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MARKOV FIELD PROPERTIES OF SOLUTIONS OF WHITE NOISE DRIVEN QUASI-LINEAR PARABOLIC PDEs

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In this paper we study a one-dimensional quasi-linear parabolic stochastic partial differential equation driven by a space-time white noise. First the germ Markov field property is proved for the solution of a Cauchy problem for this equation. Secondly, we introduce periodic (in time) boundary conditions and we study the existence and uniqueness of a solution and its Markov properties. The main result is that for the periodic solution the Markov field property only holds in the linear case.

KEY WORDS: Parabolic stochastic partial differential equations, Markov fields, Girsanov theorem.

AMS Subject classification: 60H15, 60J25

0. INTRODUCTION

Let $\{u(t,x);\ (t,x)\in[0,1]^2\}$ denote the solution of the quasi-linear stochastic partial differential equation (in short SPDE):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + \dfrac{\partial^2 W}{\partial t \partial x}(t,x), (t,x)\in[0,1]^2; \\ u(0,x) = u_0(x), \quad 0 \leqslant x \leqslant 1; \quad u(t,0) = u(t,1) = 0, \quad 0 \leqslant t \leqslant 1, \end{cases} \tag{0.1}$$

where $\{W(t,x); (t,x)\in[0,1]^2\}$ is a standard Brownian sheet, and its second order mixed derivative is a "white noise". Of course, (0.1) is a completely formal way of writing a white noise driven parabolic PDE, and we shall introduce a rigorous formulation below.

The first objective of this paper is to prove that the random field u possesses a germ Markov field property to be made precise below.

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Our second objective is to consider the SPDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + \dfrac{\partial^2 W}{\partial t \partial x}(t,x), (t,x) \in [0,1]^2; \\ u(0,x) = u(1,x), \quad 0 \leqslant x \leqslant 1; \quad u(t,0) = u(t,1) = 0, \quad 0 \leqslant t \leqslant 1, \end{cases} \tag{0.2}$$

where the initial condition in (0.1) prescribing the value of $\{u(0,x),\ 0 \leqslant x \leqslant 1\}$ is replaced by a periodicity condition, which prescribes that the values of $u(t,\cdot)$ at times $t = 0$ and $t = 1$ are identical a.s. We shall both study existence and uniqueness for (0.2), and the germ Markov field property of the solution. Our main result is that the solution of Eq. (0.2) possesses a (certain) germ Markov field property iff f is affine.

Our main tool for studying the Markov field property is the Girsanov transformation, which reduces us to the case of a linear equation. In the case of Eq. (0.1), we use the (usual) Girsanov theorem for an adapted translation of the Brownian sheet. In the case of Eq. (0.2), we must use an extension of the Girsanov theorem to nonadapted transformations, due to Kusuoka [8]. The reason for the major difference between the results concerning Eq. (0.1) and (0.2) is that in the adapted case the Radon–Nikodym derivative given by the Girsanov theorem is a multiplicative functional, and in the nonadapted case it is not.

Our negative results for the nonlinear equation with boundary conditions both in x and t should be compared with similar results obtained by the same authors in [11] and [12] for certain classes of finite dimensional SDE's with boundary conditions, as well as the results of Donati-Martin [5], [4] which concern respectively elliptic SPDEs, and versions of the results in [11], [12] and [5] for equations where the time (resp. the space) parameter varies in a (finite) discrete set.

Note that on the other hand the extension of our positive result concerning Eq. (0.1) to equations with a nonconstant diffusion coefficient is an open problem.

The paper is organized as follows. In section one, we prove a Markov field property of the solution of a Cauchy problem (initial condition) for a parabolic PDE with additive space-time white noise. In section 2 we prove existence and uniqueness of the solution of the same equation satisfying a periodic (in time) boundary condition, and we study the Markov property in the linear-Gaussian case. In section 3, we compute a Radon–Nikodym derivative, and in section 4 we prove that the Markov field property of the periodic solution implies that the equation is linear.

1. WHITE-NOISE DRIVEN PARABOLIC PDE WITH INITIAL CONDITION: MARKOV FIELD PROPERTY

We first give a precise formulation of Eq. (0.1) and state an existence and uniqueness result.

Let $\{W_{t,x}; (t,x) \in [0,1]^2\}$ be a Brownian sheet defined on the canonical probability space $(\Omega, \mathscr{F}, P)$, i.e. $\Omega = C(0,1]^2)$, $\mathscr{F}$ is the Borel σ-field over Ω completed with respect to the Wiener sheet measure P. Under P, the canonical field $W_{t,x}(\omega) \triangleq \omega(t,x)$ is a zero mean continuous Gaussian random field with covariance function given by

$$E[W_{t,x} W_{s,y}] = (t \wedge s)(x \wedge y).$$

Let $f: [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ be a measurable function of the form

$$f(t,x;z) = f_1(t,x) + f_2(t,x;z), \tag{A.0}$$

where f_1 and f_2 are jointly measurable, and the following properties are satisfied:

$$f_1 \in L^2([0,1]^2, dt \times dx), \quad f_2(t,x;0) \equiv 0; \tag{A.1}$$

(A.2) $\exists c > 0$ such that $zf_2(t,x;r) \leqslant cz^2$; for all $(t,x) \in [0,1]^2$, $z \in \mathbb{R}$.

Finally we are given an initial condition u_0 such that

$$u_0 \in C_0([0,1]), \tag{A.3}$$

where $C_0([0,1])$ denotes set of continuous functions defined on $[0,1]$ which vanish at 0 and 1.

We say that a random field u satisfying $u \in C([0,1]^2)$ a.s. is a solution of Eq. (0.1) if equivalently either for any $t \in [0,1]$, $\phi \in C([0,1])$ with $\phi(0) = \phi(1) = 0$,

$$\begin{aligned}(u(t,\cdot),\phi) = (u_0,\phi) + \int_0^t (u(s,\cdot),\phi'')\,ds \\ + \int_0^t (f(s,\cdot;u(s,\cdot)),\phi)\,ds + \int_0^t \int_0^1 \phi(x)\,W(ds,dx),\end{aligned} \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $L^2(0,1)$, or else for any $(t,x) \in [0,1]^2$,

$$\begin{aligned}u(t,x) = \int_0^1 G_t(x,y)u_0(y)\,dy + \int_0^t \int_0^1 G_{t-s}(x,y)f(s,y;u(s,y))\,dy\,ds \\ + \int_0^t \int_0^1 G_{t-s}(x,y)\,W(ds,dy),\end{aligned} \tag{1.2}$$

where $G_t(x,y)$ is the fundamental solution of the heat equation on the space interval $[0,1]$ with Dirichlet boundary conditions, i.e.

$$\varphi(t,x) \triangleq \int_0^1 G_t(x,y)u_0(y)\,dy$$

is the unique solution of the PDE:

$$\begin{cases}\dfrac{\partial\varphi}{\partial t}(t,x) = \dfrac{\partial^2\varphi}{\partial x^2}(t,x), \quad (t,x) \in (0,1)^2; \\ \varphi(0,x) = u_0(x), \quad 0 \leqslant x \leqslant 1; \quad \varphi(t,0) = \varphi(t,1) = 0, \quad 0 \leqslant t \leqslant 1.\end{cases}$$

It is shown in Walsh [16] that the weak formulation (1.1) and the integral formulation (1.2) are equivalent. The following result can be found in Gyöngy–Pardoux [6].

and H denote the Gaussian space generated by $\{W_{t,x}; (t,x)\in[0,1]^2\}$. H is in bijection with $L^2((0,1)^2)$ as follows:

$$X\in H \quad \text{iff there exists} \quad h\in L^2((0,1)^2)$$

$$\text{such that} \quad X=\int_{[0,1]^2} h\,dW.$$

Now $\mathscr{H}$ is a subset of Σ_0, which is in bijection with H (i.e. with $L^2((0,1)^2)$) as follows.

