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APPROXIMATION OF THE HEIGHT PROCESS OF A CONTINUOUS STATE BRANCHING PROCESS WITH INTERACTION

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ABSTRACT. We first show that the properly rescaled height process of the genealogical tree of a continuous time branching process converges to the height process of the genealogy of a (possibly discontinuous) continuous state branching process. We then prove the same type of result for generalized branching processes with interaction.

1. INTRODUCTION

Continuous state branching processes (or CSBP in short) are the analogues of Galton-Watson (G-W) processes in continuous time and continuous state space. Such classes of processes have been introduced by Jirina [16] and studied by many authors, including Grey [14] and Lamperti [19]. These processes are the only possible weak limits that can be obtained from sequences of rescaled G-W processes, see Lamperti [20] and Li [26], [27].

While rescaled discrete-time G-W processes converge to a CSBP, it has been shown in Duquesne and Le Gall [12] that the genealogical structure of the G-W processes converges too. More precisely, the corresponding rescaled sequence of discrete height processes, converges to the height process in continuous time that has been introduced by Le Gall and Le Jan in [22].

A lot of work has been devoted recently to generalized branching processes, which model competition within the population. This includes generalized CSBPs, see among many others Li [24], Li, Yang and Zhou [25] and the references therein. For the approximation of such generalized CSBPs by discrete time generalized GW processes, we refer to the general results in Bansaye, Caballero and Méléard [2], and for the approximation by continuous time generalized GW processes to our recent paper [11].

Some work has been also devoted recently to the description of the genealogy of such generalized CSBPs, see Le, Pardoux and Wakolbinger [23] and Pardoux [29] for the case of continuous such processes and both Berestycki, Fittipaldi and Fontbona [3] and Li, Pardoux and Wakolbinger [28] for the general case. [3] allows processes without a Brownian component unlike [28], but the latter allows more general interactions. The present paper studies the convergence of the genealogy of a generalized continuous time GW process to that of a generalized possibly discontinuous CSBP, under the same assumptions as [28].

We first give a construction of the CSBP as a scaling limit of continuous time G-W branching processes. To then give a precise meaning to the convergence of trees, we will code G-W trees

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by a continuous exploration process as already defined by Dramé et al. in [10], and we will establish the convergence of this (rescaled) continuous process to the continuous height process defined in [28], see also [12]. Each jump of our generalized CSBP corresponds to the birth of a significant proportion of the total population, whose genealogical tree needs to be explored by our height process. This gives rise to a special term in the equation for the height process, which has possibly unbounded variations and has no martingale property. It also destroys any possible Markov property of the height process. The tightness of such a term cannot be established by standard techniques. We use for that purpose a special method which has been developed in [28], see the proof of Proposition 3.4 below. The main result of this paper is Theorem 4.4 in Section 4.

The organization of the paper is as follows : in Section 2 we recall some basic definitions and notions concerning branching processes. Section 3, which is by far the longest one, considers the height process in the case without interaction. It is devoted to the description of the discrete approximation of both the population process and the height process of its genealogical tree. We prove the convergence of the height process, and of its local time. Section 4 introduces the interaction, via a Girsanov change of probability measure, and establishes the main result. We consider first the case where the interaction function has a bounded derivative, and then the general situation, which allows in particular the popular so–called logistic (i.e. quadratic) interaction.

We shall assume that all random variables in the paper are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We shall use the following notations $\mathbb{Z}_+ = \{0,1,2,...\}$, $\mathbb{N} = \{1,2,...\}$, $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{R}_+ = [0,\infty)$. For $x \in \mathbb{R}_+$, $[x]$ denotes the integer part of x.

2. The height process of a continuous state branching process

2.1. **Continuous state branching process.** A continuous state branching process (CSBP) is an $\mathbb{R}_+$-valued strong Markov process having the property $\mathbb{P}_{x+y} = \mathbb{P}_x * \mathbb{P}_y$, with $\mathbb{P}_x$ denoting the law of the process when starting from x at time $t = 0$. More precisely, a CSBP $X^x = (X_t^x,\ t \geqslant 0)$ (with initial condition $X_0^x = x$) is a Markov process taking values in $[0,\infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property; that is to say, its Laplace transform satisfies

$$\mathbb{E}\left[\exp(-\lambda X_t^x)\right] = \exp\left\{-x u_t(\lambda)\right\}, \quad \text{for } \lambda \geqslant 0,$$

for some non negative function $u_t(\lambda)$. According to Silverstein [33], the function u_t is the unique nonnegative solution of the integral equation

$$u_t(\lambda) = \lambda - \int_0^t \psi(u_r(\lambda))dr, \tag{2.1}$$

where ψ is called the branching mechanism associated with X^x and is defined by

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}})\mu(dz),$$

where $b \in \mathbb{R}$, $c \geqslant 0$ and μ is a σ-finite measure which satisfies that $(1 \wedge z^2)\mu(dz)$ is a finite measure on $(0,\infty)$. We shall in fact assume in this paper that

$$(\mathbf{H}): \quad \int_0^\infty (z \wedge z^2)\mu(dz) < \infty \quad \text{and} \quad c > 0.$$

The finiteness of the measure $(z \wedge z^2)\mu(dz)$ implies that the process X^x does not explode and allows to write the last integral in the above equation in the following form

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z)\mu(dz). \tag{2.2}$$

Let us recall that b represents a drift term, c is a diffusion coefficient and μ describes the jumps of the CSBP. The CSBP is then characterized by the triplet (b,c,μ) and can also be defined as the unique non–negative strong solution of a stochastic differential equation. More precisely, from Fu and Li [13] (see also the results in Dawson-Li [8]) we have that

$$X_t^x = x - b\int_0^t X_s^x ds + \sqrt{2c}\int_0^t\int_0^{X_s^x} W(ds,du) + \int_0^t\int_0^\infty\int_0^{X_{s-}^x} z\overline{M}(ds,dz,du), \tag{2.3}$$

where $W(ds,du)$ is a space-time white nose on $(0,\infty)^2$, $M(ds,dz,du)$ is a Poisson random measure on $(0,\infty)^3$, with intensity $ds\mu(dz)du$, and $\overline{M}$ is the compensated measure of M.

2.2. **The height process in the case without interaction.** We shall interpret below the function ψ defined by (2.2) as the Laplace exponent of a spectrally positive Lévy process Y. Lamperti [19] observed that CSBPs are connected to Lévy processes with no negative jumps by a simple time-change. More precisely, define

$$A_s^x = \int_0^s X_t^x dt, \quad \tau_s = \inf\{t>0,\, A_t^x > s\} \quad \text{and} \quad Y(s) = X_{\tau_s}^x.$$

Then, until its first hitting time of 0, $Y(s)$ is a Lévy process of the form

$$Y(s) = -bs + \sqrt{2c}B(s) + \int_0^s\int_0^\infty z\overline{\Pi}(dr,dz), \tag{2.4}$$

where B is a standard Brownian motion and $\overline{\Pi}(ds,dz) = \Pi(ds,dz) - ds\mu(dz)$, Π being a Poisson random measure on $\mathbb{R}_+^2$ independent of B with mean measure $ds\mu(dz)$. We refer the reader to [19] for a proof of that result. In order to code the genealogy of the CSBP, Le Gall and Le Jan [22] introduced the so-called height process, which is a functional of a Lévy process with Laplace exponent ψ; see also Duquesne and Le Gall [12]. In this paper, we will use the new definition of the height process H given by Li et al. in [28]. Indeed, if the Lévy process Y has the form (2.4), then the associated height process is given by

$$cH(s) = Y(s) - \inf_{0\leqslant r\leqslant s} Y(r) - \int_0^s\int_0^\infty \left(z + \inf_{r\leqslant u\leqslant s} Y(u) - Y(r)\right)^+ \Pi(dr,dz), \tag{2.5}$$

and it has a continuous modification. Note that the height process H is the one defined in Chapter 1 of [12]. i.e $cH(s) = |\{\overline{Y}^s(r);\ 0\le r\le s\}|$, where $\overline{Y}^s(r) := \inf_{r\le u\le s} Y(u)$ and $|A|$ denotes the Lebesgue measure of the set A. A graphical interpretation of (2.5) is as shown on Figure 1. Suppose that Y has a unique jump of size z at time s, and let $s' := \inf\{r>s,\ Y_r = Y_{s-}\}$. On the interval $[s,s']$, H_r equals $Y_r - z$, reflected above $Y_{s-} = Y_s - z$, while for $r \notin [s,s']$, $H_r = Y_r$.

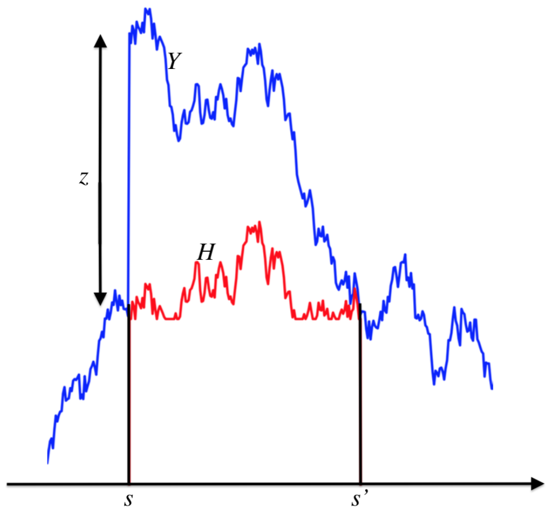

FIGURE 1. Trajectories of Y and H.

Note that we can rewrite (2.5) as

$$\begin{aligned} cH(s) &= Y(s) - \inf_{0\le r\le s} Y(r) - \mathscr{R}(s), \text{ where} \\ \mathscr{R}(s) &= \int_0^s \int_0^\infty \left(z + \inf_{r\leqslant u\leqslant s} Y(u) - Y(r)\right)^+ \Pi(dr,dz). \end{aligned} \tag{2.6}$$

Let $L_s(t)$ denote the local time accumulated by the process H at level t up to time s. The existence of $L_s(t)$ has been established in [12]. We have the following Proposition, see Li et al. [28].

Proposition 2.1. *(Itô-Tanaka formula for the local time of H) We have*

$$L_s(t) = c(H(s)-t)^+ - \int_0^s \mathbf{1}_{\{H(r)>t\}} dY(r) + \int_0^s \int_0^\infty \mathbf{1}_{\{H(r)>t\}} \left(z + \inf_{r\leqslant u\leqslant s} Y(u) - Y(r)\right)^+ \Pi(dr,dz). \tag{2.7}$$

2.3. **The height process in the case with interaction.** Now the stochastic differential equation (2.3) is replaced by

$$X_t^x = X_0^x + \int_0^t f(X_s^x)ds + \sqrt{2c}\int_0^t \int_0^{X_s^x} W(ds,du) + \int_0^t \int_0^\infty \int_0^{X_{s-}^x} z\overline{M}(ds,dz,du), \tag{2.8}$$

where f is a function $f : \mathbb{R}^+ \to \mathbb{R}$, which satisfies

$$f \in \mathscr{C}^1(\mathbb{R}^+), \quad f(0) = 0, \quad f'(z) \le \theta \tag{2.9}$$

for all $z \in \mathbb{R}$, for some $\theta \in \mathbb{R}$. In this case, the process Y will be defined as

$$Y(s) = \sqrt{2c}B(s) + \int_0^s \int_0^\infty z\overline{\Pi}(dr,dz), \tag{2.10}$$

where B is again standard Brownian motion and again $\overline{\Pi}$ denotes the compensated measure $\overline{\Pi}(ds,dz) = \Pi(ds,dz) - ds\mu(dz)$. This means that in this subsection $b = 0$.

The SDE for H reads, see [28]

$$\begin{aligned} cH(s) = Y(s) + L_s(0) + \int_0^s f'(L_r(H(r)))dr \\ - \int_0^s \int_0^\infty [z - (L_s(H(r)) - L_r(H(r)))]^+ \Pi(dr,dz). \end{aligned} \tag{2.11}$$

3. Approximation of the height process without interaction

Consider a population evolving in continuous time with m ancestors at time $t=0$, in which to each individual is attached a random vector describing her lifetime and her number of offsprings. We assume that those random vectors are independent and identically distributed (i.i.d). The rate of reproduction is governed by a finite measure ν on $\mathbb{Z}_+ = \{0,1,2,...\}$, satisfying $\nu(1)=0$. More precisely, each individual lives for an exponential time with parameter $\nu(\mathbb{Z}_+)$, and is replaced by a random number of children according to the probability $\nu(\mathbb{Z}_+)^{-1}\nu$.

We will first renormalize this model, then we will present the results of convergence of the population process, and finally we will prove the convergence of the height process of its genealogical tree, and of the local time of the height process.

Let $N \geqslant 1$ be an integer which will eventually go to infinity. In the next two sections, we choose a sequence $\delta_N \downarrow 0$ such that, as $N \to \infty$,

$$\frac{1}{N}\int_{\delta_N}^{+\infty} \mu(dz) \to 0. \tag{3.1}$$

Because of assumption (**H**) this implies in particular that

$$\frac{1}{N}\int_{\delta_N}^{+\infty} z\mu(dz) \to 0.$$

Moreover, we will need to consider the truncated branching mechanism

$$\psi_{\delta_N}(\lambda) = c\lambda^2 + \int_{\delta_N}^\infty (e^{-\lambda z} - 1 + \lambda z)\mu(dz). \tag{3.2}$$

3.1. A discrete mass approximation.

In this subsection, we obtain a CSBP as a scaling limit of continuous time Galton–Watson branching processes. In other words, the aim of this subsection is to set up a "discrete mass – continuous time" approximation of the process X^x solution of (2.3) . To this end, we set

$$h_N(s) = s + \frac{\psi_{\delta_N}((1-s)N)}{N\psi'_{\delta_N}(N)}, \qquad |s| \leqslant 1. \tag{3.3}$$

It is easy to see that $s \to h_N(s)$ is an analytic function in $(-1,1)$ satisfying $h_N(1)=1$ and

$$\frac{d^n}{ds^n}h_N(0) \geq 0, \quad n \geq 0.$$

Therefore h_N is a probability generating function. and we have

$$h_N(s) = \sum_{\ell \geq 0} \nu_N(\ell)s^\ell, \quad |s| \leqslant 1,$$

where ν_N is probability measure on $\mathbb{Z}_+$. The approximation of (2.3) will be given by the total mass $X^{N,x}$ of a population of individuals, each of which has mass $1/N$. Given an arbitrary $x>0$, the initial mass is $X_0^{N,x} = [Nx]/N$, and $X^{N,x}$ follows a Markovian jump dynamics : from its current state k/N,

$$X^{N,x} \text{ jumps to } \begin{cases} \frac{k+\ell-1}{N} \text{ at rate } \psi'_{\delta_N}(N)\nu_N(\ell)k, \text{ for all } \ell \geq 2; \\ \\ \frac{k-1}{N} \quad \text{at rate } \psi'_{\delta_N}(N)\nu_N(0)k. \end{cases}$$

In this process, each individual dies without descendant at rate

$$\frac{\psi_{\delta_N}(N)}{N} = cN + \int_{\delta_N}^{\infty} z\mu(dz) - \frac{1}{N}\int_{\delta_N}^{\infty}(1-e^{-Nz})\mu(dz);$$

it dies and leaves two descendants at rate

$$cN + \frac{1}{N}\int_{\delta_N}^{\infty}\frac{(Nz)^2}{2}e^{-Nz}\mu(dz);$$

and it dies and leaves k descendants ($k \geq 3$) at rate

$$\frac{1}{N}\int_{\delta_N}^{\infty}\frac{(Nz)^k}{k!}e^{-Nz}\mu(dz).$$

We note that $NX_t^{N,x}$ is a continuous time branching process with $m = [Nx]$ ancestors, and a rate of reproduction governed by the finite measure ν given by $\nu(0) = \psi'_{\delta_N}(N)\nu_N(0)$ and $\nu(\ell) = \psi'_{\delta_N}(N)\nu_N(\ell)$ for all $\ell \geq 2$.

Let $\mathscr{D}([0,\infty),\mathbb{R}_+)$ denote the space of functions from $[0,\infty)$ into $\mathbb{R}_+$ which are right continuous and have left limits at any $t > 0$ (as usual such a function is called càdlàg). We shall always equip the space $\mathscr{D}([0,\infty),\mathbb{R}_+)$ with the Skorohod topology. The next proposition is a consequence of Theorem 4.1 in [11].

Proposition 3.1. *Suppose that Assumption* $(\mathbf{H})$ *is satisfied. Then, as* $N \to +\infty$, $\{X_t^{N,x},\ t \geqslant 0\}$ *converges to* $\{X_t^x,\ t \geqslant 0\}$ *in distribution on* $\mathscr{D}([0,\infty),\mathbb{R}_+)$*, where* X^x *is the unique solution of the SDE* (2.3)*.*

3.2. **The approximate height process H^N.** In this subsection, we shall define $\{H^N(s),\ s \geq 0\}$, the height process associated to the population process $\{X_t^{N,x},\ t \geq 0\}$. We will use the same approximation of the height process made and detailed in [10]. We have reproduced in Figure 2 a picture from [10], which shows a typical trajectory of the approximate height process. Note that Theorem 3.3 in [10] establishes a correspondence between the law of the exploration process and the law of the associated branching process, which will be implictly exploited below. The approximating height process is constructed with the help of several mutually independent Poisson process, and a sequence of i.i.d. random variables, which after rescaling is the number of newborns for each birth event.

