Large Deviations

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LLN, CLT and LD

- Let us go back to the two last lectures. We assume N is large. $(s_N(t), i_N(t))$ is close to the solution $(s(t), i(t))$ of the ODE

$$\begin{cases} \frac{ds}{dt}(t) = -\beta s(t)i(t), \ t > 0, \\ \frac{di}{dt}(t) = \beta s(t)i(t) - \alpha i(t), \ t > 0. \end{cases}$$

- OK, but even if N is large, $N < \infty$. The CLT gives us an indication about the fluctuations, i.e. it tells us that for fixed t, $s(t) - s_N(t)$ is $N^{-1/2}$ times a Gaussian r.v., which means that it is small with a very high probability.
- Now the theory of Large Deviations, more precisely the Wentzell–Freidlin theory of "small perturbations of dynamical systems" tells us that after a long time, the small random perturbations that the process $(s_N(t), i_N(t))$ suffers, might produce a "large deviation" from its LLN limit $(s(t), i(t))$.

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- In which cases does such a phenomenon produce an important divergence from the the behaviour of $(s(t), i(t))$?
- Consider an ODE epidemiological model. Suppose the ODE has several equilibria, each being locally stable and having its basin of attraction. In the two examples to be considered below, there are 2 locally stable equilibria, one endemic equilibrium and a disease free equilibrium.
- Starting from a point close to the endemic equilibrium, the deterministic model tells us that the system remains for ever in the vicinity of the endemic equilibrium, i.e. the deterministic model predicts that the illness will continue for ever.
- The Wentzell–Freidlin theory tells us that soon or later the solution of the SDE will escape the basin of attraction of the endemic equilibrium, and reach the disease free equilibrium. However, what means "soon or later" ? 1 month, 1 year, 1 century ? More ?

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Two models with several equilibria

- We consider both the *SIV* model studied by Kribs–Zaleta and Velasco–Hernández :

$$\frac{ds}{dt}(t) = \mu(1 - s(t)) + \alpha i(t) - \beta s(t)i(t) - \eta s(t) + \theta v(t), \;\; t > 0,$$
$$\frac{di}{dt}(t) = -\mu i(t) + \beta s(t)i(t) - \alpha i(t) + r\beta v(t)i(t), \;\; t > 0,$$
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Main result

- In those two above models, one can choose the parameters in such a way that both the DFE and one of the endemic equilibria are locally stable. Denote by $\mathcal{O}$ the basing of attraction of the endemic equilibrium. Let us denote by $\tau^{N,x}$ the time it takes for the stochastic system, starting from $x \in \mathcal{O}$, to exit $\mathcal{O}$ ($\simeq$ the time to reach the DFE).
- Our main result says

Theorem

For any $x \in \mathcal{O}$, $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}(e^{(\overline{V}-\delta)N} < \tau^{N,x} < e^{(\overline{V}+\delta)N}) = 1.$$

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The case of the SIV model

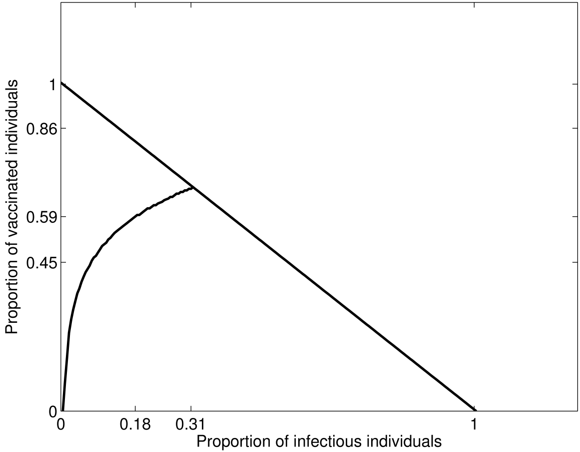

Figure: The characteristic boundary separating the domains of attractions of $\bar{x} = (0, 0.86)^\top$ (on the left) and $x^* = (0.31, 0.45)^\top$ (on the right). The unstable equilibrium $\tilde{x} = (0.18, 0.59)^\top$ is on the characteristic boundary.

The model

- We write the stochastic model as ($h_j \in \mathbb{R}^d$)

$$Z^N(t) = x + \frac{1}{N}\sum_{j=1}^{k} h_j P_j\left(N\int_0^t \beta_j(Z^N(s))ds\right),$$

- and the LLN model as

$$\frac{dY(t)}{dt} = b(Y(t)),$$

where $b(x) = \sum_{j=1}^k h_j\beta_j(x)$.

- Note that both $Z^N(t)$ and $Y(t)$ take their values in the set $A = \{x \in \mathbb{R}^d,\ x_i \geq 0, 1 \leq i \leq d,\ \sum_{i=1}^d x_i \leq 1\}$.

