Lecture 1
Bayesian inference

olivier.francois@imag.fr

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Outline of Lecture 1

- Principles of Bayesian inference
- Classical inference problems (frequency, mean, variance)
- Basic simulation algorithms

What is Bayesian data analysis?

- Model building. Build a joint distribution for both observable quantities (data) and non-observable quantities (parameters).
- Parameter inference. Compute the conditional distributions of the non-observable quantities given the data.
- Model criticism and improvement. Evaluate the fit of the model to the data and check their predictions.

Model definition.

- Parameter $\theta = (\theta_1, \dots, \theta_J)$, $J \geq 1$.
- Data $y = (y_1, \dots, y_n)$, $n \geq 1$.
- A model is a joint distribution

$$p(y, \theta) = p(y|\theta)p(\theta)$$

- $p(\theta)$ is the prior distribution.
- $p(y|\theta)$ is the likelihood or sampling distribution.

Inference.

- Use the Bayes formula to compute the posterior distribution

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

where $p(y) = \int p(y|\theta)p(\theta)d\theta$ is the marginal distribution.

- The marginal distribution is usually a highly dimensional impossible to compute integral, and we write

$$p(\theta|y) \propto p(y|\theta)p(\theta).$$

Prediction.

- The posterior predictive distribution is

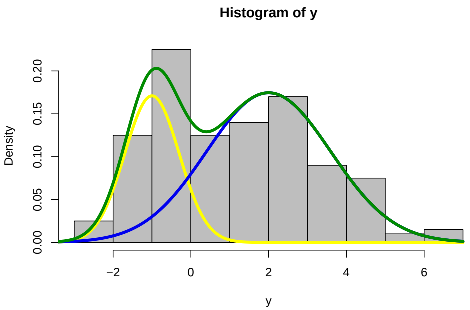
$$p(y_{\mathrm{rep}}|y) = \int p(y_{\mathrm{rep}}|\theta)p(\theta|y)d\theta.$$

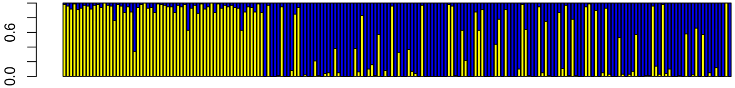
- Models are wrong, and the posterior predictive distribution can be used to evaluate aspects of the model that do not fit to the data (model checking).

Examples of application (in this course)

- Bayesian clustering: How many groups in the data?
- What are the within-group means and variances?
- For a given individual, what is the assignment probability?
- Population genetics: For an individual genome, what fraction of DNA can be assigned to putative source (ancestral) populations?

Mixture models





Example 1: Inferring allele frequencies

- Natural populations are of finite size, N.
- New genetic variants can arise from mutation or migration
- Genes frequencies at a bi-allelic locus (ancestral/derived allele) can fluctuate

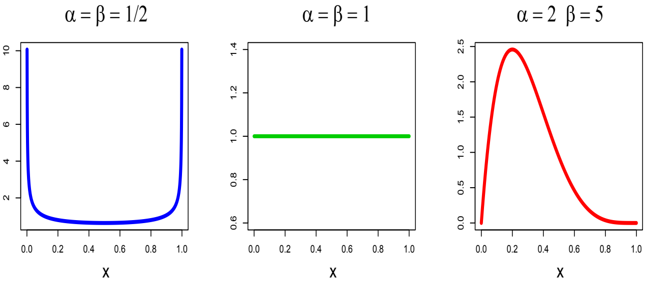
$$\frac{\#\{\text{carriers of the derived allele}\}}{N} \to \mathrm{beta}(\alpha, \beta)$$

where the beta distribution is

$$\mathrm{beta}(x, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,, \quad x \in (0, 1)$$

and α, $\beta > 0$ depend on mutation or migration rates.

Beta distribution



- Expectation and mode of the beta distribution

$$\mathrm{E}[X] = \frac{\alpha}{\alpha+\beta} \quad \mathrm{Mode}(X) = \frac{\alpha-1}{\alpha+\beta-2}$$

Model

- Prior distribution on the allele frequency: $\theta \sim$ beta(1,1) (uniform).
- Data: We observe the derived allele $y = 9$ times in a sample of size $n = 20$ genes (frequency = .45)
- Likelihood

$$p(y|\theta) = \mathrm{binom}(n, \theta)(y) \propto \theta^y (1-\theta)^{n-y}$$

- Posterior distribution (Exercise)

$$p(\theta|y) = \mathrm{beta}(y+1, n+1-y)(\theta)$$

Remarks

- Point estimate (conditional mean) different from the maximum likelihood estimate

