Monotonicity Methods for White Noise Driven Quasi-Linear SPDEs

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Abstract

We establish existence and uniqueness for nonlinear elliptic and parabolic SPDEs driven by white noise, with nonlinearities of monotone type.

1. Introduction

The aim of this paper is to establish existence and uniqueness results both for nonlinear elliptic stochastic partial differential equations of the type :

$$(1.1) \qquad \begin{cases} -\Delta u(x) + f(u(x)) = \dot{W}(x), \ x \in D \\ u|_{\partial D} = 0 \end{cases}$$

where D is an open bounded subset of $\mathbb{R}^k$ $(k = 1, 2, 3)$, $\dot{W}$ denotes white noise, and f is an increasing function; and for nonlinear parabolic stochastic partial differential equations of the type :

(1.2)
$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) + f(u)(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x) + g(t,x); \ t \geq 0, \ 0 \leq x \leq 1 \\ u(0,x) = u_0(x), \ 0 \leq x \leq 1; u(t,0) = u(t,1) = 0 \end{cases}$$

where $\frac{\partial^2 W}{\partial t \partial x}$ denotes the second order mixed derivative of the brownian sheet $\{W_{tx}\}$, i.e. space–time white noise, and

$$f = f_1 + f_2$$

where f_1 is increasing and f_2 is Lipschitz.

We shall establish existence and uniqueness of a solution for these two classes of equations.

Note that a lot is known about equation (1.1) in case of a linear function f, see e.g. Benfatto, Gallavotti, Nicolò [2], Rozanov [7]. It is known that the solution is a process in the ordinary sense iff k – the dimension of the space variable x – is less than or equal to three. This is the reason for our restriction to $k \leq 3$. There seems to be no literature on nonlinear elliptic SPDEs with the white noise on the right side. For nonlinear elliptic PDEs with measures as right side or boundary condition, we refer to Boccardo, Gallouët [1] and Röckner, Zegarlinski [6]. Our approach will consist in generalizing the use of monotonicity by Lions [4] to solve certain classes of deterministic nonlinear PDEs. Let us note that our results could be extended to more general PDE operators, and other types of boundary conditions.

Moreover, in the case $k = 1$, the nonlinear function f can depend also on $\frac{\partial u}{\partial x}$, see Nualart, Pardoux [5] where it is shown that the solution $(u, \frac{\partial u}{\partial x})$ is a Markov field iff f is linear. The positive part of that result is still true for $k = 2, 3$ (see e.g. Rozanov [8]), and we suspect that the negative part also generalizes, but we have not been able to prove it yet.

Concerning the parabolic equation (1.2), let us note that existence and uniqueness is known in case f is Lipschitz, even with a non constant diffusion coefficient, see e.g. Walsh [9]. However, our result with f_1 monotone increasing and not necessarily Lipschitz seems to be new.

2. Elliptic equations

The aim of this section is to study the equation :

$$\text{(2.1)} \qquad \begin{cases} -\Delta u(x) + f(u)(x) = g(x) + \dot{W}(x), \; x \in D \\ u|_{\partial D} = 0 \end{cases}$$

where D is a *bounded* domain of $\mathbf{R}^k$, $k = 1, 2$ or 3, whose boundary ∂D is supposed to be regular in the sense of potential theory, $g \in L^2(D)$, $\dot{W}$ denotes "white noise" and

$$f(u)(x) = f(x, u(x))$$

where $f(x,r)$ is a measurable function of $(x,r) \in D \times \mathbf{R}$ which satisfies properties to be stated below. We first note that we can and shall w.l.o.g. assume that $f(x,0) = 0$. We moreover assume that :

(2.2) $\qquad r \to f(x,r)$ is continuous and non decreasing, for any $x \in D$

and moreover

(2.3) $\qquad f$ is locally bounded

Note that (2.2) together with $f(x,0) = 0$ imply that $rf(x,r) \geq 0$, and moreover from (2.3) $f(x, u(x))$ is bounded whenever $u(x)$ is bounded.

