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A STOCHASTIC FEYNMAN-KAC FORMULA FOR ANTICIPATING SPDE'S, AND APPLICATION TO NONLINEAR SMOOTHING

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This paper establishes an anticipating stochastic differential equation of parabolic type for the expectation of the solution of a stochastic differential equation conditioned on complete knowledge of the path of one of its components. Conversely, it is shown that any appropriately regular solution of this stochastic p.d.e. must be given by the conditional expectation. These results generalize the connection, known as the Feynman-Kac formula, between parabolic equations and expectations of functions of a diffusion. As an application, we derive an equation for the *unnormalized* smoothing law of a filtering problem with observation feedback.

KEY WORDS: Stochastic partial differential equations, Feynman-Kac formula, anticipating stochastic calculus, non-linear smoothing.

1 INTRODUCTION

The point of departure for this paper is the Feynman-Kac formula, which connects solutions of linear, parabolic partial differential equations to solutions of corresponding stochastic differential equations. We first review this briefly. Let

$$\{X_{x,s}(t); (x, s, t) \in R^d \times [0, T]^2\}$$

solve the family of stochastic differential equations

$$X_{x,s}(t) = x + 1_{[t \geq s]} \int_s^t \sum_{j=0}^{l} f_j(X_{x,s}(r), r)\, d\omega_j(r), \quad (x, s, t) \in R^d \times [0, T]^2, \quad (1.1)$$

where $\omega = (\omega_1, \ldots, \omega_l)$ denotes an R^l-valued Brownian motion and $\omega_0(t) \equiv t$. Let $v(x, s)$ be a classical solution to the backward parabolic equation

$$\frac{\partial v}{\partial s}(x, s) + \frac{1}{2}\operatorname{tr}\left[\nabla^2 v(x, s) \sum_{j=1}^{l} f_j f_j^*(x, s)\right] + \nabla v(x, s) \cdot f_0(x, s) = 0, \quad s \le T,$$
$$v(x, T) = \psi(x) \quad (1.2)$$

In (1.2), $\nabla^2 v$ denotes the matrix of second partial derivatives of v with respect to x. If v and $f_0, f_1, \ldots, f_l$ also satisfy suitable growth conditions, then v admits the Feynman-Kac stochastic representation,

$$v(x, s) = E\{\psi(X_{x,s}(T))\}, \quad s \le T. \quad (1.3)$$

The proof of (1.3) involves only an application of Itô's rule.

It is also possible to reverse the direction of the argument. That is, suppose we instead *define* v by Eq. (1.3). We can then show that v is a function in the class $C^{2,0}(R^d \times [0, T])$, and that v solves (1.2), provided that the coefficients, $f_0, f_1, \ldots, f_l$, of (1.1) are suitably regular; for a treatment, see Friedman [3], Chapter 5, Section 6.

The connection between (1.2) and (1.3) can be generalized to stochastic partial differential equations. Let the Brownian motion ω be split into two components, $\omega = (W, Y)$, W taking values in R^{l_1}, and Y in R^{l_2}. In Pardoux [12], later generalized by Krylov and Rozovsky [5], it is shown how to represent solutions of certain backward stochastic partial differential equations by expressions of the form

$$v(x, s, Y) = E\{\psi(X_{x,s}(T))Z_{x,s}(T) | \mathscr{Y}_T^s\}, \quad (1.4)$$

where $\{X_{x,s}(t)\}$ solves an equation like (1.1), $\{Z_{x,s}(t)\}$ is a process of the form

$$Z_{x,s}(t) \doteq \exp\left\{1_{[t \ge s]}\left(\int_s^t \sum_{j=1}^{l_2} h_j(X_{x,s}(r), r)\, dY_j(r) - (1/2)\int_s^t |h(X_{x,s}(r), r)|^2\, dr\right)\right\}, \quad (1.5)$$

and $\mathscr{Y}_T^s = \sigma\{Y(r); s \le r \le T\}$ is the σ-algebra generated by the future of Y at time s. Expressions of the form (1.4) occur in nonlinear filtering and smoothing theory and satisfy backward stochastic p.d.e.'s adjoint to Zakai's equation; see Pardoux [13], Section 3, and [14] for an application in this context.

In this paper we consider a further generalization. We replace (1.1) by the equation

$$X_{x,s}(t) = x + 1_{[t \ge s]}\int_s^t \sum_{j=0}^{l} f_j(X_{x,s}(r), r, Y)\, d\omega_j(r), \quad (x, s, t) \in R^d \times [0, T]^2, \quad (1.6)$$

in which each coefficient $f_j(x, r, Y)$ may now depend on Y in a progressively measurable fashion. Likewise, we assume that the terms $h_j(X_{x,s}(r), r)$ in the definition of $Z_{x,s}(t)$ are replaced by $h_j(X_{x,s}(r), r, Y)$. Let ψ be given and set

$$v(x, s, Y) = E\{\psi(X_{x,s}(T))Z_{x,s}(T) | \mathscr{Y}_T\}, \quad (1.7)$$

where $\mathscr{Y}_T = \sigma\{Y(r); 0 \le r \le T\}$ is the σ-algebra generated by the entire Y-path. The goal of this paper is to find a backward stochastic partial differential equation for $v(x, s, Y)$. The special case in which the coefficients depend on Y only through its current value (i.e. $f_j(x, r, Y) = \bar{f}_j(x, r, Y(r))$) has already been treated in [14] by an enlargement of filtration technique. Here we shall treat geeneral dependence of the coefficients on the past of Y. Since $v(x, s, Y)$ depends on the entire Y-path for each s, this backward equation will necessarily involve stochastic integrals with anticipating integrands, and in this paper we shall use the Skorohod integral to handle such integrands. Our main result is Theorem 4.2, which establishes conditions on the coefficients $f_0, f_1, \dots, f_l$ and $h_1, \dots, h_{l_2}$ such that $v(x, s, Y)$ satisfies the Eq. (4.8) in Section 4, namely:

$$
\begin{aligned}
v(x, s, Y) = \psi(x) &+ \int_s^T (\nabla v(x, r, Y) f_{l_1+j}(x, r, Y) + v(x, r, Y) h_j(x, r, Y))\, dY_j(r) \\
&+ \int_s^T \left(\frac{1}{2}\operatorname{tr}\left[\nabla^2 v(x, r, Y) \sum_1^l f_j f_j^*(x, r, Y)\right] + \nabla v(x, r, Y) f_0(x, r, Y)\right) dr \\
&+ \int_s^T \sum_1^{l_2} h_i(x, r, Y) \nabla v(x, r, Y) f_{l_1+i}(x, r, Y)\, dr \\
&+ \int_s^T \left[\sum_1^{l_2} D_r^{Y_j} \nabla v(x, r+, Y) f_{l_1+j}(x, r, Y) + D_r^{Y_j} v(x, r+, Y) h_j(x, r, Y)\right] dr
\end{aligned}
\tag{4.8}
$$

Equation (4.8) is a backward stochastic p.d.e. which contains the terms expected from the theory of Eq. (1.4) plus an additional term involving the Wiener space gradients $D_r^{Y_j} v(x, r+, Y)$ and $D_r^{Y_j} \nabla v(x, r+, Y)$ of the solution v; for a precise definition of these new terms, see Definition 3.4. Conversely, we show that if v is a sufficiently regular solution of (4.8), then v admits the representation (1.7).

In this paper we first define v by (1.7) and prove (4.8) and then later prove the converse result. It is best to work in this order because there is no independent theory that provides solutions to (4.8). Moreover, in the proof of (4.8), we discover the regularity properties of v that are necessary for the analysis. Also, in Section 4, we state the Stratonovich integral version of (4.8).

This paper contains an auxiliary result that is related to the generalized Itô-Ventzell formula proved in [11]. The fundamental calculation in the proof of the main result, Eq. (4.8), requires calculating the stochastic differential of $v(X_{x,s}(t), (t), r, Y)$ for fixed r. Thus we need an Itô rule for finding the stochastic differential of a random transformation of an Itô process. We state and prove such an Itô rule in Theorem 2.3 of Section 2.

Notice that the definition of v in (1.6) contains the Girsanov transformation-type term $Z_{x,s}(T)$. We include this here because it appears in the representation of nonlinear smoothers, and this paper was motivated by the problem of constructing an equation that would be adjoint to a Zakai equation, when the Zakai equation contains coefficients which depend on the past of the observation process Y. We could try to suppress the explicit dependence on $Z_{x,s}(T)$ in (1.6) by including it in an augmentation of the process $X_{x,s}(t)$. But, we avoid doing this because we shall assume that the coefficients $f_0, \ldots, f_l$ have uniformly bounded first derivatives in the x-variable, and this property does not extend to the augmented system.

The broad outline of the paper is as follows. In Section 2 we state and prove Itô's rule for random transformations (Theorem 2.3). To apply this to $v(X_{x,s}(t), r, Y)$, it is necessary to establish conditions on the coefficients of (1.5) and (1.6) such that v satisfies the hypotheses of the modified Itô rule in Theorem 2.3. We treat this problem in Section 3, which also includes additional technical results for the proof of our non-adapted stochastic p.d.e. for v. Section 4 states and proves the main result. We should point out that Section 3 contains a lot of complex, and sometimes tedious, technical details. However, these detailed arguments are not necessary to understanding the essential ideas of the proof of Theorem 4.2, which follows a standard argument. Thus, for the reader interested in getting to the main point, we recommend going straight from Section 2 to Section 4, except for a small detour at Definitions 3.3 and 3.4, which are necessary to understand the hypotheses of Theorem 4.2.

This paper assumes familiarity with the Wiener space gradient, D, the construction of Sobolev spaces on the Wiener space, and elementary properties of the Skorohod integral. A suitable background reference is the paper Nualart & Pardoux [8], Sections 1–5. We need to specialize some of the concepts and notations of [8] for the present work, and we carry this out in the remainder of this section.

The underlying probability space, $(\Omega, \mathscr{F}, \mathbb{P})$, will be the canonical space for an R^l-valued Brownian motion $\omega = (\omega_1, \ldots, \omega_l)$ on the time interval $[0, T]$. As above, we let $\omega_0(t) \doteq t$, and we split ω into the R^{l_1}-valued component W and the R^{l_2}-valued component Y, so that $\omega = (W, Y)$. In writing stochastic integrals we shall use summation notation: thus,

$$\int g_j \, d\omega_j \doteq \sum_0^l \int g_j \, d\omega_j, \quad \int g_j \, dW_j \doteq \sum_1^{l_1} \int g_j \, dW_j,$$

and

$$\int g_j \, dY_j \doteq \sum_1^{l_2} \int g_j \, dY_j.$$

In this paper we only need to use Wiener space gradients associated to the Y-components of ω; likewise, we only need to form stochastic integrals with anticipating integrands when the integrator is dY. Therefore, we make the following definitions, in analogy to those found, for example in [8]. Let $\mathscr{S}$ denote the space of smooth Wiener functionals on Ω: that is, a functional F is in $\mathscr{S}$ if for some non-negative integers n and k, and some non-negative times $t_1, \ldots, t_n, s_1, \ldots, s_k$,

$$F(\omega) = \phi(Y(t_1), \ldots, Y(t_n), W(s_1), \ldots, W(s_k)), \quad \text{for some} \quad \phi \in C_b^\infty(R^{kl_1 + nl_2}).$$

The Y-gradient of such an $F \in \mathscr{S}$ is $D^Y_\cdot F = (D^{Y_1}_\cdot F, \ldots, D^{Y_{l_2}}_\cdot F)$, where

$$D^{Y_i}_\theta F(\omega) \doteq \sum_{j=1}^{n} \frac{\partial \phi}{\partial x^{ij}} (Y(t_1), \ldots, Y(t_n), W(s_1), \ldots, W(s_k)) 1_{[0, t_j]}(\theta), \quad 0 \le \theta \le T.$$

In this formula x^{ij} refers to the variable in ϕ corresponding to $Y_i(t_j)$. The "$\cdot$" in the notation $D_\cdot$ has the purpose of reminding us that $D^{Y_i}_\cdot F(\omega)$ is almost-surely a function in $L^2(0, T)$. If F takes values in R^m, $D^Y_\theta F$ is the matrix whose ijth component is $D^{Y_j}_\theta F_i$.

On $\mathscr{S}$ we introduce the norm

$$\|F\|^Y_{1,p} \doteq E\{|F|^p\} + \left(E\left\{\left(\int_0^T \sum_1^{l_2} (D^{Y_i}_\theta F)^2 \, d\theta\right)^{p/2}\right\}\right)^{1/p}$$

for $p \ge 1$. We denote by $\mathbb{D}^{1,2}_Y$ the closure of $\mathscr{S}$ with respect to $\|\cdot\|^Y_{1,2}$, and by $\mathbb{D}^{1,p}_Y$ the set of $F \in \mathbb{D}^{1,2}_Y$ such that $\|F\|^Y_{1,p} < \infty$, if $p > 2$. D^Y extends to a closed operator on $\mathbb{D}^{1,2}_Y$, by Lemma 2.2 in Nualart & Zakai [9]. Moreover, we define

$$\mathbb{L}^{1,p}_Y \doteq L^p((0, T), dt; \mathbb{D}^{1,p}_Y).$$

We shall use the concept of *Skorohod integrability* defined in [8]. If $u \in \mathbb{L}^{1,2}_Y$, one can argue, using Proposition 3.1 in [8], that u is Skorohod integrable with respect to Y_i for each $1 \le i \le l_2$. Briefly, this means that for $u \in \mathbb{L}^{1,2}_Y$ and $1 \le i \le l_2$, there exists a unique r.v. in $L^2(\Omega)$, denoted $\int_0^T u_s \, dY_i(s)$, having the property

$$E\left\{\int_0^T (D^{Y_i}_\theta F) u_\theta \, d\theta\right\} = E\left\{F \int_0^T u_s \, dY_i(s)\right\} \quad \forall F \in \mathbb{D}^{1,2}_Y. \tag{1.8}$$

We recall here a fundamental formula for Skorohod integrals; see Theorem 3.2 of Nualart & Pardoux [8]:

$$\int_0^T F u_s \, dY^i(s) = F \int_0^T u_s \, dY^i(s) - \int_0^T u_s D^{Y_i}_s F \, ds, \tag{1.9}$$

for $F \in \mathbb{D}^{1,2}_Y$ such that u and Fu are both Skorohod integrable.

The notation $\mathbb{D}^{1,p}$ will denote the space defined analogously to $\mathbb{D}^{1,p}_Y$, but using all the components of ω, as in [8]. In the proof of Theorem 4.1 we shall use the dual, $\mathbb{D}^{-1,2}$, of $\mathbb{D}^{1,2}$. The Skorohod integral is a continuous linear operator from $L^2((0, T) \times \Omega)$ to $\mathbb{D}^{-1,2}$; see Watanabe [17]. Thus $\int_0^T u(s) \, dY_j(s)$ may be defined for any square integrable u, but the result may be distribution-valued. When we say that u is Skorohod integrable in this convext, we mean that $\int_0^T u(s) \, dY_j(s)$ is a square-integrable random variable.

We shall need the following lemma. It is proved by verifying it first on smooth functionals and then taking completions.

LEMMA 1.1 *Let $p \geq 1$. If $F \in \mathbb{D}_Y^{1,p}$, then $E\{F|\mathscr{Y}_T\} \in \mathbb{D}_Y^{1,p}$ also, and*

$$D^Y_{\cdot} E\{F|\mathscr{Y}_T\} = E\{D^Y_{\cdot} F|\mathscr{Y}_T\}.$$

Finally, the symbols ∇, and ∂^α, where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, are reserved for differentiation in the variable x. If g takes values in R, $\nabla g(x)$ denotes the usual gradient, and $\nabla^2 g(x)$ the matrix of second partials. If g takes values in R^n, $\nabla g(x)$ denotes the differential $[\partial_{x_j} g_i(x)]_{1 \leq i,j \leq d}$. The notation $\|\cdot\|$ is reserved for the $L^2((0, T); R^{l_2})$ norm. Thus, for example,

$$\|D^Y_{\cdot} F\|^2 = \int_0^T \sum_1^{l_2} (D_\theta^{Y_i} F)^2 \, d\theta.$$

2 AN ITÔ LEMMA FOR RANDOM TRANSFORMATIONS

In this section we shall develop an Itô lemma for *random* transformations of an Itô process. That is, given a random field $\{\Phi(x, \omega); x \in R^d, \omega \in \Omega\}$ and an Itô process $\{U(t); t \geq 0\}$, we shall show how to express $\Phi(U(t, \omega), \omega)$ using stochastic integrals. Because $\Phi(x, \omega)$ depends in general on the entire history of the Brownian path ω, this generalized Itô rule necessarily requires the anticipating calculus. The result is stated in Theorem 2.3. We remark here that we have not attempted to state the most general result. In particular, we assume that the Itô process satisfies

$$E\{\sup_{s \leq T} |U(s)|^p\} < \infty,$$

for all $p > 1$. This assumption holds for our applications and helps to simplify the statements. Generalizations in which the conditions on U are relaxed are possible. In fact, Theorem 2.3 may be viewed as a special case of the generalized Itô-Ventzell formula developed in [11]. However, the hypotheses here are weaker due to the more restrictive nature of the problem.

The proof of our Itô rule requires a chain rule to compute the Wiener space gradient of the composition, $\phi(Z, \omega)$, of a Brownian functional, Z, with a random map $\phi(x, \omega)$. To establish such a chain rule, we need to know that $D\phi(x, \omega)$ exists, and we need to be able to control the moments of $\phi(Z, \omega)$ and $D\phi(x, \omega)|_{x=Z}$. We shall achieve control of moments by the imposition of polynomial growth conditions in the following definition. Recall that we decompose $\omega \in \Omega$ as $\omega = (W, Y)$, where $W \in C([0, T]; R^{l_1})$ and $Y \in C([0, T]; R^{l_2})$.

DEFINITION 2.1 We say that the random field $\phi = \{\phi(x, \omega); x \in R^d, \omega \in \Omega\}$ satisfies hypothesis (A) with moments p_1 and p_2, if ϕ is a measurable function such that:

A.1) $\phi(\cdot, \omega) \in C^1(R^d)$ almost surely, and

$$|\nabla_x \phi(x, \omega)| \leq c_1(\omega)(1 + |x|^{\beta_1}) \quad \text{for all } x, \text{ almost surely,}$$

for some $\beta_1 > 0$ and some $c_1 \in L^{p_1}$.

A.2) $\phi(x, \cdot) \in \mathbb{D}_Y^{1,p_2}$ for every x, and the map $(x, s, \omega) \to D_s^Y \phi(x, \omega) \in R^{l_2}$ admits a measurable version which is continuous in x for almost every (s, ω). Furthermore

$$|D_s^Y \phi(x, \omega)| \leq c_2(s, \omega)(1 + |x|^{\beta_2}) \quad \text{for all } x, \text{ for almost every } (s, Y),$$

for some $\beta_2 > 0$ and some $c_2 \in L^{p_2}((0, T) \times \Omega)$.