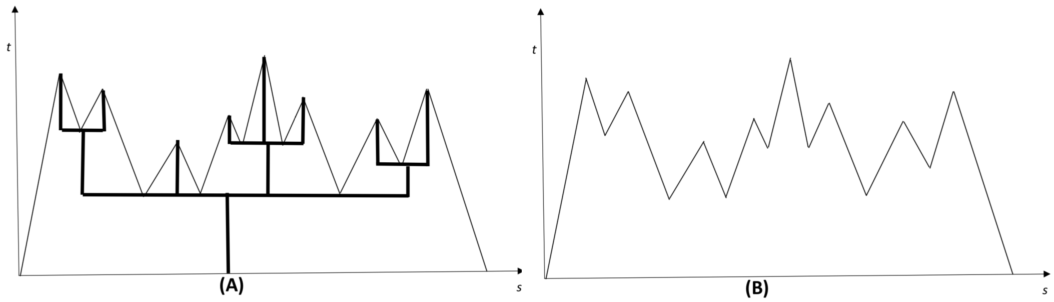

FIGURE 2. (A) The non-binary tree and its associated exploration process. (B) The exploration process. The t-axis is real time as well as exploration height, the s-axis is exploration time. The difference with the case of binary branching is that after each upward reflection, the process remembers how many additional reflections above the same level the process must experience, before being free to go below that level. In this picture, those numbers are successively 0, 2, 1 and 0. 0 additional reflections means a single reflection, i.e. a binary branching: the ancestor is replaced by two children. 2 additional reflections means $1+2=3$ reflections at the corresponding level, which means that the ancestor is replaced by four children, etc...

Before making precise the evolution of H^N, we need to define its local time $L_s^N(t)$, accumulated by H^N at level t up to time s by

$$L_s^N(t) = \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t \le H^N(r) < t+\varepsilon\}} dr. \tag{3.4}$$

$L_s^N(t)$ equals $1/N$ times the number of pairs of t-crossings of H^N between times 0 and s. In other words, $L_s^N(t)$ equals $(1/2)*(1/N)$ times the number of visits at level t. Note that this process is neither right- nor left–continuous as a function of s.

We now introduce several Poisson processes. They will be mutually independent, even if we do not repeat it. Let first $\{Q_s^N,\ s \ge 0\}$ be a Poisson process with intensity $2\int_{\delta_N}^\infty (1-e^{-Nz} - Nze^{-Nz})\mu(dz)$. This process will describe the "arrival" of multiple births. Let $\{P_s^N,\ s \ge 0\}$ and $\{P_s^{*,N},\ s \ge 0\}$, be two mutually independent Poisson processes with respective intensities $2cN^2$ and
$2\int_{\delta_N}^\infty (e^{-Nz} - 1 + Nz)\mu(dz)$. Note that the above intensities are the rates of birth and death of the population process $NX^{N,x}$, multiplied by $2N$. The slope of H^N is $\pm 2N$, which explains the factor $2N$. Let us define $P_s^{N,-} = P_s^N + P_s^{*,N}$, $\forall\, s \ge 0$. Let $\{Z_i^N,\ i \geqslant 1\}$ be a sequence of i.i.d r.v.'s taking their values in the set $\{k/N,\ k \ge 2\}$, which are independent of the Poisson processes, and whose law is precised as follows.

$$\mathbb{P}(Z_i^N = k/N) = \left(\int_{\delta_N}^\infty (1-e^{-Nz} - Nze^{-Nz})\mu(dz)\right)^{-1} \int_{\delta_N}^\infty \frac{(Nz)^k}{k!} e^{-Nz}\mu(dz), k \ge 2.$$

Let $\{V_s^N,\ s\geq 0\}$ be the càdlàg $\{-1,1\}$–valued process which is such that, $s-$almost everywhere, $dH^N(s)/ds = 2NV_s^N$. The $\mathbb{R}_+\times\{-1,1\}$–valued process $\{(H^N(s),V_s^N),\ s\geq 0\}$ solves the SDE

$$\begin{aligned}
H^N(s) =& 2N\int_0^s V_r^N dr,\\
V_s^N =& 1+2\int_0^s \mathbf{1}_{\{V_{r^-}^N=-1\}}dP_r^N + 2\int_0^s \mathbf{1}_{\{V_{r^-}^N=-1\}}dQ_r^N\\
&-2\int_0^s \mathbf{1}_{\{V_{r^-}^N=+1\}}dP_r^{N,-} + 2N\big(L_s^N(0)-L_{0^+}^N(0)\big)\\
&+2N\sum_{i>0,S_i^N\leq s}\left\{L_s^N\left(H^N(S_i^N)\right)-L_{S_i^N}^N\left(H^N(S_i^N)\right)\right\}\wedge\left(Z_i^N-\frac{1}{N}\right),
\end{aligned}\tag{3.5}$$

where the S_i^N's are the successive jump times of the process

$$\tilde{Q}_s^N = \int_0^s \mathbf{1}_{\{V_{r^-}^N=-1\}}dQ_r^N.$$

For any $i>0$, NZ_i^N-1 denotes the number of reflections of H^N above the level $H^N(S_i^N)$ before the process H^N may go below that level $H^N(S_i^N)$.

We write the first line of (3.5) as

$$cH^N(s) = 2cN\int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr - 2cN\int_0^s \mathbf{1}_{\{V_r^N=-1\}}dr.$$

Adding this to the second identity in (3.5) divided by $2N$, using the notations

$$\begin{aligned}
\mathscr{M}_s^{1,N} &= \frac{1}{N}\int_0^s \mathbf{1}_{\{V_{r^-}^N=-1\}}\big(dP_r^N - 2cN^2dr\big),\\
\mathscr{M}_s^{2,N} &= \frac{1}{N}\int_0^s \mathbf{1}_{\{V_{r^-}^N=+1\}}\big(dP_r^N - 2cN^2dr\big),\\
\mathscr{M}_s^{*,N} &= \frac{1}{N}\int_0^s \mathbf{1}_{\{V_{r^-}^N=+1\}}\big(dP_r^{*,N} - 2\int_{\delta_N}^\infty(e^{-Nz}-1+Nz)\mu(dz)dr\big),
\end{aligned}\tag{3.6}$$

and the identity $a\wedge b = b-(b-a)^+$ for $a,b>0$, we obtain

$$cH^N(s) = Y^N(s)+L_s^N(0)-\mathscr{R}^N(s),\tag{3.7}$$

where

$$Y^N(s) = \mathscr{M}_s^{1,N}-\mathscr{M}_s^{2,N}+\mathscr{Z}_s^N+\varepsilon^N(s),\tag{3.8}$$

$$\begin{aligned}
\varepsilon^N(s) =& \frac{1}{2N}-\frac{V_s^N}{2N}-\mathscr{M}_s^{*,N}-L_0^N(0)+\int_{\delta_N}^\infty z\mu(dz)\left(s-2\int_0^s\mathbf{1}_{\{V_r^N=+1\}}dr\right)\\
&+\frac{2}{N}\int_{\delta_N}^\infty(1-e^{-Nz})\mu(dz)\int_0^s\mathbf{1}_{\{V_r^N=+1\}}dr,
\end{aligned}\tag{3.9}$$

$$\mathscr{Z}_s^N = \int_0^s Z_{\tilde{Q}_r^N}^N d\tilde{Q}_r^N - s\int_{\delta_N}^\infty z\mu(dz).$$

and

$$\mathscr{R}^N(s) = \sum_{i>0,S_i^N\leq s}\left(Z_i^N-\frac{1}{N}-\left\{L_s^N\left(H^N(S_i^N)\right)-L_{S_i^N}^N\left(H^N(S_i^N)\right)\right\}\right)^+.$$

Note that $\mathscr{M}_s^{1,N}$ $\mathscr{M}_s^{2,N}$ and $\mathscr{M}_s^{*,N}$ are martingales, while $\mathscr{Z}_s^N$ is not a martingale, but the sum of a martingale and a process with bounded variations which tends to 0 as $N\to\infty$, see below (3.12) and Corallary 3.12.

We shall need the following result in the proof of Proposition 3.7, which is a semi–discrete (continuous time – discrete space) analogue of the Lévy representation of the local time of Brownian motion at 0.

Lemma 3.2. *For any* $N \geq 1, s > 0$,

$$L_s^N(0) = -\inf_{0\leq r\leq s} Y^N(r).$$

Proof. Recall (3.7). We first note that, since $L_\cdot^N(0)$ increases only when $H^N(s)=0$, $L_s^N(0) = L_{r_s}^N(0)$, where $r_s = \sup\{r \leq s; H^N(r)=0\}$. We also have $\mathscr{R}^N(r_s)=0$. Consequently

$$L_s^N(0) = L_{r_s}^N(0) = -Y^N(r_s) \leq -\inf_{0\leq r\leq s} Y^N(r).$$

To establish the converse inequality, let $0 \leq u_s \leq s$ be such that $Y^N(u_s) = \inf_{0\leq r\leq s} Y^N(r)$. Since $Y^N(u_s) + L_{u_s}^N(0) = cH^N(u_s) + \mathscr{R}^N(u_s) \geq 0$,

$$L_s^N(0) \geq L_{u_s}^N(0) \geq -Y^N(u_s) = -\inf_{0\leq r\leq s} Y^N(r)\,.$$

■

We have similarly

Lemma 3.3. *For any* $N \geq 1$, $i \geq 1$ *such that* $S_i^N \leq s$,

$$L_s^N\left(H^N(S_i^N)\right) - L_{S_i^N}^N\left(H^N(S_i^N)\right) = -\inf_{S_i^N\leq r\leq s}\left(Y^N(r) - Y^N(S_i^N)\right),$$

for $s \geq S_i^N$ *such that* $L_s^N\left(H^N(S_i^N)\right) - L_{S_i^N}^N\left(H^N(S_i^N)\right) \leq Z_i^N - 1/N$.

Proof. The argument is the same as in the previous lemma. On the considered time interval, $H^N(s)$ is reflected above the level $H^N(S_i)$, instead of being reflected above 0. ■

From the previous Lemmas, we can rewrite (3.7) in the form

$$cH^N(s) = Y^N(s) - \inf_{0\leq r\leq s} Y^N(r) - \mathscr{R}^N(s), \text{ where} \tag{3.10}$$

$$\mathscr{R}^N(s) = \sum_{i>0, S_i^N\leq s}\left(Z_i^N - \frac{1}{N} + \inf_{S_i^N\leq r\leq s}\left(Y^N(r) - Y^N(S_i^N)\right)\right)^+. \tag{3.11}$$

3.3. **Taking the limit in the SDE for H^N.** Let us first state one of the main results of this subsection.

Proposition 3.4. *As* $N \to \infty$, $H^N \Rightarrow H$ *in* $\mathscr{C}(\mathbb{R}_+)$, *where* H *is given by* (2.5), *or equivalently by* (2.6).

The main step in the proof of this Proposition is the proof of weak convergence of Y^N to Y for the topology of locally uniform convergence. For this purpose, we shall first establish an a priori estimate concerning H^N in Proposition 3.7, and then study the convergence of each term on the right hand side of (3.8). We will then need to use the same technical argument as in the proof of Proposition 3.13 in [28], and finally we conclude by using Lemma 3.16 below, which can be viewed as an extension of the second Dini theorem, and is a version of a result from [28].

Let $\Lambda^N = \sum_{i\geq 1}\delta_{(T_i^N, Z_i^N)}$, where the T_i^N's are the jump times of the Poisson process Q^N. We can couple the two point processes Λ^N and $\Pi^N = \sum_{i\geq 1}\delta_{(S_i^N, Z_i^N)}$ in such a way that

$$\Pi^N(ds, dz) = \mathbf{1}_{V_{s-}^N=-1}\Lambda^N(ds, dz).$$

It is not hard to see (exploiting e.g. Corollary VI.3.5 in Cinlar [7]) that Λ^N is a Poisson Point Process with mean measure $2ds\mu_N(dz)$, where μ_N is a measure on $(0,+\infty)$ which is supported on the set $\{k/N, k\geq 2\}$, and is specified by

$$\mu_N(\{k/N\}) = \int_{\delta_N}^{\infty} \frac{(Nz)^k}{k!} e^{-Nz}\mu(dz)\,.$$

Let us establish

Lemma 3.5. *The sequence μ_N converges to μ as $N\to\infty$, in the sense of weak convergence of measures on $(0,+\infty)$.*

Proof. It suffices to show that for any $f\in C(0,+\infty)$ with compact support, $\mu_N(f)\to\mu(f)$. But

$$\mu_N(f) = \int_{\delta_N}^{\infty}\sum_{k\geq 2} f\left(\frac{k}{N}\right)\frac{(Nz)^k}{k!}e^{-Nz}\mu(dz)\to\int_0^\infty f(z)\mu(dz),$$

as $N\to\infty$. The pointwise convergence of the integrand follows from the fact that, for fixed z, if ξ_N denotes a $\text{Poi}(Nz)$ r.v., since $f(0)=f(1/N)=0$, at least for N large enough,

$$f_N(z) := \sum_{k\geq 2} f\left(\frac{k}{N}\right)\frac{(Nz)^k}{k!}e^{-Nz} = \mathbb{E} f\left(\frac{\xi_N}{N}\right)\to f(z),$$

as $N\to\infty$ from the law of large numbers. Suppose that $\text{supp}(f)\subset[a,+\infty)$. Lebesgue's dominated convergence theorem implies that

$$\int_{a/2}^{\infty} f_N(z)\mu(dz)\to\int_{a/2}^{\infty} f(z)\mu(dz).$$

It remains to show that $\int_0^{a/2} f_N(z)\mu(dz)\to 0=\int_0^{a/2} f(z)\mu(dz)$. But for $z\leq a/2$,

$$f_N(z) = \sum_{k\geq aN} f\left(\frac{k}{N}\right)\frac{(Nz)^k}{k!}e^{-Nz},$$

$$|f_N(z)|\leq \|f\|_\infty \mathbb{P}(\xi_N > aN)\leq \frac{4\|f\|_\infty}{a^2}\frac{z}{N},$$

where we have used the fact that $\text{Var}(\xi_N/N)=z/N$. The result follows, since $N^{-1}\int_{\delta_N}^\infty z\mu(dz)\to 0$, again from assumption (3.1). ∎

Remark 3.6. From the definition of μ_N, we have

$$\int_0^\infty \mu_N(dz) = \int_{\delta_N}^\infty \mu(dz), \quad \text{and} \quad \int_0^\infty z\mu_N(dz) = \int_{\delta_N}^\infty z\mu(dz).$$

Hence, we can rewrite $\mathscr{Z}^N$, which appeared in (3.8), in the following form

$$\mathscr{Z}_s^N = \int_0^s Z^N_{\tilde{Q}_r^N}\, d\tilde{Q}_r^N - s\int_{\delta_N}^\infty z\mu(dz)$$

$$= \mathscr{M}_s^N - \int_{\delta_N}^\infty z\mu(dz)\left(s-2\int_0^s \mathbf{1}_{\{V_r^N=-1\}}dr\right), \text{ where} \tag{3.12}$$

$$\mathscr{M}_s^N = \int_0^s\int_0^\infty z\overline{\Pi}^N(dr,dz)\,. \tag{3.13}$$

We need an apriori estimate on the sequence of processes H^N.

Proposition 3.7. *For any $s>0$,*

$$\sup_{N\geq 1}\mathbb{E}\left(\sup_{0\leq r\leq s} H^N(r)\right)<\infty\,.$$

Proof. We first recall that by construction, $H^N(s) \geq 0$, for all $s \geq 0$, a.s. We note that

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} dr + \int_0^s \mathbf{1}_{\{V_r^N=-1\}} dr = s,$$

and from the identity $V_r^N = \mathbf{1}_{\{V_r^N=1\}} - \mathbf{1}_{\{V_r^N=-1\}}$ and the first line of (3.5), we deduce that

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} dr - \int_0^s \mathbf{1}_{\{V_r^N=-1\}} dr = (2N)^{-1} H^N(s).$$

It follows from those two identities that

$$(2N)^{-1} H^N(s) = 2\int_0^s \mathbf{1}_{\{V_r^N=1\}} dr - s \geq 0. \tag{3.14}$$

Moreover from Lemma 3.2 that $L_s^N(0) = -\inf_{0\leq r\leq s} Y_r^N$. From (3.7), (3.8), (3.9) and the last two identities, we deduce that

$$\begin{aligned} cH^N(s) \leq & \sup_{0\leq r\leq s} \left\{ \mathscr{M}_r^{1,N} + |\mathscr{M}_r^{2,N}| + \mathscr{Z}_r^N + |\mathscr{M}_r^{*,N}| \right\} + N^{-1} \\ & + \frac{2s}{N}\int_{\delta_N}^\infty \mu(dz) + (2N)^{-1}\int_{\delta_N}^\infty z\mu(dz) \sup_{0\leq r\leq s} H^N(r). \end{aligned} \tag{3.15}$$

Since from assumption (3.1), $(2N)^{-1}\int_{\delta_N}^\infty z\mu(dz) \to 0$ as $N\to\infty$, the proposition's assertion follows from the next four facts. Concerning $\mathscr{M}^{1,N}$, we deduce from the first line of (3.6) and Doob's L^2 inequalities for martingales that

$$\mathbb{E}\left(\sup_{0\leq r\leq s} |\mathscr{M}_r^{1,N}|\right) \leq 2\left(\mathbb{E}\left\{|\mathscr{M}_s^{1,N}|^2\right\}\right)^{1/2} \leq 2\sqrt{2cs}.$$

The same holds concerning $\mathscr{M}^{2,N}$. We next note that the expectation of the sup in s of the absolute value of the sum of the last two terms of the right hand side of (3.15) is bounded by

$$\frac{s}{N}\int_{\delta_N}^\infty \mu(dz) + (2N)^{-1}\int_{\delta_N}^\infty z\mu(dz)\, \mathbb{E}\left(\sup_{0\leq r\leq s} H^N(r)\right),$$

which we can plug in (3.15) after we have replaced on the left $H^N(s)$ by $\sup_{0\leq r\leq s} H^N(r)$ and taken the expectation. We now consider $\mathscr{Z}_s^N$. An argument similar to the one leading to (3.14) yields from (3.12)

$$\mathscr{Z}_s^N = \mathscr{M}_s^N - (2N)^{-1}\int_{\delta_N}^\infty z\mu(dz) H_s^N \leq \mathscr{M}_s^N.$$

Now we deduce from the Burkholder–David–Gundy inequality for possibly discontinuous martingales (see e.g. Theorem IV.48 in [31]) that there exists a constant $C>0$ such that

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leq r\leq s} |\mathscr{M}_r^N|\right) & \leq C\mathbb{E}\left\{\left(\int_0^s\int_0^\infty z^2 \Pi^N(dr,dz)\right)^{1/2}\right\} \\ & \leq C\mathbb{E}\left\{\left(\int_0^s\int_0^1 z^2 \Pi^N(dr,dz)\right)^{1/2}\right\} + C\mathbb{E}\int_0^s\int_1^\infty z\Pi^N(dr,dz) \\ & \leq C\left(s\int_0^1 z^2\mu_N(dz)\right)^{1/2} + Cs\int_1^\infty z\mu_N(dz), \end{aligned}$$

whose $\limsup_{N\to\infty}$ is finite. It will be shown below in Lemma 3.9 that $\mathbb{E}\sup_{0\leq r\leq s}|\mathscr{M}_r^{*,N}| \to 0$, as $N\to\infty$. This concludes the proof. ■

The first two identities in the previous proof, combined with the just obtained result, clearly yield the following essential result.