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The rate functional 1

For any $\phi \in \mathcal{AC}_{T,A}$, let $\mathcal{A}_d(\phi)$ the set of vector valued Borel measurable functions μ such that for all $1 \le j \le k$, $\mu_t^j \ge 0$ and

$$\frac{d\phi_t}{dt} = \sum_{j=1}^{k} \mu_t^j h_j, \; t \text{ a.e.}$$

We define the action function

$$I_T(\phi) = \begin{cases} \inf_{\mu \in \mathcal{A}_d(\phi)} I_T(\phi|\mu), & \text{if } \phi \in \mathcal{AC}_{T,A},\ \mathcal{A}_d(\phi) \neq \emptyset, \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$I_T(\phi|\mu) = \int_0^T \sum_{j=1}^{k} f(\mu_t^j, \beta_j(\phi_t))dt,$$

with $f(\nu, \omega) = \nu \log(\nu/\omega) - \nu + \omega$, where we use the convention $\log(\nu/0) = +\infty$ for $\nu > 0$, while $0 \log(0/0) = 0 \log(0) = 0$.

The rate functional 2

- Another equivalent definition is

$$I_T(\phi) = \int_0^T L(\phi_t, \phi_t')dt$$

where for all $z \in A$, $y \in \mathbb{R}^d$, $L(x, y) = \sup_{\theta \in \mathbb{R}^d} \ell(z, y, \theta)$, with

$$\ell(z, y, \theta) = \langle \theta, y \rangle - \sum_{j=1}^{k} \beta_j(z) \left(e^{\langle \theta, h_j \rangle} - 1 \right).$$

- Recall the definition

Definition

A rate function I_T is a semi–continuous mapping $I : D_{T,A} \to [0, \infty]$ (i.e. its level sets $\Psi_I(\alpha) = \{\phi, \; I_T(\phi) \leq \alpha\}$ are closed subsets of $D_{T,A}$). A good rate function is a rate function whose level sets are compact.

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The rate functional 3

- Our I_T above is a good rate function.
- We notice that $I_T(\phi) = 0$ iff

$$\frac{d\phi_t}{dt} = b(\phi_t).$$

- $I_T(\phi)$ could be thought of as an energy which is necessary in order to drive the function ϕ away from being a solution to the ODE.

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Girsanov's theorem 1

- Let Q denote the number of jumps of the of Z^N in the interval $[0, T]$, τ_p be the time of the p–th jump, and define

$$\delta_p(j) = \begin{cases} 1 & \text{, if the } p\text{–th jump is in the direction } h_j, \\ 0 & \text{, otherwise.} \end{cases}$$

  We shall denote $\mathcal{F}_t^N = \sigma\{Z_s^N,\ 0 \le s \le t\}$.

- Consider another set of rates $\widetilde{\beta}_j(z)$, $1 \le j \le k$. Let $\mathbb{P}^N$ denote the law of Z^N when the rates are β_j, $\widetilde{\mathbb{P}}^N$ the law of Z^N when the rates are $\widetilde{\beta}_j$

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Girsanov's theorem 2

- We have

Theorem

Assume that $\{x,\ \widetilde{\beta}_j(x)=0\}\subset\{s,\ \beta_j(x)=0\}$. *Then* $\mathbb{P}^N|_{\mathcal{F}_T^N} << \widetilde{\mathbb{P}}^N|_{\mathcal{F}_T^N}$, *and*

$$\Delta_T^N=\frac{d\mathbb{P}^N|_{\mathcal{F}_T^N}}{d\widetilde{\mathbb{P}}^N|_{\mathcal{F}_T^N}}$$

$$=\left(\prod_{p=1}^{Q}\prod_{j=1}^{k}\left[\frac{\beta_j(Z^N(\tau_p^-))}{\widetilde{\beta}_j(Z^N(\tau_p^-))}\right]^{\delta_p(j)}\right)\exp\left(N\sum_{j=1}^{k}\int_0^T[\tilde{\beta}_j(Z^N(t))-\beta_j(Z^N(t))]dt\right).$$

- Note the similarity with $f(\nu,\omega)=\nu\log(\nu/\omega)-\nu+\omega$.

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$$= \left(\prod_{p=1}^{Q} \prod_{j=1}^{k} \left[\frac{\beta_j(Z^N(\tau_p^-))}{\widetilde{\beta}_j(Z^N(\tau_p^-))} \right]^{\delta_p(j)} \right) \exp\left(N \sum_{j=1}^{k} \int_0^T [\tilde{\beta}_j(Z^N(t)) - \beta_j(Z^N(t))] dt \right).$$

- Note the similarity with $f(\nu, \omega) = \nu \log(\nu/\omega) - \nu + \omega$.