We shall show in the next section that stochastic flows associated to stochastic differential equations satisfy hypothesis (A), provided the coefficients are sufficiently regular. Hence, the choice of conditions in hypothesis (A) suits our needs.

In what follows, we shall distinguish $D_{\cdot}^Y(\phi(Z, \omega))$ from $D_{\cdot}^Y \phi(x, \omega)|_{x=Z}$ by use of the notation

$$[D_{\cdot}^Y \phi](Z, \omega) \doteq D_{\cdot}^Y \phi(x, \omega)|_{x=Z}.$$

The following result on differentiation is a refinement of Lemmas II.2.3 and II.2.4 of Ocone & Pardoux [11] to the situation of this paper.

LEMMA 2.2 *Let ϕ satisfy hypothesis (A) with $p_1 > 4$ and $p_2 > 2$. Suppose also that $Z \in (\mathbb{D}_Y^{1,p})^d$ for some $p > 4$ and $E\{|Z|^r\} < \infty$ for all $r > 1$. Then for any q such that $2 \leq q < p_2 \wedge (p_1 p/(p + p_1))$,*

$$\phi(Z(\cdot), \cdot) \in \mathbb{D}_Y^{1,q} \quad \textit{and} \quad D_{\cdot}^Y(\phi(Z, \omega)) = \nabla_x \phi(Z, \omega) D_{\cdot}^Y Z + [D_{\cdot}^Y \phi](Z, \omega). \tag{2.1}$$

If, in addition,

$$\operatorname{ess\,sup}\{|\nabla_x \phi(x, \omega)|; x \in R^d, \omega \in \Omega\} < \infty, \tag{2.2}$$

then $\phi(Z(\cdot), \cdot) \in \mathbb{D}_Y^{1,q}$ for $2 \leq q < p \wedge p_2$.

Proof From A.2), $\phi(0, \cdot) \in \mathbb{D}_Y^{1,p_2}$, and hence $\phi(0, \cdot) \in L^{p_2}$. Also, by A.1) and the mean value theorem

$$|\phi(x, \omega)| \leq K[|\phi(0, \omega)| + c_1(\omega)][1 + |x|^{1+\beta_1}],$$

where K is a deterministic constant. Since Z has moments of all orders, it follows that $\phi(Z, \omega) \in L^q$ for $q < p_1 \wedge p_2$. Also, if we let F denote the right hand side of (2.1),

$$\begin{aligned} E\{\|F\|^q\} &\leq 2^{q-1} E\{(|\nabla_x \phi(Z, \omega)| \, \|D_{\cdot}^Y Z\|)^q + \|[D_{\cdot}^Y \phi](Z, \omega)\|^q\} \\ &\leq 2^{q-1} E\{c_1^q (1 + |Z|^{\beta_1})^q \|D_{\cdot}^Y Z\|^q + c_2^q (1 + |Z|^{\beta_2})^q\}. \end{aligned} \tag{2.3}$$

Recall that, here, F takes values in $L^2((0, T); R^{l_2})$ and $\|F\|$ represents the L^2-norm of F. An application of Hölder's inequality shows that this last expression is finite if $q < p_2 \wedge (p_1 p/(p + p_1))$.

Let us first prove Lemma 2.2 under the additional assumption

$$\sup\{|\phi(x, \omega)|;\ x \in R^d,\ \omega \in \Omega\} < \infty. \tag{2.4}$$

In this case let

$$\phi_n(x, \omega) \doteq \int \rho_n(x - \lambda)\phi(\lambda, \omega)\, d\lambda, \tag{2.5}$$

where $\{\rho_n\}$ is a sequence of positive, smooth mollifiers converging to the Dirac delta function at 0. By approximating ϕ_n by a converging sequence of Riemann sum approximations to (2.5), one can show that $\phi_n(Z, \omega) \in \mathbb{D}_Y^{1,q}$ for $2 \le q < p \wedge p_2$,

$$D_\cdot^Y \phi_n(Z, \omega) = \int \rho_n(Z - \lambda)[\nabla_x \phi(\lambda, \omega) D_\cdot^Y Z + [D_\cdot^Y \phi](\lambda, Y)]\, d\lambda,$$

and

$$\lim_{n \to \infty} E\{|\phi_n(Z, \omega) - \phi(Z, \omega)|^q + \|D_\cdot^Y(\phi_n(Z, \omega)) - F\|^q\} = 0,$$

for $2 \le q < p \wedge p_2$, where F denotes the right hand side of (2.1). This completes the proof when (2.4) holds, because $D_\cdot^Y$ is closed on $\mathbb{D}_Y^{1,q}$. To remove the assumption (2.4), we consider the sequence of random fields $\psi_n(\phi(x, \omega))$, $n \ge 1$, where $\{\psi_n\}$ is a sequence of bounded, infinitely differentiable functions on R with bounded derivatives of all orders such that $\psi_n(v) = v$ if $|v| \le n$, $\psi_n(v) \le |v|$ for all v, and

$$\sup\{|\psi_n'(v)|;\ v \in R,\ n \ge 1\} < \infty.$$

Then $\psi_n(\phi(x, \omega))$ satisfies (2.4) for every n and so

$$D_\cdot^Y(\psi_n(\phi(Z, \omega))) = \psi_n'(\phi(Z, \omega))[\nabla_x \phi(Z, \omega) D_\cdot^Y Z + [D_\cdot^Y \phi](Z, \omega)].$$

It follows by dominated covergence that

$$\lim_{n \to \infty} E\{|\psi_n(\phi(x, \omega)) - \phi(x, \omega)|^q\} = 0,$$

and

$$\lim_{n \to \infty} E\{\|D_\cdot^Y(\psi_n(\phi(x, \omega))) - (\nabla_x \phi(Z, \omega) D_\cdot^Y Z + [D_\cdot^Y \phi](Z, \omega))\|^q\} = 0,$$

for $q < p_2 \wedge (p_1 p/(p + p_1))$, or ($q < p \wedge p_2$ in case of (2.2).) ■

Next we state and prove an Itô rule for random transformations.

THEOREM 2.3 *Let*

$$U(t) = u_0 + \int_0^t \alpha(s)\, ds + \int_0^t \beta_j(s)\, dW_j(s) + \int_0^t \gamma_i(s)\, dY_i(s)$$

be an R^d-valued Itô process such that $u_0 \in R^d$, and

The processes α, β_j, $1 \le j \le l_1$, and γ_i, $1 \le i \le l_2$, are progressively measurable, and they belong to $L^r((0, T) \times \Omega; R^d)$, for every $r > 1$. (2.7)

$$\gamma_i \in \mathbb{L}_Y^{1,4}, \quad \text{for} \quad 1 \le i \le l_2. \tag{2.8}$$

$$U \in \mathbb{L}_Y^{1,m} \quad \text{for some} \quad m > 8. \tag{2.9}$$

Let $\{\Phi(x, Y)\}$ be a measurable functional, defined on $R^d \times C([0, T]; R^{l_2})$, such that $\Phi(\cdot, Y) \in C^2(R^d)$ almost surely, and

For every $1 \le i \le d$, $\partial_{x_i}\Phi(\cdot, \cdot)$ satisfies hypothesis (A) with moments $p_1 = 8$ and $p_2 = 4$. (2.10)

Then, for each $t > 0$, $\nabla\Phi(U(\cdot), Y)\gamma(\cdot) \in \mathbb{L}_Y^{1,2}$, and

$$\begin{aligned}\Phi(U(t), Y) = \Phi(u_0, Y) &+ \int_0^t [\nabla\Phi(U(s), Y) \cdot \alpha(s) + \mathrm{tr}([D_s^Y \nabla\Phi](U(s), Y)\gamma(s))]\, ds \\ &+ \int_0^t \frac{1}{2} \mathrm{tr}[\nabla^2\Phi(U(s), Y)(\beta\beta^*(s) + \gamma\gamma^*(s))]\, ds \\ &+ \int_0^t \nabla\Phi(U(s), Y)[\beta(s)\, dW(s) + \gamma(s)\, dY(s)],\end{aligned} \tag{2.11}$$

where, in (2.11), $\beta(s) \doteq [\beta_1(s)|\cdots|\beta_{l_1}(s)]$ and $\gamma(s) \doteq [\gamma_1(s)|\cdots|\gamma_{l_2}(s)]$. Note that the term $[D_s^Y\nabla\Phi](U(s), Y)$ makes sense by virtue of the continuity of $U(s)$ in s and the continuity of $D_s^Y\nabla\Phi(x, Y)$ in x.

Proof $\partial_{x_i}\Phi(\cdot, \cdot)$ satisfies (A) with moments $p_1 = 8$ and $p_2 = 4$, $U(t) \in \mathbb{D}_Y^{1,m}$ for almost every t, for some $m > 8$, by (2.9) and $U(t) \in L^q(\Omega)$ for all $q > 1$ by (2.7). Therefore, Lemma 2.1 applies to $\nabla\Phi(U(t), Y)$ for almost every t;

$$\nabla\Phi(U(t), Y) \in (\mathbb{D}_Y^{1,4})^d$$

for almost every t, and

$$D_\cdot^{Y_i}\nabla\Phi(U(t), Y) = \nabla^2\Phi(U(t), Y)D_\cdot^{Y_i}U(t) + [D_\cdot^{Y_i}\nabla\Phi](U(t), Y). \tag{2.12}$$

Notice that $D_s^{Y_i}\nabla\Phi(U(t), Y) = [D_s^{Y_i}\nabla\Phi](U(t), Y)$ for $s > t$ because $D_s^{Y_i}U(t) = 0$ for $s > t$ by the progressive measurability of U. The conditions A.1) and A.2), applied to (2.12), easily yield that $\nabla\Phi(U(\cdot), \cdot) \in (\mathbb{L}_Y^{1,4})^d$. By adapting an argument of Liptser & Shiryayev [7], pp. 94–95, one can choose a sequence of partitions $\Pi^{(n)} = \{t_i^n\}$ of $[0, t]$ such that $\sup_i(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} \left\| \sum_i \nabla\Phi(U(t_i^n), \cdot)1_{[t_i^n, t_{i+1}^n)}(\cdot) - \nabla\Phi(U(\cdot), \cdot) \right\|_{L_Y^{1,4}} = 0. \tag{2.13}$$

(For a detailed argument, see the Appendix of [10].) For convenience of notation, set $\xi^n(t) \doteq \sum_i \nabla\Phi(U(t_i^n), \cdot)1_{[t_i^n, t_{i+1}^n)}(t)$. By Taylor's formula, we may write

$$\begin{aligned}\Phi(U(t), Y) - \Phi(u_0, Y) &= \sum_i \nabla\Phi(U(t_i^n), Y)[U(t_{i+1}^n) - U(t_i^n)] \\ &\quad + \frac{1}{2}\sum_i (U(t_{i+1}^n) - U(t_i^n))^*\nabla^2\Phi(\bar{U}_i, Y)(U(t_{i+1}^n) - U(t_i^n)), \end{aligned} \tag{2.14}$$

where, for each i, $\bar{U}_i$ is a point on the line segment connecting $U(t_{i+1}^n)$ and $U(t_i^n)$.

Consider the first term of (2.14). We can write it as

$$\begin{aligned}&\sum_i \nabla\Phi(U(t_i), Y)\left[\int_{t_i}^{t_{i+1}} \alpha(s)\, ds + \beta(s)\, dW(s) + \gamma(s)\, dY(s)\right] \\ &\quad = \int_0^t [\xi^n(s)\alpha(s)\, ds + \xi^n(s)\beta(s)\, dW(s)] + \sum_i \nabla\Phi(U(t_i), Y)\int_{t_i}^{t_{i+1}} \gamma(s)\, dY(s) \end{aligned} \tag{2.15}$$

Notice that in the last equality we have interchanged multiplication by $\nabla\Phi(U(t_i), Y)$ with integration with respect to dW. This is legal because Y and W are independent and $\Phi(U(t), Y)$ does not anticipate W. As a consequence of (2.13),

$$E\left\{\int_0^t |\xi^n(s) - \nabla\Phi(U(s), Y)|^2\, ds\right\} \to 0 \quad \text{as} \quad n \to 0.$$

Consequently, the first two terms of the right hand side of (2.15) converge in mean-square to

$$\int_0^t [\nabla\Phi(U(s), Y)\alpha(s)\, ds + \nabla\Phi(U(s), Y)\beta(s)]\, dW(s).$$

To handle the remaining term, we use Theorem 3.2 of Nualart & Pardoux [8], which is recalled above in Eq. (1.9), to observe that

$$\sum_i \nabla\Phi(U(t_i), Y) \int_{t_i}^{t_{i+1}} \gamma(s)\, dY(s) = \int_0^t \xi^n(s)\gamma(s)\, dY(s) + \sum_i \int_{t_i}^{t_{i+1}} \mathrm{tr}([D_s^Y \nabla\Phi](U(t_i), Y)\gamma(s))\, ds. \quad (2.16)$$

From (2.13) and (2.8) one derives that

$$\lim_{n\to\infty} \|\xi^n(\cdot)\gamma(\cdot) - \nabla\Phi(U(\cdot), \cdot)\gamma(\cdot)\|_{L_Y^2} = 0,$$

and hence, using Proposition 3.5 of [8], that $\int_0^t \xi^n(s)\gamma(s)\, dY(s)$ converges in mean-square to $\int_0^t \nabla\Phi(U(s), Y)\gamma(s)\, dY(s)$. Furthermore, the continuity in t of $U(t)$ and the continuity of $D_s^Y \nabla\phi(x, Y)$ in x guarantee that

$$\lim_{n\to\infty} \sum_i \mathrm{tr}([D_s^Y \nabla\Phi](U(t_i), Y)\gamma(s)) 1_{[t_i, t_{i+1})}(s) = \mathrm{tr}([D_s^Y \nabla\Phi](U(s), Y)\gamma(s)),$$

for almost every $s \in [0, t]$, almost surely. Since, in addition, A.2) implies

$$|[D_s^Y \nabla\Phi](U(r), Y)| \le c_2(s, Y)\left(1 + \sup_{[0,T]} |U(r)|^\beta\right),$$

where $c_2 \in L^4([0, T] \times \Omega)$, dominated convergence leads to

$$\sum_i \int_{t_i}^{t_{i+1}} \mathrm{tr}([D_s^Y \nabla\Phi](U(t_i), Y)\gamma(s))\, ds \to \int_0^t \mathrm{tr}([D_s^Y \nabla\Phi](U(s), Y)\gamma(s))\, ds, \quad (2.17)$$

almost surely. From the above argument, we find that

$$\begin{aligned} &\sum_i \nabla\Phi(U(t_i), Y)[U(t_{i+1}) - U(t_i)] \\ &\quad \to \int_0^t [\nabla\Phi(U(s), Y)\cdot\alpha(s) + \mathrm{tr}([D_s^Y \nabla\Phi](U(s), Y)\gamma(s))]\, ds \\ &\qquad + \int_0^t \nabla\Phi(U(s), Y)[\beta(s)\, dW(s) + \gamma(s)\, dY(s)] \end{aligned}$$

in probability, as $n \to \infty$.

Finally, we must deal with the term in (2.14) involving second order derivatives of Φ. First notice that $\sum_i \bar{U}_1 1_{[t_i, t_{i+1})(t)}$ converges uniformly in t to $U(t)$, almost surely, because $U(t)$ is continuous in t. Also, from (2.7), it is a standard fact about semi-martingales that

$$\sum_{t_{i+1}^n \leq t} (U(t_{i+1}^n) - U(t_i^n))(U(t_{i+1}^n) - U(t_i^n))^* \to \int_0^t [\beta\beta^*(s) + \gamma\gamma^*(s)]\, ds,$$

in probability, for each $0 \leq t \leq T$. Therefore, we can use Lemma C.2 of [8] to conclude that

$$\frac{1}{2}\sum_i (U(t_{i+1}) - U(t_i))^* \nabla^2\Phi(\bar{U}_i, Y)(U(t_{i+1}) - U(t_i))$$

converges in probability to

$$\int_0^t \frac{1}{2} \operatorname{tr}[\nabla^2\Phi(U(s), Y)(\beta\beta^*(s) + \gamma\gamma^*(s))]\, ds.$$

This completes the proof. ■

In working with fields of the form $\{\phi(x, \omega); (x, \omega) \in R^d \times \Omega\}$, it will be useful to know under what circumstances we may interchange the Wiener space gradient D and differentiation with respect to x. The next lemma states a simple sufficient condition.

LEMMA 2.4 *Let $p \geq 2$. Assume that $\{\phi(x, \omega)\}$ is a measurable random field such that*

$$\phi(x, \cdot) \in \mathbb{D}_Y^{1,p} \text{ for all } x, \text{ and } \phi(\cdot, \omega) \in C^r(R^d) \text{ almost surely}; \tag{2.18}$$

For each $1 \leq i \leq l_2$, the map $(\theta, x, t, \omega) \mapsto D_\theta^{Y_i}\phi(x, t, \omega)$ admits a measurable version such that $D_\theta^{Y_i}\phi(\cdot, t, \omega) \in C^r(R^d; R^{l_1})$ for $d\theta \times dt \times d\mathbb{P}$-almost every (θ, t, ω); (2.19)

For each compact $G \subset R^d$, there exist an $N_G \in L^p(\Omega)$, and an $M_G \in L^p((0, T) \times \Omega)$ such that

$$\sup_{x \in G} |\partial^\alpha \phi(x, \omega)| \leq N_G(\omega) \quad a.s.,$$

and

$$\sup_{x \in G} |\partial^\alpha D_\theta^Y \phi(x, \omega)| \leq M_G(\theta, \omega), \quad a.s., \tag{2.20}$$

for all $|\alpha| \leq r$.

