Probab. Theory Relat. Fields 110, 535–558 (1998)

Probability Theory and Related Fields

Generalized BSDEs and nonlinear Neumann boundary value problems

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Received: 27 September 1996 / In revised form: 1 December 1997

Summary. We study a new class of backward stochastic differential equations, which involves the integral with respect to a continuous increasing process. This allows us to give a probabilistic formula for solutions of semilinear partial differential equations with Neumann boundary condition, where the boundary condition itself is nonlinear. We consider both parabolic and elliptic equations.

Mathematics Subject Classification (1991): 60H99, 60H30, 35J60

Introduction

Backward stochastic differential equations – in short BSDEs – have been first introduced by Pardoux, Peng [7]. They provide probabilistic formulas for solutions of systems of semilinear partial differential equations, both of parabolic and elliptic type, see among others Peng [9], Pardoux, Peng [7], Hu [5], Pardoux, Pradeilles, Rao [8] and Darling, Pardoux [4]. Most of these papers treat the case of parabolic equations (or systems of equations in $[0, T] \times \mathbb{R}^d$), also elliptic equations with Dirichlet boundary condition have been treated in Darling, Pardoux [4], and with a homogeneous Neumann boundary condition in Hu [5]. The aim of the present paper is to treat the case of a nonlinear Neumann boundary condition, i.e. to give a probabilistic formula for the solution of a system of elliptic PDEs of the form:

$$Lu_i(x) + f_i(x, u(x), (\nabla u_i\sigma)(x)) = 0, \quad 1 \le i \le k, x \in G$$
$$\frac{\partial u_i}{\partial n}(x) + g_i(x, u(x)) = 0, \quad 1 \le i \le k, x \in \partial G\ ,$$

as well as for a similar parabolic system.

This requires the presence of a new term in the BSDE, namely an integral with respect to a continuous increasing process, the local time of the diffusion on the boundary.

In section 1, we study the BSDE

$$Y_t \ = \ \xi \ + \ \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s)\, dA_s \ - \ \int_t^T Z_s\, dB_s\ ,$$

where $\{A_t,\ 0\ \le\ t \le\ T\}$ is a continuous real valued increasing process. In section 2, we study the same BSDE, in the case where $T = +\infty$ and $\xi = 0$. In section 3, we introduce a class of reflected diffusion processes, and study some of its properties. In section 4, we combine the results in sections 1 and 3 and prove that a certain function of (t, x), defined through the solution of a system of forward–backward SDE, is a viscosity solution of a certain system of parabolic PDEs. In section 5, we prove the same kind of result for an elliptic PDE, by combining the results of sections 2 and 3. Throughout this paper, $\{B_t, t \ge 0\}$ will denote a d-dimensional Brownian motion, defined on a probability space $(\Omega, \mathscr{F}, P)$. For $t \ge 0$, let $\mathscr{F}_t$ denote the σ-algebra $\sigma(B_s; 0 \le s \le t)$, augmented with the P-null sets of $\mathscr{F}$.$\{A_t, t \ge 0\}$ will denote a continuous one-dimensional increasing $\mathscr{F}_t$-progressively measurable process satisfying $A_0\ =\ 0$.

1. Generalized BSDEs on a finite time interval

We are given a final time $T > 0$, a final condition

$$\xi \in L^2\big(\Omega, \mathscr{F}_T, P; \mathbb{R}^k\big)$$

such that $E(e^{\mu A_T}|\xi|^2) < \infty$, for all $\mu > 0$, and two coefficients:

$$f\colon \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \ \rightarrow\ \mathbb{R}^k \quad \text{and}$$
$$g\colon \Omega \times [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}^k\ ,$$

satisfying, for some constants $\alpha, \beta \in \mathbb{R}$, $K > 0$, some adapted processes $\{\varphi_t, \psi_t;\ 0 \le t \le T\}$ with values in $[1, +\infty)$, and all $(t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $\mu > 0$,

(i) $f(\cdot, y, z)$ and $g(\cdot, y)$ are progressively measurable;
(ii) $E(\int_0^T e^{\mu A_t}\varphi_t^2\, dt + \int_0^T e^{\mu A_t}\psi_t^2\, dA_t) < \infty$;
(iii) $\langle y - y', f(t, y, z) - f(t, y', z)\rangle \leq \alpha|y - y'|^2$;
(iv) $\langle y - y', g(t, y) - g(t, y')\rangle \leq \beta|y - y'|^2$;
(v) $|f(t, y, z) - f(t, y, z')| \leq K\|z - z'\|$;
(vi) $|f(t, y, z)| \leq \varphi_t + K(|y| + \|z\|), |g(t, y)| \leq \psi_t + K(|y| + \|z\|)$;
(vii) $y \to (f(t, y, z), g(t, y))$ is continuous for all $z, (t, \omega)$ *a.e.*

A solution of the BSDE is a pair $\{(Y_t, Z_t); 0 \leq t \leq T\}$ of progressively measurable processes with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ such that

(j) $E(\sup_{0<t\leq T} |Y_t|^2 + \int_0^T \|Z_t\|^2\, dt) < \infty$
(jj) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s)\, dA_s - \int_t^T Z_s\, dB_s,\ 0 \leq t \leq T$

We shall assume from now on that:

(viii) $\beta < 0$

Remark 1 Whenever (Y_t, Z_t) satisfies (jj), $(\bar{Y}_t, \bar{Z}_t) = (e^{\mu A_t} Y_t, e^{\mu A_t} Z_t)$ satisfies an analogous BSDE, with f and g replaced by

$$\bar{f}(t, y, z) = e^{\mu A_t} f\left(t, e^{-\mu A_t} y, e^{-\mu A_t} z\right)$$
$$\bar{g}(t, y) = e^{\mu A_t} g\left(t, e^{-\mu A_t} y\right) - \mu y$$

Hence, if g satisfies (iv) with a possibly non negative β, we can always choose μ such that $\bar{g}$ satisfies (iv) with a strictly negative $\bar{\beta}$. Consequently, (viii) is not a severe restriction. However, in case it is violated, the estimates below do not hold for the solution $\{(Y_t, Z_t)\}$ of the original equation, but for the solution $\{(\bar{Y}_t, \bar{Z}_t)\}$ of a transformed equation.

Remark 2 Condition (j) implies that $\{\int_0^t \langle Y_s, Z_s\, dB_s\rangle; 0 \leq t \leq T\}$ is a uniformly integrable martingale, which in particular has zero expectation, a fact which will be used repeatedly below.

Remark 3 In case the r.v. A_T is bounded, the condition

$$E\left(\sup_{0\leq t\leq T} |Y_t|^2\right) < \infty$$

can be deduced from (jj) and $E\int_0^T \|Z_t\|^2\, dt < \infty$.

We first establish an priori estimate on the solution.

Proposition 1.1 *Under the conditions* $(i)\ldots(viii)$, *if* $\{(Y_t, Z_t); 0 \leq t \leq T\}$ *is a solution of* (j), (jj), *then there exists a constant* C, *which depends only on* α, β, K *and* T, *such that:*

$$E\left(\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Y_t|^2 dA_t + \int_0^T \|Z_t\|^2\, dt\right) \leq CE\left(|\xi|^2 + \int_0^T |f(t,0,0)|^2\, dt + \int_0^T |g(t,0)|^2\, dA_t\right)$$

Moreover, if $\{\tilde{A}_t; 0 \leq t \leq T\}$ *is any continuous, increasing and progressively measurable* $\mathbb{R}_+$*-valued stochastic process such that*

$$\tilde{A}_0 = 0;\ E(\mathrm{e}^{\mu\tilde{A}_T}) < \infty, \forall\ \mu > 0\ ;$$

then for any $\mu > 0$, *there exists a constant*

$$C = C(\mu, \alpha, \beta, K, T)$$

such that

$$E\left(\sup_{0\leq t\leq T}\mathrm{e}^{\mu\tilde{A}_t}|Y_t|^2 + \int_0^T \mathrm{e}^{\mu\tilde{A}_t}|Y_t|^2\, d\tilde{A}_t + \int_0^T \mathrm{e}^{\mu\tilde{A}_t}\|Z_t\|^2\, dt\right) \leq CE\left(\mathrm{e}^{\mu\tilde{A}_T}|\xi|^2 + \int_0^T \mathrm{e}^{\mu\tilde{A}_t}|f(t,0,0)|^2\, dt + \int_0^T \mathrm{e}^{\mu\tilde{A}_t}|g(t,0)|^2\, dA_t\right)$$

Proof: From Itô's formula,

$$|Y_t|^2 + \int_t^T \|Z_s\|^2\, ds = |\xi|^2 + 2\int_t^T \langle Y_s, f(s, Y_s, Z_s)\rangle\, ds + 2\int_t^T \langle Y_s, g(s, Y_s)\rangle\, dA_s - 2\int_t^T \langle Y_s, Z_s\, dB_s\rangle$$

But from (iii), (iv), and (v),

$$\langle y, f(s,y,z)\rangle \leq \alpha|y|^2 + |y| \times (|f(s,0,0| + K\|z\|),$$
$$\langle y, g(s,y)\rangle \leq \beta|y|^2 + |y| \times |g(s,0)|\ .$$

Consequently

$$|Y_t|^2 + \frac{1}{2}\int_0^T \|Z_s\|^2\, ds + |\beta|\int_t^T |Y_s|^2\, dA_s \leq |\xi|^2 + c\int_t^T |Y_s|^2\, ds + c\int_t^T |f(s,0,0)|^2\, ds + c\int_t^T |g(s,0)|^2\, dA_s - 2\int_t^T \langle Y_s, Z_s\, dB_s\rangle\ . \tag{1}$$

From (j), the stochastic integral term has zero expectation. Hence, taking the expectation in (1), we deduce from Gronwall's lemma that

$$\sup_{0\le t\le T} E\left(|Y_t|^2 + \int_0^T \|Z_s\|^2\, ds + \int_0^T |Y_s|^2\, dA_s\right)$$
$$\le cE\left(|\xi|^2 + \int_0^T |f(s,0,0)|^2\, ds + \int_0^T |g(s,0)|^2 dA_s\right)$$

The first result follows from this, (1) and the Davis–Burkholder–Gundy inequality.

