Singular Homogenization with stationary in time and periodic in space coefficients

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Abstract

We study the homogenization problem for a random parabolic operator with coefficients rapidly oscillating in both the space and time variables and with a large highly oscillating nonlinear potential, in a general stationary and mixing random media, which is periodic in space. It is shown that a solution of the corresponding Cauchy problem converges in law to a solution of a limit stochastic PDE.

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1 Introduction

We study a homogenization problem for a parabolic reaction diffusion equation with a rapidly oscillating nonlinear potential, of the form

$$\begin{cases} \dfrac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\left[a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha}\Big)\nabla u^\varepsilon(x,t)\right] + \dfrac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}g\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha},u^\varepsilon(x,t)\Big) \\ \qquad\qquad + h\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha},u^\varepsilon(x,t)\Big), \qquad t>0,\ \ x\in\mathbb{R}^n \\ u^\varepsilon(0,x) = u_0(x), \quad x\in\mathbb{R}^n. \end{cases}$$

We assume that the microscopic structure is periodic in space and that the dynamics of the system is stationary and uniformly mixing. Our methods mix probabilistic arguments together with some PDE and SPDE techniques.

The same type of equation was treated in [14] and [5], under the assumptions that the coefficients depend on chance through a finite dimensional ergodic Markov process. Some of the techniques used there do not longer apply in the more general case considered here. Results similar to those of the present paper, but for a linear parabolic PDE, were obtained in [4].

In the same way as in [14], the hypothesis of centering for the nonlinear term g allows us to decompose it into the sum of the spatial average of g over the torus $\mathbf{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, denoted $\bar g$, and a process $\tilde g$ of zero spatial average on the torus. The unbounded term $\frac{1}{\varepsilon}\bar g$ requires the construction of a corrector $\bar G$ of a new type and this is related to the semigroup of conditioned shifts and its associated full generator (see for details [6], chapter 2, section 7 and in particular Lemma 2.7.5). In order to define $\bar G$ in a rigorous way and to derive some useful estimates for $\bar G$ and its first and second derivatives with respect to the solution, we impose an appropriate integrability condition for the uniform mixing coefficient. One of the key technical steps of this work is to obtain a rule of differentiation for the process $\bar G(t,u^\varepsilon(t))$, where u^ε is the unique solution of our family of equations.

Our main result consists in proving that the limiting law of the solutions of the studied family of equations in a certain function space is the solution of a martingale problem, in the case $\alpha \leq 2$, and a Dirac measure concentrated at the solution of the Cauchy problem for some deterministic parabolic equation with constant coefficients, if $\alpha > 2$. In order to prove the uniqueness of the martingale problem, when $\alpha \leq 2$, we need to construct a Lipchitzian square root of the limiting diffusion operator, while the driving Brownian motion takes values in some properly chosen Hilbert space.

While the proof of tightness is the same in the three cases which we consider, the correctors which are needed in order to take the limit are different in the three situations. When $\alpha < 2$, the time scale is slower than the natural diffusive scale, and some of the correctors are solutions of elliptic equations where time is "frozen". In that case, essentially only the "stochastic" part $\bar{g}$ of the potential g has a real contribution to the limiting covariance operator. When $\alpha > 2$, the time scale is faster than the diffusive one, and the correctors solve elliptic PDEs with averaged in time coefficients. In this case, essentially only the $\tilde{g}$ part of g remains in the limit. Finally, in the situation $\alpha = 2$, the correctors are stationnary solutions of parabolic equations. Both $\bar{g}$ and $\tilde{g}$ appear in the limiting equation.

The paper is organized as follows. The assumptions are stated in section 2. The statements of our three results are stated in section 3. Tightness of the collection $\{u^\varepsilon,\ 0 < \varepsilon \leq 1\}$ is established in section 4. Finally the convergence is proved in section 5, in the three cases $\alpha = 2$, $\alpha < 2$ and finally $\alpha > 2$.

2 Set up and assumptions

We investigate the limiting behaviour of a solution to the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}u^\varepsilon(x,t) = \operatorname{div}\Big[a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha}\Big)\nabla u^\varepsilon(x,t)\Big] + \dfrac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}g\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha},u^\varepsilon(x,t)\Big) \\ \qquad\qquad + h\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^\alpha},u^\varepsilon(x,t)\Big), \quad x\in\mathbb{R}^n, t>0; \\ u^\varepsilon(x,0) = u_0(x), \quad x\in\mathbb{R}^n \end{cases} \tag{1}$$

as $\varepsilon \searrow 0$. We assume that $u_0 \in L^2(\mathbb{R}^n)$. The assumptions on the coefficients of the equation (1) are as follows.

- **(A.1)** (Periodicity). All the coefficients $a_{ij}(z,s)$, $g(z,s,u)$ and $h(z,s,u)$ are periodic in z with period 1 in each coordinate direction.
- **(A.2)** (Randomness). For each $u\in\mathbb{R}$ the coefficients $a_{ij}(s,\cdot)$, $g(\cdot,s,u)$ and $h(\cdot,s,u)$ are stationary random processes with values in $C(\mathbf{T}^n)$, defined on a probability space $(\Omega,\mathcal{F},\mathbf{P})$.
- **(A.3)** (Smoothness and growth conditions). Uniformly in $s\in\mathbb{R}$, $z\in\mathbf{T}^n$ and $\omega\in\Omega$ the following bounds hold

$$\begin{cases} |a(z,s)| \le C, \\ |g(z,s,u)| \le C|u|, \qquad |g'_u(z,s,u)| \le C, \\ (1+|u|)|g''_{uu}(z,s,u)| \le C, \qquad |h(z,s,u)| \le C(1+|u|), \\ |h(z,s,u_1)-h(z,s,u_2)| \le C|u_1-u_2|, \end{cases} \tag{2}$$

  for any $u,u_1,u_2\in\mathbb{R}$; here and afterwards C stands for a generic positive nonrandom constant.
- **(A.4)** (Uniform ellipticity). For some $c>0$,

$$a_{ij}(z,s)\eta_i\eta_j \ge c|\eta|^2, \quad \forall\eta\in\mathbb{R}^n.$$

- **(A.5)** (Centering condition). We assume that

$$\mathbf{E}\int_{\mathbf{T}^n} g(z,s,u)dz = 0, \quad \forall u \in \mathbb{R}.$$

- **(A.6)** (Mixing condition). Let $\phi(t)$ be the uniform mixing coefficient defined by

$$\phi(t) := \sup |P(A|B) - P(A)|,$$

where the supremum is taken over all $A \in \mathcal{F}_0$ and $B \in \mathcal{F}^t$, and $\mathcal{F}_s$ and $\mathcal{F}^t$ denote

$$\mathcal{F}_s := \sigma\{a_{ij}(z,r), g(z,r,u), h(z,r,u) \mid r \le s\}$$

and

$$\mathcal{F}^t := \sigma\{a_{ij}(z,r), g(z,r,u), h(z,r,u) \mid r \ge t\}.$$

We assume that

$$\int_0^\infty \phi(t)dt < \infty.$$

The filtration of σ-algebras $\{\mathcal{F}_s\}$ is supposed to be right continuous.

- **(A.7)** The partial derivative $\frac{\partial a}{\partial s}(z,s)$ a.s. belongs to $L^p_{\rm loc}((\mathbf{T}^n \times (-\infty,+\infty))$, for some $p > n$.
- **(A. 8)**

$$|\nabla_z g(z,s,u)| \le C|u|, \qquad |\nabla_z a(z,s)| \le C$$

uniformly in z, s, u.

3 Statements of the main results

We study problem (1) on a time interval $(0,T)$, where $T > 0$ is an arbitrary fixed number. Clearly, under the assumptions **(A.3), (A.4)** this problem has a unique solution u^ε, which is an element of the space

$$V_T = L^2(0,T;H^1(\mathbb{R}^n)) \cap \mathcal{C}([0,T];L^2(\mathbb{R}^n)).$$

Denote by $\tilde{V_T}$ the space V_T endowed with the sup of the weak topology of the space $L^2(0,T;H^1(\mathbb{R}^n))$ and the strong topology of the space $\mathcal{C}([0,T];L^2_w(\mathbb{R}^n))$, where the index w indicates that the corresponding space is equipped with its weak topology. Denote by Q^ε the law of u^ε on the space $\tilde{V}_T$.

For brevity, for a generic function $f(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t,x))$ or $f(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t,x))$ we use the notation $f^\varepsilon(t)$. Also we denote $a^\varepsilon(t):=a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha})$ and $A^\varepsilon u^\varepsilon(t):=\operatorname{div}[a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha})\nabla u^\varepsilon(x,t)]$. Let $A:=A^1$.
It is convenient to decompose $g(z,s,u)$ as follows

$$g(z,s,u)=\bar{g}(s,u)+\tilde{g}(z,s,u),$$

with

$$\bar{g}(s,u):=\int_{\mathbf{T}^n} g(z,s,u)dz.$$

The asymptotic behaviour of the solution $u^\varepsilon(t)$, as $\varepsilon\to 0$, depends crucially on whether $\alpha<2, \alpha=2$ or $\alpha>2$.

3.1 The case $\alpha=2$

We first introduce two correctors. To this end, we define (see Lemma 3 below) χ and $\tilde{G}$ as stationary solutions of the following PDEs with random coefficients:

$$\frac{\partial}{\partial s}\chi_i(z,s)+\operatorname{div}\left[a(z,s)\nabla\chi_i(z,s)\right]=-\frac{\partial}{\partial z_k}a_{ik}(z,s),\quad (z,s)\in\mathbf{T}^n\times\mathbb{R}^1 \tag{3}$$

and

$$\frac{\partial}{\partial s}\tilde{G}(z,s,u)+\operatorname{div}\left[a(z,s)\nabla\tilde{G}(z,s,u)\right]=-\tilde{g}(z,s,u),(z,s)\in\mathbf{T}^n\times\mathbb{R}^1; \tag{4}$$

here $u\in\mathbb{R}$ is a parameter. Consider also the process $\bar{G}(t,u)$, defined as

$$\bar{G}(t,u)=\int_0^\infty \mathbf{E}[\bar{g}(s+t,u)|\mathcal{F}_t]ds=\int_t^\infty \mathbf{E}[\bar{g}(s,u)|\mathcal{F}_t]ds, \tag{5}$$

for $t\geq 0$ and $u\in\mathbb{R}$. Notice that $\bar{G}(t,u)$ is a stationary process for each $u\in R$.

Theorem 1 *Let $\alpha=2$. Under the assumptions* **A1**-**A6**, *for all $T>0$ the solutions $\{u^\varepsilon\}$ of problem (1) converges in law,* *as $\varepsilon \to 0$, in the space $\tilde{V}_T$, towards the unique solution of the martingale problem with drift $\hat{A}(u(t))$ and covariance operator $R(u(t))$, where*

$$\hat{A}(u) := \mathrm{div}(\bar{\mathbf{a}}\nabla u) - \mathrm{div}\mathbf{F}(u) + \mathbf{H}(u), \tag{6}$$

$$\bar{\mathbf{a}} := \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z \chi(z,s))dz,$$

$$\mathbf{F}(u) := \mathbf{E}\int_{\mathbf{T^n}} \big(a(z,s)\nabla_z \tilde{G}(z,s,u) + g(z,s,u)\chi(z,s)\big)\,dz,$$

$$\mathbf{H}(u) := \mathbf{E}\int_{\mathbf{T^n}} \big(g(z,s,u)(\tilde{G}'_u(z,s,u) + \bar{G}'_u(s,u)) + h(z,s,u)\big)\,dz,$$

and

$$\begin{aligned}(R(u)\varphi,\varphi) &:= 2\mathbf{E}[(\bar{G}(s,u(\cdot))\varphi)(\bar{g}(s,u(\cdot)),\varphi)] \\ &= 2\mathbf{E}\int_{R^n}\int_{R^n}(\bar{G}(s,u(x))\varphi(x)\bar{g}(s,u(y))\varphi(y)dxdy.\end{aligned}$$

3.2 The case $\alpha < 2$

Theorem 2 *Let $\alpha < 2$. Under the assumptions* **A.1**-**A.7**, *for all $T > 0$ u^ε converges in law, as $\varepsilon \to 0$, in the space $\tilde{V}_T$, to the unique solution of the martingale problem with drift $\tilde{A}(u(s))$ and covariance operator $R(u(t))$, where $R(u)$ has been defined in the preceding statement, and*

$$\tilde{A}(u) := \mathrm{div}(\hat{\mathbf{a}}\nabla u) + \hat{\mathbf{g}}(u),$$

$$\hat{\mathbf{a}} := \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z \chi^-(z,s))dz,$$

$$\hat{\mathbf{g}}(u) := \mathbf{E}\int_{\mathbf{T^n}} \big(\bar{G}'_u(s,u)g(z,s,u) + h(z,s,u)\big)\,dz;$$

here $\chi_i^-(z,s)$, $1 \le i \le n$, stands for a solution of elliptic equation

$$A\chi_i^-(z,s) = -\frac{\partial}{\partial z_k}a_{ki}(z,s),$$

which satisfies $\int_{\mathbf{T}^n}\chi^-(z,s)dz = 0$ for each $s \ge 0$, s being a parameter.

Theorem 3 *Let $\alpha > 2$, then under the assumptions* **A.1-A.6** *and* **A.8**, *u^ε converges in probability, in the space $\tilde{V}_T$, as $\varepsilon \to 0$, to a solution of the following Cauchy problem :*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \mathrm{div}(\tilde{\mathbf{a}}\nabla u(t,x)) + \tilde{\mathbf{h}}(u); \qquad (t,x) \in (0,T) \times \mathbb{R}^n; \\ u^\varepsilon(0,x) = u_0(x) \end{cases}$$

where

$$\tilde{\mathbf{a}} = \mathbf{E}\int_{\mathbf{T^n}} a(z,s)(I + \nabla_z \chi^+(z))dz,$$

$$\tilde{\mathbf{h}}(u) = \mathbf{E}\int_{\mathbf{T^n}} [g(z,t,u)\partial_u \tilde{G}^+(z,u) + h(z,s,u)]dz,$$

and the functions $\chi^+(z)$ and $\tilde{G}^+(z,u)$ are defined as solutions of the elliptic equations

$$\begin{aligned} \bar{A}\chi^+(z) &= -\mathrm{div}\bar{a}_i(z), \\ \bar{A}\tilde{G}^+(z,u) &= -\overline{(\tilde{g})}(z,u), \end{aligned} \tag{7}$$

where $\bar{A}$ stands for the operator A "averaged in time", i.e.

$$\bar{A}(u)(z) = \mathrm{div}(\bar{a}(\cdot)\nabla u(\cdot))(z),$$

with $\bar{a}(z) := \mathbf{E}a(z,s)$ and $\overline{(\tilde{g})}(z,u) := \mathbf{E}\tilde{g}(z,s,u) = \mathbf{E}g(z,s,u)$.

4 Auxiliary results, a priori estimates and tightness

Our first aim is to show that the family $\{Q^\varepsilon\}_{\varepsilon>0}$ of the laws of u^ε, is tight in $\tilde{V}_T$. Since

$$g(z,t,u) = \tilde{g}(z,t,u) + \bar{g}(t,u),$$

where

$$\bar{g}(t,u) = \int_{\mathbf{T}^n} g(z,t,u)dz,$$

and consequently

$$\int_{\mathbf{T}^n} \tilde{g}(z,t,u)dz = 0, \forall t \in [0,T], u \in \mathbb{R},$$

we may construct a z-periodic vector function $\tilde{H}$ such that

$$\mathrm{div}_z \tilde{H}(z,t,u) = \tilde{g}(z,t,u). \tag{8}$$

Indeed, the centering condition on $\tilde{g}$ allows us to solve on the torus $\mathbf{T}^n$ the equation

$$\Delta_z v = \tilde{g}$$

and thus we can choose

$$\tilde{H} := \nabla v$$

Under our assumptions, $\tilde{H}$ satisfies the estimates

$$|\tilde{H}(z,t,u)| \le C|u|, \qquad |\tilde{H}'_u(z,t,u)| \le C, \tag{9}$$

for any $(z,t,u) \in \mathbf{T}^n \times [0,T] \times \mathbb{R}$. From (8) it follows that

$$\begin{aligned} \mathrm{div}_x \left[\tilde{H}(\frac{x}{\varepsilon}, t, u^\varepsilon(t,x)) \right] &= \frac{1}{\varepsilon} \tilde{g}(\frac{x}{\varepsilon}, t, u^\varepsilon(t,x)) \\ &\quad + \tilde{H}'_u(\frac{x}{\varepsilon}, t, u^\varepsilon(t,x)) \nabla u^\varepsilon(t,x) \end{aligned} \tag{10}$$

We thus get a useful representation for the term $\frac{1}{\varepsilon}\tilde{g}$. In order to get rid of the big term $\frac{1}{\varepsilon}\bar{g}$, we use the process $\bar{G}(t,u)$ defined in (5). Notice that by the assumption **(A.5)**, we have $\mathbf{E}[\bar{g}(t,u)] = 0$, for all $t \ge 0$ and $u \in \mathbb{R}$. Then it follows from Proposition 7.2.6. in [6] and from **(A.3)** and **(A.6)** that

$$\mathbf{E}[\bar{g}(s+t,u)|\mathcal{F}_t] \le 2C|u|\phi(s).$$

We next deduce from **(A.3), (A.6)** that $\bar{G}(t,u)$ is well defined. and satisfies the estimates

$$|\bar{G}(t,u)| \le C|u|, \qquad |\bar{G}'_u(t,u)| \le C, \qquad |\bar{G}''_{uu}(t,u)| \le \frac{C}{1+|u|} \tag{11}$$

It is easy to see that the process $\bar{G}(t,u)$ is stationary.