Then for each x, *and each* $|\alpha| \leq r$, $\partial^\alpha \phi(x, \cdot) \in \mathbb{D}_Y^{1,p}$, *and*

$$D_\theta^Y \partial^\alpha \phi(x, \omega) = \partial^\alpha D_\theta^Y \phi(x, \omega). \tag{2.21}$$

Proof Suppose $|\alpha| = 1$. Let $\Delta_\delta^i \phi(x, \omega) \doteq \phi(x + \delta e_i, \omega) - \phi(x, \omega)$, where e_i is the unit vector along the x_i-axis. Fix x. Noting that $D^Y \Delta_\delta^i \phi(x, \omega) = \Delta_\delta^i D^Y \phi(x, \omega)$, one can use condition (2.20) and dominated convergence to show that

$$\lim_{\delta \to 0} E\{|\delta^{-1} \Delta_\delta^i \phi(x, \cdot) - \partial_{x_i} \phi(x, \cdot)|^p + \|\delta^{-1} D^Y \Delta_\delta^i \phi(x, \cdot) - \partial_{x_i} D^Y \phi(x, \cdot)\|^p\} = 0.$$

Because $\mathbb{D}_Y^{1,p}$ is closed, it follows that $\partial_{x_i} \phi(x, \cdot) \in \mathbb{D}_Y^{1,p}$ and that (2.21)) is true for each x. By applying the same argument with $\partial_{x_i} \phi$ in place of ϕ, we find that $\partial^\alpha \phi \in \mathbb{D}_Y^{1,p}$ for $|\alpha| = 2$, and so on up to $|\alpha| = r$. ■

3 DIFFERENTIABILITY PROPERTIES OF $v(x, s, Y)$.

Suppose that the process $\{X_{x,s}(t)(\omega); (x, s, t, \omega) \in R^d \times [0, T]^2 \times \Omega\}$ solves the equation

$$X_{x,s}(t) = x + 1_{[t \geq s]} \int_s^t f_j(X_{x,s}(r), r, \omega)\, d\omega_j(r) \tag{3.1}$$

for each $(x, s) \in R^d \times [0, T]$, and let

$$Z_{x,s}(t) \doteq \exp\left\{1_{[t \geq s]}\left(\int_s^t h_j(X_{x,s}(r), r, \omega)\, dY_j(r) - (1/2) \int_s^t |h(X_{x,s}(r), r, \omega)|^2\, dr\right)\right\}.$$

For each $(x, s) \in R^d \times [0, T]$, define $v(x, s, Y) \doteq E\{\psi(X_{x,s}(T)) Z_{x,s}(T) | \mathscr{Y}_Y\}$. In this section we shall establish conditions on the coefficients $\{f_j; 0 \leq j \leq l\}$ and $\{h_j; 1 \leq j \leq l_2\}$ and on ψ to ensure that, for fixed s, $v(x, s, Y)$ satisfies condition (2.10) of Theorem 2.3. This is crucial for the proof of the main result, Theorem 4.2, in which we apply the extended Itô formula using $v(x, s, Y)$ as a random transformation. Thus, we seek conditions implying that the derivatives, $\partial_{x_i} v(x, s, Y)$, $1 \leq i \leq d$, all satisfy hypothesis (A), which requires differentiability in x and Y of $\partial_{x_i} v(x, s, Y)$ and polynomial growth in x of $\partial_{x_i} v(x, s, Y)$ and its derivatives. We shall obtain such regularity by imposing the natural conditions on $\{f_j; 0 \leq j \leq l\}$ and $\{h_j; 1 \leq j \leq l_2\}$ needed for differentiability and polynomial growth of $X_{x,s}(t)$ and $Z_{x,s}(t)$, and we then show that v inherits these properties. Notice that, in this section, the coefficients $f_j(x, s, \omega)$ or $h_j(x, s, \omega)$ depend on both components, W and Y, of ω. Later on, to obtain a stochastic partial differential equation for $v(x, s, Y)$, it will be necessary to restrict the dependence to Y alone. But, for now, it is useful to allow the more general situation, and this causes no extra complication.

The section is fairly technical. In order to go on to Section 4, before negotiating all the details here, the reader needs to understand Definitions 3.1, 3.3, 3.4 and the definitions in Corollary 3.12, and the statements of Theorems 3.7, 3.10, Corollary 3.11 and Propositions 3.6 and 3.13. The other results play supporting roles. The main results are Theorems 3.7 and 3.10, and they are consequences of Propositions 3.6 and 3.9, respectively. Most of the technical work is in the proofs of the Propositions and is deferred to the end of the section. Although technical, the methods of this section are fairly standard; our treatment is inspired by the methods found in Krylov [4], Kunita [6], and Stroock [16], for example, and our results are closely related.

Notational Convention Throughout this section, K and b shall denote generic, finite, positive constants that appear in various bounds. The actual values of K and b may change from expression to expression, even within the same argument.

For our analysis it is convenient to identify several different classes of processes. In what follows, T denotes a fixed, positive time.

DEFINITION 3.1 $\mathrm{Lip}_n(p)$ shall denote the class of processes of the form

$$\gamma = \{\gamma(y, s, t, \omega); (y, s, t, \omega) \in R^n \times [0, T]^2 \times \Omega\}$$

such that $\gamma(y, s, t, \omega) = \gamma(y, s, t \vee s, \omega)$, and such that there exist constants $0 < K < \infty$ and $0 < \beta < \infty$ for which

$$E\left\{\operatorname*{ess\,sup}_{t \le T} |\gamma(y, s, t, \cdot)|^p\right\} \le K(1 + |y|^\beta) \quad \forall s \in [0, T] \tag{3.2}$$

and

$$\begin{aligned} &E\left\{\operatorname*{ess\,sup}_{t \le T} |\gamma(y', s', t, \cdot) - \gamma(y, s, t, \cdot)|^p\right\} \\ &\quad \le K(1 + |y|^\beta + |y'|^\beta)(|y - y'|^p + |s - s'|^{p/2}) \\ &\qquad \forall (s, s') \in [0, T]^2 \quad \forall (y, y') \quad \text{such that} \quad |y - y'| \le 1. \end{aligned} \tag{3.3}$$

We shall use $\mathrm{Lip}_n(\infty)$ to denote $\bigcap_{p>1} \mathrm{Lip}_n(p)$. In addition, we shall say that γ is in $\mathrm{Lip}_n(p, q)$, if γ satisfies (3.2) and, instead of (3.3),

$$\begin{aligned} &E\left\{\operatorname*{ess\,sup}_{t \le T} |\gamma(y', s', t, \cdot) - \gamma(y, s, t, \cdot)|^p\right\} \\ &\quad \le K(1 + |y|^\beta + |y'|^\beta)(|y - y'|^p + |s - s'|^{q/2}) \\ &\qquad \forall (s, s') \in [0, T]^2 \quad \forall (y, y') \quad \text{such that} \quad |y - y'| \le 1. \end{aligned} \tag{3.3'}$$

In practice, we deal only with the cases in which either $y = x \in R^d$, $(n = d)$, or $y = (x, \delta) \in R^{d+1}$, $(n = d + 1)$. Since the appropriate case will always be apparent from context, we shall usually write just Lip(q) instead of $\mathrm{Lip}_d(q)$ or $\mathrm{Lip}_{d+1}(q)$.

Suppose that $\gamma = \{\gamma(y, s, t, \omega)\}$ is almost surely continuous in t for every (y, s). Then, if $\gamma \in \mathrm{Lip}_d(q, p)$ and $p \wedge (q/2) > d + 1$, γ admits a version which is almost surely continuous in (y, s, t). This follows easily from Kolmogorov's continuity theorem; see, for example, Stroock [16]. We shall use this observation often. Also, we shall use the following fact.

LEMMA 3.2 *Let $\{\gamma(x, \omega)\}$ be a measurable random field such that $\gamma(\cdot, \omega) \in C^1(R^d)$, almost surely, and*

$$E\{|\gamma(x, \cdot)|^p\} \leq K(1 + |x|^\beta), \tag{3.4}$$

$$E\{|\nabla_x \gamma(x, \cdot)|^p\} \leq K(1 + |x|^\beta), \tag{3.5}$$

for some $K < \infty$, $\beta > 0$, and $p > d$. Then for any $b > (d + \beta)/p$, there is a $c \in L^p(\Omega)$ such that

$$|\gamma(x, \omega)| \leq c(\omega)(1 + |x|^b) \quad \textit{almost surely.} \tag{3.6}$$

Proof This is a consequence of the Sobolev inequality

$$\sup|v(x)| \leq c_p \|v\|_{1,p} \doteq c_p \left(\int (|v|^p + |\nabla v|^p)\, dx \right)^{1/p}$$

for $p > d$, where c_p is a constant independent of v: see Adams [1]. Define the random constant $c(\omega) = c_p \|\tilde{\gamma}(\cdot, \omega)\|_{1,p}$, where $\tilde{\gamma}(x, \omega) \doteq (1 + |x|^2)^{-b/2}\gamma(x, \omega)$. (3.4) and (3.5) imply that $c \in L^p(\Omega)$. (3.6) follows from applying Sobolev's inequality to $\tilde{\gamma}$. ∎

The next definition summarizes the type of regularity in x to be required of the coefficients $\{f_j(x, t, \omega)\}$ and $\{h_i(x, t, \omega)\}$.

DEFINITION 3.3 A measurable $k: R^d \times [0, T] \times \Omega \mapsto R$ is said to be in the class C^r if

$k(\cdot, t, \omega) \in C^r(R^d)$ for $dt \times d\mathbb{P}$-almost every (t, ω), and k and its derivatives are progressively measurable; (3.7)

There is a constant $K < \infty$ such that $|k(0, t, \omega)| \leq K$ for $dt \times d\mathbb{P}$-almost every (t, ω); (3.8)

There is a random variable $c \in \bigcap_{q>1} L^q(\Omega)$ and a constant $b > 0$ such that

$$\operatorname*{ess\,sup}_{t \leq T} |\partial^\alpha k(x, t, \omega)| \leq c(\omega)(1 + |x|^b), \quad \forall |\alpha| \leq r. \tag{3.9}$$

Finally, we need to impose regularity of the coefficients f_j and h_i as functionals of ω. The conditions we use are very much like those required in hypothesis (A).

DEFINITION 3.4 A measurable process $k: R^d \times [0, T] \times \Omega \to R$ is said to be in the class $\mathscr{D}_Y(r, p)$ if

$$k(x, t, \cdot) \in \mathbb{D}_Y^{1,p} \text{ for all } x \text{ and almost every } t; \tag{3.10}$$

For each $1 \leq i \leq l_2$, the map $(\theta, x, t, \omega) \mapsto D_\theta^{Y^i} k(x, t, \omega)$ admits a measurable version such that $D_\theta^{Y^i} k(\cdot, t, \omega) \in C^r(R^d)$ for $d\theta \times dt \times d\mathbb{P}$-almost every (θ, t, ω); (3.11)

There exist a r.v. $M \in L^p([0, T]^2 \times \Omega)$ and a constant $b > 0$ such that

$$|\partial^\alpha D_\theta^Y k(x, t, \omega)| \leq M(t, \theta, \omega)(1 + |x|^b) \tag{3.12}$$

for all x, for almost every (θ, t, ω) and for all $|\alpha| \leq r$.

The following lemma follows directly from Lemma 2.4 and allows us to freely interchange D and differentiation in x when dealing with sufficiently regular processes.

LEMMA 3.5 *Suppose that for some $p \geq 2$ and some integer $r \geq 1$,*

$$\{k(x, t, \omega)\} \in \mathscr{D}_Y(r, p) \cap \mathscr{C}^r. \tag{3.13}$$

Then for any $|\alpha| \leq r$, $\{\partial^\alpha k(x, t, \omega)\} \in \mathscr{D}_Y(r - |\alpha|, p)$, and

$$D_\theta^Y \partial^\alpha k(x, t, \omega) = \partial^\alpha D_\theta^Y k(x, t, \omega). \tag{3.14}$$

Now we are prepared to state the main theorems on the regularity of the process $v(x, s, Y)$. We shall first treat regularity in x only, without regard to regularity in ω.

PROPOSITION 3.6 *Fix $r \geq 2$. Assume that*

All components of f_j, $0 \leq j \leq l$, are processes in $\mathscr{C}^r$. (3.15)

All components of h_i, $1 \leq j \leq l_2$, are processes in $\mathscr{C}^r$. (3.16)

The processes ∇f_j, $0 \leq j \leq l$ and h_i, $1 \leq i \leq l_2$, are uniformly essentially bounded. (3.17)

Then $\{X_{x,s}(t)\}$ and $\{Z_{x,s}(t)\}$ admit versions such that

$\{X_{x,s}(t)\}$ and $\{Z_{x,s}(t)\}$ are in $C^{r-1,0,0}(R^d \times [0, T]^2)$ almost surely, and, (3.18)

$$\{\partial^\alpha X_{x,s}(\cdot), \partial^\alpha Z_{x,s}(\cdot); |\alpha| \leq r - 1\} \subset \mathrm{Lip}(\infty). \tag{3.19}$$

The result (3.18) on differentiability of $\{X_{x,s}(t)\}$ is standard, and its proof entails proving (3.19) also, although (3.19) is usually not so explicitly stated. See for example, Kunita [6] or Stroock [16] in the case in which the coefficient functions are not random. The statement (3.19) for $\{X_{x,s}(t)\}$ and $\alpha = 0$ goes back to Blagovescenskii & Freidlin [2]. We sketch the proof of Proposition 3.6 later.

Proposition 3.6 leads to the regularity in x of v.

THEOREM 3.7 *Assume that* $\{f_j; 0 \leq j \leq l_1 + l_2\}$ *and* $\{h_i; 1 \leq i \leq l_2\}$ *satisfy conditions* (3.15)–(3.17). *Assume in addition that* $\psi \in C^r(R^d)$ *and that* ψ *and its derivatives up to order* r *grow at most polynomially. Then* $\{v(x, s, Y)\}$ *admits a version such that*

$$v(\cdot, \cdot, Y) \in C^{r-1,0}(R^d \times [0, T]) \quad \textit{almost surely,} \tag{3.20}$$

and for all $s \leq T$ *and any* $q > 1$, *there exists a random variable* $c_{s,q}(Y)$ *and an* $a > 0$ *depending only on* q *but not on* s, *such that*

$$|\partial^\alpha v(x, s, Y)| \leq c_{s,q}(Y)(1 + |x|^a) \quad \forall |\alpha| \leq r - 2, \tag{3.21}$$

and,

$$\sup_{s \leq T} E\{c_{s,q}^q\} < \infty. \tag{3.22}$$

Finally, for $|\alpha| \leq r - 1$,

$$\partial^\alpha v(x, s, Y) = E\{\partial^\alpha(\psi(X_{x,s}(T))Z_{x,s}(T)) | \mathscr{Y}_T\}. \tag{3.23}$$

Before proving Theorem 3.7, we state a well-known technical lemma, which, in fact, also contains the basic idea behind the proof of Proposition 3.6. For this, we introduce some important notation. Let g be a function of $x \in R^d$, and let $1 \leq i \leq d$. Then, define

$$\delta^{-1}\Delta_\delta^i g(x) \doteq \begin{cases} \delta^{-1}[g(x + \delta e_i) - g(x)] & \text{if } \delta \neq 0; \\ \partial_{x_i} g(x) & \text{if } \delta = 0. \end{cases}$$

We think of $\delta^{-1}\Delta_\delta^i g(x)$ as a function of the two variables (δ, x). Notice that $\delta^{-1}\Delta_\delta^i g(x)$ can be defined for $\delta \neq 0$ even if g is not differentiable.

LEMMA 3.8 *Let* $\{\gamma(x, s, t, \omega)\}$ *be a measurable process such that for some* p, q *with* $p \wedge (q/2) > d + 2$,

$$\gamma \in \mathrm{Lip}_d(p, q), \quad \{\delta^{-1}\Delta_\delta^i \gamma(x, s, t, \omega); \delta \neq 0\} \in \mathrm{Lip}_{d+1}(p, q). \tag{3.24}$$

(To emphasize, $\delta^{-1}\Delta_\delta^i \gamma \in \mathrm{Lip}_{d+1}(p, q)$ is a random field indexed by (δ, x, s, t).) Then, it follows that

$$\gamma(\cdot, \cdot, \cdot, \omega) \in C^{1,0}(R^d \times [0, T]), \quad \textit{for almost every } (t, \omega), \textit{ and}$$
$$\{\partial^\alpha \gamma; |\alpha| \le 1\} \subset \mathrm{Lip}_d(p, q). \tag{3.25}$$

Conversely, (3.25) *implies*

$$\{\delta^{-1}\Delta_\delta^i \gamma(x, s, t, \omega)\} \in \mathrm{Lip}_{d+1}(p, q). \tag{3.26}$$

Proof (3.24) implies (3.25) by an application of Kolmogorov's continuity criterion. If we write

$$\delta^{-1}\Delta_\delta^i \gamma(x, s, t, \omega) = \int_0^1 \partial_{x_i}\gamma(x + \lambda\delta e_i, s, t, \omega)\, d\lambda,$$

we obtain (3.26) from (3.25) by a simple calculation. ∎

Proof of Theorem 3.7 Recall that K and b are generic positive constants. In all expressions the actual values of K and b will depend only on q, and on the processes $\{f_j; 0 \le j \le l\}$ and $\{h_i; 1 \le i \le l_2\}$. They will not depend on x, s, t, or ω.

First, we show that $v(x, s, Y)$ admits a version, which is almost surely continuous in (x, s). Indeed, a simple estimate using Jensen's inequality and (3.19) shows that for any $q > 1$,

$$\begin{aligned} E\{|v(x,s,\cdot) - v(x',s',\cdot)|^q\} &\le E\{|\psi(X_{x,s}(T))Z_{x,s}(T) - \psi(X_{x',s'}(T))Z_{x',s'}(T)|^q\} \\ &\le K(1 + |x|^b + |x'|^b)(|x - x'|^q + |s - s'|^{q/2}). \end{aligned}$$

Kolmogorov's continuity theorem implies the existence of an almost surely conditious version of $v(x, s, Y)$ if we choose $q > 2(d + 1)$.

Next, define

$$\rho^i(\delta, x, s, \cdot) = E\{\delta^{-1}\Delta_\delta^i[\psi(X_{x,s}(T))Z_{x,s}(T)] | \mathscr{Y}_T\}$$

for every (δ, x, s). Note that

$$\rho^i(0, x, s, \cdot) = E\{(\nabla\psi(X_{x,s}(T))\partial_{x_i}X_{x,s}(T)Z_{x,s}(T) + \psi(X_{x,s}(T))\partial_{x_i}Z_{x,s}(T) | \mathscr{Y}_T\}. \tag{3.27}$$

Result (3.19) and Lemma 3.8 imply that

$$\{\delta^{-1}\Delta_\delta^i X_{x,s}(t)\} \in \mathrm{Lip}(\infty) \quad \text{and} \quad \{\delta^{-1}\Delta_\delta^i Z_{x,s}(t)\} \in \mathrm{Lip}(\infty).$$

It follows, using the polynomial growth of ψ and its derivatives, and using (3.18), that the expression

$$\delta^{-1}\Delta_\delta^i[\psi(X_{x,s}(t))Z_{x,s}(t)] = (\delta^{-1}\Delta_\delta^i\psi(X_{x,s}(t)))Z_{x+\delta e_i,s}(t) + \psi(X_{x,s}(t))(\delta^{-1}\Delta_\delta^i Z_{x,s}(t))$$

also defines a process in Lip(∞). Hence, for any $q > 1$ there exist constants K and b such that

$$E\{|\rho^i(\delta, x, s) - \rho^i(\delta', x', s')|^q\} \le K(1 + |x|^b + |x'|^b)(|x - x'|^q + |\delta - \delta'|^q + |s - s'|^{q/2})$$

for all x, x', δ, and δ' such that $|x - x'| \le 1$, $|\delta - \delta'| \le 1$, and $s, s' \in [0, T]$. However, $\{\rho(\delta, x, s, Y); \delta \ne 0\}$ and $\{\delta^{-1}\Delta_\delta^i v(x, s, Y); \delta \ne 0\}$ are indistinguishable. Therefore the latter process satisfies (3.24), and hence Lemma 3.8 implies that

$$v(\cdot, \cdot, Y) \in C^{1,0}(R^d \times [0, T])$$

almost surely. Moreover, from (3.27)

$$\begin{aligned} \partial_{x_i} v(x, s, Y) &= \rho^i(0, x, s, Y) \\ &= E\{(\nabla\psi(X_{x,s}(T))\partial_{x_i}X_{x,s}(T)Z_{x,s}(T) + \psi(X_{x,s}(T))\partial_{x_i}Z_{x,s}(T)|\mathcal{Y}_T\}, \end{aligned} \tag{3.28}$$

thereby proving (3.23) for $|\alpha| = 1$.