An element ϕ in Σ_0 belongs to $\mathscr{H}$ iff there exists $X\in H$ such that

$$\phi(t,x)=E(X\,v(t,x)),\quad (t,x)\in[0,1]^2$$

i.e. iff there exists $\bar{\phi}\in L^2((0,1)^2)$ such that

$$\phi(t,x)=(G_{t-\cdot}(x,\cdot),\bar{\phi})_{L^2((0,1)^2)}$$

where $G_{t-s}(x,y)=0$ for $s>t$. Moreover, given ϕ, ψ in $\mathscr{H}$ with associated $\bar{\phi}$, $\bar{\psi}$ in $L^2((0,1)^2)$, then

$$(\phi,\psi)_{\mathscr{H}}=(\bar{\phi},\bar{\psi})_{L^2((0,1)^2)}.$$

In other words,

$$\mathscr{H}=\Sigma_0\cap H^1((0,1)^2)\cap L^2(0,1;dt;H^2(0,1))$$

and for $\phi,\psi\in\mathscr{H}$,

$$(\phi,\psi)_{\mathscr{H}}=\left(\frac{\partial\phi}{\partial t}-\frac{\partial^2\phi}{\partial x^2},\frac{\partial\psi}{\partial t}-\frac{\partial^2\psi}{\partial x^2}\right)_{L^2((0,1)^2)}$$

We now have the following (see Theorem 5.1 in Künsch [7]), which provides an improvement over an earlier result by Pitt ([14]):

PROPOSITION 1.4 *The Gaussian random field v given by* (1.3) *is a germ Markov field iff the RKHS $\mathscr{H}$ is local in the sense that it satisfies the two following properties*:

(i) *Whenever $\phi,\psi\in\mathscr{H}$ have disjoint supports, $(\phi,\psi)_{\mathscr{H}}=0$.*
(ii) *If $\phi\in\mathscr{H}$ is of the form $\phi=\phi_1+\phi_2$ with $\phi_1,\phi_2\in\Sigma_0$ with disjoint supports, then $\phi_1,\phi_2\in\mathscr{H}$.*

We can now prove the following result.

THEOREM 1.5 *The Gaussian random field v given by* (1.3) *is a germ Markov field.*

Proof If suffices to check conditions (i) and (ii) of Proposition 1.4. They follow easily from our definition of $\mathscr{H}$ and its scalar product. Indeed, if ϕ, ψ belong to $\mathscr{H}$ and have disjoint supports, then

$$(\phi,\psi)_{\mathscr{H}}=\left(\frac{\partial\phi}{\partial t}-\frac{\partial^2\phi}{\partial x^2},\frac{\partial\psi}{\partial t}-\frac{\partial^2\psi}{\partial x^2}\right)_{L^2((0,1)^2)}=0.$$

Finally, if ϕ is an element of $\mathscr{H}$ is of the form $\phi=\phi_1+\phi_2$, with ϕ_1, ϕ_2 having disjoint supports, then clearly if $|||\cdot|||$ denotes the norm in $H^1((0,1)^2)\cap L^2((0,1);dt;H^2(0,1))$,

$$|||\phi|||^2=|||\phi_1|||^2+|||\phi_2|||^2<\infty.$$

Hence both ϕ_1 and ϕ_2 belong to $\mathscr{H}$. □

We now turn to the study of the germ field Markov property for u, the solution of the nonlinear Eq. (0.1). We shall deduce it from Theorem 1.5 via a continuous change of probability. Consider the random variable

$$J\triangleq\exp\left[\int_{[0,1]^2}f(t,x;v(t,x))W(dt,dx)-\frac{1}{2}\int_{[0,1]^2}f^2(t,x;v(t,x))\,dt\,dx\right]$$

where the first integrand in the exponential is an Itô integral with respect to the Brownian sheet (see e.g. Walsh [16] for a rigorous definition), and v is the process defined in (1.3).

We need to formulate a new condition on f:

(A.4) $\exists c>0$ such that $|f_2(t,x;z)|\leqslant c(1+|z|), (t,x)\in[0,1]^2,\quad z\in\mathbb{R}$.

Under all the above assumptions, including (A.4), we have the following Girsanov theorem.

PROPOSITION 1.6 *$E(J)=1$, and if Q denotes the probability measure on $(\Omega,\mathscr{F})$ defined by*

$$\frac{dQ}{dP}(\omega)=J(\omega),$$

then $W_{t,x}-\int_0^t\int_0^x f(s,y;v(s,y))\,dy\,ds$ is a standard Brownian sheet under Q.

Consequently, v solves under Q an equation with the same coefficients (but a different driving Brownian sheet) as u solves under P. Hence the law of v under Q coincides with the law of u under P.

THEOREM 1.7 *Under the above assumptions (A.0), (A.1), (A.2), (A.4), on f and (A.3), the solution u of Eq. (A.1) is a germ Markov field.*

Proof From our discussion preceding this theorem, it suffices to show that v is a germ Markov field under Q. Let A be an open subset of $[0,1]^2$ with a smooth boundary. Set $\mathscr{F}_1\triangleq\mathscr{F}^v_A$ and $\mathscr{F}^2\triangleq\mathscr{F}^v_{A^c}$. We need to show that $\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent, given $\mathscr{G}^v_{\partial A}$. Let χ be any nonnegative $\mathscr{F}_1$-measurable random variable.

We shall again assume in this section that f satisfies (A.0), (A.1) and (A.2)', where:
(A.2)' There exists $0 < c < 2$ such that

$$(r-z)(f_2(t,x;r)-f_2(t,x;z)) \leqslant c(r-z)^2,$$

for all $(t,x)\in[0,1]^2$, r, $z\in\mathbb{R}$.

Define the Gaussian process

$$v(t,x) \triangleq \int_0^t \int_0^1 G_{t-s}(x,y)\,W(ds,dy).$$

Let us first see how we can solve Eq. (2.1) in case $f \equiv 0$. In that case, making the variable x implicit, we have

$$u(t) = e^{tA}u(0) + v(t).$$

Hence the periodicity condition yields

$$(I-e^A)u(0) = v(1).$$

Now since e^A is a compact operator on $L^2(0,1)$, for which 1 is not an eigenvalue, $I - e^A$ is invertible, from the Fredholm alternative. Finally, in case $f \equiv 0$, the Eq. (2.1) has the unique solution

$$u(t) = e^{tA}(I-e^A)^{-1}v(1) + v(t). \tag{2.2}$$

Note that it is easy to invert this expression, yielding

$$v(t) = u(t) - e^{tA}u(0).$$

Let us now turn to the nonlinear case. With the above notations, (2.1) can be rewritten as

$$\begin{cases} u(t) = e^{tA}u(0) + \displaystyle\int_0^t \int_0^1 G_{t-s}(\cdot,y)f(u(s,y))\,dy\,ds + v(t) \\ u(1) = u(0). \end{cases} \tag{2.3}$$

Under the above assumptions on f, we have the following:

THEOREM 2.1 *Equation* (2.3) *has a unique solution* u, *such that* $u\in C([0,1]^2)$ *a.s.*

