Spread of epidemics on random graphs

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Course 2 : Convergence of a random individual-based model to Volz' equations

SIR on a configuration model graph

★**Configuration model (CM)** : Bollobas (80), Molloy Reed (95), Durett (07), van der Hofstad (in prep.)

★ Description of an SIR epidemics spreading on a configuration model graph :

- Infinite system of denumberable equations, Ball and Neal (2008)).
- 5 ODEs, Volz (2008), Miller (2011).
- Recently: Barbour Reinert (2014), Janson Luczak Winridge (2014)

★ Individuals are separated into 3 classes :

- Susceptibles $\mathcal{S}_t$
- Infectious $\mathcal{I}_t$
- Removed $\mathcal{R}_t$

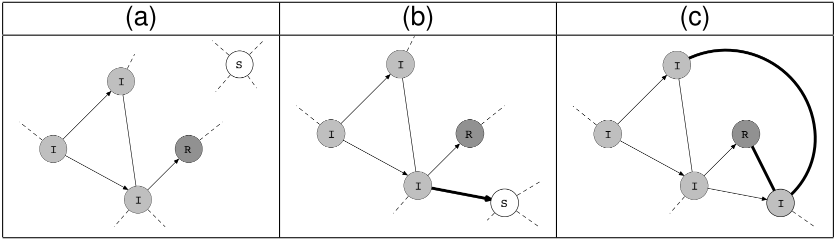
Stochastic model for a finite graph with N vertices

★ Only the edges between the $\mathcal{I}$ and $\mathcal{R}$ individuals are observed. The degree of each individual is known.

★ To each I individual is associated an exponential random clock with rate α to determine its removal.

★ To each open edge (directed to $\mathcal{S}$), we associate a random exponential clock with rate β.

★ When it rings, the edge of an $\mathcal{S}$ is chosen at random. We determine whether its remaining edges are linked with $\mathcal{S}$, $\mathcal{I}$ or $\mathcal{R}$-type individuals.



Edge-based quantities

★ The idea of Volz is to use network-centric quantities (such as the number of edges from $\mathcal{I}$ to $\mathcal{S}$) rather than node-centric quantities.

★ $\mathcal{S}_t$, $\mathcal{I}_t$, $\mathcal{R}_t$, S_t, I_t, R_t, d_i, $d_i(\mathcal{S}_t)$...
μ finite measure on $\mathbb{N}$ and f bounded or > 0 function:
$\langle \mu, f \rangle = \sum_{k\in\mathbb{N}} f(k)\mu(k)$.

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$$\mu_t^{\mathcal{S}}(dk) = \sum_{u\in\mathcal{S}_t} \delta_{d_u}(dk) \qquad \mu_t^{\mathcal{S}\mathcal{I}}(dk) = \sum_{u\in\mathcal{I}_t} \delta_{d_u(\mathcal{S}_t)}(dk)$$

$$\mu_t^{\mathcal{S}\mathcal{R}}(dk) = \sum_{u\in\mathcal{R}_t} \delta_{d_u(\mathcal{S}_t)}(dk)$$

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$$\mu_t^{\mathcal{S}\mathcal{R}}(dk) = \sum_{u\in\mathcal{R}_t} \delta_{d_u(\mathcal{S}_t)}(dk)$$

This sums up the evolution of the epidemic (but does not allow the reconstruction of the complicated graph on which the illness propagates).

$$I_t = \mathrm{Card}(\mathcal{I}_t) = \langle \mu_t^{\mathcal{S}\mathcal{I}}, 1\rangle, \quad N_t^{\mathcal{S}\mathcal{I}} = \langle \mu_t^{\mathcal{S}\mathcal{I}}, k\rangle = \sum_{u\in\mathcal{I}_t} d_u(\mathcal{S}_t)$$

Dynamics

★ Global force of infection: $\beta N_{t_-}^{\mathcal{SI}}$.

★ Choice of a given susceptible of degree k: $k/N_{t_-}^{\mathcal{S}}$.

So that the rate of infection of a given susceptible of degree k is: $\beta k p_{t_-}^{\mathcal{I}}$.

