Law of Large Numbers

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The Model

- Suppose there is no latency period ($T_1 = 0$ a.s.) and that $\Delta T \sim \text{Exp}(\alpha)$. Consider a *SIR* model with constant population size equal to N. Let $S(t)$ denote the number of susceptibles at time t, $I(t)$ the number of infectious, $R(t)$ the number of "removed" (i.e. "healed and immune").
- Hence the following equations, with $P_1(t)$ and $P_2(t)$ two standard mutually independent Poisson processes :

$$S(t) = S(0) - P_1\left(\frac{\beta}{N}\int_0^t S(s)I(s)ds\right),$$

$$I(t) = I(0) + P_1\left(\frac{\beta}{N}\int_0^t S(s)I(s)ds\right) - P_2\left(\alpha\int_0^t I(s)ds\right).$$

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The proportions

- Let us define $s_N(t) = S(t)/N$, $i_N(t) = I(t)/N$. The equations for the proportions of susceptibles and infectious are written

$$s_N(t) = s_N(0) - \frac{1}{N} P_1\left(\beta N \int_0^t s_N(r) i_N(r) dr\right),$$

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- Define the two martingales $M_1(t) = P_1(t) - t$, $M_2(t) = P_2(t) - t$. We have

$$s_N(t) = s_N(0) - \beta \int_0^t s_N(r) i_N(r) dr - \frac{1}{N} M_1\left(\beta N \int_0^t s_N(r) i_N(r) dr\right),$$

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- Consider the process

$$\mathcal{M}_N(t) := \frac{1}{N} M_1 \left(\beta N \int_0^t s_N(r) i_N(r) dr \right).$$

Let $\mathcal{F}_t = \sigma\{s_N(r), i_N(r),\ 0 \le r \le t\}$.

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Lemma

$\{\mathcal{M}_N(t),\ t \ge 0\}$ is a $\mathcal{F}_t$–martingale which satisfies

$$\mathbb{E}\left[\mathcal{M}_N(t)\right] = 0, \quad \mathbb{E}\left[|\mathcal{M}_N(t)|^2\right] = \frac{\beta}{N} \mathbb{E} \int_0^t s_N(r) i_N(r) dr.$$

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Proof of the Lemma

- For the martingale property, see the Notes. From this, $\mathbb{E}[\mathcal{M}_N(t)] = 0$.
- Let $0 = t_0 < t_1 < \cdots < t_n = t$.

$$|\mathcal{M}_N(t)|^2 = \sum_{i=1}^{n} |\mathcal{M}_N(t_i) - \mathcal{M}_N(t_{i-1})|^2 + 2\sum_{1\le i<j\le n} [\mathcal{M}_N(t_i) - \mathcal{M}_N(t_{i-1})][\mathcal{M}_N(t_j) - \mathcal{M}_N(t_{j-1})]$$

$$\mathbb{E}\left(|\mathcal{M}_N(t)|^2\right)^2 = \sum_{i=1}^{n} \mathbb{E}\left(|\mathcal{M}_N(t_i) - \mathcal{M}_N(t_{i-1})|^2\right)$$

- When $|t_i - t_{i-1}| \to 0$,

$$\sum_{i=1}^{n} |\mathcal{M}_N(t_i) - \mathcal{M}_N(t_{i-1})|^2 \to \sum_{0\le r\le t} |\Delta\mathcal{M}_N(r)|^2 = N^{-2}P_1\left(\beta N \int_0^t s_N(r) i_N(r) dr\right)$$

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- Hence

$$\mathbb{E}\left[|\mathcal{M}_N(t)|^2\right] = N^{-2}\mathbb{E}P_1\left(\beta N \int_0^t s_N(r)i_N(r)dr\right)$$
$$= \frac{\beta}{N}\mathbb{E}\int_0^t s_N(r)i_N(r)dr.$$

- This implies that $\sup_{0\le t\le T}|\mathcal{M}_N(t)| \to 0$ in probability. In fcat this is true a.s.
- This follows from

$$\sup_{0\le t\le T}\left|\frac{P_1(Nt)}{N} - t\right| \to 0 \text{ a.s. as } N\to\infty.$$

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- We prove the above results in three steps. Let $P(t)$ be a rate λ Poisson process

$$n^{-1}P(n) = n^{-1}\sum_{i=1}^{n}[P(i) - P(i-1)]$$
$$\to \lambda \text{ a.s. as } n \to \infty.$$

- Second step

$$t^{-1}P(t) = t^{-1}P([t]) + t^{-1}(P(t) - P([t]))$$
$$\left|t^{-1}P(t) - \lambda\right| \leq \left|t^{-1}P([t]) - \lambda\right| + t^{-1}P([t]+1) - t^{-1}P([t]).$$

- We have just proved that $N^{-1}P(Nt) \to \lambda t$ a.s. for all $t > 0$. We have a sequence of increasing functions which converges to a continuous function, hence by the second Dini theorem, the convergence is uniform.

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- We have the Law of Large Numbers :

$$\sup_{0\le t\le T}\{|s_N(t)-s(t)|+|i_N(t)-i(t)|\}\to 0 \text{ a.s., where}$$

$(s(t),i(t))$ solves

$$\begin{cases} \dfrac{ds}{dt}(t)=-\beta s(t)i(t), \ \ t>0, \\ \dfrac{di}{dt}(t)=\beta s(t)i(t)-\alpha i(t), \ \ t>0. \end{cases}$$

- Define $X(t)=\begin{pmatrix} s(t)\\ i(t)\end{pmatrix}$, $X_N(t)=\begin{pmatrix} s_N(t)\\ i_N(t)\end{pmatrix}$, $\overline{X}_N(t)=X(t)-X_N(t)$, $Y_N(t)=\begin{pmatrix} \mathcal{M}_N(t)\\ \mathcal{N}_N(t)-\mathcal{M}_N(t)\end{pmatrix}$, and finally $F\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} -\beta xy\\ \beta xy-\alpha y\end{pmatrix}$. For $0\le x,y,x',y'\le 1$,

$$\left\|F\begin{pmatrix} x\\ y\end{pmatrix}-F\begin{pmatrix} x'\\ y'\end{pmatrix}\right\|\le C(\alpha,\beta)\left\|\begin{pmatrix} x\\ y\end{pmatrix}-\begin{pmatrix} x'\\ y'\end{pmatrix}\right\|.$$

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- We have

$$\overline{X}_N(t) = \overline{X}_N(0) + \int_0^t [F(X(r)) - F(X_N(r))]dr + Y_N(t).$$

- We have proved that $\sup_{0 \le t \le T} \|Y_N(t)\| \to 0$ a.s. as $N \to \infty$.
- Let $\varepsilon_N(t) = \sup_{0 \le r \le t} \|Y_N(r)\|$. We have

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