If $r > 2$ in the hypotheses of Theorem 3.7, we can apply the above reasoning to (3.28) to prove that $\partial_{x_i} v(\cdot, \cdot, Y) \in C^{1,0}(R^d \times [0, T])$, and hence that $v(\cdot, \cdot, \omega)$ is almost surely in $C^{2,0}(R^d \times [0, T])$. Continuing by induction in this way, we can prove differentiability up to order $r - 1$.

To prove (3.21), we note from (3.23) and (3.18) and (3.19) that, for every $q > 1$, there is an $M_q < \infty$ and a $0 < \beta < \infty$ such that

$$E\{|\partial^\alpha v(x, s, \cdot)|^q\} \le M_q(1 + |x|^\beta) \quad \text{for all} \quad s \le T \quad \text{and} \quad |\alpha| \le r - 1.$$

For each fixed s, (3.21) follows by Lemma 3.2. Indeed, by the proof of Lemma 3.2, there is a sufficiently large deterministic K such that (3.21) is satisfied with

$$c_{s,q}(Y) = K \sup_{|\alpha| \le r-2} \left(\int (1 + |x|^2)^{-\alpha/2}(|\partial^\alpha v(x, s, Y)|^q + |\nabla\partial^\alpha v(x, s, Y)|^q)\, dx \right)^{1/q}.$$

Hence

$$E\{c_{s,q}^q\} \le K M_q \int (1 + |x|^2)^{\beta - a/2}\, dx < \infty,$$

independently of $s \le T$, if a is sufficiently large, thus proving (3.21) and (3.22). ■

Let us next consider the existence and regularity of $\{D_\theta^{Y_i}v(x, s, Y)\}$. We expect that

$$\begin{aligned} D_\theta^{Y_i}v(x, s, \cdot) &= E\{D_\theta^{Y_i}(\psi(X_{x,s}(T))Z_{x,s}(T))\,|\,\mathscr{Y}_T\} \\ &= E\{\nabla\psi(X_{x,s}(T))D_\theta^{Y_i}X_{x,s}(T)Z_{x,s}(T) + \psi(X_{x,s}(T))D_\theta^{Y_i}Z_{x,s}(T)\,|\,\mathscr{Y}_Y\} \end{aligned} \tag{3.29}$$

Proving (3.29) and using it to obtain regularity of $\{D_\theta^{Y_i}v(x, s, Y)\}$ requires proving existence and regularity of $\{D_\theta^{Y_i}X_{x,s}(t)\}$ and $\{D_\theta^{Y_i}Z_{x,s}(t)\}$. We shall study these processes through the differential equations they satisfy

$$\begin{aligned} D_\theta^{Y_i}X_{x,s}(t) = 1_{[t\ge s]}\bigg\{\int_s^t \nabla f_j(X_{x,s}(r), r, \omega)D_\theta^{Y_i}X_{x,s}(r)\, d\omega_j(r) \\ + \int_s^t [D_\theta^{Y_i}f_j](X_{x,s}(r), r, \omega)\, d\omega_j(r) + f_{l_1+i}(X_{x,s}(\theta), \theta, \omega)\chi_{(s,t]}(\theta)\bigg\} \end{aligned} \tag{3.30}$$

$$\begin{aligned} D_\theta^{Y_i}Z_{x,s}(t) = 1_{[t\ge s]}\bigg\{\int_s^t h_j(X_{x,s}(r), r, \omega)D_\theta^{Y_i}Z_{x,s}(r)\, dY_j(r) \\ + \int_s^t ([D_\theta^{Y_i}h_j](X_{x,s}(r), r, \omega)Z_{x,s}(t) \\ + \nabla h_j(X_{x,s}(r), r, \omega)D_\theta^{Y_i}X_{x,s}(r)Z_{x,s}(r))\, dY_j(r) \\ + Z_{x,s}(\theta)h_i(X_{x,s}(\theta), \theta, \omega)1_{(s,t]}(\theta)\bigg\} \end{aligned} \tag{3.31}$$

These equations may be interpreted in two ways; either as stochastic differential equations for the Hilbert space-valued random processes,

$$(x, s, t, \omega) \mapsto D_\cdot^{Y_i}X_{x,s}(t) \in L^2([0, T])$$

and $(x, s, t, \omega) \mapsto D_\cdot^{Y_i}Z_{x,s}(t) \in L^2([0, T])$, or as families of stochastic differential equations parametrized by θ. Here we want to work with the interpretation of (3.30) and (3.31) as a parametrized family, but we must take care to ensure that $D_\theta^{Y_i}X_{x,s}(t)$ is jointly measurable in $(x, s, t, \theta, \omega)$, and that $D_\cdot^{Y_i}X_{x,s}(t) \in L^2([0, T])$ for almost every (x, s, t, ω). The most convenient way to do this is to augment the probability space by defining $\tilde{\Omega} \doteq [0, T] \times \Omega$, $\tilde{\mathscr{F}} \doteq \mathscr{B}[0, T] \times \mathscr{F}$, $\tilde{\mathscr{F}}_t \doteq \mathscr{B}[0, T] \times \mathscr{F}_t$, and $d\tilde{\mathbb{P}} \doteq (1/T)\, d\theta \times d\mathbb{P}$. In other words, we think of θ as an additional, independent, uniformly distributed random variable independent of W. The use of tilde indicates when we are working on $\tilde{\Omega}$; thus, $\tilde{E}[\cdot]$ will denote the expectation with respect to $\tilde{\mathbb{P}}$, $\widetilde{\text{Lip}}(q)$ will denote the process satisfying Definition 3.1 on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$, etc. Then, for example, the rigorous interpretation of (3.29) is

$$D_\cdot^{Y_i}v(x, s, \cdot) = \tilde{E}\{D_\cdot^{Y_i}(\psi(X_{x,s}(T))Z_{x,s}(T))\,|\,\mathscr{B}([0, T]) \times \mathscr{Y}_T\},$$

which automatically gives a process measurable in (θ, ω).

We shall now interpret Eqs. (3.30) and (3.31) as stochastic differential equations on $\tilde{\Omega}$. The measurability of $\{D_\theta^{Y_i} X_{x,s}(t)\}$ will therefore be an automatic consequence of an existence theorem for (3.30). Moreover, the hypotheses we shall place on the coefficients f_j will imply that $\sup_{t \le T}\{D_\theta^{Y_i} X_{x,s}(t)\} \in L^p(\tilde{\Omega}, \tilde{\mathbb{P}})$ for some $p \ge 2$ and fixed (x, s). It follows that for fixed t

$$E\left\{\left(\int_0^T |D_\theta^{Y_i} X_{x,s}(t)|^2 \, d\theta\right)^{p/2}\right\} \le KE\left\{\int_0^T |D_\theta^{Y_i} X_{x,s}(t)|^p \, d\theta\right\} < \infty,$$

and hence we can conclude that $X_{x,s}(t)$ is an element of $\mathbb{D}_Y^{1,p}$. Thus, in discussing Wiener space gradients, we shall work on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$.

PROPOSITION 3.9 *Assume that:*

$$\text{\textit{All components of } } f_j,\ 0 \le j \le l, \text{\textit{ and of } } h_i,\ 1 \le i \le l_2, \text{\textit{ belong to }} \mathscr{C}^4, \text{\textit{ and the processes }} \nabla f_j,\ 0 \le j \le l \text{\textit{ and }} h_i,\ 1 \le i \le l_2, \text{\textit{ are uniformly essentially bounded.}} \tag{3.32}$$

$$\text{\textit{All components of }} f_j,\ 0 \le j \le l, \text{\textit{ and of }} h_i,\ 1 \le i \le l_2, \text{\textit{ belong to }} \mathscr{D}_Y(3, p), \text{\textit{ for some }} p > 2(d+3). \tag{3.33}$$

Then for any $q < p$,

$$X_{x,s}(t) \in \mathbb{D}_Y^{1,q} \quad \text{and} \quad Z_{x,s}(t) \in \mathbb{D}_Y^{1,q} \quad \text{for all} \quad (x, s, t); \tag{3.34}$$

$$\{D_\theta^{Y_i} X_{x,s}(t)\} \quad \text{is the unique strong solution of (3.30);} \tag{3.35}$$

$$\{D_\theta^{Y_i} Z_{x,s}(t)\} \quad \text{is the unique strong solution of (3.31).} \tag{3.36}$$

Moreover, for each $1 \le i \le d$, there exist processes $\{U_{x,s}^i(t, \theta, \omega)\}$ and $\{V_{x,s}^i(t, \theta, \omega)\}$ such that

$$U_{x,s}^i(t, \theta, \omega), \quad V_{x,s}^i(t, \theta, \omega) \in C^{2,0,0}(R^d \times [0, T]^2), \quad \text{in } (x, s, t),\ \tilde{\mathbb{P}}\text{-almost surely}, \tag{3.37}$$

$$\{\partial^\alpha U_{x,s}^i, \partial^\alpha V_{x,s}^i \mid |\alpha| \le 2\} \subset \widetilde{\mathrm{Lip}}(\varepsilon p, \varepsilon(p-2)), \quad \text{for any } \varepsilon < 1, \tag{3.38}$$

$$D_\theta^{Y_i} X_{x,s}(t) = U_{x,s}^i(t, \theta) 1_{[\theta \le s]} + V_{x,s}^i(t, \theta) 1_{[s < \theta \le t]} \tag{3.39}$$

Each $\{D_\theta^{Y_i} Z_{x,s}(t)\}$, $1 \le i \le d$, also admits a representation of the form (3.37)–(3.39).

THEOREM 3.10 *Assume that the coefficients f_j, $0 \le j \le l$, and h_i, $1 \le i \le l_2$, satisfy conditions* (3.32) *and* (3.33) *of Proposition* 3.9. *Assume also that $\psi \in C^3(R^d)$ and that ψ*

and its first three derivatives grow at most at a polynomial rate as $|x| \to \infty$. *Then*:

$$\text{For any } q < p,\ v(x, s, \cdot) \in \mathbb{D}_Y^{1,q} \text{ for all } (x, s) \in R^d \times [0, T], \text{ and } D_\theta^{Y_i} v(x, s, \cdot) \text{ is given by (3.29).} \tag{3.40}$$

For each $1 \le i \le l_2$, $\{D_\theta^{Y_i} v(x, s, Y)\}$ *admits a version of the form* (3.41)

$$D_\theta^{Y_i} v(x, s, Y) = 1_{[\theta \le s]} R_i(x, s, \theta, Y) + 1_{[s < \theta]} S_i(x, s, \theta, Y)$$

where $R_i(\cdot, \cdot, \theta, Y)$ *and* $S_i(\cdot, \cdot, \theta, Y)$ *are in* $C^{2,0}(R^d \times [0, T])$ *for* $\tilde{\mathbb{P}}$*-almost all* (θ, Y), *and*

$$\{\partial^\alpha R, \partial^\alpha S \mid |\alpha| \le 2\} \subset \tilde{\mathrm{L}}\mathrm{ip}(\varepsilon p, \varepsilon(p-2)) \quad \text{for any } \varepsilon < 1,$$

for all $1 \le i \le l_2$. *As a consequence,* $D_\theta^{Y_i} v(\cdot, s, Y) \in C^2(R^d)$, *almost surely for every s.*

For each $q < p$ *and each* $s \le T$, *there is a* $b < \infty$ *and an* $L_{q,s} \in \tilde{L}^q(\tilde{\Omega}, \tilde{\mathbb{P}})$ *such that* (3.42)

$$|\partial^\alpha D_\theta^{Y_i} v(x, s, Y)| \le L_{q,s}(\theta, Y)(1 + |x|^b), \quad \forall |\alpha| \le 1, \quad \forall i,\ 1 \le i \le l_2$$

Remark Statement (3.41) contains a slight abuse of notation. What we mean is that the analogues of (3.2) and (3.3′) hold for R and S and their derivatives, with the modification that, since R and S have no t-dependence, there is no ess sup in the expectation; thus, for example, (3.2) becomes $\tilde{E}\{|R(y, s, \cdot, \cdot)|\} \le K(1 + |y|^\beta)$ for all $s \le T$.

There are two important and immediate consequences of Theorem 3.7. The first will allow us to apply the extended Itô rule of Section 2 to the random transformation $v(x, s, Y)$.

COROLLARY 3.11 *Let the hypotheses of Theorem* 3.7 *be given. For each fixed s, each* $1 \le i \le d$, *each* $p_1 > 1$ *and each* $p_2 < p$, $\{\partial_{x_i} v(x, s, Y)\}$ *satisfies hypothesis* (*A*) *with moments* p_1 *and* p_2.

The second consequence is the existence of upper and lower limits of $D_\theta^{Y_i} v(x, s, Y)$ on the diagonal $\{\theta = r\}$. These are required in the statement of the stochastic p.d.e. for $v(x, s, Y)$.

COROLLARY 3.12 *Let the hypotheses of Theorem* 3.10 *be given. Then the following limits are all defined*:

$$D_r^{Y_i} v(x, r+, Y) \doteq \lim_{s \downarrow r} D_r^{Y_i} v(x, s, Y) = R(x, r, r, Y),$$

$$D_r^{Y_i} \nabla v(x, r+, Y) \doteq \lim_{s \downarrow r} D_r^{Y_i} \nabla v(x, s, Y) = \nabla R(x, r, r, Y).$$

$$D_r^{Y_i} v(x, r-, Y) \doteq \lim_{s \uparrow r} D_r^{Y_i} v(x, s, Y) = S(x, r, r, Y),$$

$$D_r^{Y_i} \nabla v(x, r-, Y) \doteq \lim_{s \uparrow r} D_r^{Y_i} \nabla v(x, s, Y) = \nabla S(x, r, r, Y).$$

The proof of Theorem 4.2 requires refined information about the random variables $L_{s,q}$ appearing in the polynomial growth bound (3.42). Define

$$\bar{L}_{s,q,\alpha,a}(\theta, Y) = \sum_i \left(\int \frac{|\partial^\alpha D_\theta^{Y_i} v(x, s, Y)|^q + |\nabla \partial^\alpha D_\theta^{Y_i} v(x, s, Y)|^q}{(1 + |x|^2)^{qa/2}} \, dx \right)^{1/q}.$$

From the proof of Lemma 3.2, there is a constant K, independent of s, such that

$$|\partial^\alpha D_\theta^{Y_i} v(x, s, Y)| \leq K \bar{L}_{s,q,\alpha,a}(\theta, Y)(1 + |x|^a). \tag{3.43}$$

Proposition 3.13 *Assume that the hypotheses of Theorem* 3.10 *hold. By Definition* 3.4, *this implies that there exist a* $b > 0$ *and* $M \in L^p([0, T] \times \tilde{\Omega})$ *such that*

$$|\partial^\beta D_\theta^{Y_i} f_j(x, r, \omega)| \vee |\partial^\beta D_\theta^{Y_i} h_i(x, r, \omega)| \leq M(r, \theta, \omega)(1 + |x|^b) \tag{3.44}$$

for all $|\beta| \leq 3$, $0 \leq j \leq l$, *and* $1 \leq i \leq l_2$. *Then, for every* $q < p$, *there exists a constant* $K_{a,q}$, *independent of* s *and* θ, *such that* $K_{a,q} < \infty$ *for sufficiently large* a *and*

$$E\{\bar{L}^q_{s,q,\alpha,a}(\theta, \cdot)\} \leq K_{a,q}\left(E\left\{ \int_s^T M^p(r, \theta, \cdot)\, dr \right\} \right)^{q/p} \tag{3.45}$$

for (Lebesgue)-almost every $\theta \leq s$, *and for all* $|\alpha| \leq 1$.

The remainder of this section provides the proofs of the results stated so far. None of the auxiliary lemmas stated after this point play a direct role in the proof of Theorem 4.2.

Proof of Theorem 3.10 from Proposition 3.9 Consider a fixed (x, s). If $1 < q < p$, then (3.34), Proposition 3.6, implies that $X_{x,s}(T) \in \mathbb{D}_Y^{1,q}$ and $Z_{x,s}(T) \in \mathbb{D}_Y^{1,q}$. Also, we know that $E[|X_{x,s}(T)|^{q'}] < \infty$ and $E[|Z_{x,s}(T)|^{q'}] < \infty$, for all $q' > 1$. It is not difficult to conclude from this that $\psi(X_{x,s}(T))Z_{x,s}(T) \in \mathbb{D}_Y^{1,q}$ for any $q < p$, and

$$D_\theta^{Y_i}(\psi(X_{x,s}(T))Z_{x,s}(T)) = \nabla\psi(X_{x,s}(T))D_\theta^{Y_i}X_{x,s}(T)Z_{x,s}(T) + \psi(X_{x,s}(T))D_\theta^{Y_i}Z_{x,s}(T). \tag{3.46}$$

In fact, (3.46) is a consequence of Lemma 2.2 applied to the non-random function $\phi(x, z) \doteq \psi(x)z$. Since D^Y commutes with conditional expectation with respect to $\mathcal{Y}_T$ (see Lemma 1.1), it follows that $E\{\psi(X_{x,s}(T))Z_{x,s}(T)|\mathcal{Y}_T\}(x, s, \cdot) \in \mathbb{D}_Y^{1,q}$, and

$$D_\theta^{Y_i} v(x, s, \cdot) = E\{D_\theta^{Y_i}(\psi(X_{x,s}(T))z_{x,s}(T))|\mathcal{Y}_T\}. \tag{3.48}$$

This is precisely Eq. (3.29).