Large Deviations

- The theory of Large Deviations is well developed for small Brownian perturbation of dynamical systems. There is much less litterature in the case of Poissonian SDEs. In particular the Wentzell–Freidlin theory is new in this context.
- One specific difficulty is that some of the rates vanish on the boundary (in order to prevent the process from exiting the set where the proportions in each compartment is non negative, and the sum of the proportions in some of the compartments is less than 1). Since the logarithm of the rates appear in the first definition of the rate function I_T, this is clearly a problem.

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The general result

- We want to prove is that for any G open subset of $D_{T,A}$,

$$\liminf_N \frac{1}{N} \log \mathbb{P}(Z^N \in G) \geq - \inf_{\phi \in G} I_T(\phi),$$

and for any F closed subset of $D_{T,A}$,

$$\limsup_N \frac{1}{N} \log \mathbb{P}(Z^N \in F) \leq - \inf_{\phi \in F} I_T(\phi).$$

- One of the difficulties is the fact that some of the rates vanish on the boundary, since otherwise the process would leave the set A. But the log of those rates appear in the first expression for the rate function !
- The way out of this is to approximate an arbitrary trajectory in A by a function which remains at distance at leats $a > 0$ from the boundary. One then have to show that with a small, the error made in the rate function is small.

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Lower Bound

- For the lower bound, it is sufficient to consider an open set G of the form $G = \{\psi,\ \|\psi - \phi\|_T < \delta\}$, for a given ϕ, and it suffices to consider ϕ absolutely continuous.
- We first approximate ϕ by a function which stays at distance a from the boundary, then by a piecewise linear function.
- Finally we use a Girsanov transformation which expresses the Radon Nikodym derivatives of the law of Z^N w.r.t. to that of a process with rates μ^j which are such that

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Upper Bound 1

- For $\phi \in D_{T,A}$, $H \subset D_{T,A}$ let $\rho_T(\phi, H) = \inf_{\psi \in H} \|\phi - \psi\|_T$. Now define $\Phi(s) = \{\phi \in D_{T,A} : I_T(\phi) \le s\}$ and for all $\delta, s > 0$,

$$H_\delta(s) = \{\phi \in D_{T,A} : \rho_T(\phi, \Phi(s)) \ge \delta\}.$$

- Essentially all we have to do is to prove that for any δ, η, $s > 0$ there exists $N_0 \in \mathbf{N}$ such that

$$(*) \quad \mathbb{P}^N(H_\delta(s)) \le \exp\{-N(s - \eta)\},$$

whenever $N \ge N_0$.

- Indeed, let $F \subset D_{T,A}$ be a closed set, choose $\eta > 0$ and put $s = \inf\{I_T(\phi) : \phi \in F\} - \eta/2$. The closed set F does not intersect the compact set $\Phi(s)$. Therefore $\delta = \inf_{\phi \in F} \inf_{\psi \in \Phi(s)} \|\phi - \psi\|_T > 0$.

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Upper Bound 2

- From $(*)$, for any $\delta, \eta, s > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$,

$$\begin{aligned}\mathbb{P}^N(F) &\le \mathbb{P}^N(H_\delta(s)) \\ &\le \exp\{-N(s - \eta/2)\} \\ &\le \exp\{-N(\inf_{\phi \in F} I_T(\phi) - \eta)\}\end{aligned}$$

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- For $y, z \in A$, we define the following functionals.

$$V(x,z,T) := \inf_{\phi \in D([0,T];A), \phi(0)=x, \phi(T)=z} I_{T,x}(\phi)$$

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Time of exit from a domain

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Main Assumptions

- We assume that For all $x \in O$, $Y^x(t) \in O$, and $Y^x(t) \to x^*$, that $\bar{V} < \infty$.
- We first assume that $Y^x(t) \to x^*$ if $x \in \partial\widetilde{O}$. This is not satisfied in our case, so we need an appoximation.
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$$\tau^{N,x} := \inf\{t > 0 | Z^{N,x}(t) \notin O\},$$
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Moreover there exists $T_1 < \infty$ and $N_1 > 0$ such that for all $N > N_1$,

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- For $T := T_0 + T_1$ and $N > N_0 \vee N_1 \vee 1/\eta$, we hence obtain

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Application to our two examples

- In the SIV model with $\beta = 3.6$, $\alpha = 1$, $\theta = 0.02$, $\mu = 0.03$, $\eta = 0.3$ and $r = 0.1$, we get $\overline{V} = 0.39$.
  This gives rather astronomical values of τ^N, even for $N = 100$!
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