Lemma 1 *For each $u \in \mathbb{R}$, the process $M_t := \bar{G}(t,u) + \int_0^t \bar{g}(s,u)ds$ is a martingale with respect to $\{\mathcal{F}_t\}$.*

Proof. This statement is a consequence of proposition 2.7.6. in [6]. All we need to check is that the family $\{\frac{1}{\delta}\mathbf{E}[\bar{G}(t+\delta,u)-\bar{G}(t,u)|\mathcal{F}_t], \delta>0, t\geq 0\}$ is uniformly integrable, and that

$$\mathbf{P}-\lim_{\delta\searrow 0}\frac{1}{\delta}\mathbf{E}[\bar{G}(t+\delta,u)-\bar{G}(t,u)|\mathcal{F}_t]=-\bar{g}(t,u), \qquad \text{for a.e. } t.$$

By the relation (5) we have

$$\frac{1}{\delta}\mathbf{E}[\bar{G}(t+\delta,u)-\bar{G}(t,u)|\mathcal{F}_t]=-\frac{1}{\delta}\int_t^{t+\delta}\mathbf{E}[\bar{g}(s,u)|\mathcal{F}_t]ds.$$

The integrand is uniformly bounded, for fixed u, and continuous with respect to s, for any t. We thus deduce the a.s. convergence of the sequence $\frac{1}{\delta}\mathbf{E}[\bar{G}(t+\delta,u)-\bar{G}(t,u)|\mathcal{F}_t]$, when $\delta\searrow 0$, for any t. The uniform integrability follows from the uniform boundedness of $\bar{g}(\cdot,u)$. □

The rule of differentiation of the expression $(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),u^\varepsilon(t))$, where $u^\varepsilon(t)$ is the solution of problem (1), is given by the following

Lemma 2 *For any test function $\varphi\in\mathcal{C}_0^\infty(R^n)$ and $\varepsilon>0$, the processes $M_t^{u,\varepsilon}$ and $M_t^{\varphi,\varepsilon}$ given by*

$$\begin{aligned}
M_t^{u,\varepsilon} &= \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}\big[\big(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),u^\varepsilon(t)\big)-\big(\bar{G}(0,u_0),u_0\big)\big]\\
&+\frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}}\int_0^t\big(\bar{g}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),u^\varepsilon(s)\big)ds\\
&+\varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}\int_0^t\big(a^\varepsilon(s)\nabla u^\varepsilon(s),\bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla u^\varepsilon(s)u^\varepsilon(s)\\
&+2\bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\nabla u^\varepsilon(s)\big)ds\\
&-\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t\big(g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))u^\varepsilon(s)+\bar{G}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\big)ds\\
&-\varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}\int_0^t\big(h(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\bar{G}'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))u^\varepsilon(s)+\bar{G}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))\big)ds
\end{aligned} \tag{12}$$

and

$$
\begin{aligned}
M_t^{\varphi,\varepsilon} :=& \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})}[(\bar{G}(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi) - (\bar{G}(0, u^\varepsilon(0)), \varphi)] \\
&+ \frac{1}{\varepsilon^{(1\wedge\frac{\alpha}{2})}} \int_0^t (\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \varphi) ds \\
&+ \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})} \int_0^t (a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)\varphi + \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla\varphi) ds \\
&- \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})} \int_0^t (g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\varphi) ds \\
&- \varepsilon^{\alpha-(1\wedge\frac{\alpha}{2})} \int_0^t (h(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\varphi) ds
\end{aligned} \tag{13}
$$

are martingales with respect to the filtration $\{\mathcal{F}_{\frac{t}{\varepsilon^\alpha}}, t \geq 0\}$.

Proof Let $[t_1, t_2]$ be an arbitrary subinterval of $[0, T]$, $\Delta = \{s_0, s_1, ..., s_m\}$ a deterministic partition of the interval $[t_1, t_2]$, and denote $h = \max_{1\leq k\leq m}\{(s_k - s_{k-1})\}$. Considering the progressive measurability of all the random functions involved, we have, for each $k = 1, \ldots, m$,

$$
\begin{aligned}
&\mathbf{E}\big[\big(\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s_k)), u^\varepsilon(s_k)\big) - \big(\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})), u^\varepsilon(s_{k-1})\big)|\mathcal{F}_{\frac{s_{k-1}}{\varepsilon^\alpha}}\big] \\
&=\mathbf{E}\big[\int_{s_{k-1}}^{s_k} \frac{d}{ds}\big(\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s)\big) ds|\mathcal{F}_{\frac{s_{k-1}}{\varepsilon^\alpha}}\big] \\
&=\mathbf{E}\big\{\int_{s_{k-1}}^{s_k} \big[\langle\bar{G}'_u(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))\frac{\partial u^\varepsilon}{\partial s}(s), u^\varepsilon(s)\rangle \\
&\quad + \langle\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s)), \frac{\partial u^\varepsilon}{\partial s}(s)\rangle\big] ds|\mathcal{F}_{\frac{s_{k-1}}{\varepsilon^\alpha}}\big\} \\
&=\mathbf{E}\big\{\int_{s_{k-1}}^{s_k} \big[-\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)u^\varepsilon(s) \\
&\quad + 2\bar{G}'_u(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)\big) \\
&\quad + \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\big(g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))\big) \\
&\quad + \big(h(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s))\big)\big] ds|\mathcal{F}_{\frac{s_{k-1}}{\varepsilon^\alpha}}\big\}
\end{aligned}
$$

and

$$\mathbf{E}\big[\big(\bar{G}(\frac{s_k}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})), u^\varepsilon(s_{k-1})\big) - \big(\bar{G}(\frac{s_{k-1}}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})), u^\varepsilon(s_{k-1})\big)|\mathcal{F}_{\frac{s_{k-1}}{\varepsilon^\alpha}}\big]$$
$$= -\frac{1}{\varepsilon^\alpha}\mathbf{E}\big[\int_{s_{k-1}}^{s_k}\big(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})), u^\varepsilon(s_{k-1})\big)ds|\mathcal{F}_{\frac{s_{k-1}}{\varepsilon^\alpha}}\big];$$

here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^1(\mathbb{R}^n)$ and $H^{-1}(\mathbb{R}^n)$. Summing up over k gives

$$\mathbf{E}\big\{\big[\big(\bar{G}(\frac{t_2}{\varepsilon^\alpha}, u^\varepsilon(t_2)), u^\varepsilon(t_2)\big) - \big(\bar{G}(\frac{t_1}{\varepsilon^\alpha}, u^\varepsilon(t_1)), u^\varepsilon(t_1)\big)$$
$$+ \int_{t_1}^{t_2}\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''_{uu}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)u^\varepsilon(s)$$
$$+ 2\bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\nabla u^\varepsilon(s)\big)ds$$
$$- \frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\int_{t_1}^{t_2}\big(g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\big)ds$$
$$- \int_{t_1}^{t_2}\big(h(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))u^\varepsilon(s) + \bar{G}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\big)ds$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon^\alpha}\int_{t_1}^{t_2}\big(\bar g(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),u^\varepsilon(s)\big)ds\big]|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\}\\
=&\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\big(-a^\varepsilon(s)\nabla u^\varepsilon(s),\nabla u^\varepsilon(s)u^\varepsilon(s)\\
&\mathbf{E}[\bar G''_{uu}(\frac{s_k}{\varepsilon^\alpha},u^\varepsilon(s))-\bar G''_{uu}(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))|\mathcal{F}_{\frac{s}{\varepsilon^\alpha}}]\big)ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}\\
&+\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}-2\big(a^\varepsilon(s)\nabla u^\varepsilon(s),\nabla u^\varepsilon(s)\\
&\mathbf{E}[\bar G'_u(\frac{s_k}{\varepsilon^\alpha},u^\varepsilon(s))-\bar G'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))|\mathcal{F}_{\frac{s}{\varepsilon^\alpha}}]\big)ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}\\
&+\frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\big(g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),u^\varepsilon(s)\\
&\mathbf{E}[[\bar G'_u(\frac{s_k}{\varepsilon^\alpha},u^\varepsilon(s))-\bar G'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))|\mathcal{F}_{\frac{s}{\varepsilon^\alpha}}]\big)ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}\\
&+\frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\big(g(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\mathbf{E}[\bar G(\frac{s_k}{\varepsilon^\alpha},u^\varepsilon(s)) \qquad (14)\\
&-\bar G(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))|\mathcal{F}_{\frac{s}{\varepsilon^\alpha}}]\big)ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}\\
&+\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\big(h(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),u^\varepsilon(s)\mathbf{E}[\bar G'_u(\frac{s_k}{\varepsilon^\alpha},u^\varepsilon(s))\\
&-\bar G'_u(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))|\mathcal{F}_{\frac{s}{\varepsilon^\alpha}}]\big)ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}\\
&+\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\big(h(\frac{\cdot}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),\mathbf{E}[\bar G(\frac{s_k}{\varepsilon^\alpha},u^\varepsilon(s))\\
&-\bar G(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s))|\mathcal{F}_{\frac{s}{\varepsilon^\alpha}}]\big)ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}\\
&-\frac{1}{\varepsilon^\alpha}\mathbf{E}\Big\{\sum_{k=1}^m\int_{s_{k-1}}^{s_k}\big[\big(\bar g(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s_{k-1})),u^\varepsilon(s_{k-1})\big)\\
&-\big(\bar g(\frac{s}{\varepsilon^\alpha},u^\varepsilon(s)),u^\varepsilon(s)\big)\big]ds|\mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\}
\end{aligned}
$$

For any $\delta>0$, using the bounds (2) and the definition (5) we derive the following estimates, for each $x\in\mathbb{R}^n$

$$
\begin{aligned}
&\mathbf{E}\big[|\bar G(t+\delta,u^\varepsilon(t,x))-\bar G(t,u^\varepsilon(t,x))|/\mathcal{F}_t\big]\le C\delta|u^\varepsilon(t,x)|,\\
&\mathbf{E}\big[|\bar G'_u(t+\delta,u^\varepsilon(t,x))-\bar G'_u(t,u^\varepsilon(t,x))|/\mathcal{F}_t\big]\le C\delta,\\
&\mathbf{E}\big[|\bar G''_{uu}(t+\delta,u^\varepsilon(t,x))-\bar G''_{uu}(t,u^\varepsilon(t,x))|/\mathcal{F}_t\big]\le C\delta\frac{1}{1+|u^\varepsilon(t,x)|}.
\end{aligned}
\qquad (15)
$$

The last term on the right hand side of (14) can be estimated as follows

$$
\begin{aligned}
&|\big(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})), u^\varepsilon(s_{k-1})\big) - \big(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s)\big)| \\
&\le \|\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})) - \bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\|\|u^\varepsilon(s_{k-1})\| \\
&+ \|\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\|\|u^\varepsilon(s_{k-1}) - u^\varepsilon(s)\| \\
&\le (1+C)\sup_{0\le t\le T}\|u^\varepsilon(t)\|\|u^\varepsilon(s_{k-1}) - u^\varepsilon(s)\|.
\end{aligned}
$$

Writing down the energy estimate for equation (1) and using Gronwall's lemma, we get, for each $\varepsilon > 0$

$$
\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla u^\varepsilon(t)\|^2 dt \le \beta_\varepsilon, \tag{16}
$$

where β_ε is a deterministic constant which depends on u_0, T and the ellipticity constants and satisfies, for all $\varepsilon_0 > 0$, the inequality $\sup\limits_{\varepsilon_0\le\varepsilon\le 1} \beta_\varepsilon < \infty$. Since the solution $u^\varepsilon(\cdot)$ is continuous on $[0,T]$ with values in $L^2(\mathbb{R}^n)$, the Lebesgue dominated convergence theorem yields

$$
\begin{aligned}
\lim_{h\searrow 0}\mathbf{E}\Big\{\sum_{k=1}^m \int_{s_{k-1}}^{s_k} \Big[&\big(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s_{k-1})), u^\varepsilon(s_{k-1})\big) \\
&- \big(\bar{g}(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), u^\varepsilon(s)\big)\Big] ds | \mathcal{F}_{\frac{t_1}{\varepsilon^\alpha}}\Big\} = 0.
\end{aligned}
$$

Using estimates (15) and a Jensen type inequality for conditional expectations, one can show that the expectation of the absolute value of all other terms on the r.h.s. of (14) is not greater than:

$$
hC\mathbf{E}\big[T\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2 + \int_0^T \|\nabla u^\varepsilon(s)\|^2 ds\big].
$$

This expression is finite, for each $\varepsilon > 0$, in view of (16). Passing to the limit as $h \searrow 0$, we obtain the first statement of the lemma. The second one can be proved in a similar way. □

We now proceed with a priori estimates.

Proposition 1 *The following bounds hold*

$$\mathbf{E}\left(\sup_{t\le T}\|u^\varepsilon(t)\|^2+\int_0^T\|\nabla u^\varepsilon(s)\|^2ds\right)\le C,$$
$$\mathbf{E}\left[\sup_{t\le T}\|u^\varepsilon(t)\|^4+\left(\int_0^T\|\nabla u^\varepsilon(s)\|^2ds\right)^2\right]\le C. \tag{17}$$

uniformly in $\varepsilon>0$.

Proof Denote $\bar G^\varepsilon(t)=\bar G\big(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)\big)$ and $\rho:=\alpha-(1\wedge\frac{\alpha}{2})$. By formula (12), considering (10) we get, after integration by parts:

$$\begin{aligned}
&d\left[\frac12\big(u^\varepsilon(t),u^\varepsilon(t)\big)+\varepsilon^\rho\big(\bar G^\varepsilon(t),u^\varepsilon(t)\big)\right]\\
&=-\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla u^\varepsilon(t)\big)dt+\big(h^\varepsilon(t),u^\varepsilon(t)\big)dt-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde H^\varepsilon(t),\nabla u^\varepsilon(t)\big)dt\\
&\quad-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde H'^{,\varepsilon}_u\nabla u^\varepsilon(t),u^\varepsilon(t)\big)dt-\varepsilon^\rho\big(a^\varepsilon(t)\nabla u^\varepsilon(t),2\bar G'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t)\\
&\quad+\bar G''^{,\varepsilon}_{uu}(t)u^\varepsilon(t)\nabla u^\varepsilon(t)\big)dt+\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\big(g^\varepsilon(t),\bar G'^{,\varepsilon}_u u^\varepsilon(t)+\bar G^\varepsilon(t)\big)dt\\
&\quad+dM^{u,\varepsilon}_t+\varepsilon^\rho\big(h^\varepsilon(t),\bar G'^{,\varepsilon}_u(t)u^\varepsilon(t)+\bar G^\varepsilon(t)\big)dt,
\end{aligned}$$

or, in the integral form

$$\begin{aligned}
&\frac12\|u^\varepsilon(t)\|^2+\textstyle\int_0^t\big(a^\varepsilon(s)\nabla u^\varepsilon(s),\nabla u^\varepsilon(s)\big)ds=\\
&\quad=\frac12\|u_0\|^2-\varepsilon^\rho\big(\bar G^\varepsilon(t),u^\varepsilon(t)\big)+\varepsilon^\rho\big(\bar G^\varepsilon(0),u_0\big)+\int_0^t\big(h^\varepsilon(s),u^\varepsilon(s)\big)\\
&\quad-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\int_0^t\Big[\big(\tilde H^\varepsilon(s),\nabla u^\varepsilon(s)\big)+\big(\tilde H'^{,\varepsilon}_u(s)\nabla u^\varepsilon(s),u^\varepsilon(s)\big)\Big]ds\\
&\quad-\varepsilon^\rho\int_0^t\big(a^\varepsilon(s)\nabla u^\varepsilon(s),2\bar G'^{,\varepsilon}_u(s)\nabla u^\varepsilon(s)+\bar G''^{,\varepsilon}_{uu}(s)u^\varepsilon(s)\nabla u^\varepsilon(s)\big)ds\\
&\quad+\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t\big(g^\varepsilon(s),\bar G'^{,\varepsilon}_u(s)u^\varepsilon(s)+\bar G^\varepsilon(s)\big)ds+M^{u,\varepsilon}_t
\end{aligned} \tag{18}$$

$$+\varepsilon^\rho\int_0^t\big(h^\varepsilon(s),\bar G'^{,\varepsilon}_u(s)u^\varepsilon(s)+\bar G^\varepsilon(s)\big)ds$$