The second result is proved with similar arguments in case $\tilde{A}_T$ is a bounded r.v. The general case then follows from Fatou's lemma. $\diamondsuit$

Let now (ξ, f, g, A) and (ξ', f', g', A') be two sets of data, each satisfying the above assumptions (i), ..., (viii). Let (Y, Z) (resp.(Y', Z')) denote a solution of the BSDE (j), (jj) with data (ξ, f, g, A) (resp. $\xi', f', g', A')$). We shall need below the following

Proposition 1.2 *Define* $(\bar{Y}, \bar{Z}, \bar{\xi}, \bar{f}, \bar{g}, \bar{A}) = (Y - Y',\ Z - Z',\ \xi - \xi',\ f - f',\ g - g', A - A')$. *Then, for any* $\mu > 0$, *there exists a constant* C *such that*

$$E\left(\sup_{0\le t\le T} e^{\mu\kappa_t}|\bar{Y}_t|^2 + \int_0^T e^{\mu\kappa_t}\|\bar{Z}_t\|^2\, dt\right)$$
$$\le CE\left(e^{\mu\kappa_T}|\bar{\xi}|^2 + \int_0^T e^{\mu\kappa_t}|f(t,Y_t,Z_t) - f'(t,Y_t,Z_t)|^2\, dt\right.$$
$$\left. + \int_0^T e^{\mu\kappa_t}|g(t,Y_t) - g'(t,Y_t)|^2\, dA'_t + \int_0^T e^{\mu\kappa_t}|g(t,Y_t)|^2 d\|\bar{A}\|_t\right)\ ,$$

where $\kappa_t := \|\bar{A}\|_t + A'_t$, $\|\bar{A}\|_t$ *denoting the total variation of the process* $\bar{A}$ *on the interval* $[0,t]$.

Proof: It suffices to prove the result in the case where κ_T is a bounded random variable, and then apply Fatou's lemma. From Itô's formula,

$$e^{\mu\kappa_t}|\bar{Y}_t|^2 \ +\ \mu\int_t^T e^{\mu\kappa_s}|\bar{Y}_s|^2\, d\kappa_s + \int_t^T e^{\mu\kappa_s}\|\bar{Z}_s\|^2\, ds$$
$$= e^{\mu\kappa_T}|\bar{\xi}|^2 + 2\int_t^T e^{\mu\kappa_s}\ \langle \bar{Y}_s, f(s,Y_s,Z_s) - f'(s,Y'_s,Z'_s)\rangle\, ds$$
$$+ 2\int_t^T e^{\mu\kappa_s}\ \langle \bar{Y}_s\ ,\ g(s,Y_s)\ -\ g'\big(s,Y'_s\big)\rangle\, dA'_s$$
$$+ 2\int_t^T e^{\mu\kappa_s}\ \langle \bar{Y}_s, g(s,Y_s)\rangle\, d\bar{A}_s - 2\int_t^T e^{\mu\kappa_s}\langle \bar{Y}_s, \bar{Z}_s\, dB_s\rangle$$

Exploiting the assumptions and using Schwarz's inequality, we deduce that

$$
\begin{aligned}
&e^{\mu\kappa_t}|\bar{Y}_t|^2 + \frac{1}{2}\int_t^T e^{\mu\kappa_s}\|\bar{Z}_s\|^2\,ds \\
&\quad \le e^{\mu\kappa_T}|\bar{\xi}|^2 + c\int_t^T e^{\mu\kappa_s}|\bar{Y}_s|^2\,ds + \int_t^T e^{\mu\kappa_s}|f(s,Y_s,Z_s) - f'(s,Y_s,Z_s)|^2\,ds \\
&\qquad + c\int_t^T e^{\mu\kappa_s}|g(s,Y_s) - g'(s,Y_s)|^2\,dA'_s + \int_t^T e^{\mu\kappa_s}|g(s,Y_s)|^2\,d\|\bar{A}\|_s \\
&\qquad - 2\int_t^T e^{\mu\kappa_s}\langle\bar{Y}_s, \bar{Z}_s\,dB_s\rangle \ .
\end{aligned}
$$

The result, without the $\sup_t$ inside the expectation, follows by taking the expectation and using Gronwall's lemma. The result then follows from the Burkholder–Davis–Gundy inequality. $\diamondsuit$

We deduce immediately from the last proposition the

Corollary 1.3 *Under the above assumptions, the BSDE* (j), (jj) *has at most one solution.*

We next prove a comparison theorem in one dimension. Suppose we are in the situation of proposition 1.2, in the particular case $A \equiv A'$.

Theorem 1.4 *We suppose now that* $k = 1$, $\xi \le \xi'$, $f(t,y,z) \le f'(t,y,z)$, $g(t,y) \le g'(t,y)$, *for all* $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $dP \times dt$, *a.s. Then* $Y_t \le Y'_t$, $0 \le t \le T$, *a.s..*

Moreover, if $Y_0 = Y'_0$, *then* $Y_t = Y'_t$, $0 \le t \le T$, *a.s. In particular, if in addition either* $P(\xi < \xi') > 0$ *or* $f(t,y,z) < f'(t,y,z)$, $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, *on a set of positive* $dt \times dP$ *measure, or* $g(t,y) < g'(t,y)$, $y \in \mathbb{R}$, *on a set of positive* $dA_t \times dP$ *measure, then* $Y_0 < Y'_0$.