Lemma 3.8. *As $N \longrightarrow \infty$,*

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}}dr \longrightarrow \frac{s}{2}; \; \int_0^s \mathbf{1}_{\{V_r^N=-1\}}dr \longrightarrow \frac{s}{2}$$

a.s., locally uniformly in s.

We need to establish the following result (recall (3.6)).

Lemma 3.9. *As $N \longrightarrow \infty$,*

$$\left(\mathscr{M}_s^{*,N},\, s \geq 0\right) \longrightarrow 0 \;\; in\ probability,\ locally\ uniformly\ in\ s.$$

Proof. Since $\mathscr{M}_s^{*,N}$ is a purely discontinuous local martingale, we deduce from (3.6) that

$$\left[\mathscr{M}^{*,N}\right]_s = \frac{1}{N^2}\int_0^s \mathbf{1}_{\{V_r^N=+1\}}dP_r^{*,N} \text{ and } \langle \mathscr{M}^{*,N}\rangle_s = \frac{2}{N^2}\int_{\delta_N}^\infty (e^{-Nz}-1+Nz)\mu(dz)\int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr.$$

From the Cauchy–Schwartz and Doob's L^2-inequality for martingales, we deduce that

$$\mathbb{E}\left(\sup_{0\leq r\leq s}|\mathscr{M}_r^{*,N}|\right) \leq \sqrt{\frac{2s}{N}\int_{\delta_N}^\infty z\mu(dz)},$$

which tends to 0 as $N \to \infty$ from assumption (3.1). ■

Recalling (3.9), we have the

Proposition 3.10. *As $N \longrightarrow \infty$, $\varepsilon^N(s) \longrightarrow 0$ in probability, locally uniformly in s.*

Proof. By using the definition of L^N in (3.4), it is easily checked that $L_0^N(0) = 1/2N$. As $N \to \infty$, $N^{-1}V_s^N \longrightarrow 0$ a.s uniformly with respect to s. However, from (3.14), we have that

$$\left|\frac{s}{2} - \int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr\right| = \frac{1}{4N}H^N(s).$$

Combining this with assumption (3.1) and Proposition 3.7, we deduce that

$$\left(\frac{s}{2} - \int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr\right)\int_{\delta_N}^\infty z\mu(dz) \longrightarrow 0 \;\text{ in probability, locally uniformly in s.}$$

The result follows by combining these arguments with (3.9) and Lemma 3.9. ■

We now deduce from Lemma 3.5 and Lemma 3.8

Proposition 3.11. *As $N \to \infty$, $\Pi^N \Rightarrow \Pi$, in the sense of weak convergence in distribution of random probability measures, where Π is a Poisson Point Process with mean measure $ds\mu(dz)$.*

Proof. In view of Lemma 3.5, all we need to show is that for any $z > 0$ such that $\mu(\{z\}) = 0$, $\Pi^N(\cdot,(z,+\infty)) \Rightarrow \Pi(\cdot,(z,+\infty))$. We first note that Λ^N converges to a PPP Λ, whose mean measure is twice that of Π. Next, since Π^N is dominated by Λ^N, it is tight (see the criterion in Lemma 16.15 of [18]), hence it converges along a subsequence to some limiting measure $\widetilde{\Pi}$, which must be a simple point measure, by comparison with Λ. We shall not distinguish the subsequence from the original one, by an abuse of notation. Let $\mathscr{H}^N$ (resp. $\mathscr{H}$) denote the filtration generated by the process H^N (resp. by the process H). We have that

$$\Pi^N((0,s],(z,\infty)) - 2\mu_N(z,\infty)\int_0^s \mathbf{1}_{\{V_r^N=-1\}}dr \;\text{ is an } (\mathscr{H}_s^N) \text{ martingale.}$$

This implies that for any $n \geq 1$, $0 < s_1 < \cdots < s_n = s < s'$, any bounded $\Phi \in C(\mathbb{R}^n;\mathbb{R})$

$$\mathbb{E}\left[\Phi(H^N_{s_1},\ldots,H^N_{s_n})\Pi^N((s,s'],(z,+\infty))\right] = 2\mu_N(z,\infty)\mathbb{E}\left[\Phi(H^N_{s_1},\ldots,H^N_{s_n})\int_s^{s'}\mathbf{1}_{\{V^N_r=-1\}}dr\right].$$

Taking the limit as $N \to \infty$ in this last identity yields that

$$\mathbb{E}\left[\Phi(H_{s_1},\ldots,H_{s_n})\widetilde{\Pi}((s,s'],(z,+\infty))\right] = \mu(z,\infty)\mathbb{E}\left[\Phi(H_{s_1},\ldots,H_{s_n})(s'-s)\right].$$

This being true for all $n \geq 1$, all $0 < s_1 < \cdots < s_n = s < s'$ and all bounded $\Phi \in C(\mathbb{R}^n;\mathbb{R})$, we have that the simple point process $\widetilde{\Pi}((0,s],(z,\infty))$ is such that $\widetilde{\Pi}((0,s],(z,\infty)) - s\mu(z,\infty)$ is a martingale. This shows that it is a Poisson process with intensity $\mu(z,\infty)$. Hence $\widetilde{\Pi}$ is a PPP with mean measure $ds\mu(dz)$, so it has the same law as Π. ■

Recall (3.12). We now deduce from the above.

Corollary 3.12. *As $N \longrightarrow \infty$, $\mathscr{Z}^N \Rightarrow \mathscr{M}$ in $\mathscr{D}([0,\infty))$, where*

$$\mathscr{M}_s = \int_0^s \int_0^\infty z\overline{\Pi}(dr,dz).$$

Proof. We already know that the second term on the right-hand side of (3.12) equals

$$-(2N)^{-1}\int_{\delta_N}^\infty z\mu(dz)H^N_s,$$

which tends to 0 as $N \to \infty$, locally uniformly in s. We now split $\mathscr{M}^N_s$ into two terms. For any $\delta > 0$ such that $\mu(\delta) = 0$, one can deduce from Proposition 3.11 that as $N \to \infty$,

$$\int_0^\cdot \int_\delta^\infty z\overline{\Pi}^N(dr,dz) \Rightarrow \int_0^\cdot \int_\delta^\infty z\overline{\Pi}(dr,dz) \text{ in } \mathscr{D}([0,\infty)).$$

On the other hand,

$$\mathbb{E}\left[\sup_{0\leq s\leq \bar{s}}\left(\int_0^s\int_0^\delta z\overline{\Pi}^N(dr,dz)\right)^2\right] \leq 8\,\mathbb{E}\int_0^{\bar{s}}\mathbf{1}_{V^N_r=-1}dr\int_0^\delta z^2\mu_N(dz)$$
$$\to 4\bar{s}\int_0^\delta z^2\mu(dz),$$

while

$$\mathbb{E}\left[\sup_{0\leq s\leq \bar{s}}\left(\int_0^s\int_0^\delta z\overline{\Pi}(dr,dz)\right)^2\right] \leq 4\bar{s}\int_0^\delta z^2\mu(dz).$$

Since $\int_0^\delta z^2\mu(dz)$ can be made arbitrarily small by choosing $\delta > 0$ small enough, the result follows from the above statements by standard arguments. ■

The following result, with the identification of the constants, is Proposition 5.3 in [23], see also Proposition 4.23 in [10].

Lemma 3.13. *As $N \longrightarrow \infty$,*

$$\left(\mathscr{M}^{1,N}_s, \mathscr{M}^{2,N}_s,\ s \geq 0\right) \Longrightarrow \left(\sqrt{c}B^1_s, \sqrt{c}B^2_s,\ s \geq 0\right) \text{ in } (\mathscr{D}([0,\infty)))^2,$$

where B^1_s and B^2_s are two mutually independent standard Brownian motions.

Let us define

$$B^N(s) = \mathscr{M}^{1,N}_s - \mathscr{M}^{2,N}_s. \tag{3.16}$$

We deduce readily

Corollary 3.14. *As $N \longrightarrow \infty$,*

$$\left(B^N(s),\ s\geq 0\right) \Longrightarrow \left(\sqrt{2c}B(s),\ s\geq 0\right) \text{ in } (\mathscr{D}([0,\infty))),$$

where B is a standard Brownian motion.

Let us rewrite (3.9) in the form

$$Y^N(s) = \varepsilon^N(s) + B^N(s) + \mathscr{L}_s^N, \tag{3.17}$$

We have proved so far that $B^N \Rightarrow \sqrt{2c}B$ and $\varepsilon^N + \mathscr{L}^N \Rightarrow \mathscr{M}$, as $N\to\infty$. Then taking the weak limit in (3.17), we have in fact the

Corollary 3.15. *As $N \longrightarrow \infty$, $Y^N \Longrightarrow Y$ for the topology of locally uniform convergence, where*

$$Y(s) = \sqrt{2c}B(s) + \int_0^s\int_0^\infty z\overline{\Pi}(dr,dz).$$

Proof. As we have taken separately the limit in B^N and in $\mathscr{L}^N$, it is clear that, at least along a subsequence, the pair converges, and the limits are independent, since one is Brownian motion, and the other is a Poisson integral, both being martingales w.r.t. the same filtration. Note that since the jump times do not move as N increases, in fact $Y^N \Rightarrow Y$ for the topology of locally uniform convergence. ∎

We can now turn to the

Proof of Proposition 3.4 : Thanks to Corollary 3.15 and to a famous theorem of Skorohod, we can assume that $Y^N(s)\to Y(s)$ a.s., locally uniformly in s. From this we will deduce that $H^N(s)\to H(s)$ in probability, locally uniformly in s. We will first show that from any subsequence, we can extract a further subsequence which converges a.s., locally uniformly in s. We will follow closely the proofs of Proposition 3.13 and Corollary 3.15 in [28]. First of all, the same argument as that of Proposition 3.13 in [28] yields that $H^N(s)\to H(s)$ in probability, for any $s\geq 0$. We fix $\bar{s}>0$ arbitrary, and let $\mathbf{D}_{\bar{s}}$ be a countable dense subset of $[0,\bar{s}]$. Along a subsequence, still denoted as H^N by an abuse of notation, $H^N(s)\to H(s)$, for any $s\in\mathbf{D}_{\bar{s}}$. We first note that, as in [28], we can rewrite (2.5) as follows. Let, for any $0\leq r<s$, $\overline{Y}_r^s := \inf_{r\leq u\leq s} Y(u)$. We have

$$cH(s) = Y(s) - \overline{Y}_0^s - \sum_{0\leq r\leq s}\Delta\overline{Y}_r^s,$$

where $\Delta\overline{Y}_r^s$ denotes the jump at r of the increasing function $r\mapsto\overline{Y}_r^s$.

Since the process Y is càdlàg, with only positive jumps,

$$\Phi_Y(h) := \sup_{0\leq r<s\leq r+h\leq\bar{s}}(Y(s)-Y(r))_-$$

is a.s. a continuous function of h on $[0,1]$, such that $\Phi_Y(0)=0$. Now

$$\begin{aligned}c(H(s+h)-H(s)) &= Y(s+h)-Y(s)-\overline{Y}_0^{s+h}+\overline{Y}_0^s-\sum_{0\leq r\leq s}(\overline{Y}_r^{s+h}-\overline{Y}_r^s)-\sum_{s<r\leq s+h}\overline{Y}_r^{s+h}\\ &\geq Y(s+h)-Y(s)-\sum_{s<r\leq s+h}\overline{Y}_r^s\end{aligned}$$

But since $Y(s+h)-\sum_{s<r\leq s+h}\overline{Y}_r^{s+h}\geq\inf_{s<r\leq s+h}Y(r)$, we have

$$c(H(s+h)-H(s)) \geq \inf_{s<r\leq s+h}Y(r)-Y(s), \text{ hence}$$

$$(H(s+h)-H(s))_- \leq \Phi_Y(h)\,.$$

It follows from (2.6) that the same formula relates H^N with Y^N, and we also have

$$(H^N(s+h)-H^N(s))_- \le \Phi_{Y^N}(h).$$

However, it is not true that $\Phi_{Y^N}(h)\to 0$, as $h\to 0$, since Y^N has negative jumps of size $-1/N$. So $\limsup_{h\to 0}\Phi_{Y^N}(h)\le 1/N$. Moreover, since $Y^N(s)\to Y(s)$ uniformy on $[0,\bar{s}]$, we have that $\limsup_{h\to 0}\sup_{N\ge N_0}\Phi_{Y^N}(h)\le 1/N_0$, for all $N_0\ge 1$. Now the fact that $H^N(s)\to H(s)$ uniformly on $\mathbf{D}_{\bar{s}}$ (hence also on $[0,\bar{s}]$) follows from the next Lemma. The result follows. ■

It remains to establish

Lemma 3.16. *Consider a sequence $\{g_N,\ N\ge 1\}$ of càlàg functions from $\mathbb{R}_+$ into $\mathbb{R}$, g a continuous function from $\mathbb{R}_+$ into $\mathbb{R}$, and $\bar{s}>0$, which are such that $g_N(s)\to g(s)$, for all $s\in\mathbf{D}_{\bar{s}}$. Assume moreover that for all $N_0\ge 1$*

$$\limsup_{h\to 0}\sup_{N\ge N_0}(g_N(s+h)-g_N(s))_- \le 1/N_0\,.$$

Then, as $N\to\infty$,

$$\sup_{s\in\mathbf{D}_{\bar{s}}}|g_N(s)-g(s)|\to 0.$$

The proof of this Lemma, which can be viewed as an extension of the second Dini theorem, is essentially the same as that of Lemma 3.16 in [28], even if its statement is slightly different, so we do not reproduce it.

We now have

Proposition 3.17. *As $N\longrightarrow\infty$, $(H^N,Y^N,\mathscr{R}^N)\Rightarrow(H,Y,\mathscr{R})$ in $\mathscr{C}([0,+\infty))\times(\mathscr{D}([0,+\infty))^2$.*

Proof. We can rewrite (3.10), in the form

$$\mathscr{R}^N(s)=Y^N(s)-\inf_{0\le r\le s}Y^N(r)-cH^N(s).$$

It follows from Proposition 3.4 that the sequence $\{H^N,\ N\ge 1\}$ is tight in $\mathscr{C}(\mathbb{R}_+)$. However, from Corollary 3.15, we deduce that the sequences $\{Y^N(s),\ s\ge 0\}_{N\ge 1}$ and $\{\inf_{0\le r\le s}Y^N(r),\ s\ge 0\}_{N\ge 1}$ are tight in $\mathscr{D}([0,+\infty))$. Moreover, the limit of the sequence $\{\inf_{0\le r\le s}Y^N(r),\ s\ge 0\}_{N\ge 1}$ is a.s. continuous. Hence the tightness of the sequence $\{\mathscr{R}^N,\ N\ge 1\}$ follows from Proposition 5.4 in the Appendix. Now since $(H^N,Y^N,\mathscr{R}^N)$ is tight, along an appropriate subsequence (which we do not distinguish notationally from the original sequence),

$$(H^N,Y^N,\mathscr{R}^N)\Rightarrow(H,Y,\mathscr{R}).$$

Moreover, from (2.6) and the fact that the law of Y is uniquely specified, we deduce that the limit is unique, which implies that the whole sequence converges. ■

We shall need the following result in the proof of Proposition 3.22 below.

Lemma 3.18. *For any $\delta>0$, as $N\to\infty$,*

$$\int_0^{\cdot}\int_0^{\delta}\left(z-\frac{1}{N}+\inf_{r\le u\le\cdot}Y^N(u)-Y^N(r)\right)^+\Pi^N(dr,dz)\Rightarrow\int_0^{\cdot}\int_0^{\delta}\left(z+\inf_{r\leqslant u\leqslant\cdot}Y(u)-Y(r)\right)^+\Pi(dr,dz)$$

in $\mathscr{D}([0,\infty))$.