Proof From Theorem 1.1, the first line of Eq. (2.3) defines a mapping from $C_0([0,1])$ into itself

$$R: u(0) \to u(1) = R(u(0)).$$

We need only show that R has a unique fixed point. If we denote by $|\cdot|$ the norm in $L^2(0,1)$, we have the following for the difference $\bar{u}=u_1-u_2$ of two solutions of the first line of (2.3) (see e.g. Dautray-Lions [3], Theorems XVIII. 3.1, 3.2 and 1.2): $\bar{u}\in L^2(0,1;H_0^1(0,1))\cap C([0,1];L^2(0,1))$, $t\to|\bar{u}(t)|^2$ is absolutely continuous and for almost all $t\in(0,1)$,

$$\frac{d}{dt}|\bar{u}(t)|^2 = 2(A\bar{u}(t),\bar{u}(t)) + 2(f(u_1(t))-f(u_2(t)),\bar{u}(t))$$

$$\leqslant -2\left|\frac{\partial\bar{u}}{\partial x}(t)\right|^2 + 2c|\bar{u}(t)|^2$$

$$\leqslant -2(2-c)|\bar{u}(t)|^2,$$

since the following holds for any $v\in H_0^1(0,1)$:

$$|v|^2 \leqslant \frac{1}{2}\left|\frac{\partial v}{\partial x}\right|^2$$

Hence $|u_1(1)-u_2(1)|\leqslant e^{-(2-c)}|u_1(0)-u_2(0)|$. From (A.2)', $c<2$, hence the mapping R is a strict contraction from $L^2(0,1)$ into itself. Since it maps $L^2(0,1)$ into $C_0([0,1])$, it has a unique fixed point in $C_0([0,1])$. □

Recall the definition of Σ_0 in section 1, and define:

$$\Sigma=\{\omega\in C([0,1]^2);\omega(0,x)=\omega(1,x),\quad 0\leqslant x\leqslant 1;\omega(t,0)=\omega(t,1)=0,\quad 0\leqslant t\leqslant 1\}.$$

We note that Theorem 2.1 associates to each $v\in\Sigma_0$ a solution $u\in\Sigma$ of (2.3). This is a mapping S_f: $\Sigma_0\to\Sigma$ such that

$$u=S_f(v).$$

In case $f\equiv 0$, we denote by S_0 the corresponding mapping, which is linear and invertible. We now show that in general S_f is a bijection from Σ_0 into Σ. Injectivity follows readily from the first line of (2.3), as well as surjectivity, since whenever $u\in\Sigma$,

$$v(t)\triangleq u(t)-e^{tA}u(0)-\int_0^t\int_0^1 G_{t-s}(\cdot,y)f(u(s,y))\,dy\,ds \tag{2.4}$$

defines an element of Σ_0.

Now the mapping

$$T\triangleq S_f^{-1}\circ S_0,$$

which is a bijection from Σ_0 onto itself, will play an important role below. If we consider v as the noise input in Eq. (2.3), T is a transformation of the noise which is such that the solution of the nonlinear equation driven by the transformed noise equals the solution of the linear equation driven by the original noise.

We need to make explicit the form of T. Note that S_0 is given by (2.2), and S_f^{-1} by (2.4). It follows:

$$
\begin{aligned}
T(v)(t,x) &= v(t,x) - \int_0^t \int_0^1 G_{t-s}(x,y) f(v(s,y) + [e^{sA}(I-e^A)^{-1}v(1)](y))\,dy\,ds \\
&= v(t,x) - \int_0^t \int_0^1 G_{t-s} f(S_0(v)(s,y))\,dy\,ds.
\end{aligned}
\tag{2.5}
$$

The following result is proved exactly as Theorem 1.5:

THEOREM 2.2 *The Gaussian random field* $\{u(t,x) = S_0(v)(t,x);\ (t,x)\in[0,1]^2\}$ *is a germ Markov field.*

Let us now consider the Markov property of the $C_0([0,1])$-valued process $\{u(t), t\in[0,1]\}$. In case u solves an initial value problem, it is clearly a $C_0([0,1])$-valued Markov process. Here, the same property cannot possibly hold, since for $t\in(0,1)$, $u(0)$ and $u(1)$ are not conditionally independent, given $u(t)$. However, we have the following (note that unlike in Theorem 2.2, we are talking here of a "sharp" —not "germ"—Markov field property):

THEOREM 2.3 *The Gaussian process* $\{u(t) = S_0(v)(t,\cdot), 0 \leqslant t \leqslant 1\}$ *is a* $C_0([0,1])$*-valued random field, i.e.* $\forall s,\ t\in[0,1]$, $\{u(r);\ r\in[s,t]\}$ *and* $\{u(\theta);\ \theta\in(s,t)^c\}$ *are conditionally independent, given* $\{u(s), u(t)\}$.

Proof Fix $0 \leqslant s \leqslant t \leqslant 1$ and let

$$\psi : C_0([0,1]) \to \mathbb{R}$$

be a bounded and measurable function. For any $r\in[s,t]$,

$$
\begin{aligned}
&E[\psi(u(r)) \mid u(\theta),\quad \theta\in(s,t)^c] \\
&= E[\psi(e^{Ar}(I-e^A)^{-1}v(1) + v(r)) \mid v(\theta),\quad \theta\in(s,t)^c] \\
&= \Phi\left(e^{Ar}(I-e^A)^{-1}v(1), \int_0^s e^{A(r-\theta)}\,dW_\theta\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi(y,z) &= E\left[\psi\left(y+z+\int_s^r e^{A(r-\theta)}\,dW_\theta\right) \Bigg| v(\theta),\quad \theta\in(s,t)^c\right] \\
&= E\left[\psi\left(y+z+\int_s^r e^{A(r-\theta)}\,dW_\theta\right) \Bigg| \int_s^t e^{A(t-\theta)}\,dW_\theta\right] \\
&= \int \psi\left(y+z+\wedge \int_s^t e^{A(t-\theta)}\,dW_\theta + \alpha\right)\mu(d\alpha),
\end{aligned}
$$

where

$$\wedge \int_s^t e^{A(t-\theta)}\,dW_\theta = E\left(\int_s^r e^{A(r-\theta)}\,dW_\theta \Big| \int_s^t e^{A(t-\theta)}\,dW_\theta\right),$$

hence

$$\wedge = (I - e^{2A(t-s)})^{-1}(e^{A(t-r)} - e^{A(t+r-2s)})$$

and μ is the centered Gaussian law on $C_0([0,1])$ with covariance equal to that of the conditional law of

$$\int_s^r e^{A(r-\theta)}\,dW_\theta, \quad \text{given} \quad \int_s^t e^{A(t-\theta)}\,dW_\theta.$$

Finally $E[\psi(u(r))/u(\theta), \theta \in (s,t)^c]$ is a measurable function of

$$e^{Ar}(I-e^A)^{-1}v(1) + (I - e^{2A(t-s)})^{-1}(e^{A(t-r)} - e^{A(t+r-2s)})\int_s^t e^{A(t-\theta)}\,dW_\theta + \int_0^s e^{A(r-\theta)}\,dW_\theta$$

$$= e^{A(r-s)}u(s) + (I - e^{2A(t-s)})^{-1}(e^{A(t-r)} - e^{A(t+r-2s)})(u(t) - e^{A(t-s)}u(s)),$$

and the result follows. □

Theorems 2.2 and 2.3 remain valid if we replace S_0 by S_f, with an affine function f. The aim of the rest of the paper is to show that whenever f is not affine, $u = S_f(v)$ does not possess a Markov property which is weaker than the two above ones.

3. COMPUTATION OF A RADON–NIKODYM DERIVATIVE

Our argument in the next section will be based on the use of the extended Girsanov theorem due to Kusuoka [8], which we will recall in the form in which we shall use it.

Let again $H = L^2((0,1)^2)$. We identify H with a subset of the Banach space Σ_0 (in fact with the Reproducing Kernel Hilbert Space $\mathscr{H}$), the injection being denoted by:

$$(Gh)(t,x) \triangleq \int_0^t \int_0^1 G_{t-s}(x,y)h(s,y)\,dy\,ds, \quad h \in H.$$

Let μ denote the law on Σ_0 of $G\dot{W}$, where

$$(G\dot{W})(t,x) \triangleq \int_0^t \int_0^1 G_{t-s}(x,y)\,W(ds,dy).$$

Then (H, Σ_0, μ) is an abstract Wiener space.