We turn next to the proof of statement (3.41). Recall the representation of the term $D_\theta^{Y_i}X_{x,s}(t)$ given in (3.39) of Proposition 3.9. $D_\theta^{Y_i}Z_{x,s}(t)$ admits a similar representation

$$D_\theta^{Y_i}Z_{x,s}(t) = \bar{U}^i_{x,s}(t, \theta)1_{[\theta \le s]} + \bar{V}^i_{x,s}(t, \theta)1_{[s < \theta \le t]}$$

Substitution of these representations into (3.48) leads to

$$D_\theta^{Y_i}v(x, s, Y) = 1_{[\theta \le s]}R(x, s, \theta, Y) + 1_{[s < \theta \le t]}S(x, s, \theta, Y),$$

where

$$R(x, s, \theta, Y) = E\{\nabla\psi(X_{x,s}(T)) \cdot U^i_{x,s}(T, \theta)Z_{x,s}(T) + \psi(X_{x,s}(T))\bar{U}^i_{x,s}(T, \theta)|\mathscr{Y}_T\},$$

and S is defined by a similar formula in which the U's are replaced by V's. The expression $\nabla\psi(X_{x,s}(T)) \cdot U^i_{x,s}(T, \theta)Z_{x,s}(T) + \psi(X_{x,s}(T))\bar{U}^i_{x,s}(T, \theta)$ inside the conditional expectation defining R satisfies conditions (3.38) and (3.39), because $U^i_{x,s}$ and $\bar{U}^i_{x,s}$ satisfy (3.38) and (3.39) and $\{X_{x,s}(t)\}$, $\{Z_{x,s}(t)\}$ and their derivatives are in Lip(∞). Thus R inherits the property (3.38) and the polynomial growth of the moments, just as v inherited the same properties of $X_{x,s}(t)$ and $Z_{x,s}(t)$ in the proof of Theorem 3.7. This involves applying Lemma 3.8, for which we need to know that for some $\varepsilon < 1$, $\varepsilon(p - 2)/2 > d + 2$; here the assumption that $p > 2(d + 3)$ is used for the application of Kolmogorov's continuity criterion. A similar claim holds for S, and this proves (3.41).

(3.42) is a consequence of Lemma 3.2 and the growth estimates of (3.41). ■

Proof of Corollary 3.11 Let $p_2 < p$. Theorem 3.7, (3.21), implies that $\partial_{x_i}v(x, s, \cdot)$ satisfies condition (A.1) of hypothesis (*A*). Theorem 3.7, (3.21) and Theorem 3.10, (3.42) allow us to apply Lemma 2.4 to conclude that $\partial_{x_i}v(x, s, \cdot) \in \mathbb{D}_Y^{1,p_2}$ for every x and s and that $D_\theta^{Y_i}\partial_{x_i}v(x, s, Y) = \partial_{x_i}D_\theta^{Y_i}v(x, s, Y)$. Condition (A.2) of hypothesis (*A*) for $\partial_{x_i}v(x, s, \cdot)$ is then an immediate consequence of (3.41) and (3.42) of Theorem 3.10. We see that $p_2 < p$ because $q < p$ in statements (3.41) and (3.42). ■

The remainder of this section is devoted to the proofs of Propositions 3.6, 3.9, and 3.13. In general, we shall only sketch the main outlines of the proofs, since a full presentation of all the technical details would be too lengthy. We begin with a key technical result which we use repeatedly. The idea of this lemma is well-known; see, for example, Stroock [16], Lemmas 3.1 and 3.2, or Krylov [4], Chapter 2, Section 3, Lemma 3. We adapt it here to our particular definitions.

Let $\gamma = \{\gamma(x, s, t, \omega)\}$ and $\eta_j = \{\eta_j(x, s, t, \omega)\}$, $0 \le j \le l$, denote, respectively, R^d- and R^{d^2}-valued processes, and assume that $1_{[t \ge s]}\gamma$ and $1_{[t \ge s]}\eta_j$ are progressively measurable in t. We are interested in the parametrized family of equations

$$\xi(x, s, t) = \gamma(x, s, t) + 1_{[t \ge s]}\int_s^t \eta_j(x, s, r)\xi(x, s, r)\, d\omega_j(r) \quad (x, s, t) \in R^d \times [0, T]^2. \tag{3.49}$$

LEMMA 3.14 *Assume that for some p and some $q \le p$, such that $p \wedge (q/2) > d + 2$, and for some integer k*

$$\gamma \in C^{k,0,0}(R^d \times [0, T]^2), \text{ almost surely, and } \eta_j(\cdot, \cdot, r) \in C^{k,0}(R^d \times [0, T]), \quad dr \times d\mathbb{P}\text{-almost everywhere}, 0 \le j \le l; \tag{3.50}$$

$$\text{The processes } \eta_j,\ 0 \le j \le l \text{ are essentially bounded;} \tag{3.51}$$

$$\{\partial^\alpha \gamma : |\alpha| \le k\} \subset \mathrm{Lip}(p, q); \tag{3.52}$$

$$\{\partial^\alpha \eta_j : |\alpha| \le k,\ 0 \le j \le l\} \subset \mathrm{Lip}(\infty). \tag{3.53}$$

Then (3.49) *admits a unique, strong solution ξ, and ξ has a version satisfying*

$$\xi(\cdot, \cdot, \cdot, \omega) \in C^{k,0,0}(R^d \times [0, T]^2)$$

almost surely, and

$$\{\partial^\alpha \xi : |\alpha| \le k\} \subset \mathrm{Lip}(\varepsilon p, \varepsilon q) \quad \text{for any } \varepsilon < 1. \tag{3.54}$$

Moreover, if $|\alpha| \le k$, $\partial^\alpha \xi$ satisfies the differential equation obtained by formally applying ∂^α to both sides of Eq. (3.49).

Proof This is proved by the same method as in Lemmas 3.1 and 3.2 of [15]. The case $k = 0$ is easily proved by an argument using the Burkholder-Gundy inequalities, the Gronwall-Bellman inequality, and Kolmogorov's continuity criterion. Once the $k = 0$ case is proved, all other cases follow. To do the $k = 1$ case, consider the process ρ defined by

$$\rho^i(\delta, x, s, t) = \bar{\gamma}^i(\delta, x, s, t) + 1_{[t \ge s]} \int_s^t \bar{\eta}^i_j(\delta, x, s, r)\rho^i(\delta, x, s, r)\, d\omega_j(r),$$

for $1 \le i \le d$, where $\bar{\eta}^i_j(\delta, x, s, t) = \int_0^1 \eta_j(x + \lambda\delta e_i, s, t)\, d\lambda$, and

$$\bar{\gamma}^i(\delta, x, s, t) = \int_0^1 \partial_{x_i}\gamma(x + \lambda\delta e_i, s, t)\, d\lambda + 1_{[t \ge s]} \int_s^t \int_0^1 \partial_{x_i}\eta_j(x + \lambda\delta e_i, s, r)[\xi(x, s, r) + \lambda\Delta^i_\delta \xi(x, s, r)]\, d\lambda\, d\omega_j(r)$$

One verifies that $\rho^i(\delta, x, s, t) = \delta^{-1}\Delta^i_\delta \xi(x, s, t)$ if $\delta \ne 0$, and $\bar{\eta}^i_j \in \mathrm{Lip}(\infty)$, $\bar{\gamma}^i \in \mathrm{Lip}(\varepsilon p, \varepsilon q)$, for any $\varepsilon > 1$. Hence, the validity of Lemma 3.14 for $k = 0$ implies $\rho^i \in \mathrm{Lip}(\varepsilon p, \varepsilon q)$, for any $\varepsilon > 1$. The continuous differentiability of ξ and statement (3.54) for $k = 1$ are then consequences of Lemma 3.8.

The general case $k > 1$ is proved by induction. Indeed, suppose we have proved Lemma 3.14 for k' and hypotheses (3.50)–(3.53) are true for $k' + 1$. Then, one shows that if $|\alpha| = k'$, $\partial^\alpha \xi$ satisfies an equation like (3.49) with the same η_j's and with a new γ' such that $\{\partial^\beta \gamma'; |\beta| \le 1\} \subset \mathrm{Lip}(\varepsilon p, \varepsilon q)$, for any $\varepsilon < 1$. By applying Lemma 3.14 with $k = 1$, we conclude that $\partial^\alpha \xi$ is once continuously differentiable, which concludes the induction step. ■

Proof of Proposition 3.6 The differentiability and moment properties of $\{X_{x,s}(t)\}$ are proved from Lemma 3.14, in a manner similar to the proof of differentiability of $\{X_{x,s}(t)\}$ in [14]. Our result differs only in including continuity in s and in concluding $\{\partial^\alpha X_{x,s}(t); |\alpha| \le r - 1\} \subset \mathrm{Lip}(\infty)$. The differentiability and moment properties of $\{Z_{x,s}(t)\}$ are proved by applying Lemma 3.14 to the equation

$$Z_{x,s}(t) = 1 + 1_{[t \ge s]} \int_s^t h_j(X_{x,s}(r), r) Z_{x,s}(r)\, dY_j(r) \tag{3.55}$$

We need only to verify that the processes $h_j(X_{x,s}(r), r, \omega)$ satisfy (3.53), and this is easily done using the differentiability properties of $\{X_{x,s}(t)\}$ and the assumption that $h_j \in \mathscr{C}^r$. ■

We turn next to the proof of Proposition 3.9 for $\{X_{x,s}(t)\}$. We shall first show that Eq. (3.30) has a unique solution with the desired regularity properties. Then we shall identify this solution with $\{D_\theta^{Y_i} X_{x,s}(t)\}$. So as not to cause any logical confusion, we shall use $J_{x,s}(t, \theta, \omega)$ to denote the solution of (3.30) until we show that it equals $D_\theta^{Y_i} X_{x,s}(t)$.

LEMMA 3.15 *Assume that the hypotheses* (3.32) *and* (3.33) *of Proposition* 3.9 *are in force. Then, for every* $1 \le i \le d$, *the equation*

$$\begin{aligned} J_{x,s}(t, \theta) = 1_{[t \ge s]} \bigg\{ &\int_s^t \nabla f_j(X_{x,s}(r), r) J_{x,s}(r, \theta)\, d\omega_j(r) \\ &+ \int_s^t [D_\theta^{Y_i} f_j](X_{x,s}(r), r)\, d\omega_j(r) + f_{l_1+i}(X_{x,s}(\theta), \theta) 1_{(s,t]}(\theta) \bigg\} \end{aligned} \tag{3.56}$$

admits a unique strong solution on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathscr{P}})$. *Moreover, we may represent this solution in the form given in Eqs.* (3.37)–(3.39) *of Proposition* 3.9.

Proof Since Eq. (3.56) is a linear equation, it is possible to represent the solution explicitly using a variation of constants formula. However, it is easier for us just to make use of Lemma 3.14 directly. Let $U_{x,s}^i(t, \theta, \omega)$ denote the solution to

$$U_{x,s}^i(t, \theta) = \gamma(x, s, t, \theta) + 1_{[t \ge s]} \int_s^t \nabla f_j(X_{x,s}(r), r) U_{x,s}^i(r, \theta)\, d\omega_j(r),$$

where $\gamma(x, s, t, \theta) \doteq 1_{[t \geq s]} \int_s^t [D_\theta^{Y_i} f_j](X_{x,s}(r), r)\, d\omega_j(r)$. The assumption that the f_j coefficients belong to $\mathscr{C}^4$ and the fact that $\{\partial^\alpha X_{x,s}(t); |\alpha| \leq 3\} \subset \mathrm{Lip}(\infty)$, which is a consequence of (3.19) in Proposition 3.6, imply that $\nabla f_j(X_{x,s}(r), r)$ satisfies condition (3.53) of Lemma 3.14 for $k = 2$. $\nabla f_j(x, s, \omega)$ is uniformly bounded by assumption (3.31). As for γ, the assumption that the f_j coefficients belong to the class $\mathscr{D}_Y(3, p)$ and the differentiability and moment properties of $\{X_{x,s}(t)\}$ imply that γ satisfies conditions (3.50) and (3.52) of Lemma 3.14 with $k = 2$, and $\widetilde{\mathrm{Lip}}(\varepsilon p, \varepsilon(p-2))$ in place of $\widetilde{\mathrm{Lip}}(p, q)$. Lemma 3.14, applied now on the probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$, then implies that $\{U^i_{x,s}\}$ satisfies the properties (3.37) and (3.38) of Proposition 3.9.

Now consider the equation

$$V^i_{x,s}(t, \theta) = \bar{\gamma}(x, s, t, \theta) + 1_{[t \geq s]} \int_s^t 1_{[\theta \leq r]} \nabla f_j(X_{x,s}(r), r) V^i_{x,s}(r, \theta)\, d\omega_j(r), \tag{3.57}$$

where $\bar{\gamma}(x, s, t, \theta, \omega) \doteq 1_{[t \geq s]} \int_s^t [D_\theta^{Y_i} f_j](X_{x,s}(r), r)\, d\omega_j(r) + f_{l_1+i}(X_{x,s}(\theta), \theta)$. Again $\bar{\gamma}$ satisfies the differentiability condition (3.51) for $k = 2$, and

$$\{\partial^\alpha \bar{\gamma}; |\alpha| \leq 2\} \subset \widetilde{\mathrm{Lip}}(\varepsilon p, \varepsilon(p-2))$$

for any $\varepsilon < 1$. Despite the fact that the term $f_{l_1+i}(X_{x,s}(\theta), \theta)$ in $\bar{\gamma}$ is not progressively measurable, $f_{l_1+i}(X_{x,s}(\theta), \theta) 1_{(s,t]}(\theta)$ is, and so Lemma 3.14 can be extended to (3.57). We therefore obtain that $V^i_{x,s}$ also satisfies the conditions (3.37) and (3.38) of Proposition 3.9. Finally, a simple calculation shows that

$$J_{x,s}(t, \theta) \doteq 1_{[\theta \leq s]} U^i_{x,s}(t, \theta) + 1_{[s < \theta \leq t]} V^i_{x,s}(t, \theta)$$

solves (3.56). Uniqueness is proved in the usual way. ∎

LEMMA 3.16 *Assume that all the components of $f_0, \ldots, f_l$ belong both to $\mathscr{C}^2$ and to $\mathscr{D}_Y(1, p_1)$ for some $p_1 > 2$. Assume that $\nabla f_0, \ldots, \nabla f_l$ are uniformly bounded. Then for every $(x, s, t) \in R^d \times [0, T]^2$ and for any $2 \leq p < p_1$, $X_{x,s}(t) \in \mathbb{D}_Y^{1,p}$. In fact, for every (x, s), $X_{x,s}(\cdot) \in \mathbb{L}_Y^{1,p}$. Furthermore, $\{D_\theta^{Y_i} X_{x,s}(t)\}$ is a solution of* (3.30).

Proof (Sketch.) The usual proof by Picard iteration works here. We set $X^{(0)}_{x,s}(t) \equiv x$, and, for $n > 0$,

$$X^{(n+1)}_{x,s} \doteq x + 1_{[t \geq s]} \int_s^s f_j(X^{(n)}_{x,s}(r), r, \omega)\, d\omega_j(r)$$

We use Proposition A.6 in Ocone & Karatzas [10] and Lemma 2.2 to verify that $X^{(n)}_{x,s}(\cdot) \in \mathbb{L}_Y^{1,p}$ for $p < p_1$ and every n, and

$$D_\theta^{Y_i} X^{(n+1)}_{x,s} = 1_{[t \geq s]} \left\{ \int_s^t \nabla f_j(X^{(n)}_{x,s}(r), r, \omega) D_\theta^{Y_i} X^{(n)}_{x,s}(r)\, d\omega_j(r) + \int_s^t [D_\theta^{Y_i} f_j](X^{(n)}_{x,s}(r), r, \omega)\, dr + f_{l_1+i}(\theta, X^{(n)}_{x,s}(\theta), \omega) 1_{(s,t]}(\theta) \right\}.$$

Finally, one shows that $\tilde{E}[\sup_{[t\le T]}|J_{x,s}(t,\theta,\omega)-D_\theta^{Y_i}X_{x,s}^{(n)}(t)(\omega)|^2]\to 0$ as $n\to\infty$ where $J_{x,s}$ is the solution to (3.30) found in Lemma 3.15. ■

Proof of Proposition 3.9 (Sketch.) All the statements in Proposition 3.9 concerning $\{D_\theta^{Y_i}X_{x,s}(t)\}$ are immediate consequences of Lemmas 3.15 and 3.16. We need to prove the analogous properties of $\{D_\theta^{Y_i}Z_{x,s}(t)\}$. To do this, let us rewrite Eq. (3.55) in the form

$$Z_{x,s}(t)=1+1_{[t\ge s]}\int_s^t \bar{h}_j(Z_{x,s}(r),r,\omega)\,dY_j(r) \tag{3.58}$$

where $\bar{h}_j(z,r,\omega)\doteq h_j(X_{x,s}(r),r,\omega)z$. Then $\bar{h}_1(z,r,\omega),\ldots,\bar{h}_{l_2}(z,r,\omega)$ can be shown to belong to $\mathscr{C}^4$ and to $\mathscr{D}_Y(1,q)$ if $q<p$. Thus the $\bar{h}_j$ coefficients satisfy the hypotheses of Lemma 3.16. This allows us to conclude that $\{D_\theta^{Y_i}Z_{x,s}(t)\}$ solves (3.31). An analysis of (3.31) similar to that undertaken on (3.30) in Lemma 3.15, allows one to prove Proposition 3.9 for $\{Z_{x,s}(t)\}$. ■

Proof of Proposition 3.13 (Sketch.) For almost every θ, $\{D_\theta^{Y_i}X_{x,s}(t);0\le t\le T\}$ satisfies (3.30) interpreted as an equation for fixed θ; a similar statement is true for $\{D_\theta^{Y_i}Z_{x,s}(t);\le t\le T\}$, and (3.31). By applying standard moment bounding arguments using the Burkholder-Gundy inequalities and the Gronwall-Bellman inequality, one finds that for any given $q'<p$, there exists constants K and b, independent of θ and s, such that for almost every $\theta\le s$,

$$E\left\{\sup_t|\partial^\alpha D_\theta^{Y_i}X_{x,s}(t)|^{q'}\vee\sup_t|\partial^\alpha D_\theta^{Y_i}Z_{x,s}(t)|^{q'}\right\}\le K(1+|x|^b)\left(E\left[\int_s^T M^p(r,\theta,\omega)\,dr\right]\right)^{q'/p} \tag{3.59}$$

for all $|\alpha|\le 2$. By using the representation (3.29) and its generalization to higher derivatives, we obtain, using (3.19) and Hölder's inequality, that

$$E\{|\partial^\alpha D_\theta^{Y_i}v(x,s,Y)|^q\}\le K(1+|x|^b)\left(E\left[\int_s^T M^p(r,\theta,\omega)\,dr\right]\right)^{q/p} \tag{3.60}$$

for $|\alpha|\le 2$ and almost every $\theta\le s$, where K and b are gain independent of s snd θ. By substituting this inequality into the definition of $\bar{L}_{s,q,\alpha,a}$ given before the statement of Proposition 3.13, we find that

$$E\{\bar{L}^q_{s,q,\alpha,a}\}\le K\left(E\left[\int_s^t M^p(r,\theta,\omega)\,dr\right]\right)^{q/p}\int((1+|x|^b)(1+|x|^{-aq/2})\,dx.$$

The proof is completed by taking a large enough to make the integral in the last expression finite. ■