The following estimates are straightforward

$$\mathbf{E}\int_0^t \big[\big|\big(\tilde H^\varepsilon(s),\nabla u^\varepsilon(s)\big)\big| + \big|\big(\tilde H'^{,\varepsilon}_u(s)\nabla u^\varepsilon(s), u^\varepsilon(s)\big)\big|\big]ds$$

$$\le 2C\mathbf{E}\int_0^t \|u^\varepsilon(s)\|\|\nabla u^\varepsilon(s)\|ds$$

$$\le \frac{C}{\gamma}\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2 ds + C\gamma\mathbf{E}\int_0^t \|\nabla u^\varepsilon(s)\|^2 ds,$$

with arbitrary $\gamma > 0$. Also, by (10) we have

$$\mathbf{E}\int_0^t \big|\big(a^\varepsilon(s)\nabla u^\varepsilon(s), 2\bar G'^{,\varepsilon}_u(s)\nabla u^\varepsilon(s) + \bar G''^{,\varepsilon}_{uu}(s)u^\varepsilon(s)\nabla u^\varepsilon(s)\big)\big|ds$$

$$\le C\mathbf{E}\int_0^t \|\nabla u^\varepsilon(s)\|^2 ds,$$

$$\mathbf{E}\int_0^t \big|\big(g^\varepsilon(s), \bar G'^{,\varepsilon}_u(s)u^\varepsilon(s) + \bar G^\varepsilon(s)\big)\big|ds \le C\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2 ds.$$

and

$$\mathbf{E}\int_0^t \big|\big(h^\varepsilon(s), \bar G'^{,\varepsilon}_u(s)u^\varepsilon(s) + \bar G^\varepsilon(s)\big)\big|ds \le C\big(1 + \mathbf{E}\int_0^t \|u^\varepsilon(s)\|^2 ds\big).$$

Choosing now ε and γ small enough, and taking the expectation in the relation (18), with the help of Gronwall's lemma we obtain

$$\sup_{t\le T}\mathbf{E}\|u^\varepsilon(t)\|^2 + \mathbf{E}\int_0^T \|\nabla u^\varepsilon(s)\|^2 ds \le C. \tag{19}$$

It is easy to see, considering the bounds (2), (10) and (16) that $M_t^{u,\varepsilon}$ is a square integrable martingale. In order to obtain an upper bound for the term $\mathbf{E}(\sup_{0\le t\le T}|M_t^{u,\varepsilon}|)$ we estimate the quadratic variation of the martingale $M_t^{u,\varepsilon}$, as well as the expectation $\mathbf{E}(\|u^\varepsilon(t)\|^4)$. To this end we consider the expression

$$
\begin{aligned}
& d\Big[\frac{1}{4}\|u^\varepsilon(t)\|^4 + \varepsilon^\rho\big(\bar{G}^\varepsilon(t), u^\varepsilon(t)\big)(u^\varepsilon(t), u^\varepsilon(t))\Big] \\
& = \frac{1}{2}\big(u^\varepsilon(t), u^\varepsilon(t)\big)d\big[\big(u^\varepsilon(t), u^\varepsilon(t)\big)\big] + d\big[\varepsilon^\rho\big(\bar{G}^\varepsilon(t), u^\varepsilon(t)\big)\big]\big(u^\varepsilon(t), u^\varepsilon(t)\big) \\
& + \varepsilon^\rho\big(\bar{G}^\varepsilon(t), u^\varepsilon(t)\big)d\big[\big(u^\varepsilon(t), u^\varepsilon(t)\big)\big] \\
& = \|u^\varepsilon(t)\|^2\big[-\big(a^\varepsilon(t)\nabla u^\varepsilon(t), \nabla u^\varepsilon(t)\big) + (h^\varepsilon(t), u^\varepsilon(t)) \\
& - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde{H}^\varepsilon(t), \nabla u^\varepsilon(t)\big) - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde{H}'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t), u^\varepsilon(t)\big) \\
& - \varepsilon^\rho\big(a^\varepsilon(t)\nabla u^\varepsilon(t), \bar{G}''^{,\varepsilon}_{uu}(t)\nabla u^\varepsilon(t)u^\varepsilon(t) + 2\bar{G}'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t)\big) \\
& + \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\big(g^\varepsilon(t), \bar{G}'^{,\varepsilon}_u(t)u^\varepsilon(t) + \bar{G}^\varepsilon(t)\big) \\
& + \varepsilon^\rho\big(h^\varepsilon(t), \bar{G}'^{,\varepsilon}_u(t)u^\varepsilon(t) + \bar{G}^\varepsilon(t)\big)\big]dt \\
& + \|u^\varepsilon(t)\|^2 dM_t^{u,\varepsilon} + \varepsilon^\rho\big(\bar{G}^\varepsilon(t), u^\varepsilon(t)\big)\big[-2\big(a^\varepsilon(t)\nabla u^\varepsilon(t), \nabla u^\varepsilon(t)\big) \\
& + 2\frac{1}{\varepsilon^{1\wedge\frac{\alpha}{2}}}\big(g^\varepsilon(t), u^\varepsilon(t)\big) + 2\big(h^\varepsilon(t), u^\varepsilon(t)\big)\big]dt,
\end{aligned}
\tag{20}
$$

where the formula of integration by parts for semimartingales has also been used. It is easy to see that the process $\int_0^t \|u^\varepsilon(s)\|^2 dM_s^{u,\varepsilon}$ is a square integrable martingale with respect to the filtration $\mathcal{F}_t$. Indeed, considering formulae (12) and (16), we have:

$$
\mathbf{E}\int_0^t \|u^\varepsilon(s)\|^4 d\langle M^{u,\varepsilon}\rangle_s \le \beta_\varepsilon^2 \mathbf{E}\left(\langle M^{u,\varepsilon}\rangle_t\right) = \beta_\varepsilon^2 \mathbf{E}\left(M_t^{u,\varepsilon}\right)^2 \le C_\varepsilon.
$$

Taking the expectation in (20) and using the same arguments as those leading to (19), one can obtain the bound

$$
\sup_{0\le t\le T}\mathbf{E}\left(\|u^\varepsilon(t)\|^4\right) + \mathbf{E}\left(\int_0^T \|u^\varepsilon(s)\|^2\|\nabla u^\varepsilon(s)\|^2 ds\right) \le C.
$$

Next, Ito's formula for the square of a semimartingale gives

$$
\begin{aligned}
&d\Big[\frac{1}{2}\|u^\varepsilon(t)\|^2+\varepsilon^\rho\big(\bar G^\varepsilon(t),u^\varepsilon(t)\big)\Big]^2\\
&\quad=\Big[\|u^\varepsilon(t)\|^2+2\varepsilon\big(\bar G^\varepsilon(t),u^\varepsilon(t)\big)\Big]\Big\{\Big[-\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla u^\varepsilon(t)\big)\\
&\quad+\big(h^\varepsilon(t),u^\varepsilon(t)\big)dt-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde H^\varepsilon(t),\nabla u^\varepsilon(t)\big)\\
&\quad-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde H_u'^{,\varepsilon}(t)\nabla u^\varepsilon(t),u^\varepsilon(t)\big)-\varepsilon^\rho\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\bar G_{uu}''^{,\varepsilon}(t)\nabla u^\varepsilon(t)u^\varepsilon(t)\\
&\quad+2\bar G_u'^{,\varepsilon}(t)\nabla u^\varepsilon(t)\big)+\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\big(g^\varepsilon(t),\bar G_u'^{,\varepsilon}(t)u^\varepsilon(t)+\bar G^\varepsilon(t)\big)\\
&\quad+\varepsilon^\rho\big(h^\varepsilon(t),\bar G_u'^{,\varepsilon}(t)u^\varepsilon(t)+\bar G^\varepsilon(t)\big)\big)\Big]dt+dM_t^{u,\varepsilon}\Big\}+d<M^{u,\varepsilon}>_t
\end{aligned}
$$

If we subtract now the last equality from (20) and integrate the result over $[0,t]$, we get after simple rearrangements

$$
\begin{aligned}
&<M^{u,\varepsilon}>_t=\varepsilon^{2\rho}\big(\bar G^\varepsilon(t),u^\varepsilon(t)\big)^2-\varepsilon^{2\rho}\big(\bar G^\varepsilon(0),u_0\big)^2\\
&\quad+2\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\int_0^t\big(\bar G^\varepsilon(s),u^\varepsilon(s)\big)\big(\bar g^\varepsilon(s),u^\varepsilon(s)\big)ds\\
&\quad+2\varepsilon^{2\rho}\int_0^t\big(\bar G^\varepsilon(s),u^\varepsilon(s)\big)\big(a^\varepsilon(s)\nabla u^\varepsilon(s),\bar G_{uu}''^{,\varepsilon}(s)\nabla u^\varepsilon(s)u^\varepsilon(s)+\bar G_u'^{,\varepsilon}(s)\nabla u^\varepsilon(s)\big)ds\\
&\quad-2\varepsilon^{2\alpha-3(1\wedge\frac{\alpha}{2})}\int_0^t\big(\bar G^\varepsilon(s),u^\varepsilon(s)\big)\big(g^\varepsilon(s),\bar G_u'^{,\varepsilon}(s)u^\varepsilon(s)+\bar G^\varepsilon(s)\big)ds\\
&\quad-2\varepsilon^{2\rho}\int_0^t\big(\bar G^\varepsilon(s),u^\varepsilon(s)\big)\big(h^\varepsilon(s),\bar G_u'^{,\varepsilon}(s)u^\varepsilon(s)+\bar G^\varepsilon(s)\big)ds\\
&\quad-2\varepsilon^\rho\int_0^t\big(\bar G^\varepsilon(s),u^\varepsilon(s)\big)dM_s^{u,\varepsilon}.
\end{aligned}
\tag{21}
$$

Hence

$$
\begin{aligned}
\mathbf{E}\Big(\langle M^{u,\varepsilon}\rangle_t\Big)&\le C\mathbf{E}\Bigg(\varepsilon^{2\rho}\|u^\varepsilon(t)\|^4+\varepsilon^{2\rho}\|u_0\|^4\\
&\quad+(2\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}+2\varepsilon^{2\alpha-3(1\wedge\frac{\alpha}{2})}+2\varepsilon^{2\rho})\int_0^t\|u^\varepsilon(s)\|^4ds\\
&\quad+\varepsilon^{2\rho}\int_0^t\|u^\varepsilon(s)\|^2ds+2\varepsilon^{2\rho}\int_0^t\|u^\varepsilon(s)\|^2\|\nabla u^\varepsilon(s)\|^2ds\Bigg)\\
&\le C\Bigg(\varepsilon^{2\rho}+(\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}+\varepsilon^{2\alpha-3(1\wedge\frac{\alpha}{2})}+\varepsilon^{2\rho})t\Bigg).
\end{aligned}
\tag{22}
$$

We next get by the Burkholder-Davis-Gundy inequality that

$$\mathbf{E}\left(\sup_{0\le t\le T}|M_t^{u,\varepsilon}|\right)\le K_T\mathbf{E}\left(\sqrt{\langle M^{u,\varepsilon}\rangle_T}\right)\le\frac{K_T}{2}+\frac{K_T}{2}\mathbf{E}\left(\langle M^{u,\varepsilon}\rangle_T\right)$$
$$\le\frac{K_T}{2}+\frac{K_T}{2}C\left(\varepsilon^{2\rho}+(\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}+\varepsilon^{2\alpha-3(1\wedge\frac{\alpha}{2})}+\varepsilon^{2\rho})T\right).\tag{23}$$

Now the first inequality of Proposition 1 follows for small ε from the relation (18), and for all other ε from (16).

The second estimate of Proposition 1 can be obtained in a similar way . We only show how to estimate the martingale term. Applying the Burkholder-Davis-Gundy inequality we get

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\le t\le T}\big|\int_0^t\|u^\varepsilon(s)\|^2dM_s^{u,\varepsilon}\big|\Big]&\le\mathbf{E}\Big(\sqrt{\int_0^T\|u^\varepsilon(s)\|^4d\langle M^{u,\varepsilon}\rangle_s}\Big)\\&\le\mathbf{E}\Big(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^2\sqrt{\langle M^{u,\varepsilon}\rangle_T}\Big)\\&\le\gamma\mathbf{E}\Big(\sup_{0\le t\le T}\|u^\varepsilon(t)\|^4\Big)+C_\gamma\mathbf{E}\Big(\langle M^{u,\varepsilon}\rangle_T\Big),\end{aligned}$$

where as before γ stands for an arbitrary positive constant. We already estimated the expectation of the quadratic variation process associated with the martingale $\{M_t^{u,\varepsilon}\}$. The desired estimate for the expression $\mathbf{E}\left(\sup_{t\le T}\|u^\varepsilon(t)\|^4\right)$ is now straightforward. □

The tightness of $\{u^\varepsilon\}$ in the space $\tilde{V}_T$ also relies on an equi-continuity result for the family of functions

$$\{t\mapsto(u^\varepsilon(t),\varphi)\}_{\varepsilon>0}$$

in $\mathcal{C}([0,T];\mathbb{R})$, where φ is an arbitrary element of $L^2(\mathbb{R}^n)$. In view of Proposition 1 it suffices to prove this equi-continuity for φ from a dense subset of $L^2(\mathbb{R}^n)$.

Proposition 2 *Under assumptions* **A.1**–**A.6**, *for any* φ *in* $\mathcal{C}_0^\infty(R^n)$

and any $\delta > 0$ there exist $\nu = \nu(\delta) > 0$ and $\varepsilon_0 > 0$ such that

$$\mathbf{P}\{\sup_{|t-s|<\nu} |(u^\varepsilon(t),\varphi) - (u^\varepsilon(s),\varphi)| > \delta\} < \delta, \quad \forall\, 0 < \varepsilon \le \varepsilon_0.$$

Proof For each $\varepsilon > 0$ the process $M_t^{\varphi,\varepsilon,\alpha}$, defined in (13), is a square integrable martingale. We deduce from (1) and (13) that

$$\begin{aligned} d\big[\big(u^\varepsilon(t),\varphi\big) + \varepsilon^\rho\big(\bar G^\varepsilon(t),\varphi\big)\big] = & \\ & -\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\big)dt - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde H^\varepsilon(t),\nabla\varphi\big)dt \\ & - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\big(\tilde H_u'^{,\varepsilon}(t)\nabla u^\varepsilon(t),\varphi\big)dt + \big(h^\varepsilon(t),\varphi\big)dt \\ & - \varepsilon^\rho\big(a^\varepsilon(t)\nabla u^\varepsilon(t), \bar G_u'^{,\varepsilon}(t)\nabla\varphi + \bar G_{uu}''^{,\varepsilon}(t)\varphi\nabla u^\varepsilon(t)\big)dt \\ & + \varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\big(g^\varepsilon(t), \bar G_u'^{,\varepsilon}(t)\varphi\big)dt \\ & + \varepsilon^\rho(h^\varepsilon(t), \bar G_u'^{,\varepsilon}(t)\varphi\big)dt + dM_t^{\varphi,\varepsilon}. \end{aligned} \tag{24}$$

In view of the inequalities

$$\left|\int_s^t \big(a^\varepsilon(r)\nabla u^\varepsilon(r),\nabla\varphi\big)dr\right| \le c\sqrt{t-s}\|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))}$$

and

$$\left|\varepsilon\int_0^t \big(\bar G_{uu}''^{,\varepsilon}(s)a^\varepsilon(s)\nabla u^\varepsilon(s),\varphi\nabla u^\varepsilon(s)\big)ds\right| \le c\varepsilon\|u^\varepsilon\|^2_{L^2(0,T;H^1(\mathbb{R}^n))},$$

and Proposition 1, the integrals $\int_0^t \big(a^\varepsilon(s)\nabla u^\varepsilon(s),\nabla\varphi\big)ds$ and $\varepsilon\int_0^t \big(\bar G_{uu}''^{,\varepsilon}(s)a^\varepsilon(s)\nabla u^\varepsilon(s),\varphi\nabla u^\varepsilon(s)\big)ds$ form tight families in $C([0,T])$. Similar estimates are valid for all other absolutely continuous terms on the right hand side of (24).