DEFINITION 3.1 *A measurable mapping* $F:\Sigma_0 \to H$ *is said to be an* $H - C^1$ *map if to each* $v \in \Sigma_0$ *we can associate a Hilbert-Schmidt operator* $DF(v) \in \mathscr{L}^2(H)$ *such that:*

(i) $\| F(v+Gh) - F(v) - DF(v)h \|_H = o(\|h\|_H)$.

(ii) *For each* $v \in \Sigma_0$, $h \to DF(v+Gh)$ *is continuous from* H *into* $\mathscr{L}^2(H)$.

The following result is proved in Kusuoka [8] (we denote below by I_{Σ_0} and I_H respectively the identity operator on Σ_0 and on H):

THEOREM 3.2 *Let* $F:\Sigma_0 \to H$ *be an* $H - C^1$ *map such that* $T = I_{\Sigma_0} - G \circ F$ *is a bijection on* Σ_0 *and* $I_H - DF(v)$ *is invertible (as an element of* $\mathscr{L}(H)$*) for any* $v \in \Sigma_0$. *Then* $\mu \circ T^{-1}$ *and* μ *are mutually absolutely continuous, and*

$$\frac{d(\mu \circ T^{-1})}{d\mu}(v) = |d_c(DF(v))| \exp\left(\delta(F(v)) - \frac{1}{2} \| F(v) \|_H^2 \right),$$

where $d_c(DF(v))$ *denotes the Carleman–Fredholm determinant of the Hilbert–Schmidt operator* $DF(v)$, *and* $\delta(F(v))$ *is the Skorohod integral of the random field* $F(v)$.

In the sequel we write S for the mapping S_0 defined in section 2. We want to apply Theorem 3.2 with $F(v) := f(S(v))$, in the sense that

$$F(v)(t,x) = f(t,x;S(v)(t,x)).$$

THEOREM 3.3 *Suppose that* f *is of class* C^1 *with respect to its third variable, the derivative* f' *being bounded, and that it satisfies the assumptions of Theorem 2.1 (namely, (A.0), (A.1) and (A.2)'). Then* $F = f \circ S$ *is an* $H - C^1$ *map which satisfies the assumptions of Theorem 3.2, and moreover, if* $Q = \mu \circ T^{-1}$,

$$\frac{dQ}{d\mu}(v) = \det[(I - e^A)^{-1}] \det(I - \psi_1 e^A) \exp\left[\int_0^1 Tr\{f'(u_t)(I - e^A)^{-1} e^A\} dt \right.$$

$$\left. + \int_0^1 \int_0^1 f(u(t,x)) W(dt,dx) - \frac{1}{2} \int_0^1 \int_0^1 f(u(t,x))^2 \, dx \, dt \right],$$

where $u = S(v)$, W *is determined by* $v = G\dot{W}$, $f'(u_t)$ *stands for the operator in* $\mathscr{L}(L^2(0,1))$ *of multiplication by the function* $f(t,x;u(t,x))$ *and* $\{\psi_t,\ 0 \leqslant t \leqslant 1\}$ *is the* $\mathscr{L}(H)$*-valued solution of the linear equation*

$$\begin{cases} \dfrac{\partial \psi_t}{\partial t} = e^{-At} f'(u_t) e^{At} \psi_t \\ \psi_0 = I. \end{cases}$$

$\{t=0\} \cup \{x=0\}$), $\mathscr{F}$ is its Borel σ-field, P the "Brownian sheet measure", and

$$W_{t,x}(\omega) = \omega(t,x), \quad (t,x) \in [0,1]^2.$$

If S denotes the set of "simple random variables" of the form

$$X = f(W(h_1), \dots, W(h_n))$$

with $n \in \mathbb{N}$, $h_1, \dots, h_n \in L^2((0,1)^2)$, $f \in C_b^\infty(\mathbb{R}^n)$ and

$$W(h_i) = \int_0^1 \int_0^1 h_i(t,x)\, W(dt,dx), \quad 1 \leqslant i \leqslant n,$$

we define the Wiener space derivative $\{D_{t,x}X, (t,x) \in [0,1]^2\}$ of X by

$$D_{t,x}X = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))\, h_i(t,x).$$

Let $\|\cdot\|_{1,2}$ denote the norm on S defined by

$$\|X\|_{1,2}^2 = E(X^2 + \|DX\|_{L^2((0,1)^2)}^2)$$

and $\mathbb{D}^{1,2}$ be the closure of S with respect to the norm $\|\cdot\|_{1,2}$. D extends to an operator (which we shall still denote D) from $\mathbb{D}^{1,2}$ into $L^2(\Omega \times [0,1]^2;\, dP\, dt\, dx)$. Following Nualart–Pardoux [10] and Bouleau Hirsch [1], we define $\mathbb{D}_{\text{loc}}^{1,2}$ as the set of those random variables X to which we can associate a so-called "localizing sequence" $\{(\Omega_n, X_n)\} \subset \mathscr{F} \times \mathbb{D}^{1,2}$ such that

(i) $$\bigcup_n \Omega_n = \Omega \quad \text{a.s.}$$

(ii) $$X_n|_{\Omega_n} = X|_{\Omega_n} \quad \text{a.s.}, \quad \forall n \geqslant 1.$$

Moreover we define DX for $X \in \mathbb{D}_{\text{loc}}^{1,2}$ by:

$$DX|_{\Omega_n} = DX_n|_{\Omega_n}, \quad n \geqslant 1,$$

where $\{(\Omega_n, X_n)\}$ is any localizing sequence.

If K denotes a Hilbert space, one can define similarly $\mathbb{D}^{1,2}(K)$ and $\mathbb{D}_{\text{loc}}^{1,2}(K)$ as spaces of K-valued random variables (see Ocone-Pardoux [13] for the details).

We shall need the following

LEMMA 4.1 *Let $X \in \mathbb{D}_{\text{loc}}^{1,2}(K)$ and $\Phi : \Omega \times K \to \mathbb{R}$ be a measurable mapping satisfying:*

(i) *for any $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is continuously Fréchet-differentiable;*

(ii) *for any* $k \in K$, $\Phi(\cdot, k) \in \mathbb{D}^{1,2}$ *and there exists a version of the mapping*

$$k \to D\Phi(\cdot, k)$$

which is a.s. continuous from K *into* $L^2((0,1)^2)$;

(iii) *for any* $a > 0$,

$$E\left(\sup_{|k| \leq a} [\Phi(k)^2 + \|D\Phi(k)\|_H^2]\right) < \infty$$

$$\sup_{\omega \in \Omega, |k| \leq a} \|\nabla\Phi(k)\|_K < \infty.$$

Then $\Phi(X) \in \mathbb{D}^{1,2}_{\text{loc}}$ *and*

$$D[\Phi(X)] = (D\Phi)(X) + (\nabla\Phi(X), DX)_K.$$

Proof The result follows easily from Lemma 2.4 in Ocone–Pardoux [13] by finite dimensional approximation, using the fact that D is a closed operator from $L^2(\Omega; K)$ into $L^2(\Omega \times (0,1)^2; K)$. □

Recall that $H = L^2(0,1)$. We consider the $\mathscr{L}(H)$-valued process $\{Y_{t,s}, 0 \leq s \leq t \leq 1\}$ solution of:

$$\begin{cases} \dfrac{\partial Y_{t,s}}{\partial t} = AY_{t,s} + f'(u_t)Y_{t,s} \\ Y_{s,s} = I_H, \end{cases} \tag{4.1}$$

where $AY_{t,s}$ denotes the composition of the bounded operator $Y_{t,s}$ on H with the unbounded operator A, and $f'(u_t)Y_{t,s}$ is the composition of the two bounded operators $Y_{t,s}$ and $f'(u_t)$. Clearly,

$$Y_{t,s} = e^{(t-s)A} + \int_s^t e^{(t-r)A} f'(u_r) Y_{r,s}\, dr.$$

It is easily shown that $Y_{t,s} \in \mathscr{L}^2(H)$, the space of Hilbert–Schmidt operators on H, for $s < t$, and the associated kernel $Y_{t,s}(\cdot,\cdot) \in L^2((0,1)^2)$ satisfies:

$$Y_{t,s}(x,y) = G_{t-s}(x,y) + \int_s^t \int_0^1 G_{t-r}(x,z) f'(r,z;u(r,z)) Y_{r,s}(z,y)\, dz\, dr.$$

We shall write Y_t for $Y_{t,0}$.