Proof. Let us decompose $cH^N(s)$ and $cH(s)$ in the form

$$cH^N(s)=\mathscr{B}^N_s+\mathscr{P}^N_s-\mathscr{R}^N_s,\quad cH(s)=\mathscr{B}_s+\mathscr{P}_s-\mathscr{R}_s,$$

where $\mathscr{R}_s^N$ (resp. $\mathscr{R}_s$) is defined by the second line of (3.10) (resp. of (2.6)),

$$\mathscr{P}_s^N = \int_0^s \int_0^\infty z\overline{\Pi}^N(dr,dz), \quad \mathscr{P}_s = \int_0^s \int_0^\infty z\overline{\Pi}(dr,dz),$$

and $\mathscr{B}_s^N$ (resp. $\mathscr{B}_s$) is the remainder of $cH^N(s)$ (resp. of $cH(s)$). First we not that $\mathscr{B}_\cdot^N \Rightarrow \mathscr{B}_\cdot$ for the topology of uniform convergence (the limit is continuous). Next we introduce the decompositions

$$\begin{aligned}
\mathscr{P}_{\delta,-}^N(s) &= \int_0^s \int_0^\delta z\overline{\Pi}^N(dr,dz), \quad \mathscr{P}_{\delta,+}^N(s) = \int_0^s \int_\delta^\infty z\overline{\Pi}^N(dr,dz),\\
\mathscr{P}_{\delta,-}(s) &= \int_0^s \int_0^\delta z\overline{\Pi}(dr,dz), \quad \mathscr{P}_{\delta,+}(s) = \int_0^s \int_\delta^\infty z\overline{\Pi}(dr,dz),\\
\mathscr{R}_{\delta,-}^N(s) &= \int_0^s \int_0^\delta \left(z - \frac{1}{N} + \inf_{r\le u\le s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz),\\
\mathscr{R}_{\delta,+}^N(s) &= \int_0^s \int_\delta^\infty \left(z - \frac{1}{N} + \inf_{r\le u\le s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz),\\
\mathscr{R}_{\delta,-}(s) &= \int_0^s \int_0^\delta \left(z + \inf_{r\le u\le s} Y(u) - Y(r)\right) \Pi(dr,dz),\\
\mathscr{R}_{\delta,+}(s) &= \int_0^s \int_\delta^\infty \left(z + \inf_{r\le u\le s} Y(u) - Y(r)\right) \Pi(dr,dz).
\end{aligned}$$

Let $\mathscr{C}^N(s) := \mathscr{P}_{\delta,+}^N(s) - \mathscr{R}_{\delta,+}^N(s)$, $\mathscr{C}(s) = \mathscr{P}_{\delta,+}(s) - \mathscr{R}_{\delta,+}(s)$. We will show in Lemma 3.19 that $\mathscr{C}^N(s)$ is tight. Moreover its limit $\mathscr{C}(s)$ is continuous since $\mathscr{P}_{\delta,+}(s)$ and $\mathscr{R}_{\delta,+}(s)$ have the same jumps and there are finitely many of those on each compact interval. Hence the convergence is (locally) uniform in s. Finally

$$\mathscr{R}_{\delta,-}^N(s) = -cH^N(s) + \mathscr{B}_s^N + \mathscr{C}^N(s) + \mathscr{P}_{\delta,-}^N(s).$$

The sum of the first three terms on the right is tight and converges locally uniformly in s towards its continuous limit $-cH(s) + \mathscr{B}_s + \mathscr{C}(s)$, while the last term can be shown (so to speak "by hand") to be tight in $\mathscr{D}(\mathbb{R}_+)$. Hence the right hand side is tight in $\mathscr{D}(\mathbb{R}_+)$, and so is the left hand side. Taking the weak limit in the last identity, we obtain that the limit of $\mathscr{R}_{\delta,-}^N(s)$ is $\mathscr{R}_{\delta,-}(s)$, which is our Lemma. ∎

We want to check the tightness of the sequence $\{\mathscr{C}^N, N \ge 1\}$ using the Aldous criterion (see section 16, page 176 in [5]). Let τ be a stopping time with value in $[0,s]$ and let $\varepsilon > 0$ be a real number which will eventually go to zero.

Lemma 3.19. *The sequence $\{\mathscr{C}^N, N \ge 1\}$ is tight in $\mathscr{D}(\mathbb{R}_+)$.*

Proof. Recall the notations used in the previous proof. We have

$$\begin{aligned}
&\mathscr{C}^N(s) = \mathscr{C}_1^N(s) - \mathscr{C}_2^N(s), \quad \text{where}\\
&\mathscr{C}_1^N(s) = \int_0^s \int_\delta^\infty \Upsilon_N(s,r,z)\Pi^N(dr,dz), \quad \mathscr{C}_2^N(s) = 2\int_0^s \mathbf{1}_{V_r^N=-1} dr \int_\delta^\infty z\,\mu_N(dz), \quad \text{with}\\
&\Upsilon_N(s,r,z) = \frac{1}{N} + \left(Y^N(r) - \inf_{r\le u\le s} Y^N(u)\right) \wedge \left(z - \frac{1}{N}\right).
\end{aligned}$$

The tightness in $\mathscr{C}(\mathbb{R}_+)$ of the sequence $\mathscr{C}_2^N$ is rather clear. We have the following a priori estimates

$$0 \le \Upsilon_N(s,r,z) \le z, \quad \text{and if } r \le s < s',$$
$$0 \le \Upsilon_N(s',r,z) - \Upsilon_N(s,r,z) \le Y^N(s) - \inf_{s \le u \le s'} Y^N(u).$$

We now verify Aldous's criterion. The first condition (see (16.22) in [5]) follows easily from the inequality

$$\mathscr{C}_1^N(s) \le \int_0^s \int_\delta^\infty z \Pi^N(dr,dz).$$

We next want to establish the second condition (see (16.23) in [5]), which will follow from the fact that for all $\eta > 0$,

$$\lim_{\varepsilon \to 0} \limsup_{N \to +\infty} \mathbb{P}\left(\left|\mathscr{C}_1^N(\tau+\varepsilon) - \mathscr{C}_1^N(\tau)\right| > \eta\right) = 0.$$

In order to verify this condition, we first note that

$$\begin{aligned} 0 \le \mathscr{C}_1^N(\tau+\varepsilon) - \mathscr{C}_1^N(\tau) &= \int_\tau^{\tau+\varepsilon} \int_\delta^\infty \Upsilon_N(\tau+\varepsilon,r,z)\Pi^N(dr,dz) \\ &+ \int_0^\tau \int_\delta^\infty \left[\Upsilon_N(\tau+\varepsilon,r,z) - \Upsilon_N(\tau,r,z)\right] \Pi^N(dr,dz) \\ &\le \int_\tau^{\tau+\varepsilon} \int_\delta^\infty z\Pi^N(dr,dz) + \left(Y^N(\tau) - \inf_{\tau \le u \le \tau+\varepsilon} Y^N(u)\right) \Pi^N([0,\tau] \times [\delta,+\infty)). \end{aligned}$$

Now using the Portmanteau theorem, Corollary 3.15 and Markov's inequality, we deduce that

$$\begin{aligned} \limsup_{N \to +\infty} \mathbb{P}\left(\left|\mathscr{C}_1^N(\tau+\varepsilon) - \mathscr{C}_1^N(\tau)\right| > \eta\right) &\le \frac{4}{\eta}\varepsilon \int_\delta^\infty z\mu(dz) \\ &+ \mathbb{P}\left(\left(Y(\tau) - \inf_{\tau \le u \le \tau+\varepsilon} Y(u)\right) \Pi([0,\tau] \times [\delta,+\infty)) > \frac{\eta}{2}\right). \end{aligned}$$

However, using the strong Markov property of Y, we obtain

$$\mathbb{E}\left\{\left(Y(\tau) - \inf_{\tau \le u \le \tau+\varepsilon} Y(u)\right) \Pi([0,\tau] \times [\delta,+\infty))\right\} = \mathbb{E}\left(-\inf_{0 \le u \le \varepsilon} Y(u)\right) \mathbb{E}\left\{\Pi([0,\tau] \times [\delta,+\infty))\right\}.$$

Combining this with the previous inequality and Markov's inequality, it follows that

$$\begin{aligned} \limsup_{N \to +\infty} \mathbb{P}\left(\left|\mathscr{C}_1^N(\tau+\varepsilon) - \mathscr{C}_1^N(\tau)\right| > \eta\right) \le &\frac{4}{\eta}\varepsilon \int_\delta^\infty z\mu(dz) \\ &+ \frac{2}{\eta}\mathbb{E}\left(-\inf_{0 \le u \le \varepsilon} Y(u)\right) \mathbb{E}\left\{\Pi([0,\tau] \times [\delta,+\infty))\right\} \\ &\xrightarrow[\varepsilon \to 0]{} 0. \end{aligned}$$

thanks to the monotone convergence theorem. ■

3.4. **Convergence of the local time of the approximate height process.** The aim of this subsection is to pass to the limit as $N \to \infty$ in the process $\{L_s^N(H^N(s)), s \ge 0\}$. This is done in Proposition 3.29. For that sake, we shall first establish Theorem 3.26, which gives the weak convergence of L^N towards L, for a topology of functions of s and t. The proof of Theorem 3.26 will rely on Proposition 3.22 which provides the tightness of the sequence $\left\{L_s^N(t), t \ge 0\right\}_{N \ge 1}$, for each $s \ge 0$ fixed, and on Lemma 3.27 which establishes that the mapping $s \mapsto L_s(t)$ is continuous, uniformly in t.

We will need the following identity in the proof of Proposition 3.22. Writing V_r^N as

$$\mathbf{1}_{\{V_{r^-}^N = +1\}} - \mathbf{1}_{\{V_{r^-}^N = -1\}}$$

and using (3.6), we deduce from (3.5):

$$
\begin{aligned}
V_s^N =&1+2N\left(\mathscr{M}_s^{1,N}-\mathscr{M}_s^{2,N}-\mathscr{M}_s^{*,N}+\mathscr{Z}_s^N\right)-4cN^2\int_0^s V_r^N dr\\
&+4\int_{\delta_N}^{\infty}(1-e^{-Nz})\mu(dz)\int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr\\
&+2N\int_{\delta_N}^{\infty} z\mu(dz)\left(s-2\int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr\right)+2N\left(L_s^N(0)-L_{0+}^N(0)\right)\\
&-2N\sum_{i>0,S_i^N\le s}\left(Z_i^N-\frac{1}{N}-\left\{L_s^N\left(H^N(S_i^N)\right)-L_{S_i^N}^N\left(H^N(S_i^N)\right)\right\}\right)^+.
\end{aligned}
\tag{3.18}
$$

We shall need the two next lemmas in the proof of Proposition 3.22.

Lemma 3.20. *For any $s\ge 0$, as $N\longrightarrow+\infty$,*

$$\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\mathbf{1}_{\{V_r^N=-1\}}dr \Rightarrow \frac{1}{2}\int_0^s \mathbf{1}_{\{H(r)>t\}}dr.$$

Proof. Imitating the proof of Lemma 3.8, we have

$$
\begin{aligned}
2\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\mathbf{1}_{\{V_r^N=-1\}}dr-\int_0^s \mathbf{1}_{\{H^N(r)>t\}}dr &= \frac{1}{2N}\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\frac{dH^N(r)}{dr}dr\\
&=\frac{1}{2N}(H^N(s)-t)^+,
\end{aligned}
$$

which clearly tends to 0 in probability, as $N\to\infty$. It thus remains to take the limit in the sequence $\{\int_0^s \mathbf{1}_{\{H^N(r)>t\}}dr\}_{N\ge1}$. For any $\delta>0$, we set

$$\Gamma_\delta=\{r\in(0,s);|H(r)-t|\le\delta\}.$$

It follows from the properties of the process H that $\text{Leb}(\Gamma_\delta)\to 0$ a.s. as $\delta\to 0$, where $\text{Leb}(A)$ denotes the Lebesgue measure of the set A. We have

$$
\begin{aligned}
\int_0^s \mathbf{1}_{\{H^N(r)>t\}}dr &= \int_{\Gamma_\delta}\mathbf{1}_{\{H^N(r)>t\}}dr+\int_{(0,s)\backslash\Gamma_\delta}\mathbf{1}_{\{H^N(r)>t\}}dr,\\
\int_0^s \mathbf{1}_{\{H(r)>t\}}dr &= \int_{\Gamma_\delta}\mathbf{1}_{\{H(r)>t\}}dr+\int_{(0,s)\backslash\Gamma_\delta}\mathbf{1}_{\{H(r)>t\}}dr.
\end{aligned}
$$

The two first terms on the right hand sides are dominated by $\text{Leb}(\Gamma_\delta)$. But for $r\in(0,s)\backslash\Gamma_\delta$, $\mathbf{1}_{\{H(r)>t\}}=g_\delta(H(r)-t)$, where g_δ is the continuous function from $\mathbb{R}$ into $[0,1]$ given as

$$g_\delta(x)=1\wedge[\delta^{-1}(x+\delta/2)^+].$$

We clearly have

$$
\begin{aligned}
\int_{(0,s)\backslash\Gamma_\delta}\mathbf{1}_{\{H(r)>t\}}dr &= \int_{(0,s)\backslash\Gamma_\delta} g_\delta(H(r)-t)dr, \quad\text{and}\\
\int_{(0,s)\backslash\Gamma_\delta} g_\delta(H^N(r)-t)dr &\Rightarrow \int_{(0,s)\backslash\Gamma_\delta} g_\delta(H(r)-t)dr.
\end{aligned}
$$

Note that $\mathbf{1}_{\{H^N(r)>t\}}$ and $g_\delta(H^N(r)-t)$ differ only when $|H^N(r)-t|\le\delta/2$. Let h_δ be a continuous function from $\mathbb{R}$ into $[0,1]$ which equals 1 on $[-\delta/2,\delta/2]$, and 0 outside $[-2\delta/3,2\delta/3]$. Let now k_δ be a function from $\mathbb{R}$ into $[0,1]$ which equals 0 on $[-2\delta/3,2\delta/3]$ and 1 outside $[-\delta,\delta]$. We have

$$\left|\int_{(0,s)\backslash\Gamma_\delta}\mathbf{1}_{\{H^N(r)>t\}}dr-\int_{(0,s)\backslash\Gamma_\delta} g_\delta(H^N(r)-t)dr\right|\le\int_0^s h_\delta(H^N(r)-t)k_\delta(H(r)-t)dr,$$

which tends to 0 in probability. ■

The proof of the next Lemma is essentially the same as that of Lemma 5.4 in [23], and is omitted.

Lemma 3.21. *For any $s>0$, $t>0$, the following identities hold a.s.*

$$(H^N(s)-t)^+ = 2N\int_0^s V_r^N \mathbf{1}_{\{H^N(r)>t\}}dr$$
$$V_s^N \mathbf{1}_{\{H^N(s)>t\}} = 2NL_s^N(t) + \int_0^s \mathbf{1}_{\{H^N(r)>t\}}dV_r^N.$$

The next proposition constitutes a first step towards the proof of Theorem 3.26.

Proposition 3.22. *For each $s\geq 0$ fixed, $\left\{L_s^N(t), t\geq 0\right\}_{N\geq 1}$ is tight in $\mathscr{D}([0,+\infty))$.*

Proof. Writing the second line of Lemma 3.21 as

$$L_s^N(t) = \frac{1}{2N}V_s^N \mathbf{1}_{\{H^N(s)>t\}} - \frac{1}{2N}\int_0^s \mathbf{1}_{\{H^N(r)>t\}}dV_r^N,$$

and using (3.18), (3.16), (3.12), Lemma 3.3 and the first line of Lemma 3.21, we deduce that for any $t\geq 0$, a. s.

$$L_s^N(t) = A_s^N(t) - \int_0^s \mathbf{1}_{\{H^N(r)>t\}}dB^N(r) + \int_0^s \mathbf{1}_{\{H^N(r)>t\}}d\mathscr{M}_r^{*,N} + D_s^N(t), \tag{3.19}$$

where

(3.20)

$$A_s^N(t) = c(H^N(s)-t)^+ + \frac{V_s^N}{2N}\mathbf{1}_{\{H^N(s)>t\}} - \frac{2}{N}\int_{\delta_N}^\infty (1-e^{-Nz})\mu(dz)\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\mathbf{1}_{\{V_r^N=+1\}}dr,$$

and

$$\begin{aligned} D_s^N(t) = &\sum_{i>0, S_i^N\leq s} \mathbf{1}_{\{H^N(S_i^N)>t\}}\left(Z_i^N - \frac{1}{N} + \inf_{S_i^N\leq r\leq s}\left(Y^N(r)-Y^N(S_i^N)\right)\right)^+ \\ &- \int_0^s \mathbf{1}_{\{H^N(r)>t\}}\int_0^\infty z\overline{\Pi}^N(dr,dz). \end{aligned} \tag{3.21}$$

The proof is organized as follows. Step 1 establishes that the sequence $\{A_s^N(t), t\geq 0\}_{N\geq 1}$ is tight, and any limit of a converging subsequence is a. s. continuous. Step 2 shows that as $N\to+\infty$,

$$\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)>t\}}d\mathscr{M}_r^{*,N}\right| \longrightarrow 0. \tag{3.22}$$

Step 3 establishes that the sequence $\left\{\int_0^s \mathbf{1}_{\{H^N(r)>t\}}dB^N(r), t\geq 0\right\}_{N\geq 1}$ is tight, and any limit of a converging subsequence is a.s. continuous. Finally step 4 shows that the sequence $\{D_s^N(t), t\geq 0\}_{N\geq 1}$ is tight as random elements of $\mathscr{D}([0,+\infty))$. The desired result follows by combining the above arguments with Proposition 5.4 below.

STEP 1. The tightness of two first terms of the right–hand side of (3.20) is established in the same way as in the proof of Proposition 5.7 in [23]. The sup over all $t>0$ of the absolute value of the last term is easily shown to go to 0, as $N\to\infty$, again thanks to (A).