Let us now give a more rigorous formulation of equation (2.1), i.e. a "weak formulation" , which is as follows. An a.s. bounded function u from D into $\mathbf{R}$ is said to satisfy the weak form of (2.1) if for any $\phi \in C_b^2(D) \cap C(\bar{D})$, which vanishes on ∂D,

(2.4)
$$-\int_D u(x)\Delta\phi(x)\,dx + \int_D f(u)(x)\phi(x)\,dx = \int_D g(x)\phi(x)\,dx + \int_D \phi(x)\,dW_x$$

where $\{W_x; x \in \mathbf{R}^k\}$ is a standard Wiener process with k–dimensional parameter, i.e. it is an a.s. continuous Gaussian random field with zero mean and covariance operator defined by

$$E[W_x W_y] = x \wedge y$$

$(x \wedge y = (x_1 \wedge y_1) \cdots (x_k \wedge y_k))$.

We shall work with another equivalent formulation, which we call an "integral formulation". Before introducing that formulation, we need to introduce the kernel associated to the linear version of the above equation. Consider the elliptic PDE :

$$\begin{cases} -\Delta v(x) = \varphi(x), \; x \in D \\ v|_{\partial D} = 0 \end{cases}$$

That equation defines a linear continuous mapping from $L^2(D)$ into itself which can be written as :

$$v(x) = \int_D K(x,y)\varphi(y)\,dy$$

We shall now give an expression for $K(x,y)$ in the three cases $k = 1, 2, 3$.

In case $n = 1$, $D = (0,t)$,

$$K(x,y) = x \wedge y - \frac{xy}{t}$$

In case $n = 2$, we have

$$K(x,y) = -\frac{1}{2\pi}\log|x-y| + \frac{1}{2\pi}E_x[\log|B_\tau - y|]$$

where $\{B_t; t \geq 0\}$ denote the standard two–dimensional Brownian motion starting from x under P_x, and τ is the exit time from D.

In case $n = 3$, we use the same notations as for $n = 2$ (but now $\{B_t; t \geq 0\}$ denotes the three–dimensional standard Brownian motion). We have :

$$K(x,y) = (4\pi|x-y|)^{-1} - E_x\left[(4\pi|B_\tau - y|)^{-1}\right]$$

Lemma 2.1. *In the three cases $k = 1, 2, 3$, the random field*

$$\left\{v(x) = \int_D K(x,y)\,dW_y,\ x \in \bar{D}\right\}$$

possesses an a.s. continuous modification.

Proof: We consider successively the three cases. If $D = (0,t)$,

$$v(x) = \frac{x}{t}\int_0^t W_s\,ds - \int_0^x W_s\,ds$$

and the results is obvious in this case.

Consider now the case $k = 2$. Let

$$\bar{v}(x) = \int_D \log|x-y|\,dW_y$$

We have for $x, z \in D$, $\varepsilon > 0$:

$$E\left[|\bar{v}(x) - \bar{v}(z)|^2\right] = \int_D \left|\log|x-y| - \log|z-y|\right|^2 dy$$
$$\leq |x-z|^{2-\varepsilon}\int_D \left|\log|x-y| - \log|z-y|\right|^{\varepsilon}\left(\int_0^1 \frac{d\theta}{\theta|x-y| + (1-\theta)|z-y|}\right)^{2-\varepsilon} dy$$
$$\leq |x-z|^{2-\varepsilon}\int_D \left|\log|x-y| - \log|z-y|\right|^{\varepsilon}\left(\frac{1}{|x-y|} + \frac{1}{|z-y|}\right)^{2-\varepsilon} dy$$
$$\leq c_\varepsilon|x-z|^{2-\varepsilon}\left(\int_D \left|\log|x-y| - \log|z-y|\right|^{q\varepsilon} dy\right)^{1/q}$$
$$\times\left(\int_D \frac{dy}{|x-y|^{(2-\varepsilon)p}} + \int_D \frac{dy}{|z-y|^{(2-\varepsilon)p}}\right)^{1/p}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < (2-\varepsilon)^{-1}2$. It is not hard to deduce from these inequalities :

$$E\left[|\bar{v}(x) - \bar{v}(z)|^2\right] \leq \bar{c}'_\varepsilon|x-z|^{1-\varepsilon},\ x, z \in D$$

But $\bar{v}(x) - \bar{v}(z)$ is a zero mean Gaussian random variable, and consequently for any integer r and $\varepsilon > 0$,

$$E\left[|\bar{v}(x) - \bar{v}(z)|^r\right] \leq c'_\varepsilon|x-z|^{(1-\varepsilon)r},\ x, z \in D$$

Choosing $r > 2$, we conclude from Kolmogorov's Lemma that $\bar{v}$ possesses an a.s. continuous modification. Having chosen that modification, we deduce that $E_x[\bar{v}(B_\tau)]$ is continuous on $\bar{D}$, and also :

$$v(x) = \int_D K(x,y)\,dW_y$$
$$= (2\pi)^{-1/2}\left(E_x[\bar{v}(B_\tau)] - \bar{v}(x)\right)$$

where E_x means here integrating with respect to the law of B_τ, $\{W_y\}$ being fixed.