The estimate of the modulus of continuity of the martingale term $M_t^{\varphi,\varepsilon}$ is based on the bound for the increment of the quadratic variation $< M^{\varphi,\varepsilon} >_t$. By the definition of u^ε and $\bar{G}^\varepsilon$ we have

$$
\begin{aligned}
&d\Big[\big(u^\varepsilon(t),\varphi\big)^2 + 2\varepsilon^\rho\big(\bar{G}^\varepsilon(t),\varphi\big)\big(u^\varepsilon(t),\varphi\big)\Big]\\
&= -2\Big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\Big)\Big(u^\varepsilon(t),\varphi\Big)dt - 2\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\Big(\tilde{H}^\varepsilon(t),\nabla\varphi\Big)\Big(u^\varepsilon(t),\varphi\Big)dt\\
&\quad - 2\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\Big(\tilde{H}'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t),\varphi\Big)\Big(u^\varepsilon(t),\varphi\Big)dt + 2\Big(h^\varepsilon(t),\varphi\Big)\Big(u^\varepsilon(t),\varphi\Big)dt\\
&\quad - 2\varepsilon^\rho\Big(\bar{G}'^{,\varepsilon}_u(t)a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\Big)\Big(u^\varepsilon(t),\varphi\Big)dt\\
&\quad - 2\varepsilon^\rho\Big(\bar{G}''^{,\varepsilon}_{uu}(t)a^\varepsilon(t)\nabla u^\varepsilon(t),\varphi\nabla u^\varepsilon(t)\Big)\Big(u^\varepsilon(t),\varphi)\Big)dt\\
&\quad + 2\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\Big(\bar{G}'^{,\varepsilon}_u(t)g^\varepsilon(t),\varphi\Big)\Big(u^\varepsilon(t),\varphi\Big)dt\\
&\quad + 2\varepsilon^\rho\Big(h^\varepsilon(t),\bar{G}'^{,\varepsilon}_u(t)\varphi\Big)\Big(u^\varepsilon(t),\varphi)\Big)dt - 2\varepsilon^\rho\Big(\bar{G}^\varepsilon(t),\varphi\Big)\Big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\Big)dt\\
&\quad + 2\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\Big(\bar{G}^\varepsilon(t),\varphi\Big)\Big(g^\varepsilon(t),\varphi\Big)dt + 2\varepsilon^\rho\Big(\bar{G}^\varepsilon(t),\varphi\Big)(h^\varepsilon(t),\varphi\Big)dt\\
&\quad + 2\Big(u^\varepsilon(t),\varphi\Big)dM_t^{\varphi,\varepsilon},
\end{aligned}
$$

On the other hand, by the Ito formula we find

$$
\begin{aligned}
d\Big[\big(u^\varepsilon(t),\varphi\big)+&\varepsilon\big(\bar{G}^\varepsilon(t),\varphi\big)\Big]^2\\
=&2\Big[\big(u^\varepsilon(t),\varphi\big) + \varepsilon\big(\bar{G}^\varepsilon(t),\varphi\big)\Big]\Big\{-\Big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\Big)dt\\
&-\varepsilon^{1-(1\wedge\frac{\alpha}{2})}\Big(\tilde{H}^\varepsilon(t),\nabla\varphi\Big)dt - \varepsilon^{1-(1\wedge\frac{\alpha}{2})}\Big(\tilde{H}'^{,\varepsilon}_u(t)\nabla u^\varepsilon(t),\varphi\Big)dt\\
&-\varepsilon^\rho\Big(\bar{G}'^{,\varepsilon}_u(t)a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\Big)dt\\
&-\varepsilon^\rho\Big(\bar{G}''^{,\varepsilon}_{uu}(t)a^\varepsilon(t)\nabla u^\varepsilon(t),\varphi\nabla u^\varepsilon(t)\Big)dt\\
&+\varepsilon^{\alpha-2(1\wedge\frac{\alpha}{2})}\Big(\bar{G}'^{,\varepsilon}_u(t)g^\varepsilon(t),\varphi\Big)dt + \Big(h^\varepsilon(t),\varphi+\varepsilon^\rho\bar{G}'^{,\varepsilon}_u(t)\varphi\Big)dt\\
&+dM_t^{\varphi,\varepsilon}\Big\} + d < M^{\varphi,\varepsilon} >_t .
\end{aligned}
$$

Opening the brackets and comparing two previous expressions allows us to write down the formula for $< M^{\varphi,\varepsilon} >_t$:

$$
\begin{aligned}
< M^{\varphi,\varepsilon} >_t =& \varepsilon^{2\rho}\big(\bar{G}(\frac{t}{\varepsilon^2}, u^\varepsilon(t)), \varphi\big)^2 - \varepsilon^{2\rho}\big(\bar{G}(0, u_0), \varphi\big)^2 \\
&+ 2\varepsilon^{\alpha - 2(1\wedge\frac{\alpha}{2})} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(\bar{g}^\varepsilon(s), \varphi\big) ds \\
&+ 2\varepsilon^{2\rho} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\nabla\varphi + \bar{G}''^{,\varepsilon}_{uu}(s)\nabla u^\varepsilon(s)\varphi\big) ds \\
&- 2\varepsilon^{2\alpha - 3(1\wedge\frac{\alpha}{2})} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(g^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\varphi\big) ds \\
&- 2\varepsilon^{2\rho} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(h^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\varphi\big) ds \\
&- 2\varepsilon^{\rho} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big) dM^{\varphi,\varepsilon}_s.
\end{aligned} \tag{25}
$$

Our aim is to estimate the quantity $\mathbf{E}\big[(< M^{\varphi,\varepsilon} >_{t_2} - < M^{\varphi,\varepsilon} >_{t_1})^2\big]$, with $0 \le t_1 < t_2 \le T$. We have

$$
\begin{aligned}
&\mathbf{E}\big[4\varepsilon^{4\rho}\big(\int_{t_1}^{t_2} \big(\bar{G}^\varepsilon(s), \varphi\big)\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\nabla\varphi + \bar{G}''^{,\varepsilon}_{uu}(s)\nabla u^\varepsilon(s)\varphi\big) ds\big)^2\big] \\
\le& C\varepsilon^{4\rho}\mathbf{E}\big[\big(\int_{t_1}^{t_2} \big(\bar{G}^\varepsilon(s), \varphi\big)\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\nabla\varphi\big) ds\big)^2 \\
+& \big(\int_{t_1}^{t_2} \big(\bar{G}^\varepsilon(s), \varphi\big)\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''^{,\varepsilon}_{uu}(s)\nabla u^\varepsilon(s)\varphi\big) ds\big)^2\big] \\
\le& C\varepsilon^{4\rho}\mathbf{E}\big[\big(\int_{t_1}^{t_2} \|u^\varepsilon(s)\|^2 ds \int_{t_1}^{t_2} \|\nabla u^\varepsilon(s)\|^2 ds\big)\big] + \mathbf{E}\big[\big(\int_{t_1}^{t_2} \|\nabla u^\varepsilon(s)\|^2 ds\big)^2\big] \\
\le& C\varepsilon^{4\rho}\big[\frac{1}{2}\mathbf{E}\big(\int_{t_1}^{t_2} (\|u^\varepsilon(s)\|)^2 ds\big)^2 + \frac{3}{2}\mathbf{E}\big((\int_{t_1}^{t_2} \|\nabla u^\varepsilon(s)\|^2 ds)^2\big)\big] \\
\le& C\varepsilon^{4\rho}\big(1 + (t_2 - t_1)^2\big)
\end{aligned}
$$

where the relation (17) and the assumptions **(A.3), (A.4)** have been used. Therefore,

$$\begin{aligned}
&\mathbf{E}\big[(< M^{\varphi,\varepsilon} >_{t_2} - < M^{\varphi,\varepsilon} >_{t_1})^2\big] \\
\leq & C\mathbf{E}\big(\varepsilon^{4\rho}\|u^\varepsilon(t_2)\|^4 + \varepsilon^{4\rho}\|u^\varepsilon(t_1)\|^4 + \big(\varepsilon^{2\alpha-(4\wedge 2\alpha)} + \varepsilon^{4\alpha-(6\wedge 3\alpha)}\big) \\
& \quad (t_2-t_1)^2 \sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^4 + \varepsilon^{4\rho}(1+(t_2-t_1)^2) \\
& \quad + \varepsilon^{4\rho}(t_2-t_1)^2(\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2 + \sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^4) \\
& \quad + C\varepsilon^{2\rho}\sup_{0\leq t\leq T}\|u^\varepsilon(t)\|^2(< M^{\varphi,\varepsilon} >_{t_2} - < M^{\varphi,\varepsilon} >_{t_1})\big) \\
\leq & C\big(\varepsilon^{4\rho} + (\varepsilon^{4\rho} + \varepsilon^{2\alpha-(4\wedge 2\alpha)} + \varepsilon^{4\alpha-(6\wedge 3\alpha)})(t_2-t_1)^2\big) \\
& \quad + \frac{\varepsilon^{2\rho}}{2}\mathbf{E}\big[(< M^{\varphi,\varepsilon} >_{t_2} - < M^{\varphi,\varepsilon} >_{t_1})^2\big].
\end{aligned}$$

This yields, for all sufficiently small ε

$$\begin{aligned}
\mathbf{E}\Big[\Big(< M^{\varphi,\varepsilon} >_{t_2} - < M^{\varphi,\varepsilon} >_{t_1} \Big)^2\Big] \leq C\Big(\varepsilon^{4\rho} + (\varepsilon^{4\rho} + \varepsilon^{2\alpha-(4\wedge 2\alpha)} \\
+ \varepsilon^{4\alpha-(6\wedge 3\alpha)})(t_2-t_1)^2\Big).
\end{aligned}$$

Finally the Burkholder-Davis-Gundy inequality gives

$$\mathbf{E}\Big[\sup_{t_1\leq s\leq t_2}|M_s^{\varphi,\varepsilon} - M_{t_1}^{\varphi,\varepsilon}|^4\Big] \leq C\mathbf{E}\Big[\Big(< M^{\varphi,\varepsilon} >_{t_2} - < M^{\varphi,\varepsilon} >_{t_1} \Big)^2\Big].$$

Combining the last two bounds, by Theorem 8.3 in [1] one can deduce the required estimate for the modulus of continuity. □

We now state

Theorem 4 *The family of measures $Q^\varepsilon = \mathcal{L}(u^\varepsilon)$ is tight in $\tilde{V_T}$.*

Proof The result follows from the above bounds by the Prokhorov criterium, whose applicability in the space $\tilde{V_T}$ has been justified in [17]. □

5 Passage to the limit and proofs of the main results

In this section we prove the convergence of u^ε, as $\varepsilon \to 0$, and describe its limit.

5.1 Case $\alpha = 2$: Proof of Theorem 1

Our goal is to introduce a limit martingale problem and to show that any accumulation point of the sequence $\{u^\varepsilon, \varepsilon > 0\}$ is a solution of this problem. We first state a lemma, whose proof can be found in [9].

Lemma 3 *The equations (3) and (4) have stationary solutions. Under the normalizations*

$$\int_{\mathbf{T}^n} \chi^k(z,s)dz = 0, \qquad \int_{\mathbf{T}^n} \tilde{G}(z,t,u)dz = 0$$

the solutions are unique and ergodic. Moreover, the following bounds hold

$$\begin{aligned} \|\chi^k\|_{L^\infty(\mathbf{T}^n\times(-\infty,+\infty)\times\Omega)} &\le C \\ \|\tilde{G}\|_{L^\infty(\mathbf{T}^n\times(-\infty,+\infty)\times\Omega)} &\le C|u| \\ \|\tilde{G}'_u\|_{L^\infty(\mathbf{T}^n\times(-\infty,+\infty)\times\Omega)} &\le C \\ \|\tilde{G}''_{uu}\|_{L^\infty(\mathbf{T}^n\times(-\infty,+\infty)\times\Omega)} &\le \frac{C}{1+|u|}. \end{aligned} \tag{26}$$

We now define two additional correctors as

$$\chi^\varepsilon(x,t) = \chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right), \quad \tilde{G}^\varepsilon(x,t,u) = \tilde{G}\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u\right).$$

Consider now the process

$$\begin{aligned} \Psi^\varepsilon(t) = \big(u^\varepsilon(t), \varphi\big) + \varepsilon\big(\bar{G}(\frac{t}{\varepsilon^2}, u^\varepsilon(t)), \varphi\big) + \varepsilon\big(\chi^\varepsilon(t)u^\varepsilon(t), \nabla\varphi\big) \\ + \varepsilon\big(\tilde{G}^\varepsilon(t, u^\varepsilon(t)), \varphi\big). \end{aligned}$$

Using equations (1), (3), (4) and the representation (13), we get, after multiple integrations by parts

$$
\begin{aligned}
& d\big[\big(u^\varepsilon(t),\varphi\big)\big] \\
& \quad = -\,\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\nabla\varphi\big)dt + \frac{1}{\varepsilon}\big(g^\varepsilon(t),\varphi\big)dt + \big(h^\varepsilon(t),\varphi\big)dt \\
& \quad = \frac{1}{\varepsilon}\big(\mathrm{div}_z a_i(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2}),u^\varepsilon(t)\frac{\partial\varphi}{\partial x_i}\big)dt + \big(a^\varepsilon(t)u^\varepsilon(t),\nabla\nabla\varphi\big)dt \\
& \qquad + \frac{1}{\varepsilon}\big(\bar g^\varepsilon(t),\varphi\big)dt + \frac{1}{\varepsilon}\big(\tilde g^\varepsilon(t),\varphi\big)dt + \big(h^\varepsilon(t),\varphi\big)dt,
\end{aligned}
$$

$$
\begin{aligned}
& d\big[\varepsilon\big(\bar G(\frac{t}{\varepsilon^2},u^\varepsilon(t)),\varphi\big)\big] \\
& = -\,\frac{1}{\varepsilon}\big(\bar g^\varepsilon(t),\varphi\big)dt - \varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\bar G'^{,\varepsilon}_u(t)\nabla\varphi + \bar G''^{,\varepsilon}_{uu}(t)\nabla u^\varepsilon(t)\varphi\big)dt \\
& \quad + \big(g^\varepsilon(t),\bar G'^{,\varepsilon}_u(t)\varphi\big)dt + \varepsilon\big(h^\varepsilon(t),\bar G'^{,\varepsilon}_u(t)\varphi\big)dt + dM_t^{\varphi,\varepsilon},
\end{aligned}
$$

$$
\begin{aligned}
& d\big[\varepsilon\big(\chi(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2})u^\varepsilon(t),\nabla\varphi\big)\big] \\
& \quad = -\,\varepsilon\big(\mathrm{div}_x(a^\varepsilon(t)\nabla_x\chi_i^\varepsilon(t)),u^\varepsilon(t)\frac{\partial\varphi}{\partial x_i}\big)dt - \frac{1}{\varepsilon}\big(\mathrm{div}_z a_i(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2}),u^\varepsilon(t)\frac{\partial\varphi}{\partial x_i}\big)dt \\
& \qquad + \varepsilon\big(\mathrm{div}_x(a^\varepsilon(t)\nabla u^\varepsilon(t)),\chi_i^\varepsilon(t)\frac{\partial\varphi}{\partial x_i}\big)dt + \big(g^\varepsilon(t)\chi^\varepsilon(t),\nabla\varphi\big)dt \\
& \qquad + \varepsilon\big(h^\varepsilon(t)\chi^\varepsilon(t),\nabla\varphi\big)dt \\
& \quad = \big(a^\varepsilon(t)u^\varepsilon(t),\nabla_z\chi(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2})\nabla\nabla\varphi\big)dt - \frac{1}{\varepsilon}\big(\mathrm{div}_z a_i(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2}),u^\varepsilon(t)\frac{\partial\varphi}{\partial x_i}\big)dt \\
& \qquad - \varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\chi^\varepsilon(t)\nabla\nabla\varphi\big)dt + \big(g^\varepsilon(t)\chi^\varepsilon(t),\nabla\varphi\big)dt + \varepsilon\big(h^\varepsilon(t)\chi^\varepsilon(t),\nabla\varphi\big)dt,
\end{aligned}
$$

$$
\begin{aligned}
& d\big[\varepsilon\big(\tilde G(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(t)),\varphi\big)\big] \\
& \quad = \big(-\,\mathrm{div}_x(a^\varepsilon(t)\nabla_x\tilde G^\varepsilon(t,u^\varepsilon(t))),\varphi\big)dt + \varepsilon\big(a^\varepsilon(t),\nabla_x\tilde G'^{,\varepsilon}(t,u^\varepsilon(t))\nabla u^\varepsilon(t),\varphi\big)dt \\
& \qquad - \frac{1}{\varepsilon}\big(\tilde g^\varepsilon(t),\varphi\big)dt + \varepsilon\big(\mathrm{div}_x(a^\varepsilon(t)\nabla u^\varepsilon(t)),\tilde G'_u(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(t))\varphi\big)dt \\
& \qquad + \big(g^\varepsilon(t),\tilde G'_u(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(t))\varphi\big)dt + \varepsilon\big(h^\varepsilon(t),\tilde G'_u(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^2},u^\varepsilon(t))\varphi\big)dt
\end{aligned}
$$

$$
\begin{aligned}
=&\big(a^\varepsilon(t)\nabla_z\tilde G(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2},u^\varepsilon(t)),\nabla\varphi\big)dt-\frac{1}{\varepsilon}\big(\tilde g^\varepsilon(t),\varphi\big)dt\\
&-\varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\tilde G'_u(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2},u^\varepsilon(t))\nabla\varphi\big)dt\\
&-\varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\tilde G''_{uu}(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2},u^\varepsilon(t))\nabla u^\varepsilon(t)\varphi\big)dt\\
&+\big(g^\varepsilon(t),\tilde G'_u(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2},u^\varepsilon(t))\varphi\big)dt+\varepsilon\big(h^\varepsilon(t),\tilde G'_u(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2},u^\varepsilon(t))\varphi\big)dt,
\end{aligned}
$$

where we denoted by $a_i(t)$ the i-th row of the matrix $a(t)$.
Summing up the above identities we get

$$
\begin{aligned}
d\Psi^\varepsilon(t)=&\big((I+\nabla_z\chi(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2}))a^\varepsilon(t)u^\varepsilon(t),\nabla\nabla\varphi\big)dt\\
&+\big(a^\varepsilon(t)\nabla_z\tilde G(\tfrac{\cdot}{\varepsilon},\tfrac{t}{\varepsilon^2},u^\varepsilon(t))+\chi^\varepsilon(t)g^\varepsilon(t),\nabla\varphi\big)dt\\
&+\big((\bar G'^{,\varepsilon}_u(t)+\tilde G'^{,\varepsilon}_u(t,u^\varepsilon(t))g^\varepsilon(t),\varphi\big)dt+\big(h^\varepsilon(t),\varphi\big)dt+dM_t^{\varphi,\varepsilon}\\
&-\varepsilon\Big[\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\tilde G''^{,\varepsilon}_{uu}(t,u^\varepsilon(t))\nabla u^\varepsilon(t)\varphi+\tilde G'^{,\varepsilon}_u(t,u^\varepsilon(t))\nabla\varphi\big)dt\\
&+\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\bar G''^{,\varepsilon}_{uu}(t)\nabla u^\varepsilon(t)\varphi+\bar G'^{,\varepsilon}_u(t)\nabla\varphi\big)dt\\
&+\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\chi^\varepsilon(t)\nabla\nabla\varphi\big)dt\\
&-\big(h^\varepsilon(t),\chi^\varepsilon(t)\nabla\varphi+\tilde G'^{,\varepsilon}_u(t,u^\varepsilon(t))\varphi+\bar G'^{,\varepsilon}_u(t)\varphi\big)dt\Big],
\end{aligned}
$$