STEP 2. We first note that

$$\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)>t\}}d\mathscr{M}_r^{*,N}\right| \leq \left|\mathscr{M}_s^{*,N}\right| + \sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)\leq t\}}d\mathscr{M}_r^{*,N}\right|.$$

Thanks Lemma 3.9, it remains to prove

$$\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)>t\}}d\mathscr{M}_r^{*,N}\right| \longrightarrow 0 \text{ as } N\to+\infty.$$

To this end, we fix $s>0$, and consider the process

$$G^N(t)=\int_0^s \mathbf{1}_{\{H^N(r)\le t\}} d\mathscr{M}_r^{*,N}.$$

Let $\mathscr{G}_t^N$ denote the σ-algebra generated by the random variables

$$\Theta_{g_N}=\int_0^s g_N(r) d\mathscr{M}_r^{*,N},$$

where g_N is bounded and $\mathscr{P}$ measurable ($\mathscr{P}$ stands for the σ-algebra of predictable subsets of $\Omega\times\mathbb{R}_+$) and satisfies $\{g_N(r)=0\}\supset\{H^N(r)>t\}$. We first establish the fact that $\{G^N(t),t\ge 0\}$ is a $\mathscr{G}_t^N$-martingale. To this end, it suffices to verify that $\mathbb{E}[(G^N(t')-G^N(t))\Theta_{g_N}]=0$ for $t<t'$ and any g_N as above. Indeed, it is a product of two stochastic integrals with respect to $\mathscr{M}^{*,N}$, with two integrands whose product is zero. Let $T>0$. In order to finally establish (3.22), we note that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T} G^N(t)\right) &\le \left(\mathbb{E}\sup_{0\le t\le T}\left(G^N(t)\right)^2\right)^{\frac{1}{2}}\\
&\le \left(4\mathbb{E}\left(\int_0^s \mathbf{1}_{\{H^N(r)\le T\}} d\mathscr{M}_r^{*,N}\right)^2\right)^{\frac{1}{2}}\\
&\le \left(\frac{8}{N}\mathbb{E}\int_0^s \mathbf{1}_{\{H^N(r)\le T\}} dr\int_{\delta_N}^\infty z\mu(dz)\right)^{\frac{1}{2}}\\
&\le 2\sqrt{2s}\left(\frac{1}{N}\int_{\delta_N}^\infty z\mu(dz)\right)^{\frac{1}{2}},
\end{aligned}$$

whose tends to 0 as $N\to\infty$, thanks to (A), where we have used Cauchy Schwarz's and Doob's inequalities. The desired result follows.

STEP 3. The tightness of $\left\{\int_0^s \mathbf{1}_{\{H^N(r)>t\}} dB^N(r), t\ge 0\right\}_{N\ge 1}$ is established in the same way as in the proof of Proposition 5.7 in [23] which we do not reproduce here. On the other hand, we will adapt the idea of this proof to treat the tightness of the sequence $\{D_s^N(t),t\ge 0\}_{N\ge 1}$.

STEP 4. Let $\delta>0$ be a real number which will eventually go to zero. Using the identity $(b-a)^+-b=-(a\wedge b)$ for $a,b>0$, we can rewrite (3.21) in the following

$$D_s^N(t)=X_s^N(t)+F_s^N(t),$$

where

$$\begin{aligned}
X_s^N(t)=&\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\int_0^\delta\left(z-\frac{1}{N}+\inf_{r\le u\le s} Y^N(u)-Y^N(r)\right)^+\Pi^N(dr,dz)\\
&-\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\int_0^\delta z\overline{\Pi}^N(dr,dz)\\
&-\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\int_\delta^\infty z\wedge\left(\frac{1}{N}-\left[\inf_{r\le u\le s} Y^N(u)-Y^N(r)\right]\right)\Pi^N(dr,dz), \quad \text{and}\\
F_s^N(t)=&2\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\mathbf{1}_{\{V_r^N=-1\}} dr\int_\delta^\infty z\mu_N(dz).
\end{aligned}$$

From Lemmas 3.5 and 3.20, it is easy to see that

$$F_s^N(t)\Longrightarrow\int_0^s \mathbf{1}_{\{H(r)>t\}} dr\int_\delta^\infty z\mu(dz) \text{ as } N\to+\infty.$$

Moreover, we have

$$\int_0^s \mathbf{1}_{\{H(r)>t\}} dr=\int_t^\infty L_s(u)du,$$

which is a.s. continuous. So according to Proposition 5.4 below, it remains only to show the sequence $\{X_s^N(t), t\geq 0\}_{N\geq 1}$ is tight as random elements of $\mathscr{D}([0,+\infty))$. To this end, we set

$$X_s^N(t) = -X_s^{N,\delta,1}(t) + X_s^{N,\delta,2}(t),$$

where

$$\begin{aligned} X_s^{N,\delta,1}(t) &= \int_0^s \mathbf{1}_{\{H^N(r)>t\}} \int_\delta^{1/\delta} z\wedge\left(\frac{1}{N} - \left[\inf_{r\leq u\leq s} Y^N(u) - Y^N(r)\right]\right)\Pi^N(dr,dz),\\ X_s^{N,\delta,2}(t) &= C_s^{N,\delta}(t) - \int_0^s \mathbf{1}_{\{H^N(r)>t\}} \int_0^\delta z\overline{\Pi}^N(dr,dz) \quad \text{and}\\ C_s^{N,\delta}(t) &= \int_0^s \mathbf{1}_{\{H^N(r)>t\}} \int_0^\delta \left(z - \frac{1}{N} + \inf_{r\leq u\leq s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz)\\ &\quad + \int_0^s \mathbf{1}_{\{H^N(r)>t\}} \int_{1/\delta}^\infty z\wedge\left(\frac{1}{N} - \left[\inf_{r\leq u\leq s} Y^N(u) - Y^N(r)\right]\right)\Pi^N(dr,dz). \end{aligned} \tag{3.23}$$

To show the tightness of the sequence $\{X_s^N(t), t\geq 0\}_{N\geq 1}$, we first show that for all $\eta > 0$,

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{t\geq 0} |X_s^{N,\delta,2}(t)| > \eta\right) \xrightarrow[\delta\to 0]{} 0. \tag{3.24}$$

Afterwards we will prove that the sequence $\{X_s^{N,\delta,1}(t), t\geq 0\}_{N\geq 1}$ is tight as random elements of $\mathscr{D}([0,+\infty))$. Finally the desired result follows by combining this with Lemma 5.6 below. In order to prove (3.24), we first note that

$$\begin{aligned} &\mathbb{E}\left(\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\int_0^\delta z\overline{\Pi}^N(dr,dz)\right|\right)\\ &\leq \mathbb{E}\left|\int_0^s\int_0^\delta z\overline{\Pi}^N(dr,dz)\right| + \mathbb{E}\left(\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)\leq t\}}\int_0^\delta z\overline{\Pi}^N(dr,dz)\right|\right)\\ &\leq \left(\mathbb{E}\left(\int_0^s\int_0^\delta z\overline{\Pi}^N(dr,dz)\right)^2\right)^{\frac12} + \left(\mathbb{E}\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)\leq t\}}\int_0^\delta z\overline{\Pi}^N(dr,dz)\right|^2\right)^{\frac12}\\ &\leq \left(2\mathbb{E}\int_0^s \mathbf{1}_{V_r^N=-1}dr\int_\delta^\infty z^2\mu_N(dz)\right)^{\frac12} + \left(8\sup_{t\geq 0}\mathbb{E}\int_0^s \mathbf{1}_{\{H^N(r)\leq t\}}\mathbf{1}_{V_r^N=-1}dr\int_0^\delta z^2\mu_N(dz)\right)^{\frac12}\\ &\leq \left(2s\int_0^\delta z^2\mu_N(dz)\right)^{\frac12} + \left(8s\int_0^\delta z^2\mu_N(dz)\right)^{\frac12} = 3\sqrt{2s}\sqrt{\int_0^\delta z^2\mu_N(dz)}, \end{aligned}$$

where we have used Cauchy Schwarz's and Doob's inequalities in the second and the third inequalities. From Markov's inequality, we deduce that

$$\limsup_{N\to+\infty}\mathbb{P}\left(\sup_{t\geq 0}\left|\int_0^s \mathbf{1}_{\{H^N(r)>t\}}\int_0^\delta z\overline{\Pi}^N(dr,dz)\right|\geq\frac{\eta}{2}\right)\leq\frac{6}{\eta}\sqrt{2s}\sqrt{\int_0^\delta z^2\mu(dz)}\xrightarrow[\delta\to 0]{} 0, \tag{3.25}$$

since as $N\to\infty$, $\int_0^\delta z^2\mu_N(dz)\to\int_0^\delta z^2\mu(dz)$, which follows from assumption (3.1) and the following formula, which is easily established by the same computation as done in the proof of Lemma 3.5,

$$\int_0^\delta z^2\mu_N(dz) = \frac{1}{N}\int_{\delta_N}^\delta z\mu(dz) + \int_{\delta_N}^\delta z^2\mu(dz).$$

However, recalling (3.23), we have

$$\sup_{t\geq 0}|C_s^{N,\delta}(t)| \leq \int_0^s\int_0^\delta\left(z - \frac{1}{N} + \inf_{r\leq u\leq s} Y^N(u) - Y^N(r)\right)^+\Pi^N(dr,dz) + \int_0^s\int_{1/\delta}^\infty z\Pi^N(dr,dz).$$

Now using the Portmanteau theorem, Lemma 3.18 and Markov's inequality, we deduce that

$$\limsup_{N\to+\infty}\mathbb{P}\left(\sup_{t\geq 0}|C_s^{N,\delta}(t)|>\frac{\eta}{2}\right)\leq \mathbb{P}\left(\int_0^s\int_0^\delta\left(z+\inf_{r\leq u\leq s}Y(u)-Y(r)\right)^+\Pi(dr,dz)>\frac{\eta}{4}\right)$$
$$+\frac{8}{\eta}s\int_{1/\delta}^\infty z\mu(dz).$$

We deduce from Corollary 3.5 in [28] and Markov's inequality that

$$\limsup_{N\to+\infty}\mathbb{P}\left(\sup_{t\geq 0}|C_s^{N,\delta}(t)|>\frac{\eta}{2}\right)\leq\frac{4}{\eta}C(s)\int_0^\delta z^2\mu(dz)+\frac{8}{\eta}s\int_{1/\delta}^\infty z\mu(dz).$$

with $C(s)=(\alpha\sqrt{s/c})\vee s$ and $\alpha=e/(e-1)$, which implies that

$$\limsup_{N\to+\infty}\mathbb{P}\left(\sup_{t\geq 0}|C_s^{N,\delta}(t)|>\frac{\eta}{2}\right)\leq\frac{4}{\eta}C(s)\int_0^\delta z^2\mu(dz)+\frac{8}{\eta}s\int_{1/\delta}^\infty z\mu(dz)\xrightarrow[\delta\to 0]{}0.$$

Consequently, we obtain (3.24) by combining this with (3.25) and (3.23). It remains to prove that the sequence $\{X_s^{N,\delta,1}(t),t\geq 0\}_{N\geq 1}$ is tight as random elements of $\mathscr{D}([0,+\infty))$. To this end, we show that the sequence $\{X_s^{N,\delta,1}(t),t\geq 0\}_{N\geq 1}$ satisfies the conditions of Proposition 5.3 below. The first condition follows easily from the fact that

$$\limsup_{N\to+\infty}\mathbb{E}\left(X_s^{N,\delta,1}(t)\right)\leq 2s\int_\delta^{1/\delta}z\,\mu(dz).$$

In order to verify the second condition, we will show that for any $T>0$, there exists $C>0,\theta>1$ such that for any $0<t<T,h>0$,

$$\mathbb{E}\left(\left|X_s^{N,\delta,1}(t+h)-X_s^{N,\delta,1}(t)\right|\left|X_s^{N,\delta,1}(t)-X_s^{N,\delta,1}(t-h)\right|\right)\leq Ch^\theta.$$

In order to simplify the notations below we let

$$a_s^{+,N}(t):=\mathbf{1}_{\{t<H^N(s)\leq t+h\}},\quad\text{ and }\quad a_s^{-,N}(t):=\mathbf{1}_{\{t-h<H^N(s)\leq t\}}.$$

An essential property, which will be crucial below, is that $a_s^{+,N}(t)\times a_s^{-,N}(t)=0$. Also $(a_s^{+,N}(t))^2=a_s^{+,N}(t)$, and similarly for $a^{-,N}$. Thus, we have

$$\begin{aligned}0\leq X_t^{N,\delta,1}-X_{t+h}^{N,\delta,1}&\leq\int_0^s a_r^{+,N}(t)\int_\delta^\infty z\Pi^N(dr,dz)\\&=\int_0^s a_r^{+,N}(t)\int_\delta^{1/\delta}z\overline{\Pi}^N(dr,dz)+2\int_0^s a_r^{+,N}(t)\mathbf{1}_{V_r^N=-1}dr\int_\delta^{1/\delta}z\,\mu_N(dz),\end{aligned}$$

and

$$\begin{aligned}0\leq X_{t-h}^{N,\delta,1}-X_t^{N,\delta,1}&\leq\int_0^s a_r^{-,N}(t)\int_\delta^{1/\delta}z\Pi^N(dr,dz)\\&=\int_0^s a_r^{-,N}(t)\int_\delta^{1/\delta}z\overline{\Pi}^N(dr,dz)+2\int_0^s a_r^{-,N}(t)\mathbf{1}_{V_r^N=-1}dr\int_\delta^{1/\delta}z\,\mu_N(dz).\end{aligned}$$

Because $a_s^{+,N}(t)\times a_s^{-,N}(t)=0$, the expectation of the product of

$$\int_0^s a_r^{-,N}(t)\int_\delta^{1/\delta}z\overline{\Pi}^N(dr,dz)\quad\text{with}\quad\int_0^s a_r^{-,N}(t)\int_\delta^{1/\delta}z\overline{\Pi}^N(dr,dz)$$

vanishes. We only need to estimate the expectations

$$\begin{aligned}&\mathbb{E}\left(\int_0^s a_r^{+,N}(t)\int_\delta^{1/\delta}z\overline{\Pi}^N(dr,dz)\times\int_0^s a_r^{-,N}(t)\mathbf{1}_{V_r^N=-1}dr\right),\\&\mathbb{E}\left(\int_0^s a_r^{-,N}(t)\int_\delta^{1/\delta}z\overline{\Pi}^N(dr,dz)\times\int_0^s a_r^{+,N}(t)\mathbf{1}_{V_r^N=-1}dr\right),\end{aligned}\tag{3.26}$$

and

$$\mathbb{E}\left(\int_0^s a_r^{+,N}(t)\mathbf{1}_{V_r^N=-1}dr \times \int_0^s a_r^{-,N}(t)\mathbf{1}_{V_r^N=-1}dr\right). \tag{3.27}$$

Since the two first equations are symmetrical, we will only estimate (3.26). To this end, we use the Cauchy–Schwarz inequality and Lemma 3.25 below,

$$\mathbb{E}\left(\int_0^s a_r^{+,N}(t)\int_\delta^{1/\delta} z\overline{\Pi}^N(dr,dz) \times \int_0^s a_r^{-,N}(t)\mathbf{1}_{V_r^N=-1}dr\right)$$
$$\leq \left(2\mathbb{E}\int_0^s a_r^{+,N}(t)dr\int_\delta^{1/\delta} z^2\mu_N(dz)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\left(\int_0^s a_r^{-,N}(t)dr\right)^2\right)\right)^{\frac{1}{2}} \leq Ch^{3/2}.$$

Finally, concerning (3.27) : Again from Lemma 3.25 with $t' = t+h$,

$$\mathbb{E}\left(\int_0^s a_r^{+,N}(t)dr \times \int_0^s a_r^{-,N}(t)dr\right) \leq Ch^2.$$

We now conclude that the sequence $\{X_s^{N,\delta,1}(t), t\geq 0\}_{N\geq 1}$ is tight as random elements of $\mathscr{D}([0,+\infty))$. The desired result follows. ∎

Recall (3.17). For $K>0$, let τ_K^N be the time of the first jump of Y^N of size greater than or equal to K.

Lemma 3.23. *Let $s,K>0$. Then there exists a constant C such that for all $N\geq 1$*

$$\mathbb{E}\left(\sup_{0\leq r<s\wedge\tau_K^N}\left|Y^N(r)\right|^2\right) \leq C.$$

Proof. By combining (3.9), (3.12) and (3.17), it is easy to obtain that

$$Y^N(s) = -\frac{V_s^N}{2N} + B^N(s) - \mathscr{M}_s^{*,N} + \int_0^s\int_{\delta_N}^\infty z\overline{\Pi}^N(dr,dz). \tag{3.28}$$

It follows that

$$\sup_{0\leq r<s\wedge\tau_K^N}\left|Y^N(r)\right|^2 \leq \frac{4}{2N^2} + 4\sup_{0\leq r<s}\left|\mathscr{M}_r^{*,N}\right|^2 + 4\sup_{0\leq r<s}\left|B^N(r)\right|^2 + 4\left|\int_0^{s\wedge\tau_K^N}\int_0^\infty z\overline{\Pi}^N(dr,dz)\right|^2.$$

From an easy adaptation of the argument used in STEP 1 in the proof of Proposition 3.22, the expectation of the last term on the right hand–side tends to 0 as $N\to+\infty$. We now use Doob's L^2 inequality for martingales, which yields that there exists constant C_2 such that for any martingale M,

$$\mathbb{E}\left(\sup_{0\leq r\leq s}|M_r|^2\right) \leq C_2\mathbb{E}\left(|M_s|^2\right).$$

Recall (3.6) and (3.16). Hence, it suffices to notice that

$$\mathbb{E}\left(\left|B^N(s)\right|^2\right) \leq 2cs, \quad \mathbb{E}\left(\left|\mathscr{M}_s^{*,N}\right|^2\right) \leq \frac{2s}{N}\int_{\delta_N}^\infty z\mu(dz),$$
$$\left|\int_0^{(s\wedge\tau_K^N)-}\int_0^\infty z\overline{\Pi}^N(dr,dz)\right| \leq \sup_{r\leq s}\left|\int_0^r\int_0^K z\overline{\Pi}^N(du,dz)\right|,$$
$$\text{and}\quad \mathbb{E}\left(\left|\int_0^s\int_0^K z\overline{\Pi}^N(du,dz)\right|^2\right) \leq 2s\int_0^K z^2\mu_N(dz).$$

The desired result follows by combining the above results. ∎

Lemma 3.24. *Let $s, K > 0$. Then there exists a constant C such that for all $N \geq 1$*

$$\sup_{t\geq 0} \mathbb{E}[(L^N_{s\wedge\tau^N_K}(t))^2] \leq C,$$

where τ^N_K is defined above.