★ The probability that its $k-1$ remaining edges are linked to $\mathcal{I}$ or $\mathcal{R}$ is:

$$p(j,\ell,m|k-1,t) = \frac{\binom{N_{t_-}^{\mathcal{SI}}-1}{j}\binom{N_{t_-}^{\mathcal{SR}}}{\ell}\binom{N_{t_-}^{\mathcal{SS}}}{m}}{\binom{N_{t_-}^{\mathcal{S}}-1}{k-1}}\mathbf{1}_{j+\ell+m=k-1}\mathbf{1}_{j<N_{t_-}^{\mathcal{SI}}}\mathbf{1}_{\ell\leq N_{t_-}^{\mathcal{SR}}}$$

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★ To modify the degree distributions $\mu_{t_-}^{\mathcal{SI}}$ (idem for $\mu_{t_-}^{\mathcal{SR}}$):
We draw a sequence $e=(e_u)_{u\in\mathcal{I}_{t_-}}$ of integers.

- e_u is the number of edges to the infectious individual u at t_-.
- not all sequences are admissible.

The probability of drawing the sequence e is

$$\rho_U(e\,|\,j,\mu_{T_-}^{\mathcal{SI}}) = \frac{\prod_{u\in\mathcal{I}_{t_-}}\binom{d_u}{e_u}}{\binom{N_{t_-}^{\mathcal{SI}}}{j+1}}\mathbf{1}_{\{\sum e_u=j+1,\ \text{e is admissible}\}}.$$

Renormalization

★ We are interested in increasing the number of vertices N without rescaling the degree distribution.
$\mu^{N,\mathcal{S}}$, $\mu^{N,\mathcal{SI}}$, $\mu^{N,\mathcal{SR}}$.

★ We now consider $\mu^{(N),\mathcal{S}}$, $\mu^{(N),\mathcal{SI}}$ and $\mu^{(N),\mathcal{SR}}$ where for ex:

$$\mu_t^{(N),\mathcal{S}}(dk) = \frac{1}{N}\mu_t^{N,\mathcal{S}}(dk) \quad \text{with} \quad \lim_{N\to+\infty} \mu_0^{(N),\mathcal{S}} = \bar{\mu}_0^{\mathcal{S}} \text{ in } \mathcal{M}_F(\mathbb{N})$$

(idem for $\mu_0^{(N),\mathcal{SI}}$ with $\bar{N}_0^{\mathcal{SI}} > \varepsilon$ and $\mu_0^{(N),\mathcal{SR}}$ with $\bar{N}_0^{\mathcal{SR}} > \varepsilon$)

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★ 3 SDE:
$\langle \mu_t^{(N),\mathcal{SI}}, f\rangle = \langle \mu_0^{(N),\mathcal{SI}}, f\rangle + A_t^{(N),\mathcal{SI},f} + M_t^{(N),\mathcal{SI},f}$,

where $M^{(N),\mathcal{SI},f}$ is a square integrable martingale started from 0 and with previsible quadratic variation in 1/N.

★ $\underline{A_t^{(N),\mathcal{IS},f}:}$

$$A_t^{(N),\mathcal{IS},f} = -\int_0^t \alpha\langle \mu_s^{(N),\mathcal{SI}}, f\rangle ds$$
$$+ \int_0^t \sum_{k\in\mathbb{N}} \beta k p_s^{(N),\mathcal{I}} \mu_s^{(N),\mathcal{S}}(k) \sum_{j+\ell+1\leq k} p_s^N(j,\ell,m|k-1,t)$$
$$\times \sum_{e\in\mathcal{U}} \rho_U(e|j+1,\mu_s^{n,\mathcal{SI}})\Big(f(m) + \sum_{u\in\mathcal{I}_s^N} \big(f(d_u-e_u)-f(d_u)\big)\Big) ds,$$

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Th: Under appropriate moment conditions,