LEMMA 4.2 *For any* $t \in [0,1]$, $B \in \mathscr{L}(H)$ *with* $\|B\| \leq 1$, *we define*

$$\Delta_t(B) = \det(I - BY_t).$$

We suppose that the assumptions of Theorem 3.3 are in force.

Then there exists $0 < t_0 < 1$ and $C > 0$ such that

$$\frac{1}{C} \leqslant \Delta_t(B) \leqslant C, \quad t_0 \leqslant t \leqslant 1.$$

Moreover for $t_0 \leqslant t \leqslant 1$, Δ_t is Fréchet differentiable on the unit ball of $\mathscr{L}(H)$, and for any $B, F \in \mathscr{L}(H)$ with $\|B\| \leqslant 1$,

$$\langle \nabla \Delta_t(B), F \rangle = -\Delta_t(B) Tr[(I - BY_t)^{-1} F Y_t].$$

Proof Note that since $f' \leqslant 2$, $\bar{Y}_t := e^{-2t} Y_t$ satisfies an equation with $f'(u_t)$ replaced by $f'(u_t) - 2$, which is negative, hence

$$\begin{aligned} Y_t(x, y) &\leqslant e^{2t} G_t(x, y) \\ &= e^{2t} \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t} \sin k\pi x \sin k\pi y \\ &\leqslant e^{2t} \sum_{n=1}^{\infty} e^{-\pi^2 nt} = \frac{e^{-(\pi^2 - 2)t}}{1 - e^{-\pi^2 t}}. \end{aligned}$$

Let $t_0 = \sup\{t > 0; \sup_{0 \leqslant x, y \leqslant \rho} Y_t(x, y) \geqslant 1\}$, with some $\frac{1}{2} \leqslant \rho < 1$. It is easily seen that $0 < t_0 < 1/2$.

Now from Simon [15, formula (5.12), page 69],

$$\det(I - BY_t) = \exp\left[-\sum_{m=1}^{\infty} \frac{1}{m} Tr(BY_t)^m\right] \tag{4.2}$$

and for $t_0 \leqslant t < 1$,

$$Tr[(BY_t)^m] \leqslant \sup_{0 \leqslant x, y \leqslant 1} Y_t(x, y)^m = \rho^m,$$

hence

$$\exp\left(-\sum_{m=1}^{\infty} \frac{\rho^m}{m}\right) \leqslant \det(I - Y_1) \leqslant \exp\left(\sum_{m=1}^{\infty} \frac{\rho^m}{m}\right)$$

Now for $\|B\| \leqslant 1$ one can interchange the derivation with respect to B and the summation in

$$\sum_{m=1}^{\infty} \frac{1}{m} Tr[(BY_t)^m].$$

Hence

$$\langle \nabla \Delta_t(B), F \rangle = \Delta_t(B) \exp\left(- \sum_{m=1}^{\infty} Tr[(BY_t)^{m-1} F Y_t] \right)$$

$$= -\Delta_t(B) Tr[(I - BY_t)^{-1} F Y_t]. \qquad \square$$

We shall use this Lemma with $B = Y_{1,t}$. Note that, from the same estimates as those in the proof of Lemma 4.2, we have $\| Y_{1,t} \|_{HS} \leqslant 1$ a.s., provided $t_0 \leqslant 1 - t \leqslant 1$.

Let us now rewrite the expression for the Radon–Nikodym derivative in Theorem 3.3. It follows from (4.2) that

$$\det(I - \psi_1 e^A) = \det(I - e^A \psi_1)$$

$$= \det(I - e^A Y_t).$$

Hence

$$J = \frac{dQ}{d\mu}(v) = \det[(I - e^A)^{-1}] \det(I - e^A Y_t) \exp\left[\int_0^1 Tr[f'((Sv)_t)(I - e^A)^{-1} e^A]\, dt \right.$$

$$\left. + \int_0^1 \int_0^1 f((Sv)(t,x))\, W(dt,dx) - \frac{1}{2} \int_0^1 \int_0^1 f((Sv)(t,x))^2\, dx\, dt \right] \tag{4.3}$$

We can now prove the main result of this section (see also Corollary 4.5 at the end of the section).

THEOREM 4.3 *Suppose that f satisfies the assumptions of Theorem 2.1 (namely, (A.0), (A.1) and (A.2)'). Suppose moreover that $f(t, x, z)$ does not depend on the variable t, and that f is of class C^2 with respect to the variable z, the two first derivatives, denoted f' and f'', being bounded functions of their variables. Then, if the solution $\{u(t), 0 \leqslant t \leqslant 1\}$ of Eq. (2.1) is a $C_0([0, 1])$-valued Markov field, $f'' \equiv 0$.*

Proof Let Q be the probability on Σ_0 defined by $Q = \mu T^{-1}$, where as in section 3 μ denotes the law of $v = G\dot{W}$. Clearly, the law of $\{u(t), 0 \leqslant t \leqslant 1\}$ under P equals the law of $\{(Sv)(t), 0 \leqslant t \leqslant 1\}$ under Q. We assume now that $\{u(t), 0 \leqslant t \leqslant 1\}$ is a $C_0([0, 1])$-valued Markov field under P, hence that $\{(Sv)(t), 0 \leqslant t \leqslant 1\}$ is a $C_0([0, 1])$-valued Markov field under Q.

For the rest of this proof, we use the notation

$$Z(t) = (Sv)(t)$$

$$= e^{At}(I - e^A)^{-1} v(1) + v(t).$$

From the two last identities and Lemma 4.1, we deduce that

$$\frac{d}{ds}\langle \nabla\Phi(Y_{1,t}), e^{As}D_{s,.}\ Y_{1,t}\rangle = 0 \quad \text{for almost all} \quad s\in[t,1],\ \omega\in\Omega. \tag{4.4}$$

It is easily seen that for $t \leqslant s \leqslant 1$

$$DY_{1,t} = \int_t^1 Y_{1,s}(f''(Z(r))DZ(r))Y_s\,ds$$

$$e^{As}D_{s,.}\ Y_{1,t} = \int_t^s Y_{1,r}f''(Z(r))e^{A(1+r)}(I-e^A)^{-1}Y_r\,dr \tag{4.5}$$

$$+\int_s^1 Y_{1,r}f''(Z(r))e^{Ar}(I-e^A)^{-1}Y_r\,dr.$$

Thus

$$\frac{d}{ds}(e^{As}D_{s,.}\ Y_{1,t}) = -Y_{1,s}f''(Z(s))e^{As}Y_s.$$

Hence (4.4) can be rewritten as

$$\int_0^1\int_0^1 (\nabla_{x,z}\Phi)(Y_{1,t})\int_0^1 Y_{1,s}(x,y')f''(Z(s,y))e^{As}(y,y')Y_s(y,z)\,dy\,dx\,dz = 0$$
$$\text{for almost all} \quad s\in[t,1],\ y'\in[0,1],\ \omega\in\Omega.$$