We consider finally the case $n = 3$, which is analogous to the last case. It suffices to check that $\bar{v}(x) = \int_D |x-y|^{-1}\,dW_y$ possesses an a.s. continuous modification. For $x, z \in D$, $0 < \varepsilon < 1/4$,

$$E\left[|\bar{v}(x) - \bar{v}(z)|^2\right] = \int_D \left|\frac{1}{|x-y|} - \frac{1}{|z-y|}\right|^2 dy$$
$$\leq \int_D \left|\frac{1}{|x-y|} - \frac{1}{|z-y|}\right|^{\frac{5}{4}+\varepsilon}\left|\frac{|z-y| - |x-y|}{|x-y||z-y|}\right|^{\frac{3}{4}-\varepsilon} dy$$
$$\leq |x-z|^{\frac{3}{4}-\varepsilon}\left(\int_D \left|\frac{1}{|x-y|} - \frac{1}{|z-y|}\right|^{\frac{5}{2}+2\varepsilon} dy\right)^{1/2}\left(\int_D \frac{dy}{(|x-y||z-y|)^{\frac{3}{2}-2\varepsilon}}\right)^{1/2}$$
$$\leq c_\varepsilon|x-z|^{\frac{3}{4}-\varepsilon}\left(\int_D \frac{dy}{|x-y|^{\frac{5}{2}+2\varepsilon}} + \int_D \frac{dy}{|z-y|^{\frac{5}{2}+2\varepsilon}}\right)^{1/2}$$
$$\times\left(\int_D \frac{dy}{|x-y|^{3-4\varepsilon}}\right)^{1/4}\left(\int_D \frac{dy}{|z-y|^{3-4\varepsilon}}\right)^{1/4}$$

It then follows :

$$E\left[|\bar{v}(x) - \bar{v}(z)|^2\right] \leq c'_\varepsilon|x-z|^{\frac{3}{4}-\varepsilon},\ x, z \in D$$

Again from well–known properties of Gaussian random variables, we deduce that for any integer p,

$$E\left[|\bar{v}(x) - \bar{v}(z)|^p\right] \leq c'_\varepsilon|x-z|^{(\frac{3}{8}-\frac{\varepsilon}{2})p},\ x, z \in D$$

We conclude using Kolmogorov's Lemma. □

Remark 2.2. *It follows from the proof of Lemma 2.1 that in case $k = 1$, v has Lipchitz paths, in case $k = 2$, the paths of v are Hölder continuous of exponent $1 - \varepsilon$, $\forall \varepsilon > 0$, and in case $k = 3$ they are Hölder continuous of exponent $\frac{3}{8} - \varepsilon$, $\forall \varepsilon > 0$.* □

We shall say that an a.s. bounded function u from D into $\mathbf{R}$ solves the "integral form" of equation (2.1) if :

(2.5)
$$u(x) + \int_D K(x,y)f(u)(y)\,dy = \int_D K(x,y)g(y)\,dy + \int_D K(x,y)\,dW_y, \quad x \in D$$

Note that if u satisfies (2.5), u is a.s. continuous on $\bar{D}$, and vanishes on ∂D.

Lemma 2.3. (2.4) *and* (2.5) *are equivalent.*

Proof: Suppose that u satisfies (2.5). Then u is a.s. continuous on $\bar{D}$. Let $\phi \in C^\infty(\mathbf{R}^k)$, with compact support in D. Multiply (2.5) by $\Delta\phi(x)$ and integrate over D. Using the identity

$$-\int_D \Delta\phi(x)K(x,y)\,dx = \phi(y)$$

we deduce (2.4) for smooth ϕ; the genaral case follows by density.