$(\mu_t^{(N),\mathcal{S}}, \mu_t^{(N),\mathcal{SI}}, \mu_t^{(N),\mathcal{SR}})_{t\in\mathbb{R}_+}$ converge to a deterministic limit $(\bar{\mu}_t^{\mathcal{S}}, \bar{\mu}_t^{\mathcal{SI}}, \bar{\mu}_t^{\mathcal{SR}})_{t\in\mathbb{R}_+}$

$$\langle \bar{\mu}_t^{\mathcal{SI}}, f\rangle = \langle \bar{\mu}_0^{\mathcal{SI}}, f\rangle - \int_0^t \alpha \langle \bar{\mu}_s^{\mathcal{SI}}, f\rangle ds$$
$$+ \int_0^t \sum_{k\in\mathbb{N}^*} \beta k \bar{p}_s^I \sum_{j+\ell+m=k-1} \left(\binom{j,\ \ell,\ m}{k-1} (\bar{p}_s^{\mathcal{I}})^j (\bar{p}_s^{\mathcal{R}})^\ell (\bar{p}_s^{\mathcal{S}})^m\right)$$
$$\times \Big(f(m) + (j+1)\sum_{k'\in\mathbb{N}^*} \big(f(k'-1) - f(k')\big) \frac{k'\bar{\mu}_s^{\mathcal{SI}}(k')}{\langle \bar{\mu}_s^{\mathcal{SI}}, k\rangle}\Big) \bar{\mu}_s^{\mathcal{S}}(k) ds$$

Deterministic limit

★ Limit equations:

$$\bar{\mu}_t^{\mathcal{S}}(k) = \bar{\mu}_0^{\mathcal{S}}(k)\theta_t^k, \qquad \theta_t = e^{-\beta \int_0^t \bar{p}_s^{\mathcal{I}} ds}$$

$$\langle \bar{\mu}_t^{\mathcal{SI}}, f \rangle = \ldots$$

$$\langle \bar{\mu}_t^{\mathcal{SR}}, f \rangle = \int_0^t \alpha \langle \bar{\mu}_s^{\mathcal{SI}}, f \rangle ds$$

$$+ \int_0^t \sum_{k \in \mathbb{N}} \beta k \bar{p}_s^{\mathcal{I}}(k-1)\bar{p}_s^{\mathcal{R}} \sum_{k' \in \mathbb{N}} \big(f(k'-1) - f(k')\big) \frac{k' \mu_s^{\mathcal{SR}}(k')}{\bar{N}_s^{\mathcal{SR}}} \bar{\mu}_s^{\mathcal{S}}(k) ds$$

★ This allows us to recover Volz'equations:

- Choosing $f \equiv 1$ gives $\bar{S}_t$, $\bar{I}_t$,
- Choosing $f(k) = k$ gives $\bar{N}^{\mathcal{S}}$, $\bar{N}^{\mathcal{SI}}$, $\bar{N}^{\mathcal{SR}}$,

from which we can deduce $\bar{p}^{\mathcal{I}} = \bar{N}^{\mathcal{SI}} / \bar{N}^{\mathcal{S}}$...

Volz'equations

Prop: let $g(z) = \sum_{k\in\mathbb{N}} \bar{\mu}_0^{\mathcal{S}}(k) z^k$ be the generating function of $\bar{\mu}_0^{\mathcal{S}}$.

$$\theta_t = \exp\Big(-\beta \int_0^t p_s^{\mathcal{I}}\, ds \Big)$$

$$\bar{S}_t = g(\theta_t), \qquad \bar{I}_t = \bar{I}_0 + \int_0^t \Big(\beta \bar{p}_s^{\mathcal{I}} \theta_s g'(\theta_s) - \alpha \bar{I}_s \Big) ds$$

$$\bar{p}_t^{\mathcal{I}} = \frac{\bar{N}_t^{\mathcal{SI}}}{\bar{N}_t^{\mathcal{S}}} = \bar{p}_0^{\mathcal{I}} + \int_0^t \Big(\beta\, \bar{p}_s^{\mathcal{I}} \bar{p}_s^{\mathcal{S}} \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} - \beta\, \bar{p}_s^{\mathcal{I}} (1 - \bar{p}_s^{\mathcal{I}}) - \alpha \bar{p}_s^{\mathcal{I}} \Big) ds.$$

$$\bar{p}_t^{\mathcal{S}} = \frac{\bar{N}_t^{\mathcal{SS}}}{\bar{N}_t^{\mathcal{S}}} = \bar{p}_0^{\mathcal{S}} + \int_0^t \beta \bar{p}_s^{\mathcal{I}} \bar{p}_s^{\mathcal{S}} \Big(1 - \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} \Big) ds.$$

