Central Limit Theorem

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- Recall $(s_N(t), i_N(t))$ and $(s(t), i(t))$. We write

$$s_N(t) = s(t) + \frac{1}{\sqrt{N}} U_N(t),$$
$$i_N(t) = i(t) + \frac{1}{\sqrt{N}} V_N(t).$$

- If we replace s_N, i_N by the above right–hand sides, exploit the $(s(t), i(t))$ equation to suppress the terms of order 1, and multiply the resulting SDEs by $\sqrt{N}$, we get

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\begin{aligned}
U_N(t) &= -\beta \int_0^t \left(s(r)V_N(r) + i(r)U_N(r) + \frac{U_N(r)V_N(r)}{\sqrt{N}} \right) dr \\
&- \frac{1}{\sqrt{N}} M_1 \left(\beta N \int_0^t \left(s(r)i(r) + \frac{s(r)V_N(r) + i(r)U_N(r)}{\sqrt{N}} + \frac{U_N(r)V_N(r)}{N} \right) dr \right), \\
V_N(t) &= \beta \int_0^t \left(s(r)V_N(r) + i(r)U_N(r) + \frac{U_N(r)V_N(r)}{\sqrt{N}} \right) dr \\
&+ \frac{1}{\sqrt{N}} M_1 \left(\beta N \int_0^t \left(s(r)i(r) + \frac{s(r)V_N(r) + i(r)U_N(r)}{\sqrt{N}} + \frac{U_N(r)V_N(r)}{N} \right) \right) \\
&- \alpha \int_0^t V_N(r) dr - \frac{1}{\sqrt{N}} M_2 \left(\alpha N \int_0^t \left(i(r) + \frac{V_N(r)}{\sqrt{N}} \right) dr \right).
\end{aligned}
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- Let $\mathcal{M}_1^N(t)$ and $\mathcal{M}_2^N(t)$ be the two martingales in the above equations.

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V_N(t) &= \beta \int_0^t \left(s(r)V_N(r) + i(r)U_N(r) + \frac{U_N(r)V_N(r)}{\sqrt{N}} \right) dr \\
&+ \frac{1}{\sqrt{N}} M_1\left(\beta N \int_0^t \left(s(r)i(r) + \frac{s(r)V_N(r) + i(r)U_N(r)}{\sqrt{N}} + \frac{U_N(r)V_N(r)}{N} \right)\right) \\
&- \alpha \int_0^t V_N(r)dr - \frac{1}{\sqrt{N}} M_2\left(\alpha N \int_0^t \left(i(r) + \frac{V_N(r)}{\sqrt{N}} \right) dr\right).
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- Let $\mathcal{M}_1^N(t)$ and $\mathcal{M}_2^N(t)$ be the two martingales in the above equations.

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$$[\mathcal{M}_2^N]_t = \sum_{0\le s\le t} |\Delta \mathcal{M}_2^N(s)|^2 = \frac{1}{N} P_2\left(\alpha N \int_0^t \left(i(r) + \frac{V_N(r)}{\sqrt{N}}\right) dr\right),$$

$$\langle \mathcal{M}_2^N\rangle_t = \alpha \int_0^t \left(i(r) + \frac{V_N(r)}{\sqrt{N}}\right) dr,$$

and similarly for $[\mathcal{M}_1^N]_t$, $\langle \mathcal{M}_1^N\rangle_t$.

- We have that $|\mathcal{M}_1^N(t)|^2 - \langle \mathcal{M}_1^N\rangle_t$ and $|\mathcal{M}_2^N(t)|^2 - \langle \mathcal{M}_2^N\rangle_t$ are martingales.
- It is plain that $|U_N(t)| \le 2\sqrt{N}$, $|V_N(t)| \le 2\sqrt{N}$. Hence

$$\mathbb{E}[(\mathcal{M}_1^N(t))^2] \le 9\beta t,\ \mathbb{E}[(\mathcal{M}_2^N(t))^2] \le 3\alpha t,$$
$$\mathbb{E}(|\mathcal{M}_1^N(t)|) \le 3\sqrt{\beta t},\ \mathbb{E}(|\mathcal{M}_2^N(t)|) \le \sqrt{3\alpha t}.$$

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- We deduce from this and the above equations that

$$\sup_{N\geq 1,\ 0\leq t\leq T} \mathbb{E}\left(|U_N(t)| + |V_N(t)|\right) \leq C_1(\alpha,\beta,T),$$
$$\sup_{N\geq 1,\ 0\leq t\leq T} \mathbb{E}\left(|U_N(t)|^2 + |V_N(t)|^2\right) \leq C_2(\alpha,\beta,T).$$

- Exploiting Doob's inequality, we deduce that for all $T>0$,

$$\sup_{N\geq 1} \mathbb{E}\left(\sup_{0\leq t\leq T}\left[|U_N(t)|^2 + |V_N(t)|^2\right]\right) < \infty.$$

- We want to take the limit in

$$U_N(t) = -\beta\int_0^t \left(s(r)V_N(r) + i(r)U_N(r) + \frac{U_N(r)V_N(r)}{\sqrt{N}}\right)dr - \mathcal{M}_1^N(t),$$
$$V_N(t) = \beta\int_0^t \left(s(r)V_N(r) + i(r)U_N(r) + \frac{U_N(r)V_N(r)}{\sqrt{N}}\right)dr$$
$$-\alpha\int_0^t V_N(r)dr + \mathcal{M}_1^N(t) - \mathcal{M}_2^N(t),$$