Consequently

$$\int_0^1\int_0^1 (\nabla_{x,z}\Phi)(Y_{1,t})Y_{1,s}(x,y)Y_s(y,z)f''(Z(s,y))\,dx\,dz = 0$$
$$\text{for almost all} \quad s\in[t,1],\ y\in[0,1],\ \omega\in\Omega. \tag{4.6}$$

Using Lemma 4.2 and the definition of $\Phi(B)$ we deduce for any square integrable kernel $h\in L^2([0,1]^2)$ the following formula

$$\langle\nabla\Phi(B),h\rangle = (E(\Delta_t(B)|\mathscr{H}_t^e))^{-2}\{-E(\xi\Delta(B)Tr[(I-BY_t)^{-1}hY_t]|\mathscr{H}_t^e)$$
$$\times E(\Delta_t(B)|\mathscr{H}_t^e) + E(\xi\Delta_t(B)|\mathscr{H}_t^e)E(\Delta_t(B)Tr[(I-BY_t)^{-1}hY_t]|\mathscr{H}_t^e)\}. \tag{4.7}$$

Define

$$B_t(x,z) = E[Y_{1,t}(x,y)Y_t(y,z)f''(Z_t(y))|\mathscr{H}_t^e].$$

By taking $s = t$ (this is possible by continuity), (4.6) implies that

$$\langle(\nabla\Phi)(Y_{1,t}),B_t\rangle = 0 \quad \text{a.s.} \tag{4.8}$$

for all $t\in[0,1]$, a.e. Applying (4.7) to (4.8) gives

$$E[\xi\Delta Tr[(I-Y_1)^{-1}B_tY_t]|\mathscr{H}_t^e]E[\Delta|\mathscr{H}_t^e] \\ = E[\xi\Delta|\mathscr{H}_t^e]E[\Delta Tr[(I-Y_1)^{-1}B_tY_t]|\mathscr{H}_t^e]. \tag{4.9}$$

This equality is clearly true if ξ is a bounded $\mathscr{H}_t^e$-measurable random variable. By a monotone class argument, (4.9) holds for any bounded random variable ξ. Consequently, we obtain that

$$Tr[(I-Y_1)^{-1}B_tY_t]$$

is $\mathscr{H}_t^e$-measurable. Using Lemma 4.4, part (i) we get

$$\frac{d}{ds}e^{As}D_s\{Tr[(I-Y_1)^{-1}B_tY_t]\}=0 \quad \text{for} \quad s\in[0,t], \quad \text{a.e.} \tag{4.10}$$

Notice that $(d/ds)e^{As}D_sB_t=0$ for $s\in[0,t]$, because B_t is $\mathscr{H}_t^e$-measurable. On the other hand, differentiating the linear Eq. (4.1) we obtain

$$D_sY_t=\int_0^t Y_{t,r}f''(Z_r)D_sZ_rY_r\,dr.$$

Therefore, (4.10) yields

$$\frac{d}{ds}e^{As}Tr\left\{(I-Y_1)^{-1}B_t\left(\int_0^t Y_{t,r}f''(Z_r)D_sZ_rY_r\,dr\right)\right. \\ \left.+(I-Y_1)^{-1}\left(\int_0^1 Y_{1,r}f''(Z_r)D_sZ_rY_r\,dr\right)(I-Y_1)^{-1}B_tY_t\right\}=0.$$

Using the expression of D_sZ_r this gives, using the same arguments as in (4.5)

$$\frac{d}{ds}Tr\left\{(I-Y_1)^{-1}B_t\left[\int_0^s Y_{t,r}f''(Z_r)\langle e^{A(1+r)}(I-e^A)^{-1}\rangle_\xi Y_r\,dr\right.\right. \\ \left.+\int_s^t Y_{t,r}f''(Z_r)\langle e^{Ar}(I-e^A)^{-1}\rangle_\xi Y_r\,dr\right] \\ +(I-Y_1)^{-1}\left[\int_0^s Y_{t,r}f''(Z_r)\langle e^{A(1+r)}(I-e^A)^{-1}\rangle_\xi Y_r\,dr\right. \\ \left.+\int_0^1 Y_{1,r}f''(Z_r)\langle e^{Ar}(I-e^A)^{-1}\rangle_\xi Y_r\,dr\right] \\ \left.\times(I-Y_1)^{-1}B_tY_t\right\}=0,$$

where if $T \in L^2([0,1]^2)$, $\langle T \rangle_\xi$ means the multiplication operator on $L^2(0,1)$ by $\{T(\xi, x), x \in [0,1]\}$. Therefore, we get

$$Tr\{(I - Y_1)^{-1} B_t Y_{t,s} f''(Z_s) \langle e^{As} \rangle_\xi Y_s$$

$$+ (I - Y_1)^{-1} B_t Y_t (I - Y_1)^{-1} Y_{1,s} f''(Z_s) \langle e^{As} \rangle_\xi Y_s\} = 0,$$

for all $s \in [0, t]$, $\xi \in [0,1]$, $\omega \in \Omega$, a.e. Consequently we obtain

$$Tr\{(I - Y_1)^{-1} B_t Y_t (I + (I - Y_1)^{-1} Y_1) Y_s^{-1} f''(Z_s) \langle e^{As} \rangle_\xi Y_s\} = 0,$$

namely, using the definition of B_t this means,

$$Tr\{(I - Y_1)^{-1} Y_1 Y_t^{-1}(\cdot, y) f''(Z_t(y)) E[Y_t(y, \cdot) | \mathscr{H}_t^e]$$

$$\times Y_t (I - Y_1)^{-1} Y_s^{-1}(\cdot, x) f''(Z_s(x)) Y_s(x, \cdot)\} = 0$$

for all $x, y \in [0,1]$, $s \in [0, t]$, $\omega \in \Omega$.
The above expression can be written as

$$[Y_s(I - Y_1)^{-1} Y_1 Y_t^{-1}](x, y) \cdot (E[Y_t | \mathscr{H}_t^e] Y_t (I - Y_1)^{-1} Y_s^{-1})(y, x) f''(Z_t(y)) f''(Z_s(x)) = 0.$$

It holds that

$$(Y_s(I - Y_1)^{-1} Y_1 Y_t^{-1})(x, y) > 0$$

$$(E(Y_t | \mathscr{H}_t^e) Y_t (I - Y_1)^{-1} Y_s^{-1})(y, x) > 0$$

for all $x, y \in [0,1]^2$ and $0 < s < t < 1$ (with $t \geqslant t_0$, $t \leqslant 1 - t_0$). This is true because the kernels Y_s, Y_t and $Y_1 Y_t^{-1}$ are strictly positive on $[0,1]^2$, which follows, for instance, from the Feynman-Kac's formula. Therefore, the preceding equality implies

$$f''(y; Z_t(y)) f''(x; Z_s(x)) = 0, \quad \text{a.s.}$$

for all $x, y \in [0,1]^2$, $0 < s < t < 1$, $t_0 \leqslant t \leqslant 1 - t_0$, which implies $f''(y; u) = 0$ for all $y \in [0,1]$, $u \in \mathbb{R}$. □

Proof of Lemma 4.4 We only prove part (i), since the proof of part (ii) is similar. By a localization argument, we can assume that $X \in \mathbb{D}^{1,2}$. Suppose first that

$$X = f(Z(0, x_1), \ldots, Z(0, x_n); Z(t_1, y_1), \ldots, Z(t_m, y_m))$$

where $n, m \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^{n+m})$, $t \leqslant t_1 < t_2 < \cdots < t_m \leqslant 1$, $x_1, \ldots, x_n$, $y_1, \ldots, y_m \in [0,1]$. In that case we have for any $s \in [0, t]$, and using the notation $Z^1 = (Z(0, x_1), \ldots, Z(0, x_n))$

and $Z^2 = (Z(t_1, y_1), \ldots, Z(t_m, y_m))$,

$$D_{s,z}X = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Z^1; Z^2) e^{A(1-s)}(I - e^A)^{-1}(x_i, z)$$
$$+ \sum_{j=1}^{m} \frac{\partial f}{\partial \xi_{n+j}}(Z^1; Z^2) e^{A(t_j - s)}(I - e^A)^{-1}(y_j, z),$$

and (i) holds. If now $X \in \mathbb{D}^{1,2}$ is arbitrary, we can find a sequence of random variables $\{X_n\}$ of the above type such that $\|X_n - X\|_{1,2} \to 0$. Now for any $n \in \mathbb{N}$ $(e^{As}D_{s,\cdot}X_n)(z)$ does not depend upon s on the interval $[0, t]$ for $\omega \in G$. Hence the same is true for X. ◻

We can finally prove the following result.