Proof. From Lemmas 3.2 and 3.3, we can rewrite (3.7) and (3.19) in the following form

$$cH^N(s) = Y^N(s) - \inf_{0\leq r\leq s} Y^N(r) - \int_0^s\int_0^\infty \left(z - \frac{1}{N} + \inf_{r\leqslant u\leqslant s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz), \tag{3.29}$$

$$\begin{aligned} L^N_s(t) =& \Gamma^N(s,t) + c(H^N(s)-t)^+ \\ &+ \int_0^s\int_0^\infty \mathbf{1}_{\{H^N(r)>t\}} \left(z - \frac{1}{N} + \inf_{r\leqslant u\leqslant s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz), \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} \Gamma^N(s,t) =& \frac{V^N_s}{2N}\mathbf{1}_{\{H^N(s)>t\}} - \int_0^s \mathbf{1}_{\{H^N(r)>t\}} dB^N(r) + \int_0^s \mathbf{1}_{\{H^N(r)>t\}} d\mathscr{M}^{*,N}_r \\ &- \int_0^s \mathbf{1}_{\{H^N(r)>t\}} \int_0^\infty z\overline{\Pi}^N(dr,dz). \end{aligned}$$

From an adaption of the argument of proof of Lemma 3.23, we have that there exists a constant C such that for all $N \geq 1$

$$\sup_{t\geq 0} \mathbb{E}\left(\sup_{0\leq r\leq s\wedge\tau^N_K} \left|\Gamma^N(r,t)\right|^2\right) \leq C. \tag{3.31}$$

We now estimate the last term on the right of (3.30). It is clear that

$$\begin{aligned} &\int_0^s\int_0^\infty \mathbf{1}_{\{H^N(r)>t\}} \left(z - \frac{1}{N} + \inf_{r\leqslant u\leqslant s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz) \\ &\leq \int_0^s\int_0^\infty \left(z - \frac{1}{N} + \inf_{r\leqslant u\leqslant s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz) \\ &\leq Y^N(s) - \inf_{0\leq r\leq s} Y^N(r) \leq 2\sup_{0\leq r\leq s} |Y^N(r)|. \end{aligned}$$

Next we observe that

$$c(H^N(s)-t)^+ \leq cH^N(s) \leq Y^N(s) - \inf_{0\leq r\leq s} Y^N(r) \leq 2\sup_{0\leq r<s} |Y^N(r)|.$$

From the last two inequalities,

$$\begin{aligned} &\int_0^{s\wedge\tau^N_K}\int_{\delta_N}^\infty \mathbf{1}_{\{H^N(r)>t\}} \left(z - \frac{1}{N} + \inf_{r\leqslant u\leqslant s} Y^N(u) - Y^N(r)\right)^+ \Pi^N(dr,dz) \\ &+ c(H^N(s\wedge\tau^N_K)-t)^+ \leq 4\sup_{0\leq r<s\wedge\tau^N_K} |Y^N(r)|. \end{aligned}$$

The desired result follows by combining this with (3.31), (3.30) and Lemma 3.23. ■

Lemma 3.25. *Let $s, h, T > 0$. Then there exists a constant C such that for all $N \geq 1$ and $0 < t, t' < T$,*

$$\mathbb{E}\left(\int_0^s a^{-,N}_r(t)dr\right) \leq Ch,$$

$$\mathbb{E}\left(\int_0^s a^{-,N}_r(t)dr \int_0^s a^{-,N}_r(t')dr\right) \leq Ch^2.$$

Proof. We will prove the second inequality, the first one follows from the second one with $t=t'$ and the Cauchy-Schwarz inequality. We have

$$\begin{aligned}\mathbb{E}\left(\int_0^s a_r^{-,N}(t)dr\int_0^s a_r^{-,N}(t')dr\right)=&\int_{t-h}^{t}\int_{t'-h}^{t'}\mathbb{E}[L_s^N(r)L_s^N(u)]drdu\\ \le& h^2\sup_{0\le r\le T}\mathbb{E}[(L_s^N(r))^2],\end{aligned}$$

where we can replace s by $s\wedge\tau_K^N$. Hence the desired result follows by combining this with Lemma 3.24. ∎

We are now ready to state the main result of this subsection. Recall (3.4) and Proposition 2.1.

Theorem 3.26. *For any $s>0$, as $N\to\infty$,*

$$\{L_s^N(t),\, t>0\}\Longrightarrow\{L_s(t),\, t>0\}\text{ in }\mathscr{D}([0,+\infty)),\text{ locally uniformly in }s,$$

where $L_s(t)$ is for any $s>0$, $t\ge 0$ the local time accumaled by H, solution of (2.7).

We shall need

Lemma 3.27. *For any $T>0$, the mapping $s\mapsto L_s(t)$ is continuous, uniformly for $t\in[0,T]$.*

Proof. We need to show that for any decreasing sequence $s_n\downarrow s$, $L_{s_n}(t)-L_s(t)\downarrow 0$ uniformly for $t\in[0,T]$, and that for any increasing sequence $s_n\uparrow s$, $L_s(t)-L_{s_n}(t)\downarrow 0$ uniformly for $t\in[0,T]$. Both statements can be proved by exactly the same argument, so we establish the first statement.

For each $n\ge 1$, $t\mapsto L_{s_n}(t)-L_s(t)$ is cadlag, with only positive jumps. Consequently it is upper semi–continuous. Moreover, since $s\mapsto L_s(t)$ is continuous and increasing for any $t\in[0,T]$, $L_{s_n}(t)-L_s(t)\downarrow 0$ for all $t\in[0,T]$. For any $\varepsilon>0$, let

$$V_n(\varepsilon):=\{t\in[0,T];\, L_{s_n}(t)-L_s(t)<\varepsilon\}.$$

Since $t\mapsto L_{s_n}(t)-L_s(t)$ is u.s.c., $V_n(\varepsilon)$ is an open subset of $[0,T]$. However, $\cup_{n\ge1}V_n(\varepsilon)=[0,T]$, hence there exists $N_\varepsilon\ge 1$ such that $\cup_{n\le N_\varepsilon}V_n(\varepsilon)=[0,T]$, and since $n\mapsto V_n(\varepsilon)$ is increasing, $V_{N_\varepsilon}(\varepsilon)=[0,T]$, and for any $n\ge N_\varepsilon$, $t\in[0,T]$, $L_{s_n}(t)-L_s(t)<\varepsilon$, which establishes the result. ∎

We are now prepared to complete the

Proof of Theorem 3.26 : For $k\ge 1$, $0\le i\le[2^k\bar{s}]$, we let $s_i^k:=i2^{-k}$. Thanks to Proposition 3.22, for each pair (i,k), $\{L^N_{s_i^k}(\cdot),\, N\ge 1\}$ is tight in $\mathscr{D}([0,T])$. Hence along a appropriate subsequence, jointly for all $k\ge 1$,

$$\left(L^N_{s_0^k}(\cdot),L^N_{s_1^k}(\cdot),\dots,L^N_{s^k_{[2^k\bar{s}]}}(\cdot))\right)\Rightarrow\left(L_{s_0^k}(\cdot),L_{s_1^k}(\cdot),\dots,L_{s^k_{[2^k\bar{s}]}}(\cdot))\right)$$

in $\mathscr{D}([0,T])^{[2^k\bar{s}]+1}$. From a theorem due to Skorohod, we can and do assume that those convergences hold a.s. This means that for any (i,k),

$$\sup_{0\le t\le T}|L^N_{s_i^k}(\lambda_N(t))-L_{s_i^k}(t)|\to 0,$$

as $N\to\infty$, where for each $N\ge 1$, $\lambda_N:[0,T]\mapsto[0,T]$ is continuous increasing, satisfies $\lambda_N(0)=0$, $\lambda_N(T)=T$, and $\sup_{0\le t\le T}|\lambda_N(t)-t|\to 0$ and $N\to\infty$. The time change λ_N is precised in Lemma 3.28 below. It displaces the jumps of $L_s^N(t)$ to those of $L_s(t)$. The t's where those jumps happen do not depend upon s, this is why we can choose λ_N independent of (i,k).

Now choose $s \in [0,\bar{s}]$ arbitrary. For any $k \geq 1$ arbitrarily large, there exists $0 \leq i \leq 2^k - 1$ such that $s_i^k \leq s \leq s_{i+1}^k$. We have

$$L^N_{s_i^k}(\lambda_N(t)) - L_{s_{i+1}^k}(t) \leq L^N_s(\lambda_N(t)) - L_s(t) \leq L^N_{s_{i+1}^k}(\lambda_N(t)) - L_{s_i^k}(t)$$

$$\begin{aligned} L^N_{s_i^k}(\lambda_N(t)) - L_{s_i^k}(t) + L_{s_i^k}(t) - L_{s_{i+1}^k}(t) &\leq L^N_s(\lambda_N(t)) - L_s(t) \\ &\leq L^N_{s_{i+1}^k}(\lambda_N(t)) - L_{s_{i+1}^k}(t) + L_{s_{i+1}^k}(t) - L_{s_i^k}(t). \end{aligned}$$

We now choose an arbitrary $\varepsilon > 0$. Thanks to Lemma 3.27, we can choose k large enough so that $L_{s_{i+1}^k}(t) - L_{s_i^k}(t) \leq \varepsilon/2$, for all $t \in [0,T]$. Hence we have

$$L^N_{s_i^k}(\lambda_N(t)) - L_{s_i^k}(t) - \frac{\varepsilon}{2} \leq L^N_s(\lambda_N(t)) - L_s(t) \leq L^N_{s_{i+1}^k}(\lambda_N(t)) - L_{s_{i+1}^k}(t) + \frac{\varepsilon}{2}$$

We can now choose N large enough so that $\sup_{0\leq i\leq 2^k} \sup_{0\leq t\leq T} \left| L^N_{s_i^k}(\lambda_N(t)) - L_{s_i^k}(t) \right| \leq \varepsilon/2$. We then deduce that for such a N, for all $0 \leq s \leq \bar{s}$, $0 \leq t \leq T$, $-\varepsilon \leq L^N_s(\lambda_N(t)) - L_s(t) \leq \varepsilon$, hence

$$\sup_{0\leq s\leq \bar{s}} \sup_{0\leq t\leq T} \left| L^N_s(\lambda_N(t)) - L_s(t) \right| \leq \varepsilon.$$

The result follows. ■

Lemma 3.28. *Fix T and $\bar{s} > 0$, $k \geq 1$ and the sequence $s_i^k = i2^{-k}$ for $0 \leq i \leq [2^k\bar{s}]$. There exists a random time change $\lambda_N(t)$ which is continuous and strictly increasing, such that, along an appropriate subsequence,*

$$(\lambda_N(t), L^N_{s_0^k}(\lambda_N(t)), \ldots, L^N_{s^k_{[2^k\bar{s}]}}(\lambda_N(t))) \Rightarrow (t, L_{s_0^k}(t), \ldots, L_{s^k_{[2^k\bar{s}]}}(t))$$

for the topology of uniform convergence on $[0,T]$.

Proof. The proof will be divided in three steps. We will first define the sequence λ_N, then establish the convergence of λ_N, and finally that of $L^N_{s_i^k}(\lambda_N(t))$ for i arbitrary. The fact that the above joint convergence holds along an appropriate subsequence then follows from the previous results.

STEP 1. We order the points of the measure Π on the set $[0,\bar{s}] \times \mathbb{R}_+$ in decreasing order of their second coordinate. This produces the sequence $\{(S_1,Z_1),(S_2,Z_2),\ldots\}$, where $Z_1 > Z_2 > \ldots$. We associate to each (S_i,Z_i) $T_i = H(S_i)$. We consider those (T_i,Z_i) for which $T_i \leq T$ (and delete the others). The corresponding sequence is still denoted by an abuse of notation $\{(T_1,Z_1),(T_2,Z_2),\ldots\}$. T_i is the values of t at which the map $t \mapsto L_{\bar{s}}(t)$ has a jump of size $\leq Z_i$. Note that for $0 < s < \bar{s}$, $t \mapsto L_s(t)$ has a jump at time T_i iff $S_i < s$, and $t \mapsto L_s(t)$ has jumps only at times where $t \mapsto L_{\bar{s}}(t)$ jumps. Moreover, for each $i \geq 1$, there exists s lare enough (possibly $> \bar{s}$) such that the jump of $t \mapsto L_{s'}(t)$ at T_i is Z_i for all $s' \geq s$.

Consider now the point measure Π^N, the associated (S_i^N, Z_i^N), and (T_i^N, Z_i^N), where $T_i^N = H^N(S_i^N)$. Again those points are ordered in decreasing order of the Z_i^N's, and only those (T_i^N, Z_i^N) for which $T_i^N < T$ are taken into account. Since $\Pi^N \Rightarrow \Pi$, for each $k \geq 1$, there exists N_k such that for all $N \geq N_k$, the order of $(T_1^N, \ldots, T_k^N)$ is the same as that of $(T_1, \ldots, T_k)$.

For each $k \geq 1, N \geq 1$ we choose as $\lambda_{N,k}$ the piecewise linear function of t whose graph joins $(0,0)$, $(T_i, T_i^N)_{1\leq i\leq k}, (T,T)$, where the T_i's are listed in increasing order. If $N \geq N_k$, then $\lambda_{N,k}(t)$ is continuous, strictly increasing and verifies $\lambda_{N,k}(0) = 0$, $\lambda_{N,k}(T_i) = T_i^N$ for $1 \leq i \leq k$ and $\lambda_{N,k}(T) = T$. For each $N \geq 1$, we let $\lambda_N(t) = \lambda_{N,\hat{k}_N}(t)$, where

$$\hat{k}_N = \sup\{k, \text{ the orders of } (T_i^N)_{1\leq i\leq k} \text{ and } (T_i)_{1\leq i\leq k} \text{ coincide}\}.$$

STEP 2. Since the limit t of $\lambda_N(t)$ is deterministic, what we want to show is in fact that $\sup_{0\le t\le T}|\lambda_N(t)-t|$ tends to 0 in probability. If $\mu(\mathbb{R}_+)<\infty$, then there are finitely many jumps, and the convergence $(T_i,T_i^N)\Rightarrow(T_i,T_i)$ is uniform w.r.t. i. The result follows. If however $\mu(\mathbb{R}_+)=+\infty$, then the S_i's are dense in $[0,\bar{s}]$, and consequently the T_i's are dense in $[0,\sup_{0\le s\le\bar{s}}H(s)\wedge T]$. For k large enough, the distance between two consecutive T_i's in the sequence $T_1,\dots,T_k$ is less than ε. Then for $N\ge N_k$, $\sup_{1\le i\le k}|T_i^N-T_i|\le\varepsilon$, and the result follows.

STEP 3 We write s for s_i^k. We assume here that all processes have been redefined in such a way that $(H^N,Y^N)\to(H,Y)$ in $\mathscr{C}(\mathbb{R}_+)\times\mathscr{D}(\mathbb{R}_+)$, and $(T_i^N,Z_i^N)_{i\ge1}\to(T_i,Z_i)_{i\ge1}$, in probability. According to (3.19), $L_s^N(t)=F_s^N(t)+D_s^N(t)$, and combining the fact that $F_s^N(t)\to F_s(t)$ in probability uniformly in t with STEP 2, we deduce that $F_s^N(\Lambda_N(t))\to F_s(t)$ uniformly in t in probability. It remains to treat the term $D_s^N(\lambda_N(t))$. We now use a similar decomposition as in the proof of Lemma 3.18. ■

Proposition 3.29. *As $N\to\infty$, $L_s^N(H^N(s))\Rightarrow L_s(H(s))$ ds a.e.*

Proof. In order to simplify the following argument, making use of a famous theorem due to Skorohod, we may and do assume that $H^N\to H$ and $L^N\to L$ a.s. From Proposition 3.4 $H^N(s)\to H(s)$ a.s., locally uniformly in s, and according to Theorem 3.26, for all $\bar{s}>0$ and $T>0$,

$$\sup_{0\le s\le\bar{s},\ 0\le t\le T}|L_s^N(\lambda_N(t))-L_s(t)|\to0\quad\text{a.s., as } N\to\infty,$$

then ds a.e., $L_s^N(H^N(s))\to L_s(H(s))$ in probability. For this purpose, for any $0\le r\le\bar{s}$, we have

$$\{t,\ t\mapsto L_r(t)\text{ is discontinuous }\}\subset\{t,\ t\mapsto L_{\bar{s}}(t)\text{ is discontinuous }\}$$

and the latter set is at most countable. Since H admits a local time, it spends zero time in such a countable set. Hence a.s., dr a.e., $t\mapsto L_r(t)$ is continuous at $H(r)$. For such an r, we have

$$\begin{aligned}|L_r^N(H^N(r))-L_r(H(r))|&\le|L_r^N(\lambda_N\circ\lambda_N^{-1}(H^N(r)))-L_r(\lambda_N^{-1}(H^N(r)))|\\&\quad+|L_r(\lambda_N^{-1}(H^N(r)))-L_r(H(r))|,\end{aligned}$$

where the random function $\lambda_N:[0,T]\mapsto[0,T]$ satisfying $\lambda_N(0)=0$, $\lambda_N(T)=T$, λ_N is continuous and strictly increasing and $\sup_{0\le t\le T}|\lambda_N(t)-t|\to0$ is such that $\sup_{0\le t\le T}|L_r^N(\lambda_N(t))-L_r(t)|\to0$. Define the event $\Omega_{\bar{s},T}=\{\sup_{0\le s\le\bar{s}}H(s)\le T-1\}$,

$$\begin{aligned}\mathbb{P}\left(\Omega_{\bar{s},T}\cap\left\{|L_r^N(H^N(r))-L_r(H(r))|>\eta\right\}\right)&\le\mathbb{P}\left(\sup_{0\le t\le T}|L_r^N(\lambda_N(t)t)-L_r(t)|>\eta/2\right)\\&\quad+\mathbb{P}\left(\sup_{0\le t\le T}|\lambda_N^{-1}(H^N(r))-H(r)|>1\right)\\&\quad+\mathbb{P}\left(|L_r(\lambda_N^{-1}(H^N(r)))-L_r(H(r))|>\eta/2\right).\end{aligned}$$

As $N\to\infty$, the first term on the right tends to 0 thanks to Lemma 3.26, the second term tends to 0 since both $H^N\to H$ uniformly on $[0,\bar{s}]$ and $\lambda_N^{-1}(t)\to t$ uniformly on $[0,T]$, and finally the third term tend to 0 since $H(r)$ is a continuity point of $L_r(\cdot)$ and again $\lambda_N^{-1}(t)\to t$ uniformly on $[0,T]$. Finally, for each $\bar{s}>0$, $\mathbb{P}(\cup_{T>0}\Omega_{\bar{s},T})=1$. The result follows. ■

4. CONVERGENCE OF THE HEIGHT PROCESS WITH INTERACTION

In the nonlinear case where the linear drift $-bz$ is replaced by a nonlinear drift $f(z)$, the approximation of (2.8) will be given by the total mass $X^{N,x}$ of a population of individuals, each of which has mass $1/N$. The initial mass is $X_0^{N,x} = [Nx]/N$, and $X^{N,x}$ follows a Markovian jump dynamics : from its current state k/N,

$$X^{N,x} \text{ jumps to } \begin{cases} \frac{k+\ell-1}{N} & \text{at rate } \psi'_{\delta_N}(N)\nu_N(\ell)k \;+\; N\mathbf{1}_{\{\ell=2\}}\sum_{i=1}^k (f(\frac{i}{N})-f(\frac{i-1}{N}))^+, \text{ for all } \ell \geq 2; \\ \frac{k-1}{N} & \text{at rate } \psi'_{\delta_N}(N)\nu_N(0)k \;+\; N\sum_{i=1}^k (f(\frac{i}{N})-f(\frac{i-1}{N}))^-. \end{cases}$$

The following result is a consequence of Theorem 4.1 in [11].