Suppose now that u satisfies (2.4). Choose

$$\phi(y) = \int_D \psi(x)K(x,y)\,dx$$

with $\psi \in C^\infty(D)$. Noting that

$$-\Delta\phi(y) = \psi(y),$$

we conclude that

$$\int_D u(x)\psi(x)\,dx + \int_D\int_D \psi(x)K(x,y)f(u)(y)\,dxdy = \int_D\int_D \psi(x)K(x,y)g(y)\,dxdy + \int_D\int_D \psi(x)K(x,y)\,dW_y dx$$

from which (2.5) follows. □

We shall rewrite equation (2.5) as :

$$u + Kf(u) = Kg + K\dot{W}$$

Let us denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and scalar product in $L^2(D)$. The existence and uniqueness proof will rely on the following :

Lemma 2.4. *There exists a constant $a > 0$ such that for any $\varphi \in L^2(D)$,*

$$(K\varphi, \varphi) \geq a\|K\varphi\|^2$$

Proof: For $\varphi \in L^2(D)$, $K\varphi$ is the unique element of the Sobolev space $H_0^1(D)$ which solves the equation :

$$\begin{cases} -\Delta v(x) = \varphi(x), \; x \in D \\ v|_{\partial D} = 0 \end{cases}$$

Multiplying the above equation by v and integrating by parts, we obtain :

$$\sum_{i=1}^k \|\frac{\partial v}{\partial x^i}\|^2 = (\varphi, v)$$

However, from Poincaré's inequality, see Gilbarg, Trudinger [3, p. 157], there exists a constant $a > 0$ such that for any $v \in H_0^1(D)$:

$$\sum_{i=1}^k \|\frac{\partial v}{\partial x^i}\|^2 \geq a\|v\|^2$$

The result follows. □

We are now in a position to establish the main result of this section :

Theorem 2.5. *Let D be a bounded domain of $\mathbf{R}^k$, $1 \leq k \leq 3$, with a regular boundary, let f satisfy (2.2) and (2.3), and $g \in L^2(D)$. Then equation (2.5) possesses a unique solution which is a.s. continuous on $\bar{D}$.*

Proof: *Uniqueness.* Let u, v be two solutions. Then

$$u - v + K[f(u) - f(v)] = 0 \tag{2.6}$$

Multiplying that equation by $f(u) - f(v)$, we obtain :

$$(u - v, f(u) - f(v)) + (K[f(u) - f(v)], f(u) - f(v)) = 0$$

It follows from Lemma 2.4, (2.2) and (2.6) that :

$$a\|u - v\|^2 \leq 0$$

from which uniqueness follows. Moreover, the argument clearly implies the following stronger uniqueness statement : if u and v satisfy (2.5) on a measurable subset $\tilde{\Omega}$ of Ω, then

$$u(x) = v(x) \quad \text{a.e. on } D \times \tilde{\Omega}.$$

Existence. Step 1. We suppose in this first step that f satisfies (2.2) and is *bounded*. Let $\{\dot W^n,\ n \in \mathbf{N}\}$ be a sequence of processes with trajectories in $L^2(D)$, which is such that :

$$K\dot W^n \to K\dot W \text{ in } L^2(\Omega \times D), \text{ as } n \to \infty$$

For each $n \in \mathbf{N}$, we consider the elliptic PDE:

$$(2.7) \qquad \begin{cases} -\Delta u^n + f(u^n) = g + \dot W^n \\ u^n|_{\partial D} = 0 \end{cases}$$

The existence of a unique solution $u^n \in H_0^1(D)$ for equation (2.7) follows from Lions [4, Theorem 2.1 p. 171]. Clearly,

$$(2.8) \qquad u^n + Kf(u^n) = Kg + K\dot W^n$$

and

$$u^n - u^m + K[f(u^n) - f(u^m)] = K[\dot W^n - \dot W^m]$$

Multiplying by $f(u^n) - f(u^m)$, we get :

$$\begin{aligned}(u^n - u^m, f(u^n) - f(u^m)) + (K[f(u^n) - f(u^m)], f(u^n) - f(u^m)) \\ = (K[\dot W^n - \dot W^m], f(u^n) - f(u^m))\end{aligned}$$

By (2.2) and Lemma 2.4, we obtain :

$$a\|u^n - u^m\|^2 \le (K[\dot W^n - \dot W^m], f(u^n) - f(u^m) + 2a(u^n - u^m))$$

Since $E[\|K(\dot W^n - \dot W^m)\|^2]$ tends to 0 as $n, m \to \infty$ and f is bounded, $\{u^n\}$ is a Cauchy sequence in $L^2(\Omega \times D)$. Define $u = \lim_n u^n$. Since f is bounded, $f(u^n) \to f(u)$ in $L^2(\Omega \times D)$ as $n \to \infty$. Existence follows by taking the limit in (2.8).