Corollary 4.5 *Suppose that f satisfies the assumptions of Theorem 4.3. If the solution $\{u(t); 0 \leqslant t \leqslant 1\}$ of Eq. (2.1) is a $C_0([0, 1])$-valued germ Markov field, $f'' \equiv 0$.*

Proof The result will be a consequence of Theorem 4.3 if we show that

$$\bigcap_{\varepsilon > 0} \mathscr{H}_t^{(\varepsilon)} = \mathscr{H}_t, \tag{4.11}$$

where

$$\mathscr{H}_t^{(\varepsilon)} = \sigma\{Z(s); s \in [0, \varepsilon] \cup [(t-\varepsilon)^+, (t+\varepsilon) \wedge 1]\}.$$

Now it is easily seen that

$$\mathscr{H}_t = \sigma\left\{\int_0^t \int_0^1 G_{t-s}(x, y) W(dy, ds), \quad x \in [0, 1]\right\}$$
$$\vee \, \sigma\left\{\int_0^1 \int_0^1 G_{1-s}(x, y) W(ds, dy), \quad x \in [0, 1]\right\},$$

while for $\varepsilon > 0$

$$\mathscr{H}_t^{(\varepsilon)} = \sigma\left\{\int_0^s \int_0^1 G_{s-r}(x, y) W(dr, dy), x \in [0, 1], s \in [0, \varepsilon] \cup [(t-\varepsilon)^+, (t+\varepsilon) \wedge 1]\right\}.$$

Let us denote by H_t the linear span in $L^2((0, 1)^2)$ generated by the functions:

$$(r, y) \to G_{t-r}(x, y) \mathbb{1}_{\{r \leqslant t\}}$$

and

$$(r, y) \to G_{1-r}(x, y)$$

for all $x \in [0, 1]$, and by $H_t^{(\varepsilon)}$ the linear span in $L^2((0, 1)^2)$ generated by the functions:

$$(r, y) \to G_{s-r}(x, y) \mathbb{1}_{\{r \leqslant s\}}$$

for all $x\in[0,1]$, $r\in[0,\varepsilon]\cap[(t-\varepsilon)^+, (t+\varepsilon)\wedge 1]$. Clearly, $H_t = \bigcap_{\varepsilon>0} H_t^{(\varepsilon)}$. Then (4.11) follows from Lemma 3.3 in Mandrekar [9]. □

APPENDIX

Proof of Proposition 3.4 We deduce from (3.1) the following formula for the kernel $DF(v)(t,x;s,y)$:

$$DF(v)(t,x;s,y) = f'(t,x;u(t,x))\left[G_{t-s}(x,y)+\int_0^1\int_0^1 G_t(x,x_1)(I-e^A)^{-1}(x_1,x_2)G_{1-s}(x_2,y)\,dx_1\,dx_2\right].$$

Note that we can approximate this kernel by the finite dimensional kernel

$$\Lambda^{n,m} = \frac{1}{nm}\sum_{i,k=0}^{n-1}\sum_{j,l=0}^{m-1} f'_{ij}(u(t_i,x_j))[G_{t_i-t_k}(x_j,x_l)\mathbb{1}_{\{k<i\}}$$

$$+ m^{-2}(G_{t_i}^m B^m G_{1-t_k}^m)(x_j,x_l)]e_i\otimes e_k\otimes v_j\otimes v_l$$

where G_t^m denotes the $m\times m$ matrix $\{G_t(x_j,x_l)\}_{1\le j,l\le m}$ and B^m the $m\times m$ matrix $\{(I-e^A)^{-1}(x_j,x_l)\}_{1\le j,l\le m}$, and

$$e_i(t)=\sqrt{n}\,\mathbb{1}_{(t_i,t_{i+1}]}(t), \qquad 0\le i\le m-1,$$

$$v_j(x)=\sqrt{m}\,\mathbb{1}_{(x_j,x_{j+1}]}(x), \quad 0\le j\le n-1.$$

Using the expression of the Carleman-Fredholm determinant for a finite matrix we obtain

$$d_c(\Lambda^{n,m}) = \det(I-\Lambda^{n,m})\exp(Tr\,\Lambda^{n,m}).$$

We have

$$\exp(Tr\,\Lambda^{n,m}) = \exp\left(\sum_{i=0}^{n-1}\sum_{j=1}^{m-1}\frac{1}{nm^3}f'(u(t_i,x_j))(G_{t_i}^m B^m G_{1-t_i}^m)(x_j,x_j)\right).$$

Letting first m tend to infinity and then $n\to\infty$ we obtain

$$\lim_{n\to\infty}\lim_{m\to\infty}\exp(Tr\,\Lambda^{n,m}) = \exp\left(\int_0^1 (Tr\,f'(u_t)(I-e^A)^{-1}e^A)\,dt\right).$$

On the other hand, we have

$$\det(I-\wedge^{n,m})$$

$$=\det\begin{bmatrix} I_m-\frac{1}{nm^3}f'_0G_0^mB^mG_1^m & -\frac{1}{nm^3}f'_0G_0^mB^mG_{1-t_1}^m & \cdots \\ -\frac{1}{nm}f'_1\left(G_{t_1}^m+\frac{1}{m^2}G_{t_1}^mB^mG_1^m\right) & I_m-\frac{1}{nm^3}f'_1G_{t_1}^mH^mG_{1-t_1}^m & \cdots \\ -\frac{1}{nm}f'_2\left(G_{t_2}^m+\frac{1}{m^2}G_{t_2}^mB^mG_1^m\right) & -\frac{1}{nm}f'_2\left(G_{t_2-t_1}^m+\frac{1}{m^2}G_{t_2}^mB^mG_{1-t_1}^m\right) & \cdots \\ -\frac{1}{nm}f'_3\left(G_{t_3}^m+\frac{1}{m^2}G_{t_3}^mB^mG_1^m\right) & -\frac{1}{nm}f'_3\left(G_{t_3-t_1}^m+\frac{1}{m^2}G_{t_3}^mB^mG_{1-t_1}^m\right) & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