Finally, it is worthwhile to note the variation of constants formula for $D_\theta^{Y_i} X_{x,s}(t)$ that is obtained by solving (3.30). This leads to expressions which will help us to reexpress the basic stochastic partial differential equation for $v(x, s, Y)$ using anticipating Stratonovich integrals instead of Skorohod integrals. Let $\Phi(x, s, t)$ denote the $n \times n$-matrix valued solution to the equation of first variation for $X_{x,s}(\cdot)$;

$$\Phi(x, s, t) = I + 1_{\{t \geq s\}} \int_s^t \nabla f_j(X_{x,s}(r), r, Y)\Phi(x, s, r)\, d\omega_j(r).$$

It may be shown that $\Phi(x, s, t)$ is invertible; moreover, it follows immediately from Proposition 3.6 and its proof that $\nabla X_{x,s}(t) = \Phi(x, s, t)$. An application of Itô's rule proves that

$$\begin{aligned} D_\theta^{Y_i} X_{x,s}(t) = 1_{[\theta \leq t]} \Phi(x, s, t)\Bigg(\int_s^t \Phi^{-1}(x, s, r)\Bigg[\sum_1^l [D_\theta^{Y_i} f_j](X_{x,s}(r), r, Y)\, d\omega_j(r) \\ + (\nabla f_j [D_\theta^{Y_i} f_j])(X_{x,s}(r), r, Y)\, dr\Bigg]\Bigg) \\ + 1_{[s < \theta \leq t]} \Phi(x, \theta, t) f_{l_1 + i}(X_{x,s}(\theta), \theta, Y) \end{aligned} \tag{3.61}$$

From this formula we obtain the identity

$$\lim_{s \uparrow \theta} D_\theta^{Y_i} X_{x,s}(T) = \lim_{s \downarrow \theta} D_\theta^{Y_i} X_{x,s}(T) + \nabla X_{x,\theta}(T) \cdot f_{l_1 + i}(x, \theta, Y). \tag{3.62}$$

A similar analysis applied to (3.31) yields

$$\lim_{s \uparrow \theta} D_\theta^{Y_i} X_{x,s}(T) = \lim_{s \downarrow \theta} D_\theta^{Y_i} Z_{x,s}(T) + Z_{x,\theta}(T) h_i(x, \theta, Y) + [\nabla Z_{x,\theta}(T)] \cdot f_{l_1 + i}(x, \theta, Y) \tag{3.63}$$

If these identities are used in (3.41) and (3.48), we obtain

$$D_r^{Y_i} v(x, r-, Y) = D_r^{Y_i} v(x, r+, Y) + v(x, r, Y) h_i(x, r, Y) + [\nabla v(x, r, Y)] \cdot f_{l_1 + i}(x, r, Y). \tag{3.64}$$

Formal differentiation of (3.64) in x yields

$$\begin{aligned} D_r^{Y_i} \nabla v(x, r-, Y) = D_r^{Y_i} \nabla v(x, r+, Y) + \nabla[v(x, r, Y) h_i(x, r, Y)] \\ + \nabla[\nabla v(x, r, Y) \cdot f_{l_1 + i}(x, r, Y)], \end{aligned} \tag{3.65}$$

which may be verified by a rigorous analysis.

4 A NON-ADAPTED STOCHASTIC PARTIAL DIFFERENTIAL EQUATION FOR $v(x, s, Y)$

In this section we derive a stochastic partial differential equation for the conditional expectation $v(x, s, Y) = E\{\psi(X_{x,s}(T))Z_{x,s}(T)|\mathscr{Y}_T\}$, where

$$X_{x,s}(t) = x + 1_{[t \geq s]} \int_s^t f_j(X_{x,s}(r), r, Y)\, d\omega_j(r). \tag{4.1}$$

and

$$Z_{x,s}(t) \doteq \exp\left\{1_{[t \geq s]}\left(\int_s^t h_j(X_{x,s}(r), r, Y)\, dY_j(r) - (1/2)\int_s^t |h(X_{x,s}(r), r, Y)|^2\, dr\right)\right\}. \tag{4.2}$$

We shall assume throughout this section that the hypotheses of Theorem 3.10 are in force. For the sake of clarity we repeat them here:

H.i) $\{f_j(x, r, Y)\}$, $0 \leq j \leq l$, and $\{h_j(x, r, Y)\}$, $1 \leq j \leq l_2$ are processes in the class $\mathscr{C}^4$.

H.ii) $\{\nabla_x f_j(x, t, Y)\}$, $0 \leq j \leq l$, and $\{h_j(x, r, Y)\}$, $1 \leq j \leq l_2$ are essentially uniformly bounded.

H.iii) $\{f_j(x, r, Y)\}$, $0 \leq j \leq l$, and $\{h_j(x, r, Y)\}$, $1 \leq j \leq l_2$ are processes in the class $\mathscr{D}_Y(3, p)$ for some $p > 2(d + 3) \vee 8$.

H.iv) $\psi \in C^3(R^d)$ and ψ and its first three derivatives admit at most polynomial growth.

Notice that in (4.1) and (4.2) the argument of f_j and of h_j is (x, r, Y) rather than (x, s, ω) as in Section 3. This restriction is made to obtain a Markov property with respect to the filtration $\{\mathscr{X}_t \vee \mathscr{Y}_T\} = \sigma\{X_s, Y_u | s \leq t, u \leq Y\}$.

LEMMA 4.1 *Let* $s < t < T$ *and assume* H.i)–H.iv). *Then*

$$E\{\psi(X_{x,s}(T))Z_{x,s}(T)|\mathscr{X}_t \vee \mathscr{Y}_T\} = Z_{x,s}(t)v(X_{x,s}(t), t, Y) \tag{4.3}$$

almost surely, for every (x, s).

Proof Fix $s < t < T$. First we establish a type of stochastic flow representation of $X_{x,s}(T)$ in terms of $X_{x,s}(t)$. Consider the space of continuous functions

$$C_0([t, T]; R^{l_1}) \times C_0([0, T] \times R^{l_2})$$

endowed with the probability measure $Q = \mathbb{P}_{t,l_1} \times \mathbb{P}_{l_2}$, where $\mathbb{P}_{l_2}$ is Wiener measure and $\mathbb{P}_{t,l_1}$ is the measure on $C_0([t, T]; R^{l_1})$ such that, if η_1 denotes the canonical process $\eta_1(\cdot - t)$ is a Wiener process. If $W \in C_0([0, T] \times R^{l_2})$, we let $W(\cdot) - W(t)$ denote the path

$$r \to W(r) - W(t), \quad r \geq t$$

in $C_0([t, T]; R^{l_1})$. Proposition 3.6 and its proof imply that there exists a measurable function $F_t(x, r, \eta_1, \eta_2)$ on $R^d \times [0, T - t) \times C_0([t, T]; R^{l_1}) \times C_0([0, T] \times R^{l_2})$ such that $\xi_x(r) \doteq F_t(x, r, W(\cdot) - W(t), Y)$ solves

$$\xi_x(r) = x + \int_t^r f_j(\xi_x(\lambda), \lambda, Y)\, d\omega_j(\lambda), \quad t \leq r \leq T, \quad x \in R^d.$$

Uniqueness of solutions to (4.1) and almost sure continuity of its solutions in x imply that

$$\xi_x(r) = F_t(x, r, W(\cdot) - W(t), Y) = X_{x,s}(r) \quad \text{for all } (x, r), \text{ almost surely.}$$

Again, using continuity in x of $\xi_x(r)$, it follows that

$$\xi_{X_{x,s}(t)}(r) = F_t(X_{x,s}(t), r, W(\cdot) - W(t), Y) \tag{4.4}$$

is a solution of

$$\xi_{X_{x,s}(t)}(r) = X_{x,s}(t) + \int_t^r f_j(\xi_{X_{x,s}(t)}(\lambda), \lambda, Y)\, d\omega_j(\lambda) \quad \forall t \leq r \leq T, \quad \text{a.s.} \tag{4.5}$$

Uniqueness of solutions to (4.5) implies that $\{\xi_{X_{x,s}(t)}(r); t \leq r \leq T\}$ is indistinguishable from $\{X_{x,s}(r); t \leq r \leq T\}$. Thus formula (4.4) helps sort out the dependence of $X_{x,s}(r)$ on W and Y. We can write a similar formula for $Z_{x,s}(r)$. Indeed, let

$$G_t(x, W, Y) = \exp\left\{\int_t^T h_j(F_t(x, r, W, Y)\, dY_j(r) - \frac{1}{2}\int |h(F_t(x, r, W, Y), r, Y)|^2\, dr\right\}$$

Then we find from (4.4) and (4.5) that

$$G_t(x, W(\cdot) - W(t), Y) = Z_{x,t}(T) \tag{4.6}$$

for all x, almost surely, and

$$Z_{x,s}(T) = Z_{x,s}(t) G_t(X_{x,s}(t), W(\cdot) - W(t), Y) \quad \text{for all } x, \text{ almost surely.} \tag{4.7}$$

Since $\{W(\cdot)-W(t)\}$ and $\mathscr{X}_t \vee \mathscr{Y}_T$ are independent and $Z_{x,s}(t)$ and $X_{x,s}(t)$ are $\mathscr{X}_t \vee \mathscr{Y}_T$-measurable, we find from (4.4)–(4.7) that

$$\begin{aligned} &E\{\psi(X_{x,s}(T))Z_{x,s}(T) \mid \mathscr{X}_t \vee \mathscr{Y}_T\} \\ &\quad = Z_{x,s}(t)E\{\psi(\xi_{X_{x,s}(t)}(T))G_t(X_{x,s}(t), W(\cdot)-W(t), Y) \mid \mathscr{X}_t \vee \mathscr{Y}_T\} \\ &\quad = Z_{x,s}(t)E\{\psi(X_{z,t}(T))Z_{z,t}(T) \mid \mathscr{Y}_T\}|_{z=X_{x,s}(t)} \\ &\quad = Z_{x,s}(t)v(X_{x,s}(t), t, Y). \qquad \blacksquare \end{aligned}$$

Remark This proof follows the usual method for proving the Markov property of solutions to stochastic differential equations. In fact we could have formulated and proved a Markov-type property with respect to the filtration $\mathscr{X}_t \vee \mathscr{Y}_T$. Furthermore, we do not really use all the conditions H.i)–H.iv) in our proof. For example, H.iii) is irrelevant and from H.i) we only used the fact that the coefficients are in $\mathscr{C}^1$.

We are now prepared to state and prove the main theorem. Recall the definition of $D_r^{Y_i}v(x, r+, Y)$ and $D_r^{Y_i}\nabla v(x, r+, Y)$ given in Corollary 3.12.

THEOREM 4.2 *Assume* H.i)–H.iv). *Then for every* x,

$$\begin{aligned} v(x, s, Y) = \psi(x) &+ \sum_1^{l_2} \int_s^T (\nabla v(x, r, Y)f_{l_1+j}(x, r, Y) + v(x, r, Y)h_j(x, r, Y))\, dY_j(r) \\ &+ \int_s^T \left(\frac{1}{2}\operatorname{tr}\left[\nabla^2 v(x, r, Y)\sum_1^{l} f_j f_j^*(x, r, Y)\right] + \nabla v(x, r, Y)f_0(x, r, Y)\right) dr \\ &+ \sum_1^{l_2} \int_s^T h_i(x, r, Y)\nabla v(x, r, Y)f_{l_1+i}(x, r, Y)\, dr \\ &+ \sum_1^{l_2} \int_s^T [f_{l_1+j}(x, r, Y)D_r^{Y_j}\nabla v(x, r+, Y) + h_j(x, r, Y)D_r^{Y_j}v(x, r+, Y)]\, dr \end{aligned} \tag{4.8}$$

for all $s \leq T$, *almost surely.*

Proof The fundamental calculation of the proof involves applying the extended Itô formula of Theorem 2.3 to $Z_{x,s}(t)v(X_{x,s}(t), t, Y)$. Therefore, we first check that the hypotheses of Theorem 2.3 are satisfied for $U(t) \doteq (X_{x,s}(t), Z_{x,s}(t))$ and $\Phi(z, x, Y) \doteq zv(x, t, Y)$. Indeed, Corollary 3.11 says that $\partial_z\Phi(\cdot,\cdot)$, $\partial_{x_1}\Phi(\cdot,\cdot)\cdots\partial_{x_d}\Phi(\cdot,\cdot)$ all satisfy hypothesis (A) with moments p_1 and p_2 for any $p_1 > 1$ and $p_2 < p$. Furthermore, (3.19), H.i) and H.ii) imply that for any (x, s)

$$f_j(X_{x,s}(r), r, \omega) \in L^q([0, T] \times \Omega; R^d) \quad \forall q > 1$$

$$h_j(X_{x,s}(r), r, \omega)Z_{x,s}(r) \in L^q([0, T] \times \Omega; R) \quad \forall q > 1$$

thus proving (2.7) for U. Equation (3.32) and (3.39) of Proposition 3.9 imply that $X_{x,s}(\cdot) \in \mathbb{L}_Y^{1,\varepsilon p}$ and $Z_{x,s}(\cdot) \in \mathbb{L}_Y^{1,\varepsilon p}$ for any $\varepsilon < 1$; since $p > 8$, we thereby obtain the validity of condition (2.9). Finally, H.ii) and H.iii) in conjunction with Lemma 2.1 imply that $\{f_j(X_{x,s}(r), r, \omega)\} \in \mathbb{L}_Y^{1,\varepsilon p}$ and $\{h_j(X_{x,s}(r), r, \omega)\} \in \mathbb{L}_Y^{1,\varepsilon p}$ for $\varepsilon < 1$, which implies that condition (2.8) on U is true. Therefore, we may apply Theorem 2.3 in the present circumstances. From formula (2.11) we get

$$\begin{aligned}
Z_{x,s}(t)v(X_{x,s}(t), t, Y) &= v(x, t, Y) \\
&\quad + \sum_1^{l_1+l_2} \int_s^t Z_{x,s}(r)\nabla v\, f_j(X_{x,s}(r), r, Y)\, d\omega_j(r) \\
&\quad + \sum_1^{l_2} \int Z_{x,s}(r)(vh_j)(X_{x,s}(r), r, Y)\, dY_j(r) \\
&\quad + \int_s^t Z_{x,s}(r)\left(\frac{1}{2}\operatorname{tr}\left[\nabla^2 v \sum_1^{l} f_j f_j^*\right]\right)(X_{x,s}(r), r, Y)\, dr \\
&\quad + \sum_1^{l_2} \int_0^t Z_{x,s}(r)(\nabla v(f_{l_1+i} h_i)(X_{x,s}(r), r, Y)\, dr \\
&\quad + \sum_1^{l_2} \int_0^t Z_{x,s}(r)([D_r^{Y_i}\nabla v] \cdot f_j + [D_r^{Y_i} v]h_j)(X_{x,s}(r), t, Y)\, dr \qquad (4.9)
\end{aligned}$$

Our strategy shall be to partition $[s, T]$, apply (4.9) over each of the increments of the partition, and then pass to the limit as the mesh size of the partition tends to 0. Thus, let $\Pi^{(n)} = \{s = s_0 < s_1^n < \cdots < s_n^n = T\}$ be a sequence of partitions of $[s, T]$ such that $\sup_i(s_{i+1}^n - s_i^n) \to 0$ as $n \to 0$. (For notational simplicity we will henceforth abbreviate s_i^n by s_i, when this poses no problem.) Now we apply Lemma 4.1 and then Eq. (4.9) in order to write

$$\begin{aligned}
\psi(x) - v(x, s, Y) &= \sum_0^{n-1} v(x, s_{i+1}, Y) - v(x, s_i, Y) \\
&= \sum_0^{n-1} v(s, s_{i+1}, Y) - E\{Z_{x,s_i}(s_{i+1})v(X_{x,s_i}(s_{i+1}), s_{i+1}, Y) | \mathscr{Y}_T\} \\
&= -\left(E\left\{\int_s^T g_{0,j}^n(r)\, dW_j(r) + g_{1,j}^n(r)\, dY_j(r) | \mathscr{Y}_T\right\}\right. \\
&\quad \left. + E\left\{\int_s^T (g_2^n(r) + g_3^n(r))\, dr | \mathscr{Y}_T\right\}\right), \qquad (4.10)
\end{aligned}$$

where

$$g^n_{0,j}(r) = \sum_i Z_{x,s_i}(r)\nabla v(X_{x,s_i}(r), s_{i+1}, Y)\cdot f_j(X_{x,s_i}(r), r, Y)1_{[s_i,s_{i+1})}(r)$$

$$g^n_{1,j}(r) = \sum_i Z_{x,s_i}(r)[v(X_{x,s_i}(r), s_{i+1}, Y)h_j(X_{x,s_i}(r), r, Y) + \nabla v(X_{x,s_i}(r), s_{i+1}, Y)\cdot f_{j+l_1}(X_{x,s_i}(r), r, Y)]1_{[s_i,s_{i+1})}(r)$$

$$g^n_2(r) = \sum_i Z_{x,s_i}(r)\Big(\frac{1}{2}\operatorname{tr}\Big[\nabla^2 v(X_{x,s_i}(r), s_{i+1}, Y)\sum_j f_j f_j^*(X_{x,s_i}(r), r, Y) + \sum_j h_j(X_{x,s_i}(r), r, Y)\nabla v(X_{x,s_i}(r), s_{i+1}, Y)f_{j+l_1}(X_{x,s_i}(r), r, Y) + \nabla v(X_{x,s_i}(r), s_{i+1}, Y)\cdot f_0(X_{x,s_i}(r), r, Y)\Big)1_{[s_i,s_{i+1})}(r)$$

$$g^n_3(r) = \sum_i Z_{x,s_i}(r)\Big[\sum_{j=1}^{l_2} D_r^{Y_j}\nabla v\cdot(X_{x,s_i}(r), s_{i+1}, Y)f_{j+l_1}(X_{x,s_i}(r), r, Y) + D_r^{Y_j}v(X_{x,s_i}(r), s_{i+1}, Y)h_j(X_{x,s_i}(r), r, Y)\Big]1_{[s_i,s_{i+1})}(r).$$

For the purposes of the limit argument, it is convenient to label the integrands of Eq. (4.8) in a manner corresponding to these last definitions. Thus, we set,

$$g_{0,j}(r) = \nabla v(x, r, Y)f_j(x, r, Y) \quad \text{for} \quad 0 \le j \le l_1,$$

$$g_{1,j}(r) = \nabla v(x, r, Y)f_{l_1+j}(x, r, Y) + v(x, r, Y)h_j(x, r, Y) \quad \text{for} \quad 1 \le j \le l_2,$$

$$g_2(r) = \frac{1}{2}\operatorname{tr}\Big[\nabla^2 v \sum_1^l f_j f_j^*(x, r, Y)\Big] + \sum_1^{l_2}(\nabla v f_{l_1+i}h_i)(x, r, Y) + \nabla v\cdot f_0(x, r, Y),$$

$$g_3(r) = \sum_1^{l_2}[(D_r^{Y_j}\nabla v(x, r+, Y))f_{l_1+j}(x, r, Y) + (D_r^{Y_j}v(x, r+, Y))h_j(x, r, Y)].$$