Existence. Step 2. We now suppose that f satisfies (2.2) and (2.3) and is *bounded from below*. Let

$$f_n(x, r) = f(x, r) \wedge n.$$

For each $n \in \mathbf{N}$, let u^n denote the unique solution (constructed in the above step) of the equation

$$u^n + Kf_n(u^n) = Kg + K\dot W.$$

It follows from Lemma 2.6 below that the sequence $\{u_n(x),\ n \in \mathbf{N}\}$ is decreasing for any $x \in D$, hence converges in $\mathbf{R} \cup \{-\infty\}$. Let

$$\Omega_n = \{\sup_x f(x, u^0(x)) \le n\}$$

On Ω_n, $f(u^m) \le f(u^0) \le n$, hence $f_m(u^m) = f(u^m)$ on Ω_n, for any $m \ge n$. Then for any $m \ge n$, u^m is the unique solution on Ω_n to our equation

$$u + Kf(u) = Kg + K\dot W.$$

Consequently, $u^m = u^n$ on Ω_n, for $m \ge n$, and clearly $u^n \to u$ where u solves our equation.

Existence. Step 3. We now assume that f satisfies (2.2) and (2.3). Let

$$f_n(x, r) = f(x, r) \vee (-n).$$

We can proceed as in the proof of the second step, constructing this time an increasing sequence $\{u^n\}$, and defining $\Omega_n = \{\inf_x f(x, u^0(x)) \ge -n\}$. □

Lemma 2.6. *Let f and h both satisfy (2.2) and (2.3), and moreover :*

$$h(x, r) \le f(x, r),\ x \in D, r \in \mathbf{R}.$$

Let u and v be a.s. continuous random fields on $\bar D$, solutions of respectively :

$$\begin{aligned} u + Kf(u) &= Kg + K\dot W \\ v + Kh(v) &= Kg + K\dot W \end{aligned}$$

Then

$$u(x) \le v(x),\ x \in D.$$

Proof: From our assumptions,

$$u - v + K(f(u) - h(v)) = 0.$$

Consequently, $u - v \in H_0^1(D)$ and

$$-\Delta(u - v) + f(u) - h(v) = 0.$$

Multiplying the above identity by $(u - v)^+$ and integrating by parts, we obtain :

$$\|\nabla(u - v)^+\|^2 + (f(u) - h(v), (u - v)^+) = 0$$

But on the set $(u - v)^+ > 0$, $f(u) \ge f(v) \ge h(v)$. Hence $(u - v)^+ = 0$. □

Remark 2.7. *Note that the Theorem is still true if f, instead of being nondecreasing, satisfies :*

$$(f(x, r) - f(x, z))(r - z) \ge -\alpha|r - z|^2,\ \forall x \in D,\ r, z \in \mathbf{R}$$

provided $\alpha < a$, a being the constant appearing in Lemma 2.4, i.e. f could be the sum of an increasing function satisfying (2.3) and a Lipschitz function with a Lipschitz constant strictly smaller than a. □

3. Parabolic equations

We now want to study the equation:

(3.1)
$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(u)(t,x) = \dfrac{\partial^2 W}{\partial t \partial x}(t,x) + g(t,x);\ t \geq 0,\ 0 \leq x \leq 1 \\ u(0,x) = u_0(x),\ 0 \leq x \leq 1; u(t,0) = u(t,1) = 0 \end{cases}$$

where $\frac{\partial^2 W}{\partial t \partial x}$ denotes the "space–time white noise", $g \in L^2_{\text{loc}}(\mathbf{R}_+ \times (0,1))$, $u_0 \in C([0,1])$ vanishes at the two endpoints, and

$$f(u)(t,x) = f_1(t,x;u(t,x)) + f_2(t,x;u(t,x))$$

satisfies properties which we now state. First note that because of the forcing term g, we can and shall assume w.l.o.g. that

$$f_1(t,x;0) = f_2(t,x,0) = 0.$$

$f_1(t,x;r)$, $f_2(t,x;r)$ are measurable functions of $(t,x,r) \in \mathbf{R}_+ \times [0,1] \times \mathbf{R}$ satisfying :

(3.2)
$r \longrightarrow f_1(t,x;r)$ is continuous and non decreasing, for any $(t,x) \in \mathbf{R}_+ \times [0,1]$,

(3.3) $\quad f_1$ is locally bounded,

and there exists a constant c s.t. for any $(t,x,r,z) \in \mathbf{R}_+ \times [0,1] \times \mathbf{R} \times \mathbf{R}$,

(3.4)
$$|f_2(t,x;r) - f_2(t,x;z)| \leq c|r-z|.$$

We shall write $f_1(u)$, $f_2(u)$ for $f_1(\cdot,u(\cdot))$, $f_2(\cdot,u(\cdot))$.