Consequently, with $B:=(I-e^A)^{-1}$,

$$\lim_{m\to\infty}\det(I-\wedge^{n,m})$$

$$=\det\begin{bmatrix} I-\frac{1}{n}f'_0Be^A & -\frac{1}{n}f'_0Be^{A(1-t_1)} & \cdots \\ -\frac{1}{n}f'_1(e^{At_1}+Be^{A(t_1+1)}) & I-\frac{1}{n}f'_1Be^A & \cdots \\ -\frac{1}{n}f'_2(e^{At_2}+Be^{A(1+t_2)}) & -\frac{1}{n}f'_2(e^{A(t_2-t_1)}+Be^{A(1+t_2-t_1)}) & \cdots \\ -\frac{1}{n}f'_3(e^{At_3}+Be^{A(1+t_3)}) & -\frac{1}{n}f'_3(e^{A(t_3-t_1)}+Be^{A(1+t_3-t_1)}) & \cdots \\ & \cdots & \end{bmatrix}$$

$$=(-1)^n\det\left[e^{A(t_1+t_2+\cdots+t_{n-1})}\right]$$

$$\times\det\begin{bmatrix} -I+\frac{1}{n}f'_0Be^A & \frac{1}{n}f'_0Be^{A(1-t_1)} & \frac{1}{n}f'_0Be^{A(1-t_2)} & \cdots \\ \frac{1}{n}f'_1(I+Be^A) & -e^{-At_1}+\frac{1}{n}f'_1Be^{A(1-t_1)} & \frac{1}{n}f'_1Be^{A(1-t_2)} & \cdots \\ \frac{1}{n}f'_2(I+Be^A) & \frac{1}{n}f'_2(e^{-At_1}+Be^{A(1-t_1)}) & -e^{-At_2}+\frac{1}{n}f'_2Be^{A(1-t_2)} & \cdots \\ \frac{1}{n}f'_3(I+Be^A) & \frac{1}{n}f'_3(e^{-At_1}+Be^{A(1-t_1)}) & \frac{1}{n}f'_3(e^{-At_2}+Be^{A(1-t_2)}) & \cdots \\ & \cdots & & \end{bmatrix}.$$

Using the fact $I + Be^A = B$, because $H = (I - e^A)^{-1}$, this determinant is equal to

$$\det\begin{bmatrix} I - \frac{1}{n}f'_0 Be^A & -\frac{1}{n}f'_0 Be^{A(1-t_1)} & -\frac{1}{n}f'_0 Be^{A(1-t_2)} & \cdots \\ -\frac{1}{n}f'_1 Be^{At_1} & I - \frac{1}{n}f'_1 Be^A & -\frac{1}{n}f'_1 Be^{A(1+t_1-t_2)} & \cdots \\ -\frac{1}{n}f'_2 Be^{At_2} & -\frac{1}{n}f'_2 Be^{A(t_2-t_1)} & I - \frac{1}{n}f'_2 Be^A & \cdots \\ -\frac{1}{n}f'_3 Be^{At_3} & -\frac{1}{n}f'_3 Be^{A(t_3-t_1)} & -\frac{1}{n}f'_3 Be^{A(t_3-t_2)} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}.$$

Now we multiply the ith column ($2 \leqslant i \leqslant n$) on the right by $e^{At_{i-1}}$, and we obtain

$$\det\left(\exp\left(-A\sum_{i=1}^{n-1} t_i\right)\right)$$

$$\times\det\begin{bmatrix} I - \frac{1}{n}f'_0 Be^A & -\frac{1}{n}f'_0 Be^A & -\frac{1}{n}f'_0 Be^A & \cdots \\ -\frac{1}{n}f'_1 Be^{At_1} & e^{At_1} - \frac{1}{n}f'_1 Be^{A(1+t_1)} & -\frac{1}{n}f'_1 Be^{A(1+t_1)} & \cdots \\ -\frac{1}{n}f'_2 Be^{At_2} & -\frac{1}{n}f'_2 Be^{At_2} & e^{At_2} - \frac{1}{n}f'_2 Be^{A(1+t_2)} & \cdots \\ -\frac{1}{n}f'_3 Be^{At_3} & -\frac{1}{n}f'_3 Be^{At_3} & -\frac{1}{n}f'_3 Be^{At_3} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}.$$

Substracting each column from the preceding one we get (we use again that $B(I - e^A) = I$)

$$\det\left(\exp\left(-A\sum_{i=1}^{n-1} t_i\right)\right)$$

$$\times\det\begin{bmatrix} I & 0 & \cdots & -\frac{1}{n}f'_0 Be^A \\ -e^{At_1} - \frac{1}{n}f'_1 e^{At_1} & e^{At_1} & \cdots & -\frac{1}{n}f'_1 Be^{A(1+t_1)} \\ 0 & -e^{At_2} - \frac{1}{n}f'_2 e^{At_2} & \cdots & -\frac{1}{n}f'_2 Be^{A(1+t_2)} \\ 0 & 0 & \cdots & \cdots \\ & & \cdots & -\frac{1}{n}f'_{n-2} Be^{A(1+t_{n-2})} \\ \cdots & \cdots & \cdots & e^{At_{n-1}} - \frac{1}{n}f'_{n-1} e^{At_{n-1}} \end{bmatrix}$$

$$= \det \begin{bmatrix} I & 0 & \cdots & -\frac{1}{n} f'_0 Be^A \\ -I - \frac{1}{n} e^{-At_1} f'_1 e^{At_1} & I & \cdots & -\frac{1}{n} e^{-At_1} f'_1 e^{At_1} Be^A \\ 0 & -I - \frac{1}{n} e^{-At_2} f'_2 e^{At_2} & \cdots & -\frac{1}{n} e^{-At_2} f'_2 e^{At_2} Be^A \\ 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ & & \cdots & -\frac{1}{n} e^{-At_{n-2}} f'_{n-2} e^{At_{n-2}} Be^A \\ & \cdots & \cdots & I - \frac{1}{n} e^{-At_{n-1}} f'_{n-1} e^{At_{n-1}} Be^A \end{bmatrix}$$

$$= \det \Big[I - \frac{1}{n} e^{-At_{n-1}} f'_{n-1} e^{At_{n-1}} Be^A$$

$$- \left(I + \frac{1}{n} e^{-At_{n-1}} f'_{n-1} e^{At_{n-1}} \right) \frac{1}{n} e^{-At_{n-2}} f'_{n-2} e^{At_{n-2}} Be^A$$

$$- \left(I + \frac{1}{n} e^{-At_{n-1}} f'_{n-1} e^{At_{n-1}} \right) \left(I + \frac{1}{n} e^{-At_{n-2}} f'_{n-2} e^{At_{n-2}} \right) \frac{1}{n} e^{-At_{n-3}} f'_{n-3} e^{At_{n-3}} Be^A$$

$$- \cdots - \left(I + \frac{1}{n} e^{-At_{n-1}} f'_{n-1} e^{At_{n-1}} \right) \left(I + \frac{1}{n} e^{-At_{n-2}} f'_{n-2} e^{At_{n-2}} \right) \cdots$$

$$\cdots \left(I + \frac{1}{n} e^{-At_1} f'_1 e^{At_1} \right) \frac{1}{n} f'_0 Be^A \Big].$$

Consider the solution $\{\psi(t), t \geqslant 0\}$ of the linear system

$$\frac{d\psi_t}{dt} = e^{-At} f'(u_t) \psi_t$$

$$\psi_0 = I.$$

Notice that $\psi(t)$ is invertible and

$$\frac{d\psi_t^{-1}}{dt} = -\psi_t^{-1} e^{-At} f'(u_t) e^{At}.$$

We can approximate each term $(I + (1/n) e^{-At_{n-j}} f'(u_{t_{n-j}}) e^{At_{n-j}})$ by $\psi_{t_{n-j+1}} \psi_{t_{n-j}}^{-1}$.

Consequently, the limit as $n \to \infty$ of the preceding determinant is

$$\det\left[I - \int_0^1 \psi_1 \psi_t^{-1} e^{-At} f'(u_t) e^{At} B e^A \, dt\right]$$
$$= \det\left[I + \psi_1(\psi_1^{-1} - I) B e^A\right]$$
$$= \det\left[I + B e^A - \psi_1 B e^A\right]$$
$$= \det\left[B - \psi_1 B e^A\right]$$
$$= \det\left[I - \psi_1 e^A\right] \det\left[(I - e^A)^{-1}\right].$$
□

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