We shall use $g_0(r)$ to denote $(g_{0,0}(r), \ldots, g_{0,l_1}(r))$, and, likewise, $g_1(r)$ to denote $(g_{1,1}(r), \ldots, g_{1,l_2}(r))$. Formula (4.8) now reads

$$v(x, s, Y) = \psi(x) + \int_s^T g_{1,j}(r)\, dY_j(r) + \int_s^T [g_2(r) + g_3(r)]\, dr.$$

We shall show that

$$\lim_{n\to\infty} g_i^n(r, Y) = g_i(r, Y) \quad \text{for } dr \times \mathbb{P}\text{-almost every } (r, Y), \text{ for all } 0 \le i \le 3. \tag{4.11}$$

$$\sup_n \int_0^T |g_i^n(r)|^q \, dr \quad \text{for every } q \le p \quad \text{and} \quad 0 \le i \le 3. \tag{4.12}$$

Let us show that this will suffice to complete the proof. Indeed, (4.12) implies that $\{|g_i^n|^q; n \ge 1\}$ is uniformly integrable on $([0, T] \times \Omega, dr \times \mathbb{P})$ for any $q < p$. Thus, because of (4.11), $\int_s^t [g_2^n(r) + g_3^n(r)] \, dr$ converges in mean square to $\int_s^t [g_1(r) + g_2(r)] \, dr$. Likewise

$$E\left\{\int_0^T |g_i^n(r) - g_i(r)|^2 \, dr\right\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad i = 0, 1. \tag{4.13}$$

For $i = 1$, (4.13) implies that

$$\int_s^T E\{g_{1,j}^n(r) | \mathcal{Y}_T\} \, dY_j(r) \to \int_s^T g_1(r) \, dY_j(r) \quad \text{in } \mathbb{D}^{-1,2},$$

as $n \to \infty$. Without going further we only know that $\int_s^T g_1(r) \, dY_j(r)$ is an element of $\mathbb{D}^{-1,2}$. However, we know that

$$E\left\{\int_s^T \sum_1^{l_1} g_{0,j}^n(r) \, dW_j(r) | \mathcal{Y}_T\right\} = 0,$$

and hence from (4.10) it follows that

$$\int_s^T E\{g_{1,j}^n(r) | \mathcal{Y}_T\} \, dY_j(r) = v(x, s, Y) - \psi(x) + \int_s^t E\{[g_{0,0}^n(r) + g_2^n(r) + g_3^n(r)] | \mathcal{Y}_T\} \, dr. \tag{4.14}$$

We have already proved that the right-hand side converges in mean square to the corresponding expression with g^n replaced throughout by g. Since convergence in mean square implies convergence in $\mathbb{D}^{-1,2}$, we can conclude that $g_1(\cdot)$ is *strictly* Skorohod integrable with respect to Y, in the sense defined in the introduction, and

$$\int_s^T g_1(r) \, dY_j(r) = v(x, s, Y) - \psi(x) - \int_s^t [g_{0,0}(r) + g_2(r) + g_3(r)] \, dr. \tag{4.15}$$

This is precisely (4.8). We should note here that it is possible to show directly that g_1 is strictly Skorohod integrable. Indeed, the results of Theorems 3.7 and 3.10 can be used to establish that $g_1 \in \mathbb{L}_Y^{1,2}$.

To complete the proof, it remains to prove (4.11) and (4.12). (4.11) is relatively easy. It follows from the fact that all the processes $v(x, s, Y)$, $\nabla v(x, s, Y)$, $\nabla^2 v(x, s, Y)$, $D_\theta^{Y_j} v(x, s, Y)$, and $D_\theta^{Y_j} \nabla v(x, s, Y)$ are almost surely continuous in (x, s), that $X_{x,s}(t)$ and $Z_{x,s}(t)$ are almost surely continuous in (x, s, t), and that the coefficients $f_j(x, r, Y)$ and $h_i(x, r, Y)$ are continuous in x for almost every (r, Y). These continuity properties are all true either by assumption, or because of Propositions 3.6, 3.9, and Theorem 3.7 and 3.10. Thus, for example,

$$\lim_{s_i \downarrow r,\, s_{i+1} \uparrow r} Z_{x,s_i}(r) \nabla v(X_{x,s_i}(r), s_{i+1}, Y) f_0(X_{x,s_i}(r), r, Y) = \nabla v(x, r, Y) f_0(x, r, Y).$$

The other terms work similarly.

The proof of (4.12) makes use of the polynomial growth bounds derived on the X and Z processes in Proposition 3.6 and on $\partial^\alpha v$ and $\partial^\alpha D^{Y_i} v$ in Theorems 3.7 and 3.10 and Proposition 3.13. We shall only do the case g_3^n in detail, as that is the most difficult. Observe first that by H.i) and Proposition 3.6,

$$\sup_r f_j(X_{x,s}(r), r, Y) Z_{x,s}(r) \quad \text{and} \quad \sup_r h_j(X_{x,s}(r), r, Y) Z_{x,s}(r)$$

admit moments of all orders which are uniformly bounded in s. Now, fix any q and q' such that $2 \le q < q' < p$, and recall the definition of $\bar{L}_{s,q,\alpha,a}(\theta, Y)$ in Proposition 3.13. From the inequality (3.45) applied to the terms $[D_\theta^{Y_j} v][X_{x,s_i}, s_{i-1}, Y)$ and $[D_\theta^{Y_j} \nabla v](X_{x,s_i}, s_{i-1}, Y)$, we obtain

$$\begin{aligned}
E\left\{\int_0^T |g_3^n(r)|^q \, dr\right\} &\le K \sum_i \sum_{|\alpha| \le 1} \int_{s_i}^{s_{i-1}} dr\, E\Big[L^q_{s_{i-1},q',\alpha,a}(r, \cdot) \\
&\qquad \times (1 + |X_{x,s_i}(r)|^{aq}) \sup_j (|f_j(X_{x,s_i}(r), r)| + |h_j(X_{x,s_i}(r), r)|)^q \Big] \\
&\le K \sum_i \int_{s_i}^{s_{i-1}} (E[L^{q'}_{s_{i-1},q',\alpha,a}(r, \cdot)])^{q/q'} \, dr \\
&\le K \sum_i \int_{s_i}^{s_{i-1}} \left(E\left[\int_0^T M^p(\lambda, r, \cdot) \, d\lambda\right]\right)^{q/p} dr \\
&\le K \left(\int_0^T E\left[\int_0^T M^p(\lambda, r, \cdot) \, dr \, d\lambda\right]\right)^{q/p}
\end{aligned}$$

if a is large enough. The last term in this string of inequalities is bounded and is independent of n; the constant K depends only on q, q', and the coefficients f_j and h_i. The second and fourth inequalities in this string are a consequence of Hölder's inequality, the third uses Proposition 3.13. ∎

Equation (4.8) may be recast in a simpler form if one employs anticipating Stratonovich rather than Skorohod integrals. We shall give a brief development of this without attempting to state all the technical regularity assumptions needed. We shall always assume that at least the hypothesis of Theorem 4.2 hold.

It is first necessary to define the anticipating Stratonovich integral. We take a formal approach, restricted to integrands u in $\mathbb{L}_Y^{1,2}$, inspired by the results of Nualart & Pardoux [8]. For a process $u \in \mathbb{L}_Y^{1,2}$ consider the following additional property: there exists a neighborhood A of the diagonal of $[0, T]^2$ such that,

the process $\{D_s^{Y_i}u(t) | 0 \leq s, t \leq T\}$ admits a version such that the map $t \to D_s^Y u(t)$ is continuous on $A \cap \{s \leq t\}$, either in an almost sure sense or as a map into $L^2(\Omega)$. (4.16a)

the process $\{D_s^{Y_i}u(t) | 0 \leq s, t \leq T\}$ admits a version such that the map $t \to D_s^Y u(t)$ is continuous on $A \cap \{s \geq t\}$, either in an almost sure sense or as a map into $L^2(\Omega)$. (4.16b)

If u satisfies (4.16), we may define

$$\begin{aligned} D_s^{Y_i}u(s+) &:= \lim_{\{t \to s, t > s\}} D_s^{Y_i}u(t), \\ D_s^{Y_i}u(s-) &:= \lim_{\{t \to s, t < s\}} D_s^{Y_i}u(t), \end{aligned} \tag{4.17}$$

in a notation consistent with that of Definition 3.4.

If $u = (u_1, \ldots, u_{l_2})$, we define

$$\text{trace}_j(u)(s) := \frac{1}{2} [D_s^{Y_i}u_j(s+) + D_s^{Y_i}u_j(s-)].$$

We then define the Stratonovich integral as

$$\int_0^t u_j(s) \circ dY_j(s) = \int_0^t u_j(s)\, dY_j(s) + \int_0^t \sum_j \text{trace}_j(u)(s)\, ds. \tag{4.18}$$

The motivation for this definition comes from Nualart and Pardoux [8], in which it is shown that, if $\{u\}$ satisfies regularity conditions similar to (4.16),

$$\int_0^t u_j(s) \circ dY_j(s) = \lim_{n \to \infty} (\text{in prob.}) \sum_1^n \tilde{u}_{k,n} \cdot [Y(t_{k+1,n}) - Y(t_{k,n})]$$

where

$$\bar{u}_{k,n} := (t_{k+1,n} - t_{k,n})^{-1} \int_{t_{k,n}}^{t_{k+1,n}} u(s)\, ds$$

and $\{\{t_{k,n}\}_{k=1}^{n};\ n \geq 1\}$ is a sequence of partitions of $[0, t]$ whose norms converge to zero as $n \mapsto \infty$.

We shall use (4.18) to rewrite the stochastic p.d.e. for $v(s, x, y)$ using the Stratonovich integral. For this, it is necessary that the integrands of the stochastic integrals in Eq. (4.8) admit traces. Therefore we shall impose an additional regularity condition; we assume that the coefficients f_{l_1+j}, and h_j, $j = 1, \ldots, l_2$; all satisfy property (4.16a) for each x. (4.16b) holds automatically because of the progressive measurability of the coefficients, so that, for example, $D_s^{Y_i} f_j(x, t, y) = 0$ for $s > t$. Now, we may use Eqs. (3.64) and (3.65) to derive

$$\begin{aligned}\sum_j (\mathrm{trace}_j(v(x,\cdot,Y)h_j(x,\cdot,Y))(r) &= v(x,r,Y)\sum_1^{l_2}\left(\frac{1}{2} D_r^{Y_j} h_j(x, r+, Y) + \frac{1}{2} h_j^2(x,r,Y)\right)\\ &\quad + \sum_1^{l_2} h_j(x,r,Y) D_r^{Y_j} v(x, r+, Y)\\ &\quad + \frac{1}{2}\sum_1^{l_2} h_j(x,r,Y)\nabla v(x,r,Y)\cdot f_{l_1+j}(x,r,Y), \end{aligned} \tag{4.19}$$

$$\begin{aligned}&\sum_j \mathrm{trace}_j(\nabla v(x,\cdot,Y) f_{l_1+j}(x,\cdot,Y))(r)\\ &\quad = \nabla v(x,r,Y)\left[\frac{1}{2}\sum_1^{l_2}(D_r^{Y_j} f_{l_1+j}(x, r+, j) + \frac{1}{2}\nabla f_{l_1+j}\cdot f_{l_1+j}(x,r,Y)\right]\\ &\quad + \frac{1}{2}\mathrm{tr}\left(\nabla^2 v(x,r,Y)\sum_1^{l_2} f_{l_1+j}\cdot f_{l_1+j}^*(x,r,Y)\right)\\ &\quad + \sum_1^{l_2}(D_r^{Y_j}\nabla v(x, r+, Y)) f_{l_1+j}(x,r,Y)\\ &\quad + \frac{1}{2}\sum_1^{l_2}(h_j(x,r,Y)\nabla v(x,r,Y) + v(x,r,Y)\nabla h_j(x,r,Y))\cdot f_{l_1+j}(x,r,Y)\end{aligned} \tag{4.20}$$

To state the Stratonovich equation, it is convenient to define the coefficients

$$\tilde{f}_0(x,r,Y) := f_0(x,r,Y) - \frac{1}{2}\left(\sum_1^{l_2}[D_r^{Y_j} f_{l_1+j}(x, r+, Y) + \nabla f_{l_1+j}\cdot f_{l_1+j}(x,r,Y)]\right)$$

$$\tilde{h}_0(x,r,Y) := \frac{1}{2}\left(\sum_1^{l_2} h_j^2(x,r,Y) + D_r^{Y_j} h(x, r+, Y) + \nabla h_j(x,r,Y) f_{l_1+j}(x,r,Y)\right)$$

Then, applying (4.19) and (4.20) in Eq. (4.8), we derive

$$v(x, s, Y) = \psi(x) + \int_s^t [\nabla v(x, r, Y) \cdot f_{l_1+j}(x, r, Y) + v(x, r, Y) h_j(x, r, Y)] \circ dY_j(r)$$
$$+ \int_s^t \left[\frac{1}{2} \operatorname{tr}\left(\nabla^2 v(x, r, Y) \sum_1^{l_1} f_j f_j^*(x, r, Y) + \nabla v \cdot \tilde{f}_0(x, r, Y)\right.\right.$$
$$\left.\left. - v(x, r, Y) \tilde{h}_0(x, r, Y)\right)\right] dr. \tag{4.21}$$

Notice in particular that in (4.21), the coefficients in the second order term

$$\operatorname{tr}\left(\nabla^2 v(x, r, Y) \sum_1^{l_1} f_j f_j^*(x, r, Y)\right)$$

involve only the vector fields $f_1, \ldots, f_{l_1}$ which multiply the dW_i, $1 \le i \le l_2$, inputs in Eq. (4.1). The reason for this, and the reason for the simple form for (4.21), becomes intuitively clear if we rewrite the equations for X and Y using the Stratonovich integral for any integral with respect to dY. Formally,

$$X_{x,s}(t) = x + 1_{[t \ge s]}\left[\int_s^t \tilde{f}_0(X_{x,s}(r), r, Y)\, dr + \int_s^t f_j(X_{x,s}(r), r, Y)\, dW_j(r)\right.$$
$$\left. + \int_s^t f_{l_1+j}(X_{x,s}(r), r, Y) \circ dY_j(r)\right] \tag{4.22}$$

$$Z_{x,s}(t) = \exp\left[1_{[t \ge s]}\left(\int_s^t h_j(X_{x,s}(r), r, Y) \circ dY_j(r) - \int_s^t \tilde{h}_0(X_{x,s}(r), r, Y)\, dr\right)\right]. \tag{4.23}$$

Heuristically speaking, (4.22) says that for each fixed Y, $X_{x,s}(t)$ is a diffusion driven by W with diffusion coefficient $\sigma(x, r, Y) = \sum_1^{l_1} f_j f_j^*(x, r, Y)$ and singular drift

$$m(x, r, Y) = \tilde{f}_0(x, r, Y) + \sum_1^{l_2} f_{l_1+j}(x, r, Y) \circ \frac{\text{“}dY_j\text{”}}{dr}$$

For each Y, (4.22) is the backward parabolic p.d.e. associated to the diffusion X with coefficients σ and m.

Remark To prove the validity of (4.22) and (4.23), in accordance with the definition (4.18) of the Stratonovich integral, one should strengthen slightly the hypotheses placed on f_{l_1+j} and h_j, $1 \le j \le l_2$. That is, one should require that for every R, there exists a neighborhood A_R of the diagonal of $[0, T]^2$ such that the map,

$$(x, s, t) \to D_s^{Y_j} f_j(x, t, T)$$

admits a version which is continuous a.s. on $[-R, R] \times [A_R \cap \{s \leq t\}]$, $1 \leq j \leq l_2$. A similar assumption is placed on h_j, $1 \leq j \leq l_2$. If these conditions are assumed,

$$D_r^{Y_j}[f_j(X_{x,s}(r+\varepsilon), r+\varepsilon, Y)] = \nabla f_j(X_{x,s}(r+\varepsilon), r+\varepsilon, Y) D_r^{Y_j} X_{x,s}(r+\varepsilon) + [D_r^{Y_j} f_j](X_{x,s}(r+\varepsilon), r+\varepsilon, Y), \tag{4.24}$$

and, hence, taking limits,

$$D_r^{Y_j}[f_j(X_{x,s}(r+), r+, Y)] = \nabla f_j(X_{x,s}(r), r, Y) \cdot f_j(X_{x,s}(r), r, Y) + [D_r^{Y_j} f_j](z, r+, Y)|_{z=X_{x,s}(r)}. \tag{4.25}$$

We have used the fact that $D_r^{Y_j} X_{x,s}(r+) = f_j(X_{x,s}(r), r, Y)$, which follows from Eq. (3.57). ■

Finally, we shall state a converse to Theorem 4.2. The derivation is simply an extension of the proof of the Feynman-Kac formula using Itô's rule.

Let u be a solution to (4.8) and fix (x, s). Given appropriate regularity assumptions on u, we will obtain the stochastic integral representation

$$Z_{x,s}(T)\psi(X_{x,s}(T)) - u(x, s) = Z_{x,s}(t)u(X_{x,s}(t), t)|_{t=s}^{t=T} = \sum_{j=1}^{l_1} \int_s^T Z_{x,s}(r)\nabla u(X_{x,s}(r), r) \cdot f_j(X_{x,s}(r), r)\, dW_j(r) \tag{4.26}$$

by developing an Itô formula for the process $(Z_{x,s}(t)u(X_{x,s}(t), t))_{s \leq t \leq T}$ and by using (4.8). When the conditional expectation operation $E\{\cdot/Y_r\}$ is applied to both sides of (4.26), we find

$$u(x, s) = E\{Z_{x,s}(T)\psi(X_{x,s}(T)) | \mathscr{Y}_T\} \tag{4.27}$$

The process $Z_{x,s}(\cdot)u(X_{x,s}(\cdot), \cdot)$ is the composition of an anticipating random field and an Itô process. Itô-Ventzell formulas for the stochastic differentiation of such compositions are developed in [11]. It is important for these formulas to impose conditions on the continuity in s of $D_r u(x, s, Y)$.

In the context of the present paper, the correct conditions to use are those proved for v in Theorem 3.10 and Proposition 3.13. These differ in a minor way from the conditions used in the theorems of [11]. Therefore we prefer to give a direct proof of (4.26). This will not require much work any way because the hard details have already been done in the proof of Theorem 4.2.

To state the result it is convenient to collect all the regularity properties which have been proved for v in Section 3, assuming hypotheses H.i)–H.iv). Henceforth we shall drop explicit dependence on Y of $u(x, s, Y)$, $f_j(x, s, Y)$, etc., for notational simplicity.