from which we derive the expression for

$$
\begin{aligned}
&\big(u^\varepsilon(t),\varphi\big)\\
&\quad=\big(u_0,\varphi\big)+\int_0^t\big(\langle a(I+\nabla\chi)\rangle u^\varepsilon(s),\nabla\nabla\varphi\big)ds\\
&\qquad+\int_0^t\Big[\big(\langle a\nabla\tilde G\rangle(u^\varepsilon(s)),\nabla\varphi\big)\Big]ds+\big(\langle g\chi\rangle(u^\varepsilon(s)),\nabla\varphi\big)\Big]ds\\
&\qquad+\int_0^t\Big[\big(\langle g\bar G'_u\rangle(u^\varepsilon(s)),\varphi\big)+\big(\langle g\tilde G'_u\rangle(u^\varepsilon(s)),\varphi\big)+\big(\langle h\rangle(u^\varepsilon(s)),\varphi\big)\Big]ds\\
&\qquad+M_t^{\varphi,\varepsilon}+A_\varepsilon(t), \qquad\qquad (27)
\end{aligned}
$$

where $M_t^{\varphi,\varepsilon}$ is the martingale introduced in (13) and

$$A_\varepsilon(t) = -\,\varepsilon\Big\{\big[\big(\bar{G}^\varepsilon(t) - \bar{G}^\varepsilon(0), \varphi\big) + \big(\tilde{G}^\varepsilon(t, u^\varepsilon(t)) - \tilde{G}^\varepsilon(0, u_0), \varphi\big)$$

$$+ \big(\chi^\varepsilon(t)u^\varepsilon(t) - \chi^\varepsilon(0)u_0, \nabla\varphi\big)\big]$$

$$+ \big[\int_0^t \big(a^\varepsilon(s)\nabla u^\varepsilon(s), \tilde{G}''^{,\varepsilon}_{uu}(s, u^\varepsilon(s))\nabla u^\varepsilon(s)\varphi + \tilde{G}'^{,\varepsilon}_u(s, u^\varepsilon(s))\nabla\varphi\big)ds$$

$$+ \int_0^t \big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}''^{,\varepsilon}_{uu}(s)\nabla u^\varepsilon(s)\varphi + \bar{G}'^{,\varepsilon}_u(s)\nabla\varphi\big)ds$$

$$+ \int_0^t \big(a^\varepsilon(s)\nabla u^\varepsilon(s), \chi^\varepsilon(s)\nabla\nabla\varphi\big)ds$$

$$- \int_0^t \big(h^\varepsilon(s), \chi^\varepsilon(s)\nabla\varphi + \tilde{G}'^{,\varepsilon}_u(s, u^\varepsilon(s))\varphi + \bar{G}'^{,\varepsilon}_u(t)\varphi\big)ds\big]\Big\}$$

$$+ \int_0^t \big(u^\varepsilon(s), (a^\varepsilon(s)(I + \nabla\chi(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^2})) - \langle a(I + \nabla\chi)\rangle)\nabla\nabla\varphi\big)ds$$

$$+ \int_0^t \big(a^\varepsilon(s)\nabla\tilde{G}^\varepsilon(s, u^\varepsilon(s)) - \langle a\nabla\tilde{G}\rangle(u^\varepsilon(s)), \nabla\varphi\big)ds$$

$$+ \int_0^t \big(g^\varepsilon(s)\chi^\varepsilon(s) - \langle g\chi\rangle(u^\varepsilon(s)), \nabla\varphi\big)ds$$

$$+ \int_0^t \big(g^\varepsilon(s)\bar{G}'^{,\varepsilon}_u(s) - \langle g\bar{G}'_u\rangle(u^\varepsilon(s)), \varphi\big)ds$$

$$+ \int_0^t \big(g^\varepsilon(s)\tilde{G}'^{,\varepsilon}_u(s, u^\varepsilon(s)) - \langle g\tilde{G}'_u\rangle(u^\varepsilon(s)), \varphi\big)ds$$

$$+ \int_0^t \big(h^\varepsilon(s) - \langle h\rangle(u^\varepsilon(s)), \varphi\big)ds.$$

Here and in what follows the notation $\langle\theta\rangle(u)$ stands for $\mathbf{E}\,{\textstyle\int}_{\mathbf{T}^n}\theta(z,t,u)dz$, with periodic in z and stationary in t random function $\theta(z,t,u)$. If θ doesn't depend on u we simply write $\langle\theta\rangle$.

Since all the terms in figure brackets have uniformly bounded expectations, the contribution of these terms vanishes as $\varepsilon \to 0$. The fact that the other terms on the right hand side tend to 0 can be proved in the same way as in proposition 7 in [14]. We conclude that

$$\lim_{\varepsilon\searrow 0}\mathbf{E}[\sup_{0\le t\le T}|A_\varepsilon(t)|] = 0. \tag{28}$$

Let Q be an accumulation point of the sequence of probability measures $\{\mathbf{P}\circ\{u^\varepsilon\}^{-1}\}$ defined on $\mathcal{B}(\tilde{V}_T)$, and denote by u a random variable with law Q.

Let $0\le s<t\le T$, and let Θ_s be a continuous bounded functional defined on $\tilde{V}_s$. If we set $\Theta_s^\varepsilon=\Theta_s(u^\varepsilon)$ and denote by F_φ the functional

$$F_\varphi(t,u):=\big(u(t),\varphi\big)-\big(u_0,\varphi\big)-\int_0^t\big(u(s),\langle a(I+\nabla\chi)\rangle\nabla\nabla\varphi\big)ds$$
$$-\int_0^t\big[\big(\langle a\nabla\tilde{G}\rangle(u(s)),\nabla\varphi\big)+\big(\langle g\chi\rangle(u(s)),\nabla\varphi\big)\big]ds$$
$$-\int_0^t\big[\big(\langle g\bar{G}'_u\rangle(u(s)),\varphi\big)+\big(\langle g\tilde{G}'_u\rangle(u(s)),\varphi\big)+\big(\langle h\rangle(u(s)),\varphi\big)\big]ds,$$

for any $u\in V_T$, then from formula (27) it follows that

$$\mathbf{E}[(F_\varphi(t,u^\varepsilon)-(F_\varphi(s,u^\varepsilon))\Theta_s^\varepsilon]=\mathbf{E}[(A_\varepsilon(t)-A_\varepsilon(s))\Theta_s^\varepsilon]$$

Using proposition 6 in [14] and taking into account (19) we can pass to the limit here as $\varepsilon\to0$ and conclude that the process F_φ is a Q-martingale with respect to the natural filtration of σ-algebras $\mathcal{B}(\tilde{V}_t)$, $0\le t\le T$.

We treat now the martingale term $M_t^{\varphi,\varepsilon}$ through its quadratic variation which was computed in (25). Notice that all the terms on the right hand side of (25) vanishes as $\varepsilon\to0$, except for the third one. Therefore,

$$\mathbf{E}\Big(\sup_{0\le t\le T}|\langle M^{\varphi,\varepsilon}\rangle_t-2\int_0^t(\bar{G}^\varepsilon(s),\varphi)(\bar{g}^\varepsilon(s),\varphi)ds|\Big)\xrightarrow[\varepsilon\to0]{}0. \qquad (29)$$

Denote by $(R(v)\varphi,\varphi)$ the quantity

$$2\mathbf{E}\big[\big(\bar{G}(t,v),\varphi\big)\big(\bar{g}(t,v),\varphi\big)\big]=2\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bar{G}(t,v(x))\bar{g}(t,v(y))\varphi(x)\varphi(y)dxdy, \qquad (30)$$

for $v\in L^2(\mathbb{R}^n)$. The bilinear form $R(v)$ does not depend on t. Using the relation (5) and the stationarity of the random field $\bar{g}(t,u)$, for each real u, we derive the following representation for

$R(v)$

$$
\begin{aligned}
(R(v)\varphi,\varphi) =&2\int_t^{\infty} ds\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bar{g}(s,v(x))\bar{g}(t,v(x'))\varphi(x)\varphi(x')dxdx'\\
=&2\int_0^{\infty} ds\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bar{g}(s+t,v(x))\bar{g}(t,v(x'))\varphi(x)\varphi(x')dxdx'\\
=&2\int_0^{\infty} ds\mathbf{E}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bar{g}(s,v(x))\bar{g}(0,v(x'))\varphi(x)\varphi(x')dxdx'
\end{aligned}
$$

for any $\varphi\in L^2(\mathbb{R}^n)$.

The mappings $\bar{G}(t,\cdot)$ and $\bar{g}(t,\cdot)$ are Lipschitz continuous uniformly with respect to t. Now, following exactly the same scheme as that in the proof of Proposition 8 in [14], we derive

$$
\mathbf{E}\big(\sup_{0\le t\le T}|\int_0^t\big[\big(\bar{G}^{\varepsilon}(s),\varphi\big)\big(\bar{g}^{\varepsilon}(s),\varphi\big)-\big(R(u^{\varepsilon}(s))\varphi,\varphi\big)\big]ds|\big)\xrightarrow[\varepsilon\searrow 0]{}0.
$$

Combining this formula with (28) and (29), we pass to the limit, as $\varepsilon\searrow 0$, in (27) and arrive at the following statement.

Proposition 3 *For every $\varphi\in C_0^{\infty}(\mathbb{R}^n)$, the process*

$$
F_{\varphi}(t,u)=\big(u(t),\varphi\big)-\big(u_0,\varphi\big)-\int_0^t\big(\hat{A}(u(s)),\varphi\big)ds,
$$

defined over the probability space $(V_T,\mathcal{B}(\tilde{V}_T),Q)$ is a square integrable martingale w.r.t. the natural filtration of σ-algebras, with the associated quadratic variation process given by

$$
\langle F_{\varphi}(\cdot,u)\rangle_t=\int_0^t\big(R(u(s))\varphi,\varphi\big)ds,
$$

where

$$
\begin{aligned}
\hat{A}(v):=&\mathrm{div}\langle a(I+\nabla\chi)\rangle\nabla v)-\mathrm{div}\langle a\nabla\tilde{G}\rangle(v)-\mathrm{div}\langle g\chi\rangle(v)\\
&+\langle g(\tilde{G}'_u+\bar{G}'_u)\rangle(v)+\langle h\rangle(v)
\end{aligned}
\tag{31}
$$

and $R(u)$ is defined in (30).

We prove now that the martingale problem we just stated has a unique solution. To this end we apply the well-known result of Yamada-Watanabe which specifies that the uniqueness of the solution of a martingale problem is a consequence of the pathwise uniqueness for a corresponding SDE. This result may be adapted to our type of SPDE.

We define the Hilbert space K of real valued stationary random processes as follows. We first denote by W the set of all processes $\{\bar{g}(t,u), t \geq 0\}$, where u varies in $\mathbb{R}$ and let Span(W) be the linear set generated by W. All the processes in Span(W) are stationary and adapted to the filtration $\{\mathcal{F}_t\}$. The space Span(W) may be endowed with the bilinear form:

$$< \bar{g}_u, \bar{g}_v >:= \int_0^\infty \mathbf{E}[\bar{g}(0,u)\bar{g}(t,v) + \bar{g}(0,v)\bar{g}(t,u)]dt,$$

where for instance $\bar{g}_u$ stands for $\bar{g}(\cdot,u)$. In view of assumption **(A.6)** this form is well defined. Also it is easy to see that $< \cdot,\cdot >$ is pre-Hilbertian, as considered on the quotient space Span$(W)/\mathcal{N}$, where $\mathcal{N}$ is the null set $\{h \in \text{Span}(W)/ < h,h >= 0\}$.

Set now K the closure of Span$(W)/\mathcal{N}$ under $< \cdot,\cdot >$. In this way K becomes a Hilbert space. We now define , for each fixed w in $L^2(\mathbb{R}^n)$, the mapping

$$\mathcal{C}^*(w) : L^2(\mathbb{R}^n) \mapsto K$$

as

$$[\mathcal{C}^*(w)\varphi](t) := \int_{\mathbb{R}^n} \bar{g}(t,w(x))\varphi(x)dx,$$

and denote by $\mathcal{C}(w)$ the adjoint of $\mathcal{C}^*(w)$.

It is easy to see that $\mathcal{C}(w)$ is a linear operator, for each w, and is Lipschitz with respect to the parameter w, according to assumption **(A.3)**, i.e. $\|C(w_2) - C(w_1)\|_{\mathcal{L}(K;L^2(\mathbb{R}^n))} \leq \|w_2 - w_1\|_{L^2(\mathbb{R}^n)}$. The following relations are straightforward

$$\begin{aligned} < \mathcal{C}(w)\mathcal{C}^*(w)\varphi,\varphi > &= < \mathcal{C}^*(w)\varphi, \mathcal{C}^*(w)\varphi > \\ &= \int_0^\infty \mathbf{E}[\mathcal{C}^*(w)\varphi](0)[\mathcal{C}^*(w)\varphi](t)dt \\ &= \int_0^\infty \mathbf{E}\big[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \bar{g}(0,w(x))\varphi(x)\bar{g}(t,w(y))\varphi(y)dxdy\big]dt \\ &= (R(w)\varphi,\varphi). \end{aligned}$$

Consider now the following SPDE in $L^2(\mathbb{R}^n)$

$$\begin{cases} du(t) = \hat{A}(u(t))dt + \mathcal{C}(u(t))dB_t, \\ u(0) = u_0, \end{cases} \tag{32}$$

where B_t is a standard cylindrical Brownian motion on K, i.e. for any $h \in K, < h, B_t >$ is a real valued Brownian motion with covariance $t\sqrt{< h, h >}$, and $\hat{A}(u)$ is defined in (6). By theorem 1.1, page 83, in [12], the equation (32) has a unique solution u in V_T. $\square$

5.2 Case $\alpha < 2$: Proof of Theorem 2

We consider first the following elliptic PDEs written in divergence form

$$A\chi_i^-(z,s) = -\mathrm{div} a_i(z,s), \quad z \in \mathbf{T}^n, \tag{33}$$

$1 \leq i \leq n$, $s \in [0, \infty)$ being a parameter. For each $s \in \mathbb{R}$ this equation has a unique up to an additive constant solution, $\chi^-(z,s)$ denotes the solution which satisfies:

$$\int_{\mathbf{T}^n} \chi_i^-(z,s)dz = 0.$$

Combining now the theorems 8.3, 8.8 and 8.34 from [7], we deduce that $\chi_i^-(\cdot,s) \in W^{2,2}(\mathbf{T}^n) \cap \mathcal{C}^{1,\gamma}(\mathbf{T}^n)$, $\gamma \in (0,1)$ being a deterministic constant. The assumption **A.7** and Theorem 8.22 from [7] tells us now that $\psi_i^-(z,s) := \frac{\partial \chi_i^-}{\partial s}(;s) \in W^{1,2}(\mathbf{T}^n) \cup \mathcal{C}^{\gamma}(\mathbf{T}^n)$ and satisfies the equation:

$$A\psi_i^-(z,s) = -\frac{\partial \mathrm{div} a_i}{\partial s}(z,s) - \mathrm{div}\left(\frac{\partial a}{\partial s}(z,s)\nabla_z \chi^-(z,s)\right). \tag{34}$$

It is obvious that

$$\int_{\mathbf{T}^n} \psi_i^-(z,s)dz = \frac{\partial}{\partial s}\int_{\mathbf{T}^n} \chi_i^-(z,s)dz = 0.$$

Now, like in (8), one can find $E_i(z,s) \in \mathcal{C}^{1,\gamma}(\mathbf{T}^n)$ such that,

$$\frac{\partial \chi_i^-}{\partial s} = \mathrm{div} E_i(z,s) \tag{35}$$

Denote $\chi^{-,\varepsilon}(x,t) := \chi^-\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)$ and $E^\varepsilon(x,t) = E\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)$.