□

Recall the limit for mixing models:

$$\frac{d\bar{S}_t}{dt} = -\beta\, \bar{S}_t\, \bar{I}_t, \qquad \frac{d\bar{I}_t}{dt} = \beta\, \bar{S}_t\, \bar{I}_t - \alpha \bar{I}_t.$$

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$$\bar{p}_t^{\mathcal{I}} = \frac{\bar{N}_t^{\mathcal{S}\mathcal{I}}}{\bar{N}_t^{\mathcal{S}}} = \bar{p}_0^{\mathcal{I}} + \int_0^t \Big(\beta\, \bar{p}_s^{\mathcal{I}} \bar{p}_s^{\mathcal{S}} \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} - \beta\, \bar{p}_s^{\mathcal{I}}(1 - \bar{p}_s^{\mathcal{I}}) - \alpha \bar{p}_s^{\mathcal{I}}\Big) ds.$$

$$\bar{p}_t^{\mathcal{S}} = \frac{\bar{N}_t^{\mathcal{S}\mathcal{S}}}{\bar{N}_t^{\mathcal{S}}} = \bar{p}_0^{\mathcal{S}} + \int_0^t \beta \bar{p}_s^{\mathcal{I}} \bar{p}_s^{\mathcal{S}} \Big(1 - \theta_s \frac{g''(\theta_s)}{g'(\theta_s)}\Big) ds.$$

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Here:

$$\frac{d\bar{S}_t}{dt} = g'(\theta_t)\dot{\theta}_t = -\beta g'(\theta_t)\theta_t \bar{p}_t^{\mathcal{I}} = -\beta \bar{N}_t^{\mathcal{S}} \bar{p}_t^{\mathcal{I}} = -\beta \bar{N}_t^{\mathcal{S}\mathcal{I}}.$$

Sketch of the proof

Assumption: $\sup_{N\in\mathbb{N}^*}\left(\langle\mu_0^{(N),\mathcal{S}},1+k^5\rangle+\langle\mu_0^{(N),\mathcal{SI}},1+k^5\rangle\right)<+\infty$,

★ Tightness: topology on $\mathcal{M}_F(\mathbb{N})$. Roelly's criterion.
Aldous-Rebolledo criterion.

$$\mathbb{P}\big(|A_{\tau_N}^{(N),\mathcal{SI},f}-A_{\sigma_N}^{(N),\mathcal{SI},f}|>\varepsilon\big)\leq\varepsilon$$
$$\mathbb{P}\big(|\langle M^{(N),\mathcal{SI},f}\rangle_{\tau_N}-\langle M^{(N),\mathcal{SI},f}\rangle_{\sigma_N}|>\varepsilon\big)\leq\varepsilon.$$

★ Convergence of the generators.

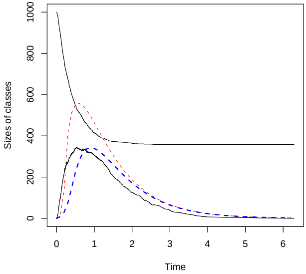
- The identification of the limit is **OK on** $[0,T]$ **IF** $T<\tau_\varepsilon^N$ where

$$\tau_\varepsilon^N=\inf\{t\geq 0,\ N_t^{(N),\mathcal{SI}}<\varepsilon\}.$$

★ Uniqueness:

- Gronwall's lemma gives that solutions of the limiting equation have same mass and same moments of order 1 and 2.
- Uniqueness of the generating function of $\bar{\mu}^{\mathcal{IS}}$ which solves a transport equation.

Degree distribution of the “initial condition”



Prop: For $\varepsilon > 0$, when $N \to +\infty$, the degree distribution when after $[\varepsilon N]$ infections converges to:

$$\frac{1}{1-\varepsilon} \sum_{k \geq 0} p_k (1 - z^{\varepsilon})^k \delta_k$$

where z^{ε} is the solution of $1 - \varepsilon = f(1 - z)$, f being the generating function of the original degree distribution.