It is shown in Walsh [9] that the two following formulations of equation (3.1) are equivalent (at least under the assumption that u – and then also $f(u)$ – is locally bounded) :

(3.5)
$$\begin{cases} \displaystyle\int_0^1 u(t,x)\phi(x)\,dx - \int_0^t\int_0^1 u(s,x)\frac{\partial^2\phi}{\partial x^2}(x)\,dxds + \int_0^t\int_0^1 f(u)(s,x)\phi(x)\,dxds \\ \displaystyle\quad = \int_0^1 u_0(x)\phi(x)\,dx + \int_0^t\int_0^1 \phi(x)\,dW_{sx} + \int_0^t\int_0^1 g(s,x)\phi(x)\,dxds, \\ \qquad\qquad \forall t \geq 0,\ \phi \in C^2([0,1]) \cap C_0([0,1]) \end{cases}$$

($C_0([0,1])$ denotes the set of continuous functions from $[0,1]$ into $\mathbf{R}$ which vanish at 0 and 1.)

(3.6)
$$\begin{cases} \displaystyle u(t,x) + \int_0^t\int_0^1 G_{t-s}(x,y)f(u)(s,y)\,dyds = \int_0^1 G_t(x,y)u_0(y)\,dy \\ \displaystyle + \int_0^t\int_0^1 G_{t-s}(x,y)\,dW_{sy} + \int_0^t\int_0^1 G_{t-s}(x,y)g(s,y)\,dtds,\ t \geq 0,\ 0 \leq x \leq 1 \end{cases}$$

where $(W_{tx};\ t \geq 0,\ 0 \leq x \leq 1)$ denotes the standard Brownian sheet, defined on a probability space $(\Omega, \mathcal{F}, P)$, and $G_t(x,y)$ is the fundamental solution of the heat equation with Dirichlet boundary conditions, i.e. for any $\varphi \in C_0([0,1])$,

$$v(t,x) = \int_0^1 G_t(x,y)\varphi(y)\,dy$$

is the unique solution of

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) - \dfrac{\partial^2 v}{\partial x^2}(t,x) = 0,\ t > 0,\ 0 < x < 1 \\ v(0,x) = \varphi(x),\ 0 \leq x \leq 1;\ v(t,0) = v(t,1) = 0,\ t \geq 0 \end{cases}$$

It is shown in Walsh [8] that the random field

$$\left\{\int_0^t\int_0^1 G_{t-s}(x,y)\,dW_{sy};\ t \geq 0;\ 0 \leq x \leq 1\right\}$$

possesses an a.s. continuous version.

We shall rewrite (3.6) as follows:

(3.7)
$$u + Gf(u) = Hu_0 + G\dot{W} + Gg$$

Note that for $\varphi \in L^2_{\text{loc}}(\mathbf{R}_+ \times (0,1))$, $v = G\varphi$ is the unique solution of :

(3.8)
$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) - \dfrac{\partial^2 v}{\partial x^2}(t,x) = \varphi(t,x),\ t > 0,\ 0 < x < 1 \\ v(0,x) = 0,\ v(t,0) = v(t,1) = 0 \end{cases}$$

Let $(\cdot,\cdot)$ and $|\cdot|$ denote the usual scalar product and norm on $L^2(0,1)$, $((\cdot,\cdot))_t$ and $\|\cdot\|_t$ the usual scalar product and norm on $L^2((0,t)\times(0,1))$, and $|||\cdot|||_t$ denote the norm on $C([0,t];L^2(0,1))$ defined by :

$$|||v|||_t^2 = \frac{1}{2}|v(t,\cdot)|^2 + \|v\|_t^2.$$

We shall make repeated use of the following:

Lemma 3.1. *For any* $\varphi \in L^2((0,t)\times(0,1))$,

$$((G\varphi,\varphi))_t \geq |||G\varphi|||_t^2$$

Proof : $G\varphi$ is the unique $v \in L^2(0,t;H_0^1(0,1))$ which solves (3.8). Multiplying (3.8) by $v(t,x)$ and integrating by parts, we obtain:

$$\frac{1}{2}|v(t,\cdot)|^2 + \|\frac{\partial v}{\partial x}\|_t^2 = ((\varphi,v))_t$$

But elementary inequalities yield:

$$u \in H_0^1(0,1) \Rightarrow |u| \leq \left|\frac{du}{dx}\right|$$

The result follows. □

We can now establish the

Theorem 3.2. *Under conditions (3.2), (3.3), and (3.4), equation (3.6) has a unique solution*

$$u \in C(\mathbb{R}_+ \times [0,1]) \text{ a.s.}$$

Proof : *Uniqueness* Let u and v be two solutions. Then the difference satisfies :

$$u - v + G[f(u) - f(v)] = 0$$

Multiplying by $f(u) - f(v)$, and using the monotonicity of f_1 and Lemma 3.1, we obtain :

$$((u - v, f_2(u) - f_2(v)))_t + |||u - v|||_t^2 \leq 0$$

from which we deduce, with the help of (3.3),

$$|u(t) - v(t)|^2 \leq k \int_0^t |u(s) - v(s)|^2 \, ds$$

The result follows from Gronwall's Lemma. As in the elliptic case, we note that the same proof shows that whenever u and v are two solutions of the equation on $[0,t] \times [0,1] \times \bar{\Omega}$, then $u = v$ a.e. on $[0,t] \times [0,1] \times \bar{\Omega}$.

Existence. Step 1. We suppose, in addition to the above assumptions, that f_1 is *bounded*. Let $\{W_{sy}^n,\, n \in \mathbb{N}\}$ denote a sequence of smooth random fields which are such that

$$\int_0^t \int_0^1 G_{t-s}(x,y)\, dW_{sy}^n \to \int_0^t \int_0^1 G_{t-s}(x,y)\, dW_{sy}$$

in $L^2(\Omega \times (0,1))$ for each $t > 0$ and in $L^2(\Omega \times (0,T) \times (0,1))$ for any $T > 0$, as $n \to \infty$. If we replace W by W^n, equation (3.1) has a.s. a unique solution u^n in $L^2_{\text{loc}}(\mathbb{R}_+; H^1(0,1))$, see Lions [4, Theorem 1.2, page 162], which satisfies in particular :

$$u^n + Gf(u^n) = Hu_0 + G\dot{W}^n + Gg. \tag{3.7$_n$}$$

Let us show that $\{u^n\}$ is a Cauchy sequence in $L^2(\Omega \times (0,t) \times (0,1))$. We clearly have :

$$((u^n - u^m, f(u^n) - f(u^m)))_t + ((G[f(u^n) - f(u^m)], f(u^n) - f(u^m)))_t$$
$$= ((G[\dot{W}^n - \dot{W}^m], f(u^n) - f(u^m)))_t$$

We deduce from Lemma 3.1, the monotonicity of f_1 and (3.4) :

$$|||u^n - u^m|||_t^2 \leq (u^n(t) - u^m(t), G[\dot{W}^n - \dot{W}^m](t))$$
$$+ ((G[\dot{W}^n - \dot{W}^m], f(u^n) - f(u^m) + 2(u^n - u^m)))_t$$
$$+ c\|u^n - u^m\|_t^2$$

From the assumption of the sequence $\{W^n\}$ and the boundedness of f_1,

$$\varepsilon(n,m) = \frac{1}{2} E(|G[\dot{W}^n - \dot{W}^m](t)|^2) + E\left(((f_1(u^n) - f_1(u^m), G[\dot{W}^n - \dot{W}^m]))_t\right)$$
$$+ E(\|G[\dot{W}^n - \dot{W}^m]\|_t^2)$$

tends to zero as $n, m \to \infty$. But from the above estimate and (3.3), there exists a constant c such that :

$$E\left(|u^n(t) - u^m(t)|^2\right) \leq c\left(\varepsilon(n,m) + \int_0^t E\left(|u^n(s) - u^m(s)|^2\right) ds\right)$$

Then from Gronwall's lemma,

$$E\left(|u^n(t) - u^m(t)|^2\right) \leq c\varepsilon(n,m)e^{ct}$$

We know have that there exists $u \in L^2(\Omega \times (0,t) \times (0,1))$ for all $t > 0$, such that $u^n \to u$ and $f(u^n) \to f(u)$ in $L^2(\Omega \times (0,t) \times (0,1))$. It remains to take the limit in (3.7_n). Note that it is shown in Walsh [9] that $u \in C(\mathbb{R}_+ \times [0,1])$ a.s.