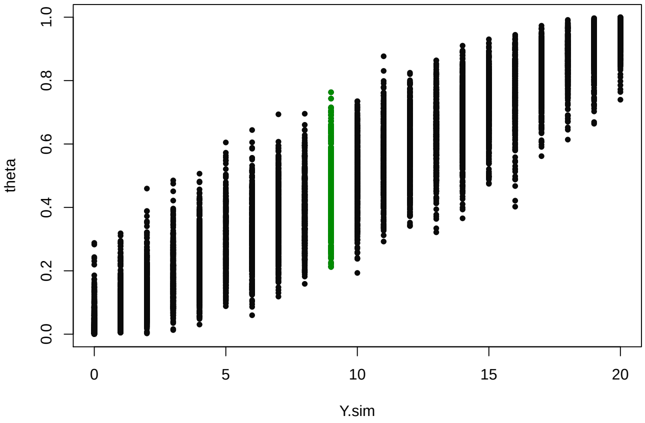
$$\mathrm{E}[\theta|y] = \frac{y+1}{n+2} \approx \frac{y}{n}, \quad \text{as } n \to \infty$$

- Credible interval I so that $\Pr(\theta \in I|y) = .95$ (R command `quantile`)

$$I = (0.25, 0.65)$$

  Not a confidence interval!

Joint distribution



Computing the posterior distribution from simulations

- Rejection algorithm

```
Repeat
theta <- unif(0,1)
y.s <- binom(n,theta)
Until (y.s == y)
return(theta)
```

- It generates samples from the posterior distribution $p(\theta|y)$ (Exercise).

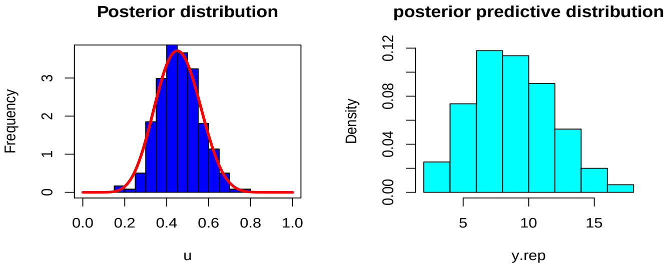
R scripts

- Rejection in R (sample of random size)

```
y = 9 ; n = 20
theta <- runif(10000)
y.s <- rbinom(10000, n, theta)
theta.post <- theta[ y.s == y ]
```

- Exercise: Compute a 95% credible interval for θ and a histogram of the posterior predictive distribution given y.

Is the rejection algorithm efficient?



- The acceptance rate is only $\approx 4.5\%$. It leaves room for improvement (Lecture 2).

Gaussian model: $\theta = m$ (σ^2 known)

- Case 1: One-dimensional data: $y \in \mathbb{R}$
- Prior distribution

$$p(\theta) \propto \exp\left(-\frac{1}{2\sigma_0^2}(\theta - m_0)^2\right), \quad \beta_0 = \frac{1}{\sigma_0^2}$$

- Sampling distribution

$$p(y|\theta) \propto \exp\left(-\frac{1}{2\sigma^2}(y - \theta)^2\right), \quad \beta = \frac{1}{\sigma^2}$$

- Posterior distribution (Exercise)

$$\theta|y \sim N(m_1, \sigma_1^2)$$

with $1/\sigma_1^2 = \beta_1 = \beta_0 + \beta$, and $m_1 = (\beta_0 m_0 + \beta y)/\beta_1$.

Gaussian model: $\theta = m$ (σ^2 known)

- Non-informative prior distribution

$$p(\theta) \propto 1\,, \quad \beta_0 \to 0\ (\sigma_0^2 = \infty)\,.$$

- Posterior distribution

$$\theta | y \sim N(y, \sigma^2)$$

- Exercise: Posterior predictive distribution

$$\tilde{y} | y \sim N(m_1, \sigma^2 + \sigma_1^2) = N(m_1, 2\sigma^2)$$

Gaussian model: $\theta = m$ (σ^2 known)

- Case 2: n data, $y = (y_1, \ldots, y_n)$
- Sampling distribution

$$p(y_1, \ldots, y_n|\theta) \propto \prod_{i=1}^{n} \exp\left(-\frac{1}{2\sigma^2}(y_i - \theta)^2\right),$$

- $\bar{y} = \sum_{i=1}^{n} y_i/n$ is sufficient

$$p(\theta|y) = p(\theta|\bar{y})$$

- Posterior distribution (Uninformative prior)

$$\theta|y \sim N(\bar{y}, \sigma^2/n).$$

Gaussian model: $\theta = \sigma^2$ (m known)