Proof: Define

$$
\alpha_t = \begin{cases} (Y'_t - Y_t)^{-1}\big(f(t,Y'_t,Z_t) - f(t,Y_t,Z_t)\big) & \text{if } Y_t \neq Y'_t; \\ 0 & \text{if } Y_t = Y'_t \ ; \end{cases}
$$

the $\mathbb{R}^d$-valued process $\{\beta_t;\ 0 \le t \le T\}$ as follows. For $1 \le i \le d$, let $Z_t^{(i)}$ denote the d-dimensional vector whose i first components are equal to those of Z'_t, and whose $d - i$ last components are equal to those of Z_t. With this notation, we define for each $1 \le i \le d$,

$$
\beta_t^i = \begin{cases} (Z_t'^i - Z_t^i)^{-1}\Big(f\big(t,Y_t,Z_t^{(i)}\big) - f\big(t,Y_t,Z_t^{(i-1)}\big)\Big) & \text{if } Z_t^i \neq Z_t'^i; \\ 0 & \text{if } Z_t^i = Z_t'^i \ , \end{cases}
$$

and

$$\gamma_t = \begin{cases} \left(Y'_t - Y_t\right)^{-1}\left(g(t, Y'_t) - g(t, Y_t)\right) & \text{if } Y_t \neq Y'_t; \\ 0 & \text{if } Y_t = Y'_t \end{cases} .$$

We note that $\{\alpha_t;\ 0 \le t \le T\}$, $\{\beta_t;\ 0 \le t \le T\}$ and $\{\gamma_t;\ 0 \le t \le T\}$ are progressively measurable, $\alpha_t \le \alpha$, $|\beta| \le K$ and $\gamma_t \le \beta$.

For $0 \le s \le t \le T$, let

$$\Gamma_{s,t} = \exp\left[\int_s^t (\alpha_r - 1/2|\beta|^2)dr + \int_s^t \gamma_s\, dA_s + \int_s^t \langle \beta_r, dB_r\rangle\right] .$$

Define $(\overline{Y}_t, \overline{Z}_t) = (Y'_t - Y_t, Z'_t - Z_t)$, $\bar{\xi} = \xi' - \xi$, $U_t = f'(t, Y'_t, Z'_t) - f(t, Y'_t, Z'_t)$, $V_t = g'(t, Y'_t) - g(t, Y'_t)$.

Then

$$\begin{aligned} \overline{Y}_t = \bar{\xi} &+ \int_t^T \left(\alpha_s \overline{Y}_s + \langle \beta_s, \overline{Z}_s\rangle\right) ds + \int_t^T \gamma_s \overline{Y}_s\, dA_s \\ &+ \int_t^T (U_s\, ds + V_s\, dA_s) - \int_t^T \overline{Z}_s\, dB_s . \end{aligned}$$

It is not hard to see that for $0 \le s \le t \le T$,

$$\overline{Y}_s = \Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r}(U_r\, dr + V_r\, dA_r) - \int_s^t \Gamma_{s,r}\left(\overline{Z}_r + \overline{Y}_r \beta_r\right) dB_r$$

$$\overline{Y}_s = E\left(\Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r}(U_r\, dr + V_r\, dA_r)/\mathscr{F}_s\right) .$$

The result follows from this formula and the positivity of $\bar{\xi}$, U and V.

Remark 1.5 Suppose that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z)_s\, ds + \int_t^T g(s, Y_s)\, dA_s - \int_t^T Z_s\, dB_s$$

$$Y'_t = \xi' + \int_t^T U_s\, ds + \int_t^T V_s\, dA_s - \int_t^T Z'_s\, dB_s ,$$

and $\xi \le \xi'$, $f(t, Y'_t, Z'_t) \le U_t$, $g(t, Y'_t) \le V_t$. Then we can apply theorem 1.4. Indeed, we can define

$$\begin{aligned} f'(t,y,z) &= f(t,y,z) + \left(U_t - f'\left(t, Y'_t, Z'_t\right)\right) \\ g'(t,y) &= g(t,y) + \left(V_t - g(t, Y'_t)\right) . \end{aligned}$$

If moreover either $f(t, Y'_t, Z'_t) < U_t$ on a set of $dt \times dP$ positive measure, or $g(t, Y'_t) < V_t$ on a set of $dA_t \times dP$ positive measure, then $Y_0 < Y'_0$. $\diamondsuit$

We now prove a first existence and uniqueness theorem, under an additional assumption. Namely, we assume that f and g are Lipschitz in y, i.e. there exists a constant K such that

(ix) $|f(t,y,z) - f(t,y',z)| + |g(t,y) - g(t,y')| \ \leq K|y - y'|,\ t > 0,\ y,$ $y' \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$, a.s.

Theorem 1.6 *Under the assumptions* (i), (ii), (v), (vi), (ix), *the BSDE* (j), (jj) *has a unique solution* (Y,Z).

Proof: For $\mu \geq 0$, let $M^2_\mu(A)$ denote the set of progressively measurable processes $\{X_t, 0 \leq t \leq T\}$, which are such that

$$E\left(\int_0^T \mathrm{e}^{\mu A_t}|X_t|^2\,dt + \int_0^T \mathrm{e}^{\mu A_t}|X_t|^2 dA_t\right) < \infty \ ,$$

and M^2_μ the same space without the second term. We define

$$\mathscr{B}^2_\mu \triangleq \left(M^2_\mu(A)\right)^k \times \left(M^2_\mu\right)^{k\times d}$$

Let Φ be the mapping from $\mathscr{B}^2_\mu$ into itself, which is defined as follows. Given $(U,V) \in \mathscr{B}^2_\mu$, $(Y,Z) = \Phi(U,V)$ where

$$Y_t = E\left[\xi + \int_t^T f(s,U_s,V_s)\,ds + \int_t^T g(s,U_s)\ dA_s/\mathscr{F}_t\right],\ \ 0 \leq t \leq T \ ,$$

and $\{Z_t; 0 \leq t \leq T\}$ is given by Itô's representation theorem applied to the square integrable r.v.

