_x - \tau_y$, hence for any $\varepsilon > 0$,

$$0 \leq \limsup_N \mathbb{P}\big(\tau_x'^N - \tau_x^N \geq \varepsilon\big) \leq \limsup_N \mathbb{P}\big(\tau_x'^N - \tau_y'^N \geq \varepsilon\big) \leq \mathbb{P}\big(\tau_x - \tau_y \geq \varepsilon\big).$$

The result follows, since $\tau_y \rightarrow \tau_{x^-}$ as $y \rightarrow x$, $y < x$, and $\tau_{x^-} = \tau_x$ a. s. ∎

References

[1] Jean–François Delmas, Height process for super-critical continuous state branching process, *Markov Proc. and Rel. Fields* **14**, 309–326, 2008.

[2] Stuart Ethier, Thomas Kurtz : *Markov processes, Characterization and convergence*, John Wiley and sons, Inc., New York, 1986.

[3] J. Geiger, G. Kersting, Depth–first search of random trees nd Poisson point processes, in *Classical and modern branching processes*, IMA Vol. Math. Appl., 84,, 111–126 Springer, New York, 1997.

[4] Anatole Joffe, Michel Métivier, Weak convregence of sequences of semimartingales with applications to multitype branching processes, *Adv. Appl. Prob.* **18**, 20–65, 1986.

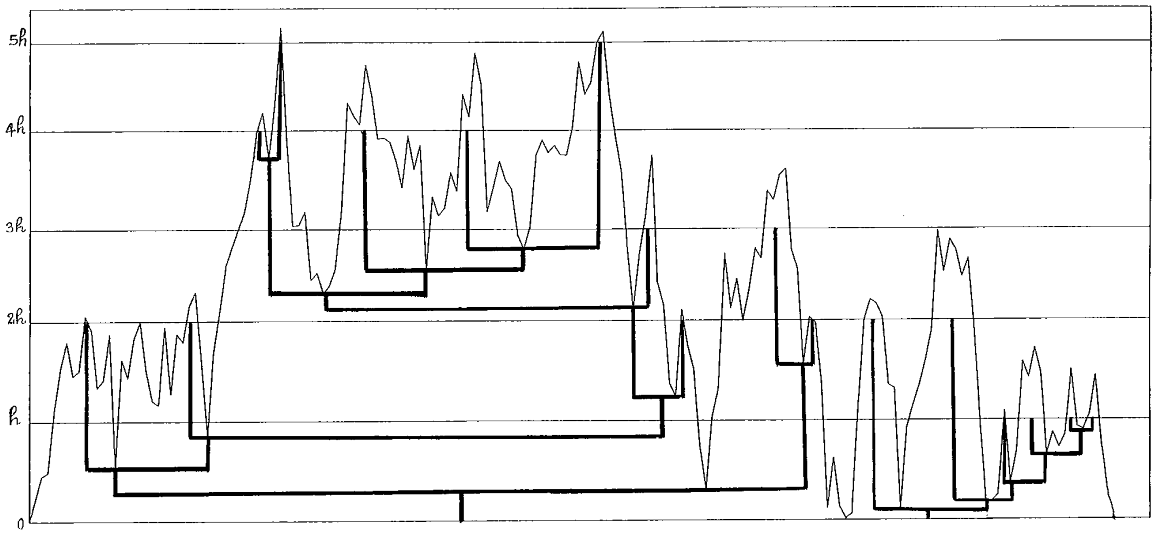

Figure 3: The subtree of progenies which live at least for a duration h

[5] Jean–François Le Gall, Marches aléatoires, mouvement brownien et processus de branchement, *Séminaire de Probabilités XXIII*, Lecture Notes in Math. **1372**, 258–274, Springer, Berlin 1989.

[6] Amaury Lambert, The contour of splitting trees is a Lévy process, submitted.

[7] Jim Pitman, Matthias Winkel, Growth of the Brownian forest, *Ann. Probab.* **33**, 2188–2211, 2005.

[8] D. Revuz, M. Yor: Continuous Martingales and Brownien Motion, 3d edition, *Grundlehren der mathematischen Wiisenschaften* **293**, Springer Verlag, Berlin, 1999.

[9] Daniel W. Stroock, S. R. S. Varadhan : Diffusion processes with boundary conditions, *Comm. Pure and Appl. Math.* **24**, 147–225, 1971.