For an arbitrary $\varphi \in \mathcal{C}_0^\infty$ set

$$\Phi^\varepsilon(t) = (u^\varepsilon(t), \varphi) + \varepsilon(\chi^{-,\varepsilon}(t)u^\varepsilon(t), \nabla\varphi) + \varepsilon^{\frac{\alpha}{2}}\left(\bar{G}^\varepsilon(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi\right)$$

We have

$$\begin{aligned}
d\big(u^\varepsilon(t), \varphi\big) =& -\big(a^\varepsilon(t)\nabla u^\varepsilon(t), \nabla\varphi\big)dt + \frac{1}{\varepsilon^{\frac{\alpha}{2}}}\big(g^\varepsilon(t), \varphi\big)dt + \big(h^\varepsilon(t), \varphi\big)dt \\
=& \frac{1}{\varepsilon}\big(u^\varepsilon(t), \mathrm{div} a_i(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha})\frac{\partial\varphi}{\partial x_i}\big)dt + \big(u^\varepsilon(t), a^\varepsilon(t)\nabla\nabla\varphi\big)dt \\
&+ \varepsilon^{1-\frac{\alpha}{2}}\big(\mathrm{div}_x\left[\tilde{H}(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t))\right], \varphi\big)dt \\
&- \varepsilon^{1-\frac{\alpha}{2}}\big(\tilde{H}'_u(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t))\nabla u^\varepsilon(t), \varphi\big)dt \\
&+ \varepsilon^{-\frac{\alpha}{2}}(\bar{g}^\varepsilon(t), \varphi)dt + (h^\varepsilon(t), \varphi)dt,
\end{aligned}$$

$$\begin{aligned}
d\left[\varepsilon^{\frac{\alpha}{2}}\left(\bar{G}^\varepsilon(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi\right)\right] =& -\varepsilon^{-\frac{\alpha}{2}}(\bar{g}^\varepsilon(t), \varphi)dt \\
&- \varepsilon^{\frac{\alpha}{2}}\big(a^\varepsilon(t)\nabla u^\varepsilon(t), \bar{G}'^{,\varepsilon}_u(t)\nabla\varphi + \bar{G}''^{,\varepsilon}_{uu}(t)\nabla u^\varepsilon(t)\varphi\big)dt \\
&+ \left(g(\frac{\cdot}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon(t))\bar{G}'_u(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi\right)dt \\
&+ \varepsilon^{\frac{\alpha}{2}}\left(h^\varepsilon(t)\bar{G}'_u(\frac{t}{\varepsilon^\alpha}, u^\varepsilon(t)), \varphi\right)dt + dM_t^{\varphi,\varepsilon},
\end{aligned}$$

$$\begin{aligned}
d[\varepsilon(\chi^{-,\varepsilon}(t)u^\varepsilon(t), \nabla\varphi)] =& \varepsilon^{1-\alpha}\left(\frac{\partial\chi^{-,\varepsilon}}{ds}(t), u^\varepsilon(t)\nabla\varphi\right)dt \\
&+ \varepsilon^1\left(\frac{\partial u^\varepsilon(t)}{dt}(t), \chi^{-,\varepsilon}(t)\nabla\varphi\right)dt \\
=& \varepsilon^{2-\alpha}(E^\varepsilon(t), u^\varepsilon(t)\nabla\nabla\varphi)dt + \varepsilon^{2-\alpha}(E^\varepsilon(t), \nabla u^\varepsilon(t)\nabla\varphi)dt \\
&+ \frac{1}{\varepsilon}\left(\nabla_z(a^\varepsilon(t)\nabla_z(\chi^{-,\varepsilon}(t))u^\varepsilon(t), \nabla\varphi\right)dt \\
&+ \left(a^\varepsilon(t)\nabla_z\chi^{-,\varepsilon}(t), u^\varepsilon(t)\nabla\nabla\varphi\right)dt \\
&- \varepsilon\left(a^\varepsilon(t)\nabla u^\varepsilon(t), \chi^{-,\varepsilon}(t)\nabla\nabla\varphi\right)dt \\
&+ \varepsilon^{1-\frac{\alpha}{2}}(g^\varepsilon(t), \chi^{-,\varepsilon}(t)\nabla\varphi)dt + \varepsilon(h^\varepsilon(t), \chi^{-,\varepsilon}(t)\nabla\varphi)dt,
\end{aligned}$$

where $M_t^{\varphi,\varepsilon}$ is a square integrable martingale. Summing up gives

$$\Phi^\varepsilon(t) = \Phi^\varepsilon(0) + \int_0^t \Big(u^\varepsilon(s), a(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}) \Big(I + \nabla_z \chi^-(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}\Big) \nabla\nabla\varphi\Big)\, ds$$
$$+ \int_0^t (h^\varepsilon(s), \varphi) ds + \int_0^t \Big(g(\frac{\cdot}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s))\bar{G}'_u(\frac{s}{\varepsilon^\alpha}, u^\varepsilon(s)), \varphi\Big)\, ds$$
$$+ M_t^{\varphi,\varepsilon} + R_t^\varepsilon,$$

where

$$\mathbf{E}(\sup_{0\le t\le T} |R_t^\varepsilon|) \to 0,$$

as $\varepsilon \to 0$. The quadratic variation of the martingale term $M_t^{\varphi,\varepsilon}$ was computed in (25)

$$< M^{\varphi,\varepsilon} >_t = \varepsilon^\alpha\big(\bar{G}(\frac{t}{\varepsilon^2}, u^\varepsilon(t)), \varphi\big)^2 - \varepsilon^\alpha\big(\bar{G}(0, u_0), \varphi\big)^2$$
$$+ 2\int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(\bar{g}^\varepsilon(s), \varphi\big) ds$$
$$+ 2\varepsilon^\alpha \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(a^\varepsilon(s)\nabla u^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\nabla\varphi + \bar{G}''^{,\varepsilon}_{uu}(s)\nabla u^\varepsilon(s)\varphi\big) ds$$
$$- 2\varepsilon^{\frac{3\alpha}{2}} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(g^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\varphi\big) ds$$
$$- 2\varepsilon^\alpha \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big)\big(h^\varepsilon(s), \bar{G}'^{,\varepsilon}_u(s)\varphi\big) ds$$
$$- 2\varepsilon^{\frac{\alpha}{2}} \int_0^t \big(\bar{G}^\varepsilon(s), \varphi\big) dM_s^{\varphi,\varepsilon}$$

We now pass to the limit in the last two expressions in the same way as we did in the proof of Theorem 1 and the required statement follows. □

5.3 *Case $\alpha > 2$: Proof of Theorem 3*

The proof of convergence is slightly more involved in this last case, comparing to the two other ones. The general strategy is the same as in the previous subsections, however we shall need to introduce and study new types of correctors, and prove some averaging lemmas adapted to those. In order to try to clarify our strategy, we split this subsection into smaller units.

5.3.1 Definition and properties of new correctors

We now define correctors which are obtained via stationary and ergodic solutions of linear parabolic PDEs with large parameters in front of the time derivative. Let $\chi_i^{+,\varepsilon}(z,s), \tilde{G}^{+,\varepsilon}(z,s,u)$ be stationary and ergodic solutions of the equations

$$\frac{1}{\varepsilon^{\alpha-2}}\frac{\partial \chi_i^{+,\varepsilon}}{\partial s}(z,s) + \mathrm{div}_z\big(a(z,s)\nabla_z \chi_i^{+,\varepsilon}(z,s)\big) = -\frac{\partial a_{ik}}{\partial z_k}(z,s),$$

$$\frac{1}{\varepsilon^{\alpha-2}}\frac{\partial \tilde{G}^{+,\varepsilon}}{\partial s}(z,s,u) + \mathrm{div}_z\big(a(z,s)\nabla_z \tilde{G}^{+,\varepsilon}(z,s,u)\big) = -\tilde{g}(z,s,u),$$

for $s \in (-\infty,+\infty)$, $z \in \mathbf{T}^n$, $u \in \mathbb{R}$.

As was proved in [8], for each $\varepsilon > 0$ these equations have stationary solutions which are unique under the centering conditions

$$\int_{\mathbf{T}^n} \chi^{+,\varepsilon}(z,s)dz = 0 \qquad \text{and} \qquad \int_{\mathbf{T}^n} \tilde{G}^{+,\varepsilon}(z,s,u)dz = 0.$$

Define

$$v_i^\varepsilon(x,t) := \chi_i^{+,\varepsilon}(x, \frac{t}{\varepsilon^{\alpha-2}}), \qquad w^\varepsilon(x,t,u) := \tilde{G}^{+,\varepsilon}(x, \frac{t}{\varepsilon^{\alpha-2}}, u). \tag{36}$$

These functions satisfy the parabolic PDEs:

$$\frac{\partial v_i^\varepsilon}{\partial t}(x,t) + \mathrm{div}\left[a(x,\frac{t}{\varepsilon^{\alpha-2}})\nabla v_i^\varepsilon(x,t)\right] = -b_i(x,\frac{t}{\varepsilon^{\alpha-2}}), \tag{37}$$

$$\frac{\partial w^\varepsilon}{\partial t}(x,t,u) + \mathrm{div}\left[a(x,\frac{t}{\varepsilon^{\alpha-2}})\nabla w^\varepsilon(x,t,u)\right] = -\tilde{g}(x,\frac{t}{\varepsilon^{\alpha-2}},u), \tag{38}$$

where $u \in \mathbb{R}$ is a parameter, and for brevity we have denoted

$$b_i(z,s) = \sum_k \frac{\partial a_{ik}}{\partial z_k}(z,s).$$

The solutions of these equations are periodic in x and stationary ergodic in t, for each fixed u.

Proposition 4 *For any* $(t,x) \in (0,\infty) \times \mathbf{T}^n$,

$$\begin{aligned} (a) \quad & v_i^\varepsilon(t,x) \to \chi_i^+(x), \\ (b) \quad & \chi_i^{+,\varepsilon}(x,t) \to \chi_i^+(x), \end{aligned} \tag{39}$$

a.s., as $\varepsilon \to 0$*, where* χ_i^+ *is a solution to the first equation in (7). Moreover these convergences are uniform on* $\mathbf{T}^n \times [0,T]$*, for any* $T > 0$*.*

Proof. First we are going to show that uniformly in ε

$$\|v_i^\varepsilon\|_{L^\infty(\mathbf{T}^n \times (-\infty,\infty))} \le C \tag{40}$$

To this end we consider the following Cauchy problems

$$\frac{\partial}{\partial t} v_i^{N,\varepsilon} + \operatorname{div}\left[a(x, \frac{t}{\varepsilon^{\alpha-2}}) \nabla v_i^{N,\varepsilon}\right] = -\mathbf{1}_{\{N-1 \le t < N\}}\, b_i(x, \frac{t}{\varepsilon^{\alpha-2}}) \quad \text{in } \mathbf{T}^n \times (-\infty, N),$$
$$v_i^{N,\varepsilon}|_{t \ge N} = 0$$

with $N = 0, \pm 1, \pm 2, \ldots$. As was shown in the proof of Lemma 4 in [9], the functions $v^{N,\varepsilon}$ satisfy the bound

$$\|v_i^{N,\varepsilon}(t,\cdot)\|_{L^\infty(\mathbf{T}^n)} \le C e^{-\kappa(N-t)} \|b_i\|_{W^{-1,\infty}(\mathbf{T}^n \times (-\infty,\infty))}, \tag{41}$$

with constants $\kappa > 0$ and $C > 0$ which only depend on the ellipticity constants of matrix $a(x,t)$.

By the same Lemma 4 in [9], the function v^ε admits the representation

$$v^\varepsilon = \sum_{N=-\infty}^{\infty} v^{N,\varepsilon}.$$

Summing up the estimates (41) we obtain the desired bound (40).

Next, combining (40) with the Nash estimate for solutions of parabolic equations, we conclude that the family of functions $\{v^\varepsilon,\ \varepsilon > 0,\ \omega \in \Omega\}$ is Hölder continuous in $\mathbf{T}^n \times (-\infty,\infty)$ and, moreover for any $a \in \mathbb{R}$,

$$\|v^\varepsilon\|_{C^\nu(\mathbf{T}^n \times [a,a+1])} \le C$$

for some $\nu > 0$ and $C > 0$ which do not depend on a. Hence the same estimate holds for the $C^\nu(\mathbf{T}^n \times (-\infty,\infty))$ norm. Therefore, for each $\omega \in \Omega$ the function v^ε converges along a subsequence, as $\varepsilon \to 0$, and the convergence is uniform on compact sets. Denote the limit function by $v^0 = v^0(x,t)$.

Now denote $V_k^\varepsilon = \frac{\partial}{\partial x_k} v^\varepsilon$. Approximating the coefficients $\{a_{ij}\}$ by smooth ones, one can easily show that V_k^ε solves the equation

$$\frac{\partial}{\partial t} V_k^\varepsilon + \mathrm{div}\left[a(x, \frac{t}{\varepsilon^{\alpha-2}})\nabla V_k^\varepsilon\right] = -\frac{\partial}{\partial x_k} b(x, \frac{t}{\varepsilon^{\alpha-2}}) - \mathrm{div}\big[(\frac{\partial}{\partial x_k} a(x, \frac{t}{\varepsilon^{\alpha-2}}))\nabla v^\varepsilon\big]$$

We want to show that for any $a \in \mathbb{R}$, $\omega \in \Omega$, $V_k^\varepsilon(\omega)$ is a compact family (indexed by $\varepsilon > 0$) of elements of $L^2(\mathbf{T}^n \times (a, a+1))$. The function on the right hand side of the V_k^ε–equation is uniformly bounded in
$L^2((a-1, a+1); H^{-1}(\mathbf{T}^n))$. Moreover there exists $C > 0$ such that

$$\|V_k^\varepsilon\|_{L^2((a-1,a)\times\mathbf{T}^n)} \le C,$$

hence for each $\varepsilon > 0$ and $\omega \in \Omega$, there exists $t_0 \in (a-1, a)$ such that

$$\|V_k^\varepsilon(t_0)\|_{L^2(\mathbf{T}^n)} \le C.$$

Now from standard parabolic estimate,

$$\|V_k^\varepsilon\|_{L^2(a,a+1;H^1(\mathbf{T}^n))} + \left\|\frac{\partial V_k^\varepsilon}{\partial t}\right\|_{L^2(a,a+1;H^{-1}(\mathbf{T}^n))} \le C',$$

which implies the whished compactness in $L^2(\mathbf{T}^n \times (a, a+1))$.

Let $\varphi = \varphi(x,t)$ be a $C^\infty(\mathbf{T}^n \times (-\infty, \infty))$ function with a compact support. Using φ as a test function in the integral identity of the first equation in (7), we get

$$\int\limits_{-\infty}^{\infty} \int_{\mathbf{T}^n} v^\varepsilon \frac{\partial \varphi}{\partial t} dx dt + \int\limits_{-\infty}^{\infty} \int_{\mathbf{T}^n} a(x, \frac{t}{\varepsilon^{\alpha-2}}) \nabla v^\varepsilon \cdot \nabla \varphi dx dt =$$

$$= -\int\limits_{-\infty}^{\infty} \int_{\mathbf{T}^n} a(x, \frac{t}{\varepsilon^{\alpha-2}}) \nabla \varphi dx dt.$$

By the Birkhoff ergodic theorem $a(x, \frac{t}{\varepsilon^{\alpha-2}})$ converges a.s., as $\varepsilon \to 0$, towards $\bar{a}(x) = \mathbf{E}a(x,s)$ weakly in $L^2_{\mathrm{loc}}(\mathbf{T}^n \times (-\infty, \infty))$. Passing to the limit in the above integral relation we find

$$\int_{-\infty}^{\infty} \int_{\mathbf{T}^n} v^0 \varphi dx dt + \int_{-\infty}^{\infty} \int_{\mathbf{T}^n} \bar{a}(x) \nabla v^0 \cdot \nabla \varphi dx dt =$$

$$= - \int_{-\infty}^{\infty} \int_{\mathbf{T}^n} \bar{a}(x) \nabla \varphi dx dt.$$

Therefore, v^0 is a bounded zero spatial average solution of the equation

$$\frac{\partial}{\partial t} v^0 + \operatorname{div} \left[\bar{a}(x) \nabla v^0 \right] = -\operatorname{div} \bar{a}(x).$$

By the uniqueness of a bounded solution, v^0 does not depend on t and solves the elliptic equation $\operatorname{div} \left[\bar{a}(x) \nabla v^0 \right] = -\operatorname{div} \bar{a}(x)$. Thus $v(x) = \chi^+(x)$, and the entire family v^ε converges a.s. to $\chi^+(x)$, as $\varepsilon \to 0$.

The second convergence in (39) is the evident consequence of the first one. □

We have also proved the following statement.

Lemma 4 *The sequences* $\{\chi_i^{+,\varepsilon}, \varepsilon > 0\}$ *and* $\left\{ \frac{\partial \chi_i^{+,\varepsilon}}{\partial x_j}, \varepsilon > 0 \right\}$ *are bounded in* $L^\infty(\mathbf{T}^n \times [0, \infty))$*, uniformly in* $\omega \in \Omega$*.*

Similar results hold for the process $\tilde{G}^{+,\varepsilon}(x,t,u)$, as well as $\tilde{G}_u^{+,\varepsilon}(x,t,u)$.

Lemma 5 *(a) For any* $(t,x) \in (0,\infty) \times \mathbf{T}^n$, $u \in \mathbb{R}$*, the following convergence takes place:*

$$\begin{aligned} w^\varepsilon(x,t,u) &\to \tilde{G}^+(x,u), \\ \tilde{G}^{+,\varepsilon}(x,t,u) &\to \tilde{G}^+(x,u), \\ \tilde{G}_u^{+,\varepsilon,\prime}(x,t,u) &\to \tilde{G}_u^{+,\prime}(x,u). \end{aligned}$$

in probability, as $\varepsilon \to 0$*.*
(b) The function $\tilde{G}^{+,\varepsilon}(x,t,u)$ *is differentiable in* x *and its partial derivatives* $\dfrac{\partial \tilde{G}^{+,\varepsilon}}{\partial x_j}(x,t,\cdot)$ *are Lipschitz, uniformly with respect to* $\varepsilon, x, t, \omega$*.*

(c) The following bounds hold

$$|\tilde{G}^{+,\varepsilon}(x,t,u)| \le C|u|, \qquad \left|\frac{\partial \tilde{G}^{+,\varepsilon}}{\partial x_j}(x,t,u)\right| \le C\,|u|\,,$$

for any $(t,x) \in [0,\infty) \times \mathbf{T}^n$, $u \in \mathbb{R}$.