$$\xi + \int_0^T f(t,U_t,V_t)\,dt + \int_0^T g(t,U_t)\ dA_t \ ,$$

i.e.

$$\xi + \int_0^T f(t,U_t,V_t)\,dt + \int_0^T g(t,U_t)\ dA_t$$
$$= E\left[\xi + \int_0^T f(t,U_t,V_t)\,dt + \int_0^T g(t,U_t)\ dA_t\right] + \int_0^T Z_t\,dB_t \ .$$

Taking $E(\cdot/\mathscr{F}_t)$ of the last identity yields

$$Y_t = \xi + \int_t^T f(s,U_s,V_s)\,ds + \int_t^T g(s,U_s)\ dA_s - \int_t^T Z_s\,dB_s \ .$$

The fact that $(Y,Z) \in \mathscr{B}^2_\mu$ follows from computations similar to those in the proof of proposition 1.1, and we have that $(Y,Z) \in \mathscr{B}^2_\mu$ solves (jj) iff it is a fixed point of Φ. We note also that whenever $(U,V) \in \mathscr{B}^2_\mu$ and $(Y,Z) = \Phi(U,V)$,

$$E\left(\sup_{0\le t\le T} \mathrm{e}^{\mu A_t}|Y_t|^2\right) < \infty \ ,$$

and $\{\int_0^t \mathrm{e}^{\mu A_s}\langle Y_s, Z_s\, dB_s\rangle, 0 \le t \le T\}$ is a uniformly integrable martingale. Let $(U,V),(U',V') \in \mathscr{B}^2_\mu$, $(Y,Z) = \Phi(U,V)$, $(Y',Z') = \Phi(U',V')$, $(\bar U,\bar V) = (U-U', V-V')$, $(\bar Y,\bar Z) = (Y-Y', Z-Z')$. For each $\gamma \in \mathbb{R}$, we have from Itô's formula that

$$\begin{aligned}
&E\left(\mathrm{e}^{\gamma t+\mu A_t}|\bar Y_t|^2\right) + E\int_t^T \mathrm{e}^{\gamma s+\mu A_s}\left[\gamma|\bar Y_s|^2\, ds + \mu|\bar Y_s|^2 dA_s + \|\bar Z_s\|^2\, ds\right] \\
&\quad \le 2KE\int_t^T \mathrm{e}^{\gamma_s+\mu A_s}[|\bar Y_s| \times (|\bar U_s| + \|\bar V_s\|)\, ds + |\bar Y_s| \times |\bar U_s|\, dA_s] \\
&\quad \le E\int_t^T \mathrm{e}^{\gamma_s+\mu A_s}|\bar Y_s|^2\left(4K^2\, ds + 2K^2\, dA_s\right) \\
&\qquad + \frac{1}{2}E\int_t^T \mathrm{e}^{\gamma s+\mu A_s}\left[\left(|\bar U_s|^2 + \|\bar V_s\|^2\right) ds + |\bar U_s|^2\, dA_s\right] \ .
\end{aligned}$$

We choose $\gamma = 1+4K^2$, $\mu = 1+2K^2$, and deduce

$$\begin{aligned}
&E\int_0^T \mathrm{e}^{\gamma t+\mu A_t}\left[\left(|\bar Y_t|^2 + \|\bar Z_t\|^2\right) dt + |\bar Y_t|^2\, dA_t\right] \\
&\quad \le \frac{1}{2}E\int_0^T \mathrm{e}^{\gamma t+\mu A_t}\left[\left(|\bar U_t|^2 + \|\bar V_t\|^2\right) dt + |\bar V_t|^2\, dA_t\right] \ ,
\end{aligned}$$

from which it follows that Φ is a strict contraction on $\mathscr{B}^2_\mu$ equipped with the norm

$$|||(Y,Z)|||_{\gamma,\mu} = \left(E\int_0^T \mathrm{e}^{\gamma t+\mu A_t}\left[\left(|Y_t|^2 + \|Z_t\|^2\right) dt + |Y_t|^2 dA_t\right]\right)^{\frac{1}{2}}$$

provided $\gamma \ge 1+4K^2$, $\mu \ge 1+2K^2$. Hence Φ has a unique fixed point. $\diamondsuit$

We next prove existence and uniqueness for the BSDE (j), (jj), under the conditions (i), (ii), (iii), (iv), (v), (vi), (vii), (viii).

Theorem 1.7 *Under the conditions* (i), (ii), (iii), (iv), (v), (vi), (vii), *the BSDE* (j), (jj) *has a unique solution.*

Proof: Uniqueness has been established in corollary 1.4. For the proof of existence, let us first admit the (with the notation $M^2 \stackrel{\triangle}{=} M^2_0$).

Proposition 1.8 *Given* $V \in (M^2)^{k\times d}$, *there exists a unique pair of progressively measurable processes* $\{(Y_t,Z_t);\ 0\le t\le T\}$ *such that*

(j) $E(\sup_{0\le t\le T}|Y_t|^2+\int_0^T\|Z_t\|^2\,dt)<\infty$;
(jj′) $Y_t=\xi+\int_t^T f(s,Y_s,V_s)ds+\int_t^T g(s,Y_s)dA_s-\int_t^T Z_s\,dB_s, 0\le t\le T$.