Proposition 4.1. *Suppose that Assumptions* (**H**) *and* (2.9) *are satisfied. Then, as $N \to +\infty$, $\{X_t^{N,x},\ t \geqslant 0\}$ converges to $\{X_t^x,\ t \geqslant 0\}$ in distribution on $\mathscr{D}([0,\infty),\mathbb{R}_+)$, where X^x is the unique solution of the SDE* (2.8).

The process H^N is piecewise linear, continuous with derivative $\pm 2N$: at any time $s \geq 0$, the rate of appearance of minima (giving rise to births, i.e. to the creation of new branches) is equal to

$$2cN^2 + 2\int_{\delta_N}^{\infty}(1-e^{-Nz}-Nze^{-Nz})\mu(dz) + 2N^2\left[f\left(L_s^N(H^N(s))+\frac{1}{N}\right) - f\left(L_s^N(H^N(s))\right)\right]^+,$$

and the rate of of appearance of maxima (describing deaths of branches) is equal to

$$2cN^2 + 2\int_{\delta_N}^{\infty}(e^{-Nz}-1+Nz)\mu(dz) + 2N^2\left[f\left(L_s^N(H^N(s))+\frac{1}{N}\right) - f\left(L_s^N(H^N(s))\right)\right]^-.$$

We now want to use Girsanov's theorem, in order to reduce the present model to the one studied in Section 3. To this end, for $s > 0$, define

$$\mathscr{P}_s^{1,N} = \int_0^s \mathbf{1}_{\{V_{r^-}^N=-1\}} dP_r^N \text{ and } \mathscr{P}_s^{2,N} = \int_0^s \mathbf{1}_{\{V_{r^-}^N=+1\}} dP_r^N, \tag{4.1}$$

recall that P^N is a Poisson point process with intensity $2cN^2$ under the probability measure $\mathbb{P}$, so that $\mathscr{P}_s^{1,N}$, (*resp.*, $\mathscr{P}_s^{2,N}$) has the intensity

$$\lambda_s^{1,N} = 2cN^2\mathbf{1}_{\{V_{s^-}^N=-1\}}, \quad \text{resp.} \quad \lambda_s^{2,N} = 2cN^2\mathbf{1}_{\{V_{s^-}^N=+1\}}.$$

Recall (3.6). We now define the collection of σ–algebras $\mathscr{F}_s^N := \sigma\{H^N(r),\ 0 \leq r \leq s\}$ and we introduce a Girsanov–Radon–Nikodym derivative

$$U_s^N = 1 + \int_0^s U_{r^-}^N\left[(f_N')^+\left(L_{r^-}^N(H^N(r))\right)d\mathscr{M}_r^{1,N} + (f_N')^-\left(L_{r^-}^N(H^N(r))\right)d\mathscr{M}_r^{2,N}\right], \tag{4.2}$$

with $f_N'(x) = N[f(x+1/N)-f(x)]$. Under the additional assumption that f' is bounded, it is clear that U^N is a martingale, hence $\mathbb{E}[U_s^N] = 1$ for all $s \geq 0$. In this case, we define $\widetilde{\mathbb{P}}^N$ as the probability such that for each $s > 0$,

$$\frac{d\widetilde{\mathbb{P}}^N}{d\mathbb{P}}\Big|_{\mathscr{F}_s^N} = U_s^N.$$

It follows from Proposition 5.2 below with

$$\mu_r^{1,N} = 1 + \frac{1}{cN^2}\int_{\delta_N}^{\infty}(1-e^{-Nz}-Nze^{-Nz})\mu(dz) + \frac{1}{cN}(f_N')^+\left(L_{r^-}^N(H^N(r))\right) \quad \text{and}$$
$$\mu_r^{2,N} = 1 + \frac{1}{cN^2}\int_{\delta_N}^{\infty}(e^{-Nz}-1+Nz)\mu(dz) + \frac{1}{cN}(f_N')^-\left(L_{r^-}^N(H^N(r))\right)$$

that under $\widetilde{\mathbb{P}}^N$, $\mathscr{P}_s^{1,N}$ (resp., $\mathscr{P}_s^{2,N}$) has the intensity

$$\begin{aligned} &\left[2cN^2 + 2\int_{\delta_N}^{\infty}(1-e^{-Nz}-Nze^{-Nz})\mu(dz) + 2N(f_N')^+\left(L_{r^-}^N(H^N(r))\right)\right]\mathbf{1}_{\{V_{r^-}^N=-1\}}, \\ \text{resp. } &\left[2cN^2 + 2\int_{\delta_N}^{\infty}(e^{-Nz}-1+Nz)\mu(dz) + 2N(f_N')^-\left(L_{r^-}^N(H^N(r))\right)\right]\mathbf{1}_{\{V_{r^-}^N=+1\}}. \end{aligned} \tag{4.3}$$

4.1. ***The Case where $|f'|$ is bouned.*** We assume in this subsection that $|f'(x)| \leq \beta$ for all $x \geq 0$ and some $\beta > 0$. This constitutes the first step of the proof of convergence of H^N. As explained at above, in this case we can use Girsanov's theorem to bring us back to the situation studied in section 3.

Recalling equations (3.16) and (3.17), we can rewrite (3.10) in the form

$$cH^N(s) = \mathscr{M}_s^{1,N} - \mathscr{M}_s^{2,N} + \mathscr{M}_s^N + \varepsilon^N(s) - \inf_{0\leq r\leq s} Y^N(r) - \mathscr{R}^N(s).$$

Moreover, from (3.6), (4.1) and (4.2), we have

$$\begin{aligned}
\left[\mathscr{M}^{1,N}\right]_s &= \frac{1}{N^2}\mathscr{P}_s^{1,N}, \quad \left[\mathscr{M}^{2,N}\right]_s = \frac{1}{N^2}\mathscr{P}_s^{2,N} \\
\langle\mathscr{M}^{1,N}\rangle_s &= 2c\int_0^s \mathbf{1}_{\{V_r^N=-1\}}dr, \quad \langle\mathscr{M}^{2,N}\rangle_s = 2c\int_0^s \mathbf{1}_{\{V_r^N=+1\}}dr, \\
\left[U^N\right]_s &= \frac{1}{N^2}\int_0^s \left|U_{r-}^N\right|^2 \left[\left|(f_N')^+\left(L_{r-}^N(H^N(r))\right)\right|^2 d\mathscr{P}_r^{1,N}\right. \\
&\qquad \left. + \left|(f_N')^-\left(L_{r-}^N(H^N(r))\right)\right|^2 d\mathscr{P}_r^{2,N}\right], \\
\langle U^N\rangle_s &= 2c\int_0^s \left|U_r^N\right|^2 \left[\left|(f_N')^+\left(L_r^N(H^N(r))\right)\right|^2 \mathbf{1}_{\{V_r^N=-1\}}\right. \\
&\qquad \left. + \left|(f_N')^-\left(L_r^N(H^N(r))\right)\right|^2 \mathbf{1}_{\{V_r^N=+1\}}\right] dr, \\
\left[U^N,\mathscr{M}^{1,N}\right]_s &= \frac{1}{N^2}\int_0^s U_{r-}^N (f_N')^+\left(L_{r-}^N(H^N(r))\right) d\mathscr{P}_r^{1,N}, \\
\left[U^N,\mathscr{M}^{2,N}\right]_s &= \frac{1}{N^2}\int_0^s U_{r-}^N (f_N')^-\left(L_{r-}^N(H^N(r))\right) d\mathscr{P}_r^{2,N}, \\
\langle U^N,\mathscr{M}^{1,N}\rangle_s &= 2c\int_0^s U_r^N (f_N')^+\left(L_r^N(H^N(r))\right) \mathbf{1}_{\{V_r^N=-1\}}dr, \\
\langle U^N,\mathscr{M}^{2,N}\rangle_s &= 2c\int_0^s U_r^N (f_N')^-\left(L_r^N(H^N(r))\right) \mathbf{1}_{\{V_r^N=+1\}}dr,
\end{aligned}$$

while

$$\left[\mathscr{M}^{1,N},\mathscr{M}^{2,N}\right]_s = \langle\mathscr{M}^{1,N},\mathscr{M}^{2,N}\rangle_s = 0.$$

From Corollary 3.12, Lemma 3.13 and Proposition 3.17, we deduce that $\{(H^N,\mathscr{M}^{1,N},\mathscr{M}^{2,N},\mathscr{M}^N,\mathscr{R}^N),\ N\geq 1\}$ is a tight sequence in $\mathscr{C}([0,\infty))\times(\mathscr{D}([0,\infty)))^4$. Since f' is bounded, the same is true for $f_N'(x) = N[f(x+1/N)-f(x)]$, uniformly with respect to N. It easy to deduce from (4.2) and Proposition 5.5 that the sequence $\{U^N,\ N\geq 1\}$ is tight and as a consequence $\{(H^N,\mathscr{M}^{1,N},\mathscr{M}^{2,N},\mathscr{M}^N,\mathscr{R}^N,U^N),\ N\geq 1\}$ is a tight sequence in $\mathscr{C}([0,\infty))\times(\mathscr{D}([0,\infty)))^5$. Therefore at least along a subsequence (but we do not distinguish between the notation for the subsequence and for the sequence),

$$(H^N,\mathscr{M}^{1,N},\mathscr{M}^{2,N},\mathscr{M}^N,\mathscr{R}^N,U^N) \Rightarrow (H,\mathscr{M}^1,\mathscr{M}^2,\mathscr{M},\mathscr{R},U)$$

as $N\to\infty$ in $\mathscr{C}([0,\infty))\times(\mathscr{D}([0,\infty)))^5$.

Moreover, from Lemma 3.8 and Proposition 3.29, we deduce that

$$\begin{aligned}\langle \mathscr{M}^{1,N}\rangle_s &\Rightarrow cs,\\ \langle \mathscr{M}^{2,N}\rangle_s &\Rightarrow cs,\\ \langle U^N\rangle_s &\Rightarrow c\int_0^s \left|U_r\right|^2 \times \left|f'\left(L_r^{\Gamma}(H(r))\right)\right|^2 dr,\\ \langle U^N, \mathscr{M}^{1,N}\rangle_s &\Rightarrow c\int_0^s U_r f'^{+}\left(L_r(H(r))\right)dr\\ \langle U^N, \mathscr{M}^{2,N}\rangle_s &\Rightarrow c\int_0^s U_r f'^{-}\left(L_r(H(r))\right)dr.\end{aligned}$$

Recall Corollary 3.12, Lemma 3.13 and (2.6). It follows from the above that Proposition 3.17 can be enriched as follows

Proposition 4.2. *As $N\to\infty$,*

$$\left(H^N, \mathscr{M}^{1,N}, \mathscr{M}^{2,N}, \mathscr{M}^N, \mathscr{R}^N, U^N\right) \Longrightarrow \left(H, \sqrt{c}B_s^1, \sqrt{c}B_s^2, \mathscr{M}, \mathscr{R}, U\right),$$

in $(\mathscr{C}([0,\infty)))\times(\mathscr{D}([0,\infty)))^6$, where B^1 and B^2 are two mutually independent standard Brownian motions. Moreover

$$\begin{aligned}cH(s) &= \sqrt{c}\left(B_s^1 - B_s^2\right) + \int_0^s\int_0^\infty z\overline{\Pi}(dr,dz) - \inf_{0\le r\le s} Y(r)\\ &\quad - \int_0^s\int_0^\infty \left(z + \inf_{r\leqslant u\leqslant s} Y(u) - Y(r)\right)^+ \Pi(dr,dz),\\ \textit{and}\quad U_s &= 1 + \frac{1}{\sqrt{c}}\int_0^s U_r\left[f'^{+}\left(L_r(H(r))\right)dB_r^1 + f'^{-}\left(L_r(H(r))\right)dB_r^2\right].\end{aligned}$$

We clearly have

$$U_s = \exp\left(\frac{1}{\sqrt{c}}\int_0^s \left\{f'^{+}\left(L_r(H(r))\right)dB_r^1 + f'^{-}\left(L_r(H(r))\right)dB_r^2\right\} - \frac{c}{2}\int_0^s \left|f'\left(L_r(H(r))\right)\right|^2 dr\right).$$

Since f' is bounded $\mathbb{E}[U_s]=1$ for all $s\ge 0$. Let now $\widetilde{\mathbb{P}}$ denote the probability measure such that

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathscr{F}_s} = U_s,$$

where $\mathscr{F}_s := \sigma\{H(r),\ 0\le r\le s\}$. It follows from Girsanov's theorem (see Proposition 5.1 below) that there exist two mutually independent standard $\widetilde{\mathbb{P}}$-Brownian motions $\widetilde{B}^1$ and $\widetilde{B}^2$ such that

$$\begin{aligned}B_s^1 &= \frac{1}{\sqrt{c}}\int_0^s f'^{+}\left(L_r(H(r))\right)dr + \widetilde{B}_s^1\\ B_s^2 &= \frac{1}{\sqrt{c}}\int_0^s f'^{-}\left(L_r(H(r))\right)dr + \widetilde{B}_s^2.\end{aligned}$$

Consequently

$$\sqrt{c}\left(B_s^1 - B_s^2\right) = \sqrt{2c}B_s + \int_0^s f'\left(L_r(H(r))\right)dr,$$

where

$$B_s = \frac{1}{\sqrt{2}}\left(\widetilde{B}_s^1 - \widetilde{B}_s^2\right)$$

is a standard Brownian motion under $\widetilde{\mathbb{P}}$. Consequently H is a weak solution of the SDE
(4.4)
$$cH(s) = Y(s) + \int_0^s f'(L_r(H(r)))\,dr - \inf_{0\le r\le s} Y(r) - \int_0^s \int_0^\infty \left(z + \inf_{r\leqslant u\leqslant s} Y(u) - Y(r)\right)^+ \Pi(dr,dz).$$
Proposition 4.2 tells us that $(H^N,U^N) \Rightarrow (H,U)$ under $\mathbb{P}$. Fix an arbitrary $s>0$, and recall that $\frac{d\widetilde{\mathbb{P}}^N}{d\mathbb{P}}|_{\mathscr{F}_s^N} = U_s^N$, $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathscr{F}_s} = U_s$. Lemma 24 page 92 of [29] deduces from Proposition 4.2 that the law of $\{H_r^N,\ 0\le r\le s\}$ under $\widetilde{\mathbb{P}}^N$ converges as $N\to\infty$ to the law of $\{H_r,\ 0\le r\le s\}$ under $\widetilde{\mathbb{P}}$. This being true for any $s>0$, we conclude the main result of this subsection.

Theorem 4.3. *Assume that $f\in\mathscr{C}^1(\mathbb{R}_+)$, $f(0)=0$ and f' is bounded. Then the law of the approximate height process H^N, defined under $\widetilde{\mathbb{P}}^N$ (i.e. with the Poisson processes having the intensities specified by (4.3)) converges towards the law of the height process H under $\widetilde{\mathbb{P}}$ (i.e. specified by (4.4)).*

Note that the weak uniqueness of (4.4) will be established in the proof of Theorem 4.4 below.

4.2. **The general case ($f\in\mathscr{C}^1$ and $f'\le\theta$).** We first note that condition (2.9) guarantees only local boundedness of f'. Thus, in order to make sure that Girsanov's Theorem is applicable, we use a localization procedure and associate to each $n\in(0,\infty)$ a function $f_n\in\mathscr{C}^1(\mathbb{R}_+)$, f_n' is uniformly continuous on $\mathbb{R}_+$, and
$$f_n(x) = \begin{cases} f(x), & \text{if}\quad 0<x\le n,\\[2ex] f(n)+f'(x)(x-n), & \text{if}\quad x>n.\end{cases}$$
From this definition, it is easy to see that $f_n'(x) = f'(x\wedge n)$, which implies $\sup_{x\in(0,\infty)}|f_n'(x)| = \sup_{0\le x\le n}|f'(x)|$.

Now we define the processes $\{U_n^N(s), U_n(s),\ s\ge 0\}$ exactly as the process U^N, U, except that f is replaced by f_n. Let us now state our final result.