We now define

$$\chi^{+,\varepsilon,\alpha}(x,t) = \chi^{+,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right), \qquad \tilde{G}^{+,\varepsilon,\alpha}(x,t,u) = \tilde{G}^{+,\varepsilon}\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u\right).$$

It is easy to see that these processes satisfy the equations

$$\varepsilon^2\frac{\partial \chi_i^{+,\varepsilon,\alpha}}{\partial t}(x,t) + \varepsilon^2 \mathrm{div}_x\big(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha})\nabla_x \chi_i^{+,\varepsilon,\alpha}(x,t)\big) = -\frac{\partial a_{ik}}{\partial z_k}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}),$$

$$\varepsilon^2\frac{\partial \tilde{G}^{+,\varepsilon,\alpha}}{\partial t}(x,t,u)+\varepsilon^2 \mathrm{div}_x\big(a(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha})\nabla_x \tilde{G}^{+,\varepsilon,\alpha}(x,t,u)\big) = -\tilde{g}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u).$$

Moreover $\tilde{G}_u^{+,\varepsilon,\alpha,\prime}$ and $\tilde{G}_{uu}^{+,\varepsilon,\alpha,\prime\prime}$ satisfy

$$\varepsilon^2\frac{\partial \tilde{G}_u^{+,\varepsilon,\alpha,\prime}}{\partial t}(x,t,u)+\varepsilon^2 \mathrm{div}_x\big(a^\varepsilon(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha})\nabla_x \tilde{G}_u^{+,\varepsilon,\alpha,\prime}(x,t,u)\big) = -\tilde{g}_u'(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u),$$

$$\varepsilon^2\frac{\partial \tilde{G}_{uu}^{+,\varepsilon,\alpha,\prime\prime}}{\partial t}(x,t,u)+\varepsilon^2 \mathrm{div}_x\big(a^\varepsilon(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha})\nabla_x \tilde{G}_{uu}^{+,\varepsilon,\alpha,\prime\prime}(x,t,u)\big) = -\tilde{g}_{uu}''(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha},u),$$

5.3.2 *Preparation for taking the limit*

Consider now the process:

$$\Psi^{\varepsilon,\alpha}(t) = \big(u^\varepsilon(t),\varphi\big) + \varepsilon^{\alpha-1}\big(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\varphi\big) + \varepsilon\big(\chi^{+,\varepsilon,\alpha}(t)u^\varepsilon(t),\nabla\varphi\big) + \varepsilon\big(\tilde{G}^{+,\varepsilon,\alpha}(t,u^\varepsilon(t)),\varphi\big).$$

Differentiating the terms on the right hand gives

$$\begin{aligned}&\varepsilon^{\alpha-1}d\big(\bar{G}(\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\varphi\big)\\ &= -\frac{1}{\varepsilon}\big(\bar{g}^\varepsilon(t),\varphi\big)dt - \varepsilon^{\alpha-1}\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\bar{G}_u^{\prime,\varepsilon}(t)\nabla\varphi + \bar{G}_{uu}^{\prime\prime,\varepsilon}(t)\nabla u^\varepsilon(t)\varphi\big)dt\\ &+ \varepsilon^{\alpha-2}\big(g^\varepsilon(t),\bar{G}_u^{\prime,\varepsilon}(t)\varphi\big)dt + \varepsilon^{\alpha-1}\big(h^\varepsilon(t),\bar{G}_u^{\prime,\varepsilon}(t)\varphi\big)dt + dM_t^{\varphi,\varepsilon},\end{aligned}$$

$$\varepsilon d\big(\chi^{+,\varepsilon}(t)u^\varepsilon(t),\nabla\varphi\big)$$
$$= \big(a^\varepsilon(t)u^\varepsilon(t),\nabla_z\chi^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha})\nabla\nabla\varphi\big)dt - \frac{1}{\varepsilon}\big(\mathrm{div}_z a_i(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha}),u^\varepsilon(t)\frac{\partial\varphi}{\partial x_i}\big)dt$$
$$-\,\varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\chi^{+,\varepsilon,\alpha}(t)\nabla\nabla\varphi\big)dt + \big(g^\varepsilon(t)\chi^{+,\varepsilon,\alpha}(t),\nabla\varphi\big)dt$$
$$+\,\varepsilon\big(h^\varepsilon(t)\chi^{+,\varepsilon,\alpha}(t),\nabla\varphi\big)dt,$$

$$\varepsilon d\big(\tilde G^{+,\varepsilon}(\cdot,t,u^\varepsilon(t)),\varphi\big)$$
$$= \big(a^\varepsilon(t)\nabla_z\tilde G^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)),\nabla\varphi\big)dt - \frac{1}{\varepsilon}\big(\tilde g^\varepsilon(t),\varphi\big)dt$$
$$-\,\varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\tilde G_u^{+,\varepsilon,\prime}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))\nabla\varphi\big)dt$$
$$-\,\varepsilon\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\tilde G_{uu}^{+,\varepsilon,\prime\prime}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))\nabla u^\varepsilon(t)\varphi\big)dt$$
$$+\,\big(g^\varepsilon(t),\tilde G_u^{+,\varepsilon,\prime}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))\varphi\big)dt + \varepsilon\big(h^\varepsilon(t),\tilde G_u^{+,\varepsilon,\prime}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))\varphi\big)dt.$$

Summing up the above relations we get

$$\begin{aligned}
d\Psi^\varepsilon(t) =& \big((I+\nabla_z\chi^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha}))a^\varepsilon(t)u^\varepsilon(t),\nabla\nabla\varphi\big)dt \\
&+ \big(a^\varepsilon(t)\nabla_z\tilde G^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t)) + \chi^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha})g^\varepsilon(t),\nabla\varphi\big)dt \\
&+ \big(\tilde G_u^{+,\varepsilon,\prime}(\frac{\cdot}{\varepsilon},\frac{t}{\varepsilon^\alpha},u^\varepsilon(t))g^\varepsilon(t),\varphi\big)dt + \big(h^\varepsilon(t),\varphi\big)dt + dM_t^{\varphi,\varepsilon} \\
&- \varepsilon\Big[\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\tilde G_{uu}^{+,\varepsilon,\alpha,\prime\prime}(t,u^\varepsilon(t))\nabla u^\varepsilon(t)\varphi + \tilde G_u^{+,\varepsilon,\alpha,\prime}(t,u^\varepsilon(t))\nabla\varphi\big)dt \\
&+ \big(a^\varepsilon(t)\nabla u^\varepsilon(t),\chi^{+,\varepsilon}(t)\nabla\nabla\varphi\big)dt \\
&- \big(h^\varepsilon(t),\chi^{+,\varepsilon}(t)\nabla\varphi + \tilde G_u^{+,\varepsilon,\alpha,\prime}(t,u^\varepsilon(t))\varphi\big)dt\Big] \\
&- \varepsilon^{\alpha-1}\big(a^\varepsilon(t)\nabla u^\varepsilon(t),\bar G_u^{\prime,\varepsilon}(t)\nabla\varphi + \bar G_{uu}^{\prime\prime,\varepsilon}(t)\nabla u^\varepsilon(t)\varphi\big)dt \\
&+ \varepsilon^{\alpha-2}\big(g^\varepsilon(t),\bar G_u^{\prime,\varepsilon}(t)\varphi\big)dt + \varepsilon^{\alpha-1}\big(h^\varepsilon(t),\bar G_u^{\prime,\varepsilon}(t)\varphi\big)dt
\end{aligned} \tag{42}$$

Denote, as in the case $\alpha = 2$, by Q a limit point of the sequence $Q^\varepsilon = \mathcal{L}(u^\varepsilon)$.

5.3.3 Some averaging lemmas

The following statements will allow us to pass to the limit in (42).

Lemma 6 *Let $E^\varepsilon(z,t,u)$ be a sequence of continuous z-periodic random fields, such that*

$$|E^\varepsilon(z,t,u)| \le C|u|,$$

$$|E^\varepsilon(z,t,u_2) - E^\varepsilon(z,t,u_1)| \le C|u_2 - u_1|,$$

for any u, u_1, $u_2 \in \mathbb{R}$, uniformly with respect to ε, t, z, ω. Moreover, suppose that for each $\varepsilon > 0$, $s > 0$, $u \in \mathbb{R}$, the function E^ε has a.s. zero spatial average : $\int_{\mathbf{T}^n} E^\varepsilon(z,s,u)dz = 0$, and for each $\varepsilon > 0$, $s > 0$ the function $u \to E^\varepsilon(z,t,u)$ is a.s. of class C^1. Then for any $t \in [0,T]$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, the following convergence holds true:

$$\int_0^t \int_K E^\varepsilon\left(\frac{x}{\varepsilon}, \frac{s}{\varepsilon^\alpha}, u^\varepsilon(s,x)\right)\varphi(x)dxds \to 0,$$

in $L^1(\Omega)$, as $\varepsilon \to 0$, where K stands for supp(φ).

Proof. Making use of the representation

$$E^\varepsilon(z,t,u) = \mathrm{div}_z\left[\kappa^\varepsilon(z,t,u)\right],$$

where $u \in \mathbb{R}$ is a parameter and $\kappa^\varepsilon(z,t,u)$ is a z-periodic function satisfying the estimates

$$|\kappa^\varepsilon(x,t,u)| \le C|u|,$$

$$|\kappa_u^{\prime,\varepsilon}(x,t,u)| \le C,$$

we obtain

$$\mathrm{div}_x\left[\kappa^\varepsilon(\frac{x}{\varepsilon}, t, u^\varepsilon(t,x))\right] = \frac{1}{\varepsilon}E^\varepsilon(\frac{x}{\varepsilon}, t, u^\varepsilon(t,x)) + \kappa_u^{\prime,\varepsilon}(\frac{x}{\varepsilon}, t, u^\varepsilon(t,x))\nabla u^\varepsilon(t,x).$$

By Proposition 1, we get

$$
\begin{aligned}
&\mathbf{E}|\int_0^t \int_K E^\varepsilon\left(\frac{x}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s,x)\right)\varphi(x)dxds| \\
&= \mathbf{E}|\varepsilon\int_0^t \int_K \kappa^\varepsilon\left(\frac{x}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s,x)\right)\nabla\varphi(x)dxds \\
&+\varepsilon\int_0^t \int_K \kappa_u^{\prime,\varepsilon}\left(\frac{x}{\varepsilon},\frac{s}{\varepsilon^\alpha},u^\varepsilon(s,x)\right)\nabla u^\varepsilon(s,x)\varphi(x)dxds| \\
&\le \varepsilon C[t\mathbf{E}\sup_{0\le s\le t}\|u^\varepsilon(s)\|_{L^2(K)}+\mathbf{E}\int_0^t\|\nabla u^\varepsilon(s)\|_{L^2(K)}ds] \\
&\le \varepsilon C(t+1).
\end{aligned}
$$

□

Let $d(z,r)$ and $c(z,r)$ be stationary, continuous, periodic in z random fields, which are measurable w.r.t. $\sigma\{a(z,r),g(z,r,u),h(z,r,u),z\in\mathbf{T}^n,u\in\mathbb{R}\}$, and satisfy, for some $C>0$,

$$
\int_{\mathbf{T}^n} c(z,r)dz=0, \qquad |d(z,r)|\le C.
$$

Define

$$
f^\varepsilon(r)=\int_{\mathbf{T}^n} d(z,\frac{r}{\varepsilon^{\alpha-2}})F^\varepsilon(z,r)dz, \quad f_1^\varepsilon(r)=\int_{\mathbf{T}^n} d(z,\frac{r}{\varepsilon^{\alpha-2}})\nabla F^\varepsilon(z,r)dz,
$$

where F^ε stands for a stationary zero average solution of the following parabolic equation:

$$
\frac{\partial F^\varepsilon}{\partial t}(z,t)+\operatorname{div}\left[a(z,\frac{t}{\varepsilon^{\alpha-2}})\nabla F^\varepsilon(z,t)\right]=c(z,\frac{t}{\varepsilon^{\alpha-2}}).
$$

Lemma 7 *For any $t>0$, the following convegences hold in $L^2(\Omega)$, as $\varepsilon\to 0$,*

$$
\frac{1}{t}\int_0^t f^\varepsilon(\frac{r}{\varepsilon^2})dr-\mathbf{E}f^\varepsilon(0)\to 0, \qquad \frac{1}{t}\int_0^t f_1^\varepsilon(\frac{r}{\varepsilon^2})dr-\mathbf{E}f_1^\varepsilon(0)\to 0.
$$

Proof. Denote by $F^{N,\varepsilon}(z,t)$ the solution of the Cauchy problem

$$
\frac{\partial F^{N,\varepsilon}}{\partial t}(z,t)+\operatorname{div}\left[a(z,\frac{t}{\varepsilon^{\alpha-2}})\nabla F^{N,\varepsilon}(z,t)\right]=c(z,\frac{t}{\varepsilon^{\alpha-2}}),
$$

$$
F^{N,\varepsilon}(z,N)=0, \quad (z,t)\in\mathbf{T}^n\times(-\infty,N),
$$

where N is an arbitrary real number. The difference $(F^{N,\varepsilon}(z,t) - F^{\varepsilon}(z,t))$ decays exponentially, as $(N-t) \to \infty$, uniformly in ε, that is

$$\sup_{z \in \mathbf{T}^n,\, t \in [k,k+1]} |F^{N,\varepsilon}(z,t) - F^{\varepsilon}(z,t)| \leq C e^{-\gamma(N-k)},$$

for any $k \leq N$, with nonrandom constants C and γ. Denote $f^{N,\varepsilon}(r) = \int_{\mathbf{T}^n} d(z, \frac{r}{\varepsilon^{\alpha-2}}) F^{N,\varepsilon}(z,r)dz$. By integrating the latter inequality over $\mathbf{T}^n$ we get

$$|f^{N,\varepsilon}(t) - f^{\varepsilon}(t)| \leq C e^{-\gamma(N-t)}, \tag{43}$$

Since $f^{\frac{t}{2},\varepsilon}(0)$ is measurable with respect to the events before $\frac{t}{2\varepsilon^{\alpha-2}}$, and $f^{\varepsilon}(t)$ is measurable with respect to the events after time $\frac{t}{\varepsilon^{\alpha-2}}$, for each $t > 0$,

$$\begin{aligned}
&|\mathbf{E}[(f^{\varepsilon}(t)f^{\varepsilon}(0) - \mathbf{E}(f^{\varepsilon}(t)\mathbf{E}f^{\varepsilon}(0)]| \\
&\quad = |\mathbf{E}[(f^{\varepsilon}(t) - \mathbf{E}(f^{\varepsilon}(t))f^{\varepsilon}(0)]| \\
&\quad = |\mathbf{E}\{(f^{\varepsilon}(0) - f^{\frac{t}{2},\varepsilon}(0))(f^{\varepsilon}(t) - \mathbf{E}f^{\varepsilon}(t)) + f^{\frac{t}{2},\varepsilon}(0)(f^{\varepsilon}(t) - \mathbf{E}f^{\varepsilon}(t))\}| \\
&\quad \leq 2Ce^{-\gamma\frac{t}{2}}\mathbf{E}|f^{\varepsilon}(t)| + C\phi(\frac{t}{2\varepsilon^{\alpha-2}})\sqrt{\mathbf{E}(f^{\frac{t}{2},\varepsilon}(0))^2\mathbf{E}(f^{\varepsilon}(0))^2} \\
&\quad \leq C(e^{-\gamma\frac{t}{2}} + \phi(\frac{t}{2\varepsilon^{\alpha-2}})),
\end{aligned}$$

where $\phi(t)$ denotes again the uniform mixing coefficient (for further details see Lemmas 3 and 4 in [8]). Hence

$$\begin{aligned}
&\mathbf{E}\big[\big(\frac{\varepsilon^2}{t}\int_0^{\frac{t}{\varepsilon^2}}(f^{\varepsilon}(s) - \mathbf{E}f^{\varepsilon}(s))ds\big)^2\big] \\
&\quad = \frac{\varepsilon^4}{t^2}\int_0^{\frac{t}{\varepsilon^2}}\int_0^{\frac{t}{\varepsilon^2}}\mathbf{E}[f^{\varepsilon}(s)f^{\varepsilon}(r) - \mathbf{E}(f^{\varepsilon}(s))\mathbf{E}(f^{\varepsilon}(r))]dsdr \\
&\quad = 2\frac{\varepsilon^4}{t^2}\int_0^{\frac{t}{\varepsilon^2}}\int_0^{s}\mathbf{E}[f^{\varepsilon}(s)f^{\varepsilon}(r) - \mathbf{E}(f^{\varepsilon}(s))\mathbf{E}(f^{\varepsilon}(r))]drds \\
&\quad \leq 2C\frac{\varepsilon^4}{t^2}\int_0^{\frac{t}{\varepsilon^2}}\int_0^{s}[e^{-\gamma\frac{s-r}{2}} + \phi(\frac{s-r}{2\varepsilon^{\alpha-2}})]drds \\
&\quad \leq C'\frac{\varepsilon^2}{t}.
\end{aligned}$$

where we have used the assumption **(A.6)** and the stationarity of the random field $f^{\varepsilon}(s)$. The first result follows. The second one can be proved similarly. □

Lemma 8 *The following convergence holds, as* $\varepsilon \to 0$, *for any* $r \geq 0$

$$(a) \quad \mathbf{E}\int_{\mathbf{T}^n} \Big(a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla_z v^\varepsilon(z,r)\Big)\, dz - \mathbf{E}\int_{\mathbf{T}^n} \big(a(z,0)(I + \nabla\chi^+(z))\big)\, dz \to 0$$

$$(b) \quad \mathbf{E}\int_{\mathbf{T}^n} \Big(a(z, \frac{r}{\varepsilon^{\alpha-2}})\nabla_z w^\varepsilon(z, r, u^\varepsilon(\varepsilon z, \varepsilon^2 r))\Big)\, dz - \mathbf{E}\int_{\mathbf{T}^n} \langle a\nabla_z \tilde{G}^+\rangle(u^\varepsilon(\varepsilon z, \varepsilon^2 r))dz \to$$

where for each $\varepsilon > 0$, u^ε *is the solution of the problem (1),* v^ε *and* w^ε *are defined in* (36).