We shall use the notation $\mathscr{B}^2=(M^2)^k\times(M^2)^{k\times d}$. With the help of proposition 1.8, we construct a mapping Φ from $\mathscr{B}^2$ into itself, which to $(U,V)\in\mathscr{B}^2$ associates $(Y,Z)=\Phi(U,V)$, the solution of (j), (jj). Although the first component U of the pair (U,V) is not needed in order to construct (Y,Z), it is convenient to consider the mapping $(U,V)\to(Y,Z)$ in making a fixed point argument. Let (U,V), $(U',V')\in\mathscr{B}^2$,

$$(Y,Z)=\Phi(U,V),(Y',Z')=\Phi(U',V'),$$
$$(\bar U,\bar V)=(U-U',V-V'),(\bar Y,\bar Z)=(Y-Y',Z-Z')\ .$$

It follows from Itô's formula that

$$\begin{aligned}
&e^{\gamma t}E|\bar Y_t|^2+E\int_t^T e^{\gamma s}\left(\gamma|\bar Y_s|^2+\|\bar Z_s\|^2\right)ds\\
&\le 2E\int_t^T e^{\gamma s}\left(\alpha|\bar Y_s|^2+K|\bar Y_s|\times\|\bar V_s\|\right)ds\\
&\le\frac12 E\int_t^T e^{\gamma s}\left(4\left(\alpha+K^2\right)|\bar Y_s|^2+\|\bar V_s\|^2\right)ds\ .
\end{aligned}$$

Choosing $\gamma=1+2(\alpha+K^2)$, we deduce that

$$E\int_0^T e^{\gamma t}\left(|\bar Y_t|^2+\|\bar Z_t\|^2\right)dt\le\frac12 E\int_t^T e_s^{\gamma t}\left(|\bar U_t|^2+\|\bar V_t\|^2\right)dt\ ,$$

and we have proved that Φ has a unique fixed point, which satisfies (j), (jj′).

Proof of proposition 1.8 We shall write $f(s,y)$ for $f(s,y,V_s)$. Note that $f(s,y)$ satisfies

(ii′) $E\int_0^T|f(t,0)|^2\,dt<\infty$
(iii′) $\langle y-y',f(t,y)-f(t,y')\rangle\le\alpha|y-y'|^2$
(vi′) $|f(t,y)|\le\varphi'_t+K|y|$, where $E\int_0^T e^{\mu A_t}(\varphi'_t)^2\,dt<\infty$
(vii′) $y\longrightarrow f(t,y)$ is continuous, $dP\times dt$ *a.e.*

We define

$$f_n(t,y)\stackrel{\triangle}{=}(\rho_n*f(t,\cdot))(y),\quad g_n(t,y)\stackrel{\triangle}{=}(\rho_n*g(t,\cdot))(y)\ ,$$

where $\rho_n:\mathbb{R}^k\longrightarrow\mathbb{R}_+$ is a sequence of smooth functions which approximates the Dirac measure at 0 and satisfies

$$\int \rho_n(z)\,dz = 1, \quad \sup_n \int |z|\rho_n(z)\,dz < \infty \ .$$

f_n satisfies again (iii′), (vi′), and g_n satisfies (iv), (vi), with the same constants α, β and K and possibly larger (but uniform in n) φ'_t and ψ_t. For each n, f_n and g_n are Lipschitz in y, uniformly with respect to (t,ω), hence from Theorem 1.6 the BSDE

$$Y^n_t = \xi + \int_t^T f_n(s, Y^n_s)\,ds + \int_t^T g_n(s, Y^n_s)\,dA_s - \int_t^T Z^n_s\,dB_s, \ \ 0 \le t \le T \ ,$$

has a unique solution which satisfies

$$E\left(\sup_{0\le t\le T} |Y^n_t|^2 + \int_0^T \|Z^n_t\|^2\,dt\right) < \infty \ ,$$

from theorem 1.6. Moreover

$$\begin{aligned} |Y^n_t|^2 + \int_t^T \|Z^n_s\|^2\,ds = |\xi|^2 &+ 2\int_t^T \langle Y^n_s, f_n(s, Y^n_s)\rangle\,ds \\ &+ 2\int_t^T \langle Y^n_s, g_n(s, Y^n_s)\rangle\,dA_s \\ &- 2\int_t^T \langle Y^n_s, Z^n_s\,dB_s\rangle \end{aligned}$$

$$\begin{aligned} &E|Y^n_t|^2 + E\int_t^T \|Z^n_s\|^2\,ds = \\ &\quad E|\xi|^2 + 2E\int_t^T \langle Y^n_s, f_n(s, Y^n_s)\rangle\,ds + 2E\int_t^T \langle Y^n_s, g(s, Y^n_s\rangle)\rangle\,dA_s \\ &\le E|\xi|^2 + 2\alpha E\int_t^T |Y^n_s|^2\,ds + 2E\int_t^T |Y^n_s| \times |f_n(s,0)|\,ds \\ &\quad + 2\beta\int_t^T |Y^n_s|^2 dA_s + 2E\int_t^T |Y^n_s| \times |g_n(s,0)|\,dA_s \ , \end{aligned}$$

hence

$$E|Y^n_t|^2 + E\int_t^T \|Z^n_s\|^2\,ds + |\beta|E\int_t^T |Y^n_s|^2\,dA_s \le C\left(1 + E\int_t^T |Y^n_s|^2\,ds\right),$$

and from this, Gronwall's lemma and the Burkholder inequality we have that

$$\sup_n E\left(\sup_{0\le t\le T} |Y^n_t|^2 + \int_0^T |Y^n_t|^2\,dA_t + \int_0^T \|Z^n_t\|^2\,dt\right) < \infty$$