Assume H.i)–H.iv) are in force. We shall say u satisfies C.i) if (see Theorem 3.7)

C.i) u admits a version such that $u(\cdot, \cdot) \in C^{2,0}(R^d \times [0, T])$ almost surely and for all $s \leq T$ and any $q > 1$, there exists a random variable $C_{s,q}$ and an $a > 0$, depending on q but not on s, such that $|\partial^\alpha u(x, s)| \le C_{s,q}(1 + |x|^a)$, $\forall |\alpha| \le 2$ and

$$\sup_{s \le T} E\{C^q_{s,q}\} < \infty.$$

We shall say u satisfies C.ii) if (see Theorem 3.10 and Proposition 3.13):

C.ii) u satisfies conditions (3.40)–(3.42) of Theorem 3.10; and, for p as in H.iii), there exists an $\bar{M} \in L^p([0, T]^2 \times \Omega)$ such that for every $q < p$, there exist a constant a and $L_s(\theta, Y)$ with

$$|\partial^\alpha D^{Y_i}_\theta u(x, s, Y)| \le L_s(\theta, Y)(1 + |x|^a), \quad |\alpha| \le 1, \quad 1 \le i \le l_2$$

where

$$E\{L^q_s(\theta, \cdot)\} \le K_{a,q}\left(E \int_s^T \bar{M}^p(\eta, \theta, \cdot)\, d\eta\right)^{q/p},$$

for almost every s, and almost every $\theta \le s$.

We note the following consequences of C.ii), which are often used. For $q < p$ and $F \in \bigcap_{q' > 1} L^{q'}$

$$E\{|D^{Y_i}_r u(F, r+)|^q\} \le K(F)\left(E \int_r^T \bar{M}^p(\eta, r, \cdot)\, d\eta\right)^{q/p}. \tag{4.28}$$

For $q < p$ and $\sup_{[0,T]} F(r) \in \bigcap_{q' > 1} L^{q'}$,

$$E \int_0^T |D^{Y_i}_r u(F(r), r+)|^q\, dr \le K(F)\left(E \int_0^T \int_0^T \bar{M}^p(\eta, r, \cdot)\, d\eta\, dr\right)^{q/p}. \tag{4.29}$$

THEOREM 4.3 *Assume* H.i)–H.iv). *Let* $\{u(x, s, Y); x \in R^d, 0 \le s \le T\}$ *be a solution to* (4.8) *and assume* u *satisfies conditions* C.i) *and* C.ii). *Then*

$$u(x, s) = E\{Z_{x,s}(T)\psi(X_{x,s}(T))|\mathscr{Y}_T\} \quad a.s. \tag{4.30}$$

for every (x, s), $s \le T$.

Proof As we have shown above, it suffies to prove the stochastic integral representation (4.26).

Let $g_{0,j}(x, r)$, $g_{1,j}(x, r)$, $g_2(x, r)$, and $g_3(x, r)$ be defined as in the proof of Theorem 4.2, but with v replaced by u. Thus, for example

$$g_{0,j}(x, r) = \nabla u(x, r) \cdot f_j(x, rY) \quad 1 \le j \le l_1$$

$$g_{1,j}(x, r) = \nabla u(x, r) \cdot f_{l_1+j}(x, r, Y) + u(x, r)h_j(x, r) \quad 1 \le j \le l_2,$$

With this notation, (4.8) is written

$$u(x, s) = \psi(x) + \sum_{1}^{l_2} \int_s^T g_{1,j}(x, r)\, dY_j(r) + \int_s^T [g_2(x, r) + g_3(x, r)]\, dr \tag{4.31}$$

Fix (x, s) and let $\Pi^{(n)} = \{S_i^{(n)}, 0 \le i \le n\}$ be a sequence of partitions of $[s, T]$ as in the proof of Theorem 4.2.

Then

$$\begin{aligned} Z_{x,s}(T)\psi(X_{x,s}(T)) - u(x, s, Y) &= \sum_{i=0}^{n-1} [Z_{x,s}(s_{i+1})u(X_{x,s}(s_{i+1}), s_{i+1}) - Z_{x,s}(s_i)u(X_{x,s}(s_i), s_{i+1})] \\ &\quad + \sum_{i=0}^{n-1} Z_{x,s}(s_i)[u(X_{x,s}(s_i), s_{i+1}) - u(X_{x,s}(s_i), s_i)] \\ &=: I_1^{(n)} + I_2^{(n)} \end{aligned} \tag{4.32}$$

$I_1^{(n)}$ is analyzed by the same method applied to

$$\sum_{0}^{n-1} v(x, s_{i+1}, Y) - Z_{x,s}(s_{i+1})v(X_{x,s}(s_{i+1}), s_{i+1}, Y)$$

of (4.10) in the proof of Theorem 4.2. Indeed, by following this proof step by step, we obtain

$$\begin{aligned} \lim_{n\to 0} E[I_1^n F] = E\Bigg[F\Bigg(&\sum_{j=1}^{l_1} \int_x^T Z_{x,s}(r) g_{0,j}(X_{x,s}(r), r)\, dW_j(r) \\ &+ \sum_{j=1}^{l_2} \int_s^T Z_{x,s}(r) g_{1,j}(X_{x,s}(r), r)\, dY_j(r) \\ &+ \int_s^T Z_{x,s}(r)[g_2(X_{x,s}(r), r) + g_3(X_{x,s}(r), r)]\, dr\Bigg)\Bigg] \end{aligned} \tag{4.33}$$

for every $F \in \mathbb{D}_Y^{1,2}$. We have expressed the limit (4.33) in weak form because the method of Theorem 4.2 is to derive convergence in $\mathbb{D}^{-1,2}$ of the stochastic integral term. To establish convergence of the sequence

$$\sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} Z_{x,s}(r)\Bigg[\sum_{j=1}^{l_2} (D^{y_j}\nabla u(X_{x,s}(r), s_{i+1}) f_{l_1+j}(X_{x,s}(r), r) + D_r^{Y_j} u(X_{x,s}(r), s_{i+1}) h_j(X_{x,s}(r), r)\Bigg]\, dr$$

to

$$\int_s^T Z_{x,s}(r)g_3(X_{x,s}(r), r)\,dr = \int_s^T Z_{x,s}(r)\Bigg[\sum_1^{l_2} (D^{Y_j}\nabla u(X_{x,s}(r), r+)f_{l_1+j}(X_{x,s}(r), r) + D_r^{Y_j}u(X_{x,s}(r), r+)h_j(X_{x,s}(r), r)\Bigg]\,dr$$

one uses C-ii), (4.28), and (4.29), as in Theorem 4.2.

Turning now to the analysis of $I_2^{(n)}$, we first note that

$$\begin{aligned} &Z_{x,s}(s_i)[u(X_{x,s}(s_i), s_{i+1}) - u(X_{x,s}(s_i), s_i)] \\ &= -\Bigg[\sum_{j=1}^{l_2} \int_{s_i}^{s_{i+1}} zg_{1,j}(x, r)\,dY_j(r) + \int_{s_i}^{s_{i+1}} z[g_2(x, r) + g_3(x, r)]\,dr\Bigg|_{\substack{z=z_{x,s}(s_i)\\ x=X_{x,s}(s_i)}} \\ &= -\Bigg[\sum_{j=1}^{l_2} \int_{s_i}^{s_{i+1}} Z_{x,s}(s_i)g_{1,j}(X_{x,s}(s_i), r)\,dY_j(r) \\ &\quad + \int_{s_i}^{s_{i+1}} Z_{x,s}(s_i)[g_2(X_{x,s}(s_i), r) + g_3(X_{x,s}(s_i), r)]\,dr\Bigg] \end{aligned} \tag{4.34}$$

Normally, a correction term appears when replacing a parameter in a Skorohod integral (see [8], Section 4) with a random variable, as we have done in the last equality of (4.34). In this case the correction term is 0 because $X_{x,s}(s_i)$ and $Z_{x,s}(s_i)$ are adapted to the past at s_i and the integral is over the interval $[s_i, s_{i+1}]$.

(4.34) is easy to prove by standard methods of anticipating calculus. First, one proves that

$$F\int_s^T a\,dY = \int_s^T Fa\,dY \tag{4.35}$$

whenever F is $\{\mathscr{F}\}_s^{Y,X}$-measurable, $F \in L^4$, $a1_{[s,T]}$ is Skorohod integrable,

$$E\int_s^T a^4\,dr < \infty, \quad \text{and} \quad E\bigg(\int_s^T a\,dY\bigg)^4 < \infty.$$

Then one approximates $Z_{x,s}(s_i)$ and $X_{x,s}(s_i)$ by step functions, applies (4.35) and passes to the limit.

Given (4.34)

$$\begin{aligned} I_2^n = &-\Bigg[\sum_{j=1}^{l_2} \int_s^T \sum_{i=0}^{n-1} Z_{x,s}(s_i)g_{1,j}(X_{x,s}(s_i), r)1_{[s_i,s_{i+1})}(r)\,dY_j(r) \\ &+ \int_s^T \sum_{i=0}^{n-1} Z_{x,s}(s_i)[g_2(X_{x,s}(s_i), r) + g_3(X_{x,s}(s_i), r)1_{[s_i,s_{i+1})}(r)\,dr\Bigg]. \end{aligned}$$

Arguing as in Theorem 4.2 and making use of C.i), C.ii) and (4.28), (4.29), we find

$$\lim_{n\to\infty} E[I_2^n F] = -E\left[F\left[\sum_{j=1}^{l_2}\int_s^T Z_{x,s}(r)g_{1,j}(X_{x,s}(r), r)\, dY_j(r) + \int_s^T Z_{x,s}(r)[g_2(X_{x,s}(r), r) + g_3(X_{x,s}(r), r)]\, dr\right]\right]. \tag{4.36}$$

for all $F \in \mathbb{D}_Y^{1,2}$. Combining (4.33) and (4.36) gives

$$E[F(Z_{x,s}(T)\psi(X_{x,s}(T)) - u(x, s))] = E\left[F\left(\sum_{j=1}^{l_1}\int_s^T Z_{x,s}(r)g_{0,j}(X_{x,s}(r), r)\, dW_j(r)\right)\right].$$

for all $F \in \mathbb{D}_Y^{1,2}$. This implies (4.26) immediately and completes the proof.

5 APPLICATION TO NONLINEAR SMOOTHING

We finally want to show in this section that in the particular case $\psi \equiv 1$, the quantity $v(x, t, Y)$, for which an SPDE was derived in Section 4, is the Randon–Nikodym derivative of a "smoothing conditional law" with respect to a "filtering conditional law" for a certain partially observed diffusion model, thus extending the results in Pardoux [14] and [15] (Section 2.5).

Consider the following system of SDEs:

$$\begin{aligned} X(t) &= X_0 + \sum_{j=0}^{l_1}\int_0^t f_j(X(r), r, Y)\, d\omega_j(r) + \sum_{i=1}^{l_2}\int_0^t f_{l_1+i}(X(r), r, Y)\, dY_i(r) \\ Y(t) &= \int_0^t h(X(r), r, Y)\, dr + V(r), \quad 0 \le t \le T \end{aligned} \tag{5.1}$$

where now $\omega = (W, V)$ is a standard $l_1 + l_2$ dimensional Wiener process in (Ω, F, Q), and we assume w.l.o.g. that $\Omega = R^d \times C([0, T]; R^{l_1+l_2})$, $\Omega = \Omega_0 \times \Omega_1 \times \Omega_2$, with $\Omega_0 = R^d$, $\Omega_1 = C([0, T], R^{l_1})$, $\Omega_2 = C([0, T], R^{l_2})$.

We define moreover

$$Z(t) = \exp\left[\int_0^t (h(X(r), r, Y), dY(r)) - \frac{1}{2}\int_0^t |h(X(r), r, Y)|^2\, dr\right], \quad 0 \le t \le T$$

Our assumptions are as follows. We suppose that the coefficients $f_0, f_1, \ldots, f_{l_1+l_2}$, $h_1, \ldots, h_{l_2}$ are $\mathscr{B}_d \otimes \mathscr{P}_2$ measurable, where $\mathscr{B}_d$ is the Borel σ-field over R^d, and $\mathscr{P}_2$ is the σ-field of $\mathscr{Y}_t$-progressively measurable subsets of $[0, T] \times \Omega_2$, and locally bounded on $R^d \times [0, T] \times \Omega_2$. We moreover assume that (5.1) is satisfied, with X_0 a

d-dimensional random vector independent of (W, V), and further more that the SDE appearing on the first line of (5.1) has a unique strong solution $\{X_{x,s}(t), s \le t \le T\}$ for any starting point $(x, s) \in R^d \times [0, T]$. We finally suppose that:

$$E[Z(T)^{-1}] = 1 \tag{5.2}$$

It follows from (5.2) and Girsanov's theorem that there exists a probability measure P on (Ω, F) which is equivalent to Q, such that

$$\left.\frac{dQ}{dP}\right|_{F_t} = Z(t), \quad 0 \le t \le T$$

and the law of (X_0, W, Y) under P equals the law of (X_0, W, V) under Q. Under P, we are back in the situation of the previous sections (but with much weaker assumptions).

We are interested in the so-called "fixed observation interval smoothing" problem, i.e. we want to derive equations for the evolution of the conditional law under Q of X_t given $\mathscr{Y}_T$, where t varies and T is fixed.

It follows from a well-known formula about conditional expectations that for any $\varphi \in C_b(R^d)$,

$$E_Q[\varphi(X(t)) | \mathscr{Y}_T] = \frac{E_P[\varphi(X(t))Z(T) | \mathscr{Y}_T]}{E_P[Z(T) | \mathscr{Y}_T]}$$

Hence, if we define the random measure μ_t on R^d by

$$\mu_t(\varphi) = E_P[\varphi(X(t))Z(T) | Y_T], \quad \varphi \in C_b(R^d),$$

the smoothing conditional law at time t is $[\mu_t(1)]^{-1}\mu_t$, in other words μ_t is an "unnormalized version" of the smoothing conditional law.

Consider now the "filtering conditional law" at time t, i.e. the conditional law under Q of X_t, given $\mathscr{Y}_t$. Again, for $\varphi \in C_b(R^d)$,

$$E_Q[\varphi(X(t)) | \mathscr{Y}_t] = \frac{E_P[\varphi(X(t))Z(t) | \mathscr{Y}_t]}{E_P[Z(t) | \mathscr{Y}_t]}$$

Now the "unnormalized filtering conditional law" σ_t defined as

$$\sigma_t(\varphi) = E_P[\varphi(X(t))Z(t) | y_t], \quad \varphi \in C_b(R^d), \quad t \ge 0,$$

is known to satisfy the Zakai equation under the present assumptions (see Pardoux [15], Th. 2.3.3)

$$\sigma_t(\varphi) = \sigma_0(\varphi) + \int_0^t \sigma_s(L_{sy}^0 \varphi)\, ds + \sum_{i=1}^{l_2} \int_0^t \sigma_s(L_{sy}^i \varphi)\, dY^i(s), \quad 0 \le t \le T, \quad \varphi \in C_b^2(R^d),$$

where σ_0 is the law of X_0,

$$(L_{sy}^0\varphi)(x) = \frac{1}{2}\,\mathrm{tr}\left[\nabla^2\varphi(x)\sum_1^l f_j f_j^*(x, s, Y)\right] + \nabla\varphi(x) f_0(x, s, Y)$$

$$(L_{sy}^i\varphi)(x) = h_i(x, s, Y)\varphi(x) + \nabla\varphi(x)\sum_1^{l_2} f_{l_1+j}(x, s, Y).$$

For the rest of this section, we redefine v from previous sections, with $\psi = 1$, i.e.

$$v(x, t, Y) = E[Z_{x,s}(T)|\mathscr{Y}_T],$$

where

$$Z_{x,s}(t) = \exp\left[\int_s^t (h(X_{x,s}(r), r, Y), dY_r) - \frac{1}{2}\int_s^t |h(X_{x,s}(r), r, Y)|^2\,dr\right], \quad 0 \le s \le t \le T.$$

We can now state the main result of this section.

THEOREM 5.1 *For any $0 \le t \le T$, the "unnormalized smoothing law" μ_t is absolutely continuous with respect to the "unnormalized filtering law" σ_t, and*

$$\frac{d\mu_t}{d\sigma_t}(x) = v(x, t, Y) \quad \sigma_t\text{-a.e.}, \quad \text{a.s.}$$

In other words,

$$\mu_t(\varphi) = \sigma_t(\varphi)v(\cdot, t, Y) \quad \text{a.s.}$$

Proof We first note that for $\varphi \in C_b(R^d)$,

$$\mu_t(\varphi) = E_P[\varphi(X(t))Z(T)|\mathscr{Y}_T] = E_P\left[\varphi(X(t))Z(t)E_P\left[\frac{Z(T)}{Z(t)}\middle|\mathscr{Y}_T \vee \mathscr{X}_t\right]\middle|\mathscr{Y}_T\right]$$

$$= E_P[\varphi(X(t))Z(t)c(X(t), t, Y)|\mathscr{Y}_T],$$

since $\sigma(X(u), u \ge t) \vee \mathscr{Y}_T$ and $\sigma(X(s), s \le t) \vee \mathscr{Y}_T$ are conditionally independent given $\sigma(X(t)) \vee \mathscr{Y}_T$. Indeed

$$\sigma(X(s);\, s \le t) \subset \sigma(X(0)) \vee \sigma(W(s);\, s \le t) \vee \mathscr{Y}_T$$

and

$$\sigma(X(u);\, u \ge t) \subset \sigma(X(t)) \vee \sigma(W(u) - W(t);\, u \ge t) \vee \mathscr{Y}_T$$

It now remains to show that for any $B_d \otimes \mathscr{Y}_T$ measurable mapping

$$G: R^d \times \Omega_2 \to R_+,$$

any $\varphi \in C_b(R^d)$,

$$E_P[G(X(t), Y)\varphi(X(t))Z(t) | \mathscr{Y}_T] = \sigma_t(G(\cdot, Y)\varphi).$$

From the monotone class theorem, it sufficies to prove the result when $G(x, Y) = \lambda(x)\psi(Y)$, where $\lambda \in C_b(R^d)$, ψ is measurable and non negative, in which case the required identity follows easily from the definition of σ. ■

If we now return to the assumptions of Section 4, then the Radon–Nikodym derivative v satisfies the backward SPDE (4.8). Under certain assumptions, we can show that $\{\sigma_t\}$ is the unique solution of the Zakai equation (see Pardoux [15] Th. 3.4.3] and also that v is the unique solution of the backward SPDE (4.8), hence the unnormalized (and also the normalized) smoothing law is characterized through the solutions of a forward and a backward SPDE.

COROLLARY 5.2 *Let the assumptions* H.i)–H.iv) *of Section* 4 *hold. Then the smoothing density v satisfies Eq.* (4.8) *with final condition* $v(x, T, Y) = 1$. *It is the unique solution in the class of these solutions satisfying conditions* C.i) *and* C.ii) *in Section* 4.

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