- χ^2_n distribution

$$p(x) \propto x^{n/2-1} e^{-x/2}\,, \quad x > 0$$

- $\mathrm{Inv}\chi^2(\nu, s^2)$ distribution: $X = \frac{\nu s^2}{\chi^2_\nu}$ (Exercise)

$$p(x) \propto \frac{1}{x^{\nu/2+1}} e^{-\frac{\nu s^2}{2x}}\,, \quad x > 0\,.$$

Gaussian model: $\theta = \sigma^2$ (m known)

- Prior distribution (not a density)

$$p(\theta) \propto \frac{1}{\theta}, \quad p(\log(\theta)) \propto 1 \,.$$

- Sampling distribution

$$p(y_1, \ldots, y_n | \theta) \propto \frac{1}{\theta^{n/2}} exp\left(-\frac{n}{2\theta} s_n^2\right)$$

where

$$s_n^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - m)^2$$

- Posterior distribution (Exercise)

$$\sigma^2 | y \sim \mathrm{Inv}\chi^2(n, s_n^2)$$

Joint inference $\theta = (m, \sigma^2)$

- Prior distribution (not a density)

$$p(m, \sigma^2) \propto \frac{1}{\sigma^2}\,.$$

- Posterior distribution

$$p(m, \sigma^2 | y) \propto \frac{1}{(\sigma^2)^{n/2+1}} \exp\left(-\frac{1}{2\sigma^2}((n-1)s_{n-1}^2 + n(\bar{y} - m)^2)\right)$$

where the unbiased empirical variance is

$$s_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

- The (marginal) posterior distribution of σ^2 is (exercise)

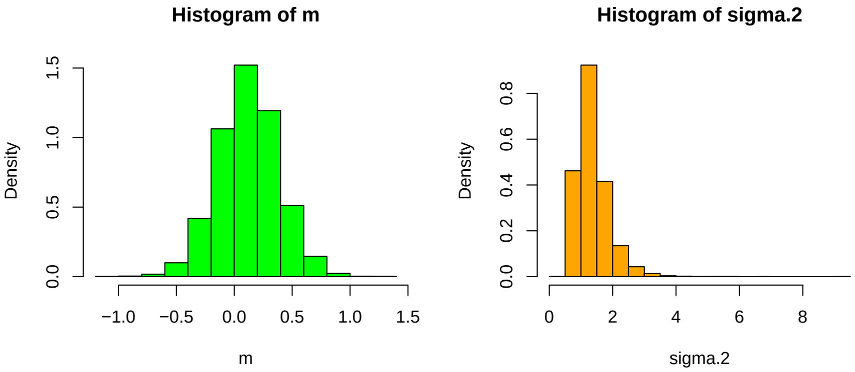
$$\sigma^2 | y \sim \mathrm{Inv}\chi^2(n-1, s_{n-1}^2)$$

Simulation $\theta = (m, \sigma^2)$

1. $\sigma^2 | y \sim (n-1)\mathrm{var}(y)/\chi^2_{n-1}$
2. $m | \sigma^2, y \sim N(\mathrm{mean}(y), \sigma^2/n)$

```
# simulated data
n = 20; y = rnorm(n)
# Posterior distribution sampling
sigma.2 = (n-1)*var(y)/rchisq(10000, n-1)
m = rnorm(10000, mean(y), sd = sqrt(sigma.2/n))
```

Posterior distribution (Gaussian model)



Model checking

- Our data are perhaps not from a Gaussian model

```
# Example
y = rcauchy(n)
sigma.2 = (n-1)*var(y)/rchisq(10000, n-1)
m = rnorm(10000, mean(y), sd = sqrt(sigma.2/n))
```

- Use a test statistic (skewness)

```
post.pred = NULL
for (i in 1:1000) {
ind = sample(10000, 1)
post.pred[i] = skewness(rnorm(20, m[ind],
sqrt(sigma.2[ind]))) }
hist(post.pred)
skewness(y)
```

Take-home messages

- Bayesian inference is about computing the conditional distribution of a parameter given the data.
- This can be achieved by using computational Monte Carlo methods
- More to come in lecture 2.

Exercises

Ex1. Find the posterior distribution in the beta-binomial model (answer: $\mathrm{beta}(y+\alpha, n+\beta-y)$).

Ex2. Prove the rejection algorithm.

Ex3. Compute the 95% credible interval for θ and the posterior predictive distribution given y from the rejection algorithm

Ex4. Simulate from the posterior distribution in the Gaussian model (two parameters). Use your own statistic for model checking.

Ex5. Run inference for the sepal length in `data(iris)`

Bibliography and resources

- Gelman A, Carlin JB, Stern HS, Rubin DB (2004) Bayesian Data Analysis 2nd ed. Chapman & Hall, New-York.
- E. Paradis (2005) R pour les débutants. Univ. Montpellier II.
- R website: `http://cran.r-project.org/`