Moreover, if we define

$$U_t^n \triangleq f_n\big(t, Y_t^n\big), \quad V_t^n \triangleq g_n\big(t, Y_t^n\big) \ ,$$

we have that

$$\sup_n E\bigg(\int_0^T |U_t^n|^2\, dt + \int_0^T |V_t^n|^2\, dA_t\bigg) < \infty \ .$$

Consequently, after extraction of a subsequence, we have that

$$(Y^n, Z^n, U^n) \longrightarrow (Y, Z, U)$$

weakly in

$$L^2\big(\Omega \times (0,T), dP \times\, dt, \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k\big)$$

and also

$$V^n \longrightarrow V$$

weakly in $L^2(\Omega \times (0,T),\ P(d\omega) \times A(\omega, dt),\ \mathbb{R}^k)$. It is then easy to deduce that

$$Y_t = \xi + \int_t^T U_s\, ds + \int_t^T V_s\, dA_s - \int_t^T Z_s\, dB_s,\ 0 \le t \le T \ .$$

It remains to show that $U_t = f(t, Y_t)$ and $V_t = g(t, Y_t)$. For each $n \in \mathbb{N}$, X, X' progressively measurable k–dimensional processes such that

$$E\int_0^T |X_t|^2\, dt < \infty,\ E\int_0^T |X_t'|^2\, dA_t < \infty \ ,$$

$$E\int_0^T \mathrm{e}^{\alpha t}\langle Y_t^n - X_t, f_n\big(t, Y_t^n\big) - f_n(t, X_t) - \alpha(Y_t^n - X_t)\rangle\, dt$$
$$+ E\int_0^T \mathrm{e}^{\alpha t}\langle Y_t^n - X_t', g_n\big(t, Y_t^n\big) - g_n(t, X_t')\rangle\, dA_t \le 0 \ .$$

Since, as $n \to \infty$,

$$E\bigg(\int_0^T |f_n(t, X_t) - f(t, X_t)|^2\, dt + \int_0^T |g_n\big(t, X_t'\big) - g\big(t, X_t'\big)|^2\, dA_t\bigg) \to 0 \ ,$$

$$\limsup_{n\to\infty} E\int_0^T \mathrm{e}^{\alpha t}\langle Y_t^n - X_t, f_n\big(t, Y_t^n\big) - f(t, X_t) - \alpha(Y_t - X_t)\rangle\, dt$$
$$+ E\int_0^T \mathrm{e}^{\alpha t}\langle Y_t^n - X_t', g_n(t, Y_t^n) - g(t, X_t')\rangle\, dA_t \ \le\ 0 \ .$$

Moreover

$$2E\int_0^T e^{\alpha t}\langle Y_t^n, f_n(t, Y_t^n) - \alpha Y_t^n\rangle\, dt + 2E\int_0^T e^{\alpha t}\langle Y_t^n, g_n\big(t, Y_t^n\big)\rangle\, dA_t$$
$$= |Y_0^n|^2 - e^{\alpha T}E|\xi|^2 + E\int_0^T e^{\alpha t}\|Z_t^n\|^2\, dt \ ,$$

$Y_0^n \to Y_0$ in $\mathbb{R}^k$, and the mapping

$$Z \longrightarrow E\int_0^T e^{\alpha t}\|Z_t\|^2\, dt$$

being convex and continuous for the strong topology of $(M^2)^{k\times d}$, it is weakly l.s.c., hence

$$\liminf_{n\to\infty}\ 2E\int_0^T e^{\alpha t}\langle Y_t^n, f_n(t, Y_t^n) - \alpha Y_t^n\rangle\, dt + 2E\int_0^T e^{\alpha t}\langle Y_t^n, g_n\big(t, Y_t^n\big)\rangle\, dA_t$$
$$\geq |Y_0|^2 - e^{\alpha T}E|\xi|^2 + E\int_0^T e^{\alpha t}\|Z_t|^2\, dt$$
$$= E\int_0^T e^{\alpha t}\langle Y_t, U_t - \alpha Y_t\rangle\, dt + E\int_0^T e^{\alpha t}\langle Y_t, V_t\rangle\, dA_t \ .$$

Combining this with weak convergence and the above inequality, we deduce that

$$E\int_0^T e^{\alpha t}\langle Y_t - X_t, U_t - f(t, X_t) + \alpha(Y_t - X_t)\rangle\, dt$$
$$+ E\int_0^T e^{\alpha t}\langle Y_t - X_t', V_t - g(t, X_t')\rangle\, dA_t \leq 0 \ .$$

We finally choose $X_t = Y_t - \varepsilon(U_t - f(t, Y_t))$, $X_t' = Y_t - \varepsilon(V_t - g(t, Y_t))$, $\varepsilon > 0$, divide by ε the resulting inequality, and let $\varepsilon \to 0$, yielding

$$E\int_0^T e^{\alpha t}|U_t - f(t, Y_t)|^2\, dt + E\int_0^T e^{\alpha t}|V_t - g(t, Y_t)|^2\, dA_t \leq 0 \ .$$

The result follows.