Theorem 4.4. *Assume that $f\in\mathscr{C}^1(\mathbb{R}_+)$, $f(0)=0$ and $f'(z)\le\theta$, for some given $\theta>0$. Then, as $N\to\infty$, the law of H^N, specified by (3.5) with the intensities of the Poisson processes specified by (4.3) converges to the law of H, the unique (in law) solution of equation (4.4).*

Proof. Let us first note that the uniqueness in law of the solution of (4.4) follows readily fro Girsanov's theorem. We work on the probability space $(\Omega,\mathscr{F},\mathbb{P})$. We consider the processes $H^N(r)$ and $H(r)$ restricted to an arbitrary time interval $[0,s]$. Suppose we have two interaction functions f^1 and f^2 which both satisfy the above assumption 2.9, and which coincide on the interval $[0,K]$. It is then plain that the corresponding processes $H^{N,1}$ and $H^{N,2}$ (resp. H^1 and H^2) have the same law on the time interval $[0,S_K^N]$ (resp. $[0,S_K]$), where
$$S_k^N = \inf\{s>0,\ H_s^{N,1}\vee H_s^{N,2} > K\},\ \text{resp. } S_K = \inf\{s>0,\ H_s^1\vee H_s^2 > K\,.$$

For each $m,n\ge 1$, consider the event
$$A_{m,n} = \{\sup_{0\le r\le s} H(r)\le m;\ \sup_{0\le r\le s;\ 0\le t\le m} L_r(t)\le n\}.$$
On the event $A_{m,n}$, $\sup_{0\le r\le s} L_r(H_r)\le n$. On the event $A_{m-1,n-1}$, from Proposition 3.4 and Proposition 3.29, for N large enough, $\sup_{0\le r\le s} H^N(r)\le m$ and $\sup_{0\le r\le s,\ 0\le t\le m} L_r^N(t)\le n$. Consequently on the event $A_{m,n}$, for such an N, $\sup_{0\le r\le s} L_r^N(H_r^N)\le n$, and from Theorem 4.3 with f replaced by f_n tells us that H^N with the intensities specified by (4.3) (but with f replaced by f_n) converges towards H, the weak solution of (4.4), but with f replaced by f_n. But on the event

$A_{m-1,n-1}$, and uniformly for N large enough, the intensities in (4.3) with f_n and with f coincide, and similarly for the equation (4.4). Since $\cup_{m,n\geq 2}A_{m,n}=\Omega$, the result follows. ■

5. Appendix

In this section we recall few important notions and give some results used in this work. We do not give proofs of most of the following statements.

5.1. **Two Girsanov Theorems.** We state two versions of the Girsanov theorem, one for the Brownian and one for the point process case. The first one can be found, e.g., in [30] and the second one combines Theorems T2 and T3 from [6], pages 165-166. We assume here that our probability space $(\Omega,\mathbb{P},\mathscr{F})$ is such that $\mathscr{F}=\sigma(\cup_{t>0}\mathscr{F}_t)$.

Proposition 5.1. *Let $\{B_s,\ s\geq 0\}$ be a standard $d-$dimensional Brownian motion (i.e., its coordinates are mutually independent standard scalar Brownian motions) defined on the filtered probability space $(\Omega,\mathbb{P},\mathscr{F})$. Let moreover ϕ be an $\mathscr{F}$-progressively measurable $d-$dimensional process satisfying $\int_0^s|\phi(r)|^2dr<\infty$ for all $s\geq 0$. Let*

$$U_s=\exp\left\{\int_0^s\langle\phi(r),dB_r\rangle-\frac{1}{2}\int_0^s|\phi(r)|^2dr\right\}.$$

If $\mathbb{E}(U_s)=1$, $s\geq 0$, then $\widetilde{B}_s:=B_s-\int_0^s\phi(r)dr$, $s\geq 0$, is a standard Brownian motion under the unique probability measure $\widetilde{\mathbb{P}}$ on $(\Omega,\mathscr{F})$ which is such that $d\widetilde{\mathbb{P}}|_{\mathscr{F}_s}/d\mathbb{P}|_{\mathscr{F}_s}=U_s$, for all $s\geq 0$.

Proposition 5.2. *Let $\{(Q_s^{(1)},...,Q_s^{(d)}),\ s\geq 0\}$ be a d-variate point process adapted to some filtration $\mathscr{F}$, and let $\{\lambda_s^{(i)},\ s\geq 0\}$ be the predictable $(\mathbb{P},\mathscr{F})$-intensity of $Q^{(i)}$, $1\leq i\leq d$. Assume that none of the $Q^{(i)},Q^{(j)}$, $i\neq j$, jump simultaneously. Let $\{\alpha_r^{(i)},\ r\geq 0\}$, $1\leq i\leq d$, be nonnegative $\mathscr{F}$-predictable processes such that for all $s\geq 0$ and all $1\leq i\leq d$*

$$\int_0^s\alpha_r^{(i)}\lambda_r^{(i)}dr<\infty\quad \mathbb{P}-a.s.$$

For $i=1,...,d$ and $s\geq 0$ define, $\{T_k^i,\ k=1,2...\}$ denoting the jump times of $Q^{(i)}$,

$$U_s^{(i)}=\left(\prod_{k\geq 1:T_k^i\leq s}\alpha_{T_k^i}^{(i)}\right)\exp\left\{\int_0^s(1-\alpha_r^{(i)})\lambda_r^{(i)}dr\right\}\quad \textit{and}\quad U_s=\prod_{i=1}^dU_s^{(i)},\quad s\geq 0.$$

If $\mathbb{E}(U_s)=1$, $s\geq 0$, then, for each $1\leq i\leq d$, the process $Q^{(i)}$ has the $(\widetilde{\mathbb{P}},\mathscr{F})$-intensity $\widetilde{\lambda}_s^{(i)}=\alpha_s^{(i)}\lambda_s^{(i)}$, $s\geq 0$, where the probability measure $\widetilde{\mathbb{P}}$ is defined by $d\widetilde{\mathbb{P}}|_{\mathscr{F}_s}/d\mathbb{P}|_{\mathscr{F}_s}=U_s$, for all $s\geq 0$.

5.2. **Tightness criteria in $\mathscr{D}([0,+\infty))$.** We denote by $\mathscr{D}([0,\infty))$, the space of functions from $[0,\infty)$ into $\mathbb{R}$ which are right continuous and have left limits at any $t>0$ (as usual such a function is called càdlàg).We briefly write $\mathbb{D}$ for the space of adapted, càdlàg stochastic processes. We shall always equip the space $\mathscr{D}([0,\infty))$ with the Skorohod topology, for the definition of which we refer the reader to Billingsley [5] or Joffe, Métivier [17].

We first state a tightness criterion, which is Theorem 13.5 from [5] :

Proposition 5.3. *Let $(X_t^n,\ t\geq 0)_{n\geq 0}$ be a sequence of random elements of $\mathscr{D}([0,+\infty);\mathbb{R}^d)$. A sufficient condition of $(X_t^n,\ t\geq 0)_{n\geq 0}$ to be tight is that the two conditions 1 and 2 be satisfied :*

1. For each t, the sequence of random variables $(X_t^n,\ n\geq 0)$ is tight in $\mathbb{R}^d$.

2. For each $T>0$, there exists $\beta,C>0$ and $\theta>1$ such that

$$\mathbb{E}(|X_{t+h}^n-X_t^n|^\beta|X_t^n-X_{t-h}^n|^\beta)\leq Ch^\theta,$$

for all $0 < t < T$, $0 < h < t$, $n \geq 0$.

The convergence in $\mathscr{D}([0,+\infty);\mathbb{R}^d)$ is not additive in general. The next proposition gives a sufficient condition to have this additivity, which is Lemma 7.1 of [23].

Proposition 5.4. *Let* $\{X_t^n,\ t \geq 0\}_{n\geq 0}$ *and* $\{Y_t^n,\ t \geq 0\}_{n\geq 0}$ *be two tight sequences of random elements of* $\mathscr{D}([0,\infty);\mathbb{R}^d)$ *such that any limit of a weakly converging sub–sequence of the sequence* $\{X_t^n,\ t \geq 0\}_{n\geq 0}$ *is a.s. continuous. Then* $\{X_t^n + Y_t^n,\ t \geq 0\}_{n\geq 0}$ *is tight in* $\mathscr{D}([0,\infty);\mathbb{R}^d)$.

Consider a sequence $\{X_t^n,\ t \geq 0\}_{n\geq 1}$ of one-dimensional semi-martingales, which is such that for each $n \geq 1$,

$$X_t^n = X_0^n + \int_0^t \varphi_s^n ds + M_t^n, \quad t \geq 0;$$

where for each $n \geq 1$, M^n is a locally square–integrable martingale such that

$$\langle M^n \rangle_t = \int_0^t \psi_s^n ds, \quad t \geq 0;$$

φ^n and ψ^n are Borel measurable functions with values into $\mathbb{R}$ and $\mathbb{R}_+$ respectively. We define $V_t^n = X_0^n + \int_0^t \varphi_n(X_s^n) ds$.

The following statement can be deduced from Theorem 13.4 and 16.10 of [5].

Proposition 5.5. *A sufficient condition for the above sequence* $\{X_t^n, t \geq 0\}_{n\geq 1}$ *of semi–martingales to be tight in* $\mathscr{D}([0,\infty))$ *is that both*

$$the\ sequence\ of\ r.v.'s\ \{X_0^n, n \geq 1\}\ is\ tight;$$

and for some $p > 1$,

$$\forall T > 0,\ the\ sequence\ of\ r.v.'s\ \left\{ \int_0^T [|\varphi_n(X_s^n)| + \psi_n(X_t^n)]^p dt, n \geq 1 \right\}\ is\ tight.$$

Those conditions imply that both the bounded variation parts $\{V^n, n \geq 1\}$ *and the martingale parts* $\{M^n, n \geq 1\}$ *are tight, and that the limit of any converging subsequence of* $\{V^n\}$ *is a.s. continuous.*

If moreover, for any $T > 0$, *as* $n \longrightarrow \infty$,

$$\sup_{0\leq t\leq T} |M_t^n - M_{t^-}^n| \longrightarrow 0\ in\ probability,$$

then any limit X of a converging subsequence of the original sequence $\{X^n\}_{n\geq 1}$ *is a.s. continuous.*

Lemma 5.6. *Let* $\{X_t^N,\ t \geq 0\}_{N\geq 1}$ *be a sequence of processes whose trajectories belong to* $\mathscr{D}([0,+\infty))$ *and satisfy*

$$\sup_{N\geq 1} \mathbb{E}\left(\sup_{t\geq 0} |X_t^N| \right) < \infty. \tag{5.1}$$

We assume that for each $\delta > 0$, *there exists a decomposition* $X_t^N = X_t^{N,\delta,1} + X_t^{N,\delta,2}$ *such that* $\{X^{N,\delta,1}\}_{N\geq 1}$ *is tight as random elements of* $\mathscr{D}([0,+\infty))$, *and moreover, for all* $\eta > 0$

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{t\geq 0} |X_t^{N,\delta,2}| > \eta \right) \to 0,\ \ as\ \delta \to 0. \tag{5.2}$$

Then the sequence $\{X_t^N,\ t \geq 0\}_{N\geq 1}$ *is tight as random elements of* $\mathscr{D}([0,+\infty))$.

Proof. We shall exploit Theorem 13.2 from [5]. We will establish tightness in $D([0,T])$, for $T>0$ arbitrary. The moduli of continuity below are understood to be defined on the time interval $[0,T]$. Condition (i) follows from our assumption (5.1). Hence it suffices to verify (ii), namely that for each $\varepsilon,\rho>0$, there exists $\eta>0$ such that

$$\mathbb{P}\left(w'_{X^N}(\eta)\geq\varepsilon\right)\leq\rho\,. \tag{5.3}$$

We first note that from the definitions of w (resp. w') (see (7.1) (resp. (12.6)) in [5]), for each $\eta>0$,

$$w'_{X^N}(\eta)\leq w'_{X^{N,\delta,1}}(\eta)+w_{X^{N,\delta,2}}(\eta)\,. \tag{5.4}$$

But since $w_{X^{N,\delta,2}}(\eta)\leq 2\sup_{t\geq 0}|X_t^{N,\delta,2}|$, for all $\eta>0$,

$$\limsup_{N\to\infty}\mathbb{P}\left(w_{X^{N,\delta,2}}(\eta)\geq\varepsilon/2\right)\leq\limsup_{N\to\infty}\mathbb{P}\left(\sup_{t\geq 0}|X_t^{N,\delta,2}|\geq\varepsilon/4\right).$$

Hence from (5.2), we can choose $\delta_{\varepsilon,\rho}>0$ such that

$$\limsup_{N\to\infty}\mathbb{P}\left(w_{X^{N,\delta_{\varepsilon,\rho},2}}(\eta)\geq\varepsilon/2\right)\leq\frac{\rho}{2},\forall\eta>0\,. \tag{5.5}$$

Since $\{X^{N,\delta_{\varepsilon,\rho},1}\}_{N\geq 1}$ is tight, again from Theorem 13.2 from [5], we can choose $\eta>0$ small enough such that

$$\limsup_{N\to\infty}\mathbb{P}\left(w'_{X^{N,\delta_{\varepsilon,\rho},1}}(\eta)\geq\varepsilon/2\right)\leq\frac{\rho}{2}\,. \tag{5.6}$$

A combination of (5.4), (5.5) and (5.6) yields (5.3). ■

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BIBLIOGRAPHY

1. Athreya, K. B. and Ney, P.E. *Branching processes*, Springer- Verlag, New York, 1972.
2. Bansaye, V., Caballero, M.E. and Méléard, S. Scaling limits of population and evolution processes in random environment. *Electron. J. Probab.* **24**, Paper No. 19, 2019.
3. Berestycki, J., Fittipaldi, M.C. and Fontbona, J. Ray–Knight representation of flows of branching processes with competition by pruning of Lévy trees, *Probab. Theory and Rel. Fields*, **172**, no. 3–4, 725–788, 2018.
4. Bertoin, J. *Lévy processes*, Cambridge University Press, 1996.
5. Billingsley, P. *Convergence of Probability Measures*, 2nd ed. John Wiley, New York, 1999.
6. Brémaud, P. *Point processes and queues: martingale dynamics*, Springer 1981.
7. Çinlar, E. *Probability and Stochastics*, Graduate Texts in Mathematics,vol. **261**. Springer Science & Business Media, 2011.
8. Dawson, D. A. and Li, Z. Stochastic equations, flows and measure-valued processes. *Ann. Probab.*, **40**(2):813–857, 2012.
9. Delmas, J.-F. Height process for super-critical continuous state branching process. *Markov Process. Relat. Fields* **14**, 309-326, 2008.
10. Dramé, I. Pardoux, E. and Sow, A.B. Non–binary branching process and non–markovian exploration process. *ESAIM Prob. Stat.* **21**, 1–33, 2017.
11. Dramé, I. and Pardoux, E. Approximation of a generalized continuous-state branching process with interaction. *Electron. Commun. Probab.*, **23**, 2018.
12. Duquesne, T. and Le Gall, J.-F. *Random trees, Lévy processes and spatial branching processes*, Asrérisque vol. **281**. Société mathématique de France, 2002.
13. Fu, Z. and Li, Z. Stochastic equations of non-negative processes with jumps. *Stoch. Proc. Appl.*, **120**(3):306–330, 2010.
14. Grey, D. Asymptotic behaviour of continuous time, continuous state-space branching processes. *J. Appl. Probab.* **11**, 669–677, 1974.
15. Grimvall, A. On the convergence of sequences of branching processes. *Ann. Probab.* **2**, 1027–1045, 1974.

16. Jiřina, M. Stochastic branching processes with continuous state space. *Czechoslovak Mathematical Journal*, **8**(2):292–313, 1958.
17. Joffe, A. and Métivier, M. Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Adv. Appl. Prob.* **18**, 20–65, 1986
18. Kallenberg, O. *Foundations of Modern Probability*, 2d ed. Springer 2002.
19. Lamperti, J. Continuous-state branching processes. *Bull. Amer. Math. Soc*, **73**(3):382–386, 1967.
20. Lamperti, J. The limit of a sequence of branching processes. *Probab. Theory and Rel. Fields*, **7**(4):271–288, 1967.
21. Le Gall, J.-F. *Spatial branching processes, random snakes, and partial differential equations*. Springer Science & Business Media, 1999.
22. Le Gall, J.-F. and Le Jan, Y. Branching processes in Lévy processes: the exploration process. *Ann. Probab.* **26**, 213–252, 1998.
23. Le, V., Pardoux, E. and Wakolbinger, A. (2013). "Trees under attack": a Ray–Knight representation of Feller's branching diffusion with logistic growth. *Probab. Theory and Rel. Fields*, **155** (3-4): 583–619.
24. Li, P.S. A continuous–state polynomial branching process, *Stoch. Proc. Appl.* **129**, 2941–2967, 2019.
25. Li, P.S., Yang, X. and Zhou, X. A general continuous–state nonlinear branching process *Ann. Appl. Probab.* **29**, 2523–2555, 2019.
26. Li, Z. *Measure-valued branching Markov processes*. Springer, Berlin, 2011.
27. Li, Z. Continuous-state branching processes, *arXiv preprint arXiv:1202.3223*, 2012.
28. Li, Z., Pardoux. E., and Wakolbinger, A. The Height process of a general CSBP with interaction. *arXiv preprint arXiv:1904.04151*, 2019.
29. Pardoux, E. *Probabilistic models of population evolution scaling limits and interactions*, Springer, 2015
30. Pardoux, E. and Rascanu, A. *Stochastic Differential Equations, Backward SDEs, Partial Differential Equations*, Springer, 2014.
31. Protter, P. *Stochastic integration and differential equations*, Springer, 2004
32. Revuz, D. and Yor, M. *Continuous martingales and Brownian motion. 3rd edn.* Springer Verlag, New York, 1999.
33. Silverstein, M. L. A new approach to local times (local time existence and property determination using real valued strong markov process basis). *Journal of Mathematics and Mechanics*, **17**, 1023–1054, 1968.

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