Proof Denote

$$\eta^\varepsilon = \mathbf{E}\int_{\mathbf{T}^n} a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla_z v^\varepsilon(z,r))dz.$$

In view of the stationarity of the integrand, we may write :

$$\eta^\varepsilon = \mathbf{E}\int_{\mathbf{T}^n}\int_0^1 a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla_z v^\varepsilon(z,r))drdz.$$

By the definition of v^ε we have

$$\begin{aligned}
&\mathbf{E}\int_0^1\int_{\mathbf{T}^n} a(z, \frac{r}{\varepsilon^{\alpha-2}})\nabla\chi^+(z)(I + \nabla_z v^\varepsilon(z,r))dzdr \\
&\quad = -\mathbf{E}\int_0^1\int_{\mathbf{T}^n} \mathrm{div}[a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla_z v^\varepsilon(z,r))]\,\chi^+(z)dzdr \\
&\quad = \mathbf{E}\int_{\mathbf{T}^n}\int_0^1 \frac{\partial v^\varepsilon}{\partial s}(z,s)\chi^+(z)dsdz = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\eta^\varepsilon &= \mathbf{E}\int_0^1\int_{\mathbf{T}^n} a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla\chi^+(z))(I + \nabla_z v^\varepsilon(z,r))drdz \\
&= \mathbf{E}\int_0^1\int_{\mathbf{T}^n} a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla\chi^+(z))dz \\
&- \mathbf{E}\int_0^1\int_{\mathbf{T}^n} \mathrm{div}\left[a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla\chi^+(z))\right] v^\varepsilon(z,s)dzds \\
&= \mathbf{E}\int_{\mathbf{T}^n} a(z,0)(I + \nabla\chi^+(z))dz - \mathbf{E}\int_{\mathbf{T}^n} \mathrm{div}\left[a(z,0)(I + \nabla\chi^+(z))\right]\chi^+(z)dz \\
&- \mathbf{E}\int_0^1\int_{\mathbf{T}^n} \mathrm{div}\left[a(z, \frac{r}{\varepsilon^{\alpha-2}})(I + \nabla\chi^+(z))\right](v^\varepsilon(z,r) - \chi^+(z))dzdr.
\end{aligned}$$

The second term on the r. h. s. is equal to 0 by (7). The third term tends to 0 by Proposition 4, Lemma 4 and the boundedness of the first factor of the integrand. Statement (b) can be proved in an analogous way, using Proposition 1, Lemma 5, and the argument developed in the last part of the proof of Lemma 9 below (the study of the term I_2^ε). □

5.3.4 Passage to the limit

Let $\{u^\varepsilon\}$ denote a subsequence of solutions of (1) that converges in law, as $\varepsilon \to 0$, in the space $\tilde{V}_T$. We denote by Q the limit law, and by u the generic element of $\tilde{V}_T$.

Proposition 5 *For any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, the process M_t^φ defined on the probability space $(V_T, \mathcal{B}(\tilde{V}_T), Q)$ by the formula:*

$$M_t^\varphi := (u(t),\varphi) - (u_0,\varphi) - \int_0^t (u(s), \tilde{\mathbf{a}}\nabla\nabla\varphi)\, ds + \int_0^t \Big(\hat{\mathbf{b}}(u(s)), \nabla\varphi\Big)\, ds - \int_0^t \Big(\hat{\mathbf{h}}(u(s)), \varphi\Big)\, ds$$

is a martingale with respect to the natural filtration of σ-algebras $\mathcal{B}(\tilde{V}_t)$, $0 \le t \le T$.

Proof Fix $0 \le s < t \le T$ and let again Θ_s denote an arbitrary bounded continuous functional defined on $\tilde{V}_s$. We denote $\Theta_s^\varepsilon = \Theta_s(u^\varepsilon)$, and write Θ_s for $\Theta_s(u)$. By the formula (42), we have:

$$\begin{aligned}
0 &= \mathbf{E}\big[(M_t^{\varphi,\varepsilon} - M_s^{\varphi,\varepsilon})\Theta_s^\varepsilon\big] = \mathbf{E}\big[(u^\varepsilon(t),\varphi) - (u^\varepsilon(s),\varphi)\big]\Theta_s^\varepsilon \\
&- \mathbf{E}\Big[\int_s^t \big(a^\varepsilon(r)(I + \nabla_z\chi^{+,\varepsilon}(\tfrac{\cdot}{\varepsilon}, \tfrac{r}{\varepsilon^\alpha}))u^\varepsilon(r), \nabla\nabla\varphi\big)dr\Big]\Theta_s^\varepsilon \\
&- \mathbf{E}\Big[\int_s^t \big(a^\varepsilon(r)\nabla_z\tilde{G}^{+,\varepsilon}(\tfrac{\cdot}{\varepsilon}, \tfrac{r}{\varepsilon^\alpha}, u^\varepsilon(r)) + \chi^{+,\varepsilon}(\tfrac{\cdot}{\varepsilon}, \tfrac{r}{\varepsilon^\alpha})g^\varepsilon(r), \nabla\varphi\big)dr\Big]\Theta_s^\varepsilon \\
&- \mathbf{E}\Big[\int_s^t \big(\tilde{G}_u^{+,\varepsilon,\prime}(\tfrac{\cdot}{\varepsilon}, \tfrac{r}{\varepsilon^\alpha}, u^\varepsilon(r))g^\varepsilon(r), \varphi\big)dr\Big]\Theta_s^\varepsilon \\
&- \mathbf{E}\Big[\int_s^t \big(h^\varepsilon(r), \varphi\big)dr\Big]\Theta_s^\varepsilon + \varepsilon^{\alpha-2}R_\varepsilon,
\end{aligned}$$

where R_ε is bounded. We proceed with the following statement.

Lemma 9 *For any test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $0 < s < t < T$, the following convergence takes place, as $\varepsilon \to 0$,*

(a) $\mathbf{E}\int_s^t \Big(a(\tfrac{\cdot}{\varepsilon}, \tfrac{r}{\varepsilon^\alpha})(I + \nabla_z\chi^{+,\varepsilon}(\tfrac{\cdot}{\varepsilon}, \tfrac{r}{\varepsilon^\alpha}))u^\varepsilon(r), \nabla\nabla\varphi\Big)\Theta_s^\varepsilon dr$

$$\to \mathbf{E}^Q \int_s^t \Big(\langle a(I+\nabla_z\chi^+)\rangle u(r), \nabla\nabla\varphi\Big)\Theta_s dr,$$

where $\langle a(I+\nabla_z\chi^+)\rangle = \mathbf{E} \int_{\mathbf{T}^n} a(z,r)(I+\nabla\chi^+(z))dz$*, and* $\mathbf{E}^Q$ *denotes expectation with respect to the measure* Q*;*

(b) $$\mathbf{E}\int_s^t \Big(a(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha})(\nabla_z\tilde{G}^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha},u^\varepsilon(r)),\nabla\varphi\Big)\Theta_s^\varepsilon dr \to \mathbf{E}^Q\int_s^t \Big(\langle a\nabla_z\tilde{G}^+\rangle(u(r)),\nabla\varphi\Big)\Theta_s dr,$$

(c) $$\mathbf{E}\int_s^t \Big(\chi^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha})g(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha},u^\varepsilon(r)),\nabla\varphi\Big)\Theta_s^\varepsilon \to \mathbf{E}^Q\int_s^t \langle\chi^+ g\rangle(u(r)),\nabla\varphi\Big)\Theta_s dr,$$

(d) $$\mathbf{E}\int_s^t \Big(\tilde{G}_u^{+,\varepsilon,\prime}(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha},u^\varepsilon(s))g(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha},u^\varepsilon(r))+h^\varepsilon(r),\varphi\Big)\Theta_s^\varepsilon dr \to \mathbf{E}^Q\int_s^t \langle\tilde{G}_u^{+,\prime}g+h\rangle(u(r)),\varphi\Big)\Theta_s dr.$$

Proof. We prove the first statement only. Essentially similar arguments apply to the others. Denote $K := supp(\varphi)$. In (a), we only consider the most complex term :

$$\begin{aligned}
&\Big|\mathbf{E}\int_s^t \big(a(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha})\nabla_z\chi^{+,\varepsilon}(\frac{\cdot}{\varepsilon},\frac{r}{\varepsilon^\alpha})u^\varepsilon(r),\nabla\nabla\varphi\big)\Theta_s^\varepsilon dr - \mathbf{E}^Q\int_s^t\langle a\nabla_z\chi^+\rangle u(r),\nabla\nabla\varphi\big)\Theta_s dr\Big| \\
&\quad\le \mathbf{E}\int_s^t\Big|\int_K\Big[a(\frac{x}{\varepsilon},\frac{r}{\varepsilon^\alpha})\nabla_z\chi^{+,\varepsilon}(\frac{x}{\varepsilon},\frac{r}{\varepsilon^\alpha}) \\
&\qquad\quad - \int_{\mathbf{T}^n} a(z,\frac{r}{\varepsilon^\alpha})\nabla\chi^{+,\varepsilon}(z,\frac{r}{\varepsilon^\alpha})dz\Big]u^\varepsilon(x,r)\nabla\nabla\varphi(x)dx\Big|dr\,|\Theta_s^\varepsilon| \\
&\quad + \mathbf{E}\int_K\Big|\int_s^t\Big[\int_{\mathbf{T}^n} a(z,\frac{r}{\varepsilon^\alpha})\nabla\chi^{+,\varepsilon}(z,\frac{r}{\varepsilon^\alpha})dz \\
&\qquad\quad - \mathbf{E}\int_{\mathbf{T}^n} a(z,0)\nabla_z v^\varepsilon(z,0)dz\Big]u^\varepsilon(x,r)\nabla\nabla\varphi(x)dr\Big|dx|\Theta_s^\varepsilon| \\
&\quad + \Big|\mathbf{E}\int_K\Big\{\int_s^t\Big[\mathbf{E}\int_{\mathbf{T}^n} a(z,0)\nabla_z v^\varepsilon(z,0)dz - \langle a\nabla_z\chi^+\rangle\Big] \\
&\qquad\quad \times u^\varepsilon(x,r)\nabla\nabla\varphi(x)dr\Big\}dx\Big||\Theta_s^\varepsilon| \\
&\quad + \Big|\int_K\mathbf{E}\int_s^t\langle a\nabla_z\chi^+\rangle u^\varepsilon(x,r)\nabla\nabla\varphi(x)drdx\Theta_s^\varepsilon \\
&\qquad\quad - \int_K\mathbf{E}^Q\int_s^t\langle a\nabla_z\chi^+\rangle u(x,r)\nabla\nabla\varphi(x)drdx\Theta_s\Big| \\
&\quad = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon.
\end{aligned}$$

From the convergence in law of the uniformly integrable subsequence $\{u^\varepsilon\}$ it follows that $I_4^\varepsilon \to 0$, as $\varepsilon \to 0$. The uniform integrability is a direct consequence of Proposition 1. The convergence of I_1^ε to 0 follows from Lemma 6 and Lebesgue's dominated convergence theorem, while the same result for the integral I_3^ε is proved in Lemma 8.

It remains to consider the second term I_2^ε. Fix a small $\delta > 0$. The tightness of the sequence (u^ε) allows us to choose, for any $t > 0$ and any compact set K, the step functions $q_j(r,x)$, $1 \le j \le N$, defined on $(0,T) \times K$, such that: $\mathbf{P}\left(\cap_j (B_j^\varepsilon)^c\right) < \delta$, $\forall \varepsilon > 0$, where the events B_j^ε, $1 \le j \le N$, are disjoint and such that

$$B_j^\varepsilon \subseteq \{\|u^\varepsilon(x,r) - q_j(r,x)\|_{L^2((0,t)\times K)} < \delta\}.$$

This δ–net can be chosen in such a way that all its elements q_j have the form:

$$q_j(r,x) = \sum_i \alpha_i^j \mathbf{1}_{[t_{i-1}^j, t_i^j]\times K_i^j}(r,x),$$

where $\{t_i^j,\ 1 \le i \le N\}$, is a partition of $[0,t]$ and the sets $\{K_i^j,\ 1 \le i \le N\}$ are disjoint and such that their union contains K. Denote

$$A^\varepsilon := \bigcap_j \{\|u^\varepsilon(x,r) - q_j(r,x)\|_{L^2((0,t)\times K)} > \delta\},$$

and

$$e^\varepsilon(r) := \int_{\mathbf{T}^n} a(z, \frac{r}{\varepsilon^{\alpha-2}}) \nabla_z v^\varepsilon(z,r) dz$$

We then obtain:

$$\begin{aligned}
I_2^\varepsilon &\le C \int_{A^\varepsilon} \int_K \int_s^t \left| e^\varepsilon(\frac{r}{\varepsilon^2}) - \mathbf{E}(e^\varepsilon(0)) \right| \, |u^\varepsilon(x,r)| \, dr dx \, d\mathbf{P} \\
&+ C \sum_j \int_{B_j^\varepsilon} [\sum_i | \int_{t_{i-1}}^{t_i} \left[e^\varepsilon(\frac{r}{\varepsilon^2}) - \mathbf{E}(e^\varepsilon(0)) \right] dr |\alpha_i^j] \, d\mathbf{P} \\
&+ C \sum_j \int_{B_j^\varepsilon} \int_s^t \int_K |u^\varepsilon(r,x) - q_j(r,x)| \, dr dx \, d\mathbf{P} \\
&= J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon.
\end{aligned}$$

It is clear that $J_1^\varepsilon < C\delta$, due to the fact that $\mathbf{P}(A^\varepsilon) < \delta$. Lemma 7 implies that $J_2^\varepsilon \to 0$. Finally, J_3^ε satisfies the estimate

$$J_3^\varepsilon < C\sqrt{t}\sum_j \int_{B_j^\varepsilon} \|u^\varepsilon - q_j\|_{L^2((0,t)\times K)} < C\sqrt{T}\delta.$$

The convergence of I_2^ε to 0 is now obvious. □

The quadratic variation of the martingale term $M^{\varphi,\varepsilon}$ was computed in the formula (21) from which it easily follows that

$$\lim_{\varepsilon\to 0} \mathbf{E}\big(\langle M^{\varphi,\varepsilon}\rangle_t\big) = 0.$$

This implies

$$\langle M^\varphi\rangle_t = 0, \quad 0 \le t \le T.$$

□

Combining this with Proposition 5, we conclude that, on the probability space $(V_T, \mathcal{B}(\tilde{V}_T), Q)$, we have:

$$\begin{aligned}(v(t),\varphi) - (u_0,\varphi) - \int_0^t (v(s), \tilde{\mathbf{a}}\nabla\nabla\varphi)\,ds \\ + \int_0^t \Big(\hat{\mathbf{b}}(v(s)), \nabla\varphi\Big)\,ds - \int_0^t \Big(\hat{\mathbf{h}}(v(s)), \varphi\Big)\,ds = 0,\end{aligned}$$

Q a.s. In the latter relation we have used the notation

$$\hat{\mathbf{b}}(u) := \mathbf{E}\int_{\mathbf{T^n}} [a(z,s)\nabla_z \tilde{G}^+(z,u) + g(z,s,u)\chi^+(z)]dz, \quad u \in \mathbb{R}.$$

Let us show that $\hat{\mathbf{b}} = 0$. Indeed, by the definition of χ^+ and $\tilde{G}^+$ one has:

$$\begin{aligned}&\mathbf{E}\int_{\mathbf{T^n}} [a(z,s)\nabla_z \tilde{G}^+(z,u) + g(z,s,u)\chi^+(z)]dz \\ &\quad = \int_{\mathbf{T^n}} [\bar{a}(z)\nabla_z \tilde{G}^+(z,u) + \overline{(\tilde{g})}(z,u)\chi^+(z)]dz \\ &\quad = -\int_{\mathbf{T^n}} [\operatorname{div}(\bar{a}(z)) \quad \tilde{G}^+(z,u)]dz + \int_{\mathbf{T^n}} [\bar{A}(\tilde{G}^+(\cdot,u))(z)\chi^+(z)]dz \\ &\quad = -\big(\bar{A}\chi^+, \tilde{G}^+(u)\big)_{L^2(\mathbf{T}^n)} + \big(\bar{A}\tilde{G}^+(u), \chi^+\big)_{L^2(\mathbf{T}^n)} = 0,\end{aligned}$$

where we have also used the assumption **(A.5)**. Hence Q is the Dirac mass concentrated at a solution of the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \mathrm{div}(\tilde{\mathbf{a}}\nabla u(t,x)) + \hat{\mathbf{h}}(u); 0 < t < T, x \in \mathbf{R}^n; \\ u^\varepsilon(0,x) = u_0(x), x \in \mathbf{R}^n. \end{cases}$$

Since this problem has a unique solution, the whole sequence $\{u^\varepsilon,\ \varepsilon > 0\}$ converges in probability to the solution of the above Cauchy problem.

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