Existence. Step 2 and 3. Those steps are completely analogous to the corresponding steps in the proof of Theorem 2.5, given the following Lemma. □

Lemma 3.3. *Let φ and ψ be measurable mappings : $\mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ which satisfy (3.2) and (3.3) and such that*

$$\varphi(t,x,r) \geq \psi(t,x,r) \quad a.e. \ on\ \mathbb{R}_+ \times [0,1] \times \mathbb{R}.$$

Let u and v belong to $C(\mathbb{R}_+ \times [0,1])$ a.s. and satisfy respectively :

$$u + G\varphi(u) + Gf_2(u) = Hu_0 + G\dot{W} + Gg$$

$$v + G\psi(v) + Gf_2(v) = Hu_0 + G\dot{W} + Gg$$

Then $\quad u(t,x) \leq v(t,x) \quad a.e.\ on\ \mathbb{R}_+ \times [0,1] \times \Omega.$

Proof : It follows from the assumptions that

$$u - v + G[\varphi(u) - \psi(v)] + G[f_2(u) - f_2(v)] = 0.$$

Then $u-v \in L^2(O,T;H_0^1(0,1)) \cap C([0,T];L^2(0,1))$ for any $T > 0$ and solves :

$$\begin{cases} \frac{\partial}{\partial t}(u-v) - \frac{\partial^2}{\partial x^2}(u-v) + \varphi(u) - \psi(v) + f_2(u) - f_2(v) = 0 \\ (u-v)(0,x) = 0,\ (u-v)(t,0) = (u-v)(t,1) = 0 \end{cases}$$

Multiplying this equation by $(u-v)^+$, we deduce that :

$$\frac{1}{2}\frac{d}{dt}\|(u-v)^+(t)\|^2 + \|\frac{\partial}{\partial x}(u-v)^+(t)\|^2 + (\varphi(u(t)) - \psi(v(t)), (u-v)^+(t)) + (f_2(u(t)) - f_2(v(t)), (u-v)^+(t)) = 0.$$

However, if $(u-v)^+ \neq 0$, $\varphi(u) \geq \varphi(v) \geq \psi(v)$. Hence the second and third terms in the above identity are non negative, and from the Lipschitz property of f_2,

$$\frac{d}{dt}\|(u-v)^+(t)\|^2 \leq 2c\|(u-v)^+(t)\|^2, \qquad (u-v)^+(0) = 0.$$

Hence $(u-v)^+(t) = 0$, $\forall t \geq 0$.

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Note added in proof. *In a recent preprint, Dembo and Zeitouni study an equation similar to our elliptic SPDE, under the assumption that the nonlinear perturbation f is Lipschitz with a Lipschitz constant smaller than the constant α of Lemma 2.4 (see Remark 2.7 above). In a forthcoming paper by I. Gyöngy and E. Pardoux, the above results will be generalized in the parabolic case.*

Bibliography

[1] L. Boccardo, T. Gallouët : Nonlinear elliptic and parabolic equations involving measure data, *J. of Funct. Anal.* **87**, 149–169, 1989.

[2] G. Benfatto, G. Gallavotti, F. Nicolò : Elliptic equations and Gaussian processes, *J. of Funct. Anal.* **36**, 343–400, 1980.

[3] D. Gilbarg, N.S. Trudinger : *Elliptic partial differential equations of second order*, Springer 1977.

[4] J.L. Lions : *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod 1969.

[5] D. Nualart, E. Pardoux : Second order stochastic differential equations with boundary conditions, *Preprint.*

[6] M. Röckner, B. Zegarlinski : The Dirichlet problem for quasi-linear partial differential operators with boundary data given by a distribution, *Bibos preprint* 188/86.

[7] Y. Rozanov : *Markov Random Fields*, Springer 1982.

[8] Y. Rozanov : Markov random fields and stochastic partial differential equations, *Mat. Sb.* **103**, 590–613, 1977.

[9] J. Walsh : An introduction to stochastic partial differential equations, in *Ecole d'été de Probabilités de St Flour XIV*, Lecture Notes in Math. **1180**, Springer 1986.

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