The Sellke construction

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The standard epidemiological model

- Consider a SEIR model. We start with a few infected individuals and many susceptibles. At each time $t > 0$, $S(t)$ denotes the number of susceptibles in the population, while $I(t)$ denotes the number of infectious individuals.
- Each Infectious individual meets other persons from the population at rate c. For the encounter to result in an infection, we need that the encountered individual is susceptible.
- We assume that the population is "fully mixed", which means that when an infected individual meets someone, that person can be considered as being chosen at random uniformly in the population.

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The standard epidemiological model, cont'd

- Consequently, the probability that an encounter at time t is with a susceptible is $S(t)/N$. Moreover, the probability that an encounter between an infected and a susceptible results in a new infection is $p < 1$.
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Sellke's construction

- We number the individuals from 0 to N :

$$0\ 1\ 2\ 3\ \ldots\ N.$$

0 denotes the initially infected individual, and the individuals numbered from 1 to N are all susceptible at time 0.

- Let

$Q_1, Q_2, \ldots, Q_N$ be i.i.d. r.v.'s, with the law $\mathrm{Exp}(1)$; $(T_{1,0}, \Delta T_0), (T_{1,1}, \Delta T_1), \ldots, (T_{1,N}, \Delta T_N)$ i.i.d. r.v.'s, with the law $\mathbb{P}_L \otimes \mathbb{P}_I$, where $\mathbb{P}_L$ is the law of the latency period and $\mathbb{P}_I$ that of the infectious period.

- We define the cumulated force of infection experienced by an individual, between times 0 and t as

$$\Lambda_C(t) = \frac{cp}{N} \int_0^t I(s)ds.$$

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- We define the cumulated force of infection experienced by an individual, between times 0 and t as

$$\Lambda_C(t) = \frac{cp}{N} \int_0^t I(s)ds.$$

- For $i = 1, \ldots, N$, individual i is infected at the time when $\Lambda_C(t)$ achieves the value Q_i (which might be considered as the "level of resistance to infection of individual i").
- The epidemic stops when there is no more individual in either latent of infectious state. Then $\Lambda_C(t)$ does not grow any more, $\Lambda_C(t) = \Lambda_C(\infty)$.
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- We put the Q_i's in increasing order : $Q_{(1)} < Q_{(2)} < \cdots < Q_{(N)}$. It is the order in which individuals are infected in Sellke's model. Note that Sellke's model respects the durations of latency and infection. In order to show that Sellke's construction gives a process which has the same law as the process defined above, it remains to verify that the rates at which infections happen are the correct ones.
- At time t, $S(t)$ susceptibles have not yet been infected.
- Each of those corresponds to a $Q_i > \Lambda_C(t)$. At time t, the slope of the curve which represents the function $t \mapsto \Lambda_C(t)$ is $cpI(t)/N$.

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- If $Q_i > \Lambda_C(t) = x$, then

$$\mathbb{P}(Q_i > x + y | Q_i > x) = e^{-y}, \text{ hence}$$

$$\mathbb{P}(Q_i > \Lambda_C(t+s) | Q_i > \Lambda_C(t)) = \exp\left(-\frac{cp}{N}\int_t^{t+s} I(r)dr\right)$$

$$= \exp\left(-\frac{cp}{N} I(t)s\right),$$

  if I is constant on the interval $[t, t+s]$.

- Consequently, conditionally upon $Q_i > \Lambda_C(t)$,

$$Q_i - \Lambda_C(t) \sim \text{Exp}\left(\frac{cp}{N} I(t)\right).$$

- The same is true for those $S(t)$ Q_i which are $> \Lambda_C(t)$. Then the first Q_i to come is the minimum of those, hence the waiting time after $\Lambda_C(t)$ for the next infection follows the law $\text{Exp}\left(\frac{cp}{N} I(t)S(t)\right)$, if no removal of an infectious individual happens in the mean time.
- Then in Sellke's construction, at time t the next infection comes at the correct rate :

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