- We deduce from this and the above equations that

$$\sup_{N\geq 1,\ 0\leq t\leq T} \mathbb{E}\left(|U_N(t)|+|V_N(t)|\right) \leq C_1(\alpha,\beta,T),$$
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- where both $\mathcal{M}_1^N(t)$ and $\mathcal{M}_2^N(t)$ are of the form $N^{-1/2}M(Nt+\sqrt{N}t_N)$, where t_N's are random variables, with $N^{-1/2}\mathbb{E}[t_N]\to 0$ as $N\to\infty$.
- We want to show

Proposition

Under the above assumptions,

$$\left\{\frac{M(Nt+\sqrt{N}t_N)}{\sqrt{N}},\ t\ge 0\right\}\Rightarrow\{B(t),\ t\ge 0\},$$

where $B(t)$ is a standard Brownian motion.

- Let us first prove the result in case t_N is a deterministic sequence s.t. $N^{-1/2}t_N\to 0$ as $N\to\infty$.

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Proof of Proposition

- We have to prove two things. As $N \to \infty$,

$$(1)\quad N^{-1/2}M(Nt) \Rightarrow B(t),$$
$$(2)\quad N^{-1/2}[M(Nt+\sqrt{N}t_N) - M(Nt)] \to 0 \text{ in probability.}$$

- Proof of (1). It follows from the usual CLT. Indeed

$$\frac{M(Nt)}{\sqrt{[Nt]}} = \frac{1}{\sqrt{[Nt]}}\sum_{i=1}^{[Nt]}[M(i)-M(i-1)] + \frac{M(Nt)-M([Nt])}{\sqrt{[Nt]}},$$

- The r.v.'s $M(i) - M(i-1)$ are i.i.d. centered with variance 1, and the last term above converges in probability to 0 as $N \to \infty$, hence

$$\frac{M(Nt)}{\sqrt{[Nt]}} \Rightarrow \mathcal{N}(0,1),$$
$$\frac{M(Nt)}{\sqrt{N}} = \frac{\sqrt{[Nt]}}{\sqrt{N}} \times \frac{M(Nt)}{\sqrt{[Nt]}}$$
$$\Rightarrow B(t),$$

where $B(t) \simeq \mathcal{N}(0,t)$.

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Proof of (2)

$$
\begin{aligned}
\mathbb{P}\left(\left|\frac{M(Nt+\sqrt{N}t_N)-M(Nt)}{\sqrt{N}}\right|>\varepsilon\right)
&\le \frac{1}{N\varepsilon^2}\mathrm{Var}\left(M(Nt+\sqrt{N}t_N)-M(Nt)\right)\\
&= \frac{\sqrt{N}\,|t_N|}{N\varepsilon^2}\\
&\to 0,
\end{aligned}
$$

provided $N^{-1/2}t_N \to 0$ as $N\to\infty$.

The case t_N random

- Note that
$$\mathbb{P}\left(\frac{|t_N|}{\sqrt{N}} > \eta\right) \le \frac{1}{\eta}\frac{\mathbb{E}|t_N|}{\sqrt{N}}.$$
- We split the event
$$\left\{\frac{|M(Nt+\sqrt{N}t_N)-M(Nt)|}{\sqrt{N}} > \varepsilon\right\}$$
into three pieces, intersecting with the three events (which constitute a partition of Ω) $\left\{0 \le t_N \le \eta\sqrt{N}\right\}$, $\left\{-\eta\sqrt{N} \le t_N \le 0\right\}$ and $\left\{\frac{|t_N|}{\sqrt{N}} > \eta\right\}$.

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- The probability of the first event is dominated by

$$\mathbb{P}\left(\sup_{0\le s\le N\eta}\frac{|M(Nt+s)-M(Nt)|}{\sqrt{N}}>\varepsilon\right)$$
$$\le \frac{4}{N\varepsilon^2}\mathbb{E}\left(|M(N(t+\eta))-M(Nt)|^2\right)\le\frac{4\eta}{\varepsilon^2}.$$

- The probability of the second event is estimated analogously. As for the third event, its proabbility is dominated by $\frac{1}{\eta}\frac{\mathbb{E}|t_N|}{\sqrt{N}}$.
- It remains to choose $\eta=\varepsilon^3/8$ to deduce that

$$\limsup_N \mathbb{P}\left(\frac{|M(Nt+\sqrt{N}t_N)-M(Nt)|}{\sqrt{N}}>\varepsilon\right)\le\varepsilon.$$

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Weak convergence

- Moreover one can rather easily show that the sequence $\{(U_N(t), V_N(t)),\ t \geq 0\}$ is tight as a process whose trajectories belong to $C([0,+\infty);\mathbb{R}^2)$. Hence along a subsequence

$$\{(U_N(t), V_N(t)),\ t \geq 0\} \Rightarrow \{(U(t), V(t)),\ t \geq 0\},$$

- where the limit satisfies

$$U(t) = -\beta \int_0^t [s(r)V(r) + i(r)U(r)]\, dr + \sqrt{\beta} \int_0^t \sqrt{s(r)i(r)} dB_1(r),$$

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- The process $\{(U(t), V(t)),\ t \geq 0\}$ is a Gaussian process of the Ornstein–Uhlenbeck type.
  The law of the limit is uniquely determined. Hence the whole sequence converges.

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