2. Infinite horizon generalized BSDEs

In this section, we want to solve the BSDE

(jj′) $\qquad Y_t = \int_t^\infty f(s, Y_s, Z_s)\, ds + \int_t^\infty g(s, Y_s)\, dA_s - \int_t^\infty Z_s\, dB_s,$

where essentially $\{Y_t\}$ starts from 0 at time $+\infty$. We shall assume that f and g satisfy (i), (iii), (iv), (v), (vi) and (vii), and moreover that there exists

(viii) $\lambda > 2\alpha + K^2$, $\mu > 2\beta$,

such that

(ii″) $E\int_0^\infty e^{\lambda t+\mu A_t}(\varphi_t^2\, dt + \psi_t^2\, dA_t) < \infty$.

We have the

Theorem 2.1 *Under conditions* (i), (ii″), (iii), (iv), (v), (vi), *and* (vii), *there exists a unique progressively measurable process* $\{(Y_t, Z_t); t \geq 0\}$ *with values in* $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ *such that*

$$E\left(\sup_t e^{\lambda t+\mu A_t}|Y_t|^2 + \int_0^\infty e^{\lambda t+\mu A_t}\left[\left(|Y_t|^2 + \|Z_t\|^2\right) dt + |Y_t|^2\, dA_t\right]\right) < \infty \ , \tag{2}$$

and for any $0 \leq t \leq T$,

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s,)\, dA_s - \int_t^T Z_s\, dB_s \ . \tag{3}$$

Proof: Proof of uniqueness: Let (Y_t, Z_t) and (Y_t', Z_t') be two solutions of (2), (3), and $(\bar{Y}_t, \bar{Z}_t) = (Y_t - Y_t', Z_t - Z_t')$. It follows from Itô's formula, the assumptions (iii), (iv) and (v) that

$$\begin{aligned} e^{\lambda t+\mu A_t}|\bar{Y}_t|^2 &+ \int_t^T e^{\lambda s+\mu A_s}\left(\lambda|\bar{Y}_s|^2\, ds + \mu|\bar{Y}_s|^2 dA_s + \|\bar{Z}_s\|^2\, ds\right) \\ &\leq e^{\lambda T+\mu A_T}|\bar{Y}_T|^2 + 2\int_t^T e^{\lambda s+\mu A_s}\left(\alpha|\bar{Y}_s|^2 + K|\bar{Y}_s| \times \|\bar{Z}_s\|\right) ds \\ &\quad + 2\beta\int_t^T e^{\lambda s+\mu A_s}|\bar{Y}_s|^2\, dA_s - 2\int_t^T e^{\lambda s+\mu A_s}\langle \bar{Y}_s, \bar{Z}_s\, dB_s\rangle \ . \end{aligned}$$

Hence, with $\rho < 1$, $\bar{\lambda} = \lambda - 2\alpha - \rho^{-1}K^2 > 0$, $\bar{\mu} = \mu - 2\beta > 0$,

$$\begin{aligned} &E e^{\lambda t+\mu A_t}|\bar{Y}_t|^2 + E\int_t^T e^{\lambda s+\mu A_s}\left[\bar{\lambda}|\bar{Y}_s|^2\, ds + \bar{\mu}|\bar{Y}_s|^2\, dA_s + (1-\rho)\|\bar{Z}_s\|^2\, ds\right] \\ &\leq E e^{\lambda T+\mu A_T}|\bar{Y}_T|^2 \ , \end{aligned}$$

and consequently

$$E\left(e^{\lambda t+\mu A_t}|\bar{Y}_t|^2\right) \leq E\left(e^{\lambda T+\mu A_T}|\bar{Y}_T|^2\right) \ .$$

The same result holds true with λ replaced by λ' such that $2\alpha + K^2 < \lambda' < \lambda$. Hence, from (2),

$$E\left(e^{\lambda' t+\mu A_t}|\bar{Y}_t|^2\right) \leq Ce^{(\lambda'-\lambda)T} \ ,$$

and this tends to zero as $T \to \infty$.

Proof of existence: For each $n \in \mathbf{N}$, let $\{(Y_t^n, Z_t^n) \ ; \ t \geq 0\}$ denote the solution of

$$Y_t^n = \int_t^n f\left(s, Y_s^n, Z_s^n\right) ds + \int_t^n g\left(s, Y_s^n\right) dA_s - \int_t^n Z_s^n \, dB_s, \ 0 \leq t \leq n,$$
$$Y_t^n = 0, \ Z_t^n = 0, \ t > n,$$

given by theorem 1.7.

We first prove that there exists a constant C such that for all $s \geq 0$,

$$E\left(\sup_{t\geq s} e^{\lambda t+\mu A_t}|Y_t^n|^2 + \int_s^\infty e^{\lambda t+\mu A_t}\left[\left(|Y_t^n|^2 + \|Z_t^n\|^2\right) dt + |Y_t^n|^2 \, dA_t\right]\right)$$
$$\leq CE\int_s^\infty e^{\lambda t+\mu A_t}\left[|f(t,0,0)|^2 \, dt + |g(t,0,0)|^2 \, dA_t\right] \ . \tag{4}$$

We shall use the fact that for any arbitrarily small $\varepsilon > 0$ and any $\rho < 1$ arbitrarily close to one, there exists a constant C such that for all $t > 0$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$2\langle y, f(t,y,z)\rangle \leq \left(2\alpha + \rho^{-1}K^2 + \varepsilon\right)|y|^2 + \rho\|z\|^2 + c|f(t,0,0)|^2,$$
$$2\langle y, g(t,y,z)\rangle \leq (2\beta + \varepsilon)|y|^2 + c|g(t,0)|^2$$

From these and Itô's formula, we deduce that

$$e^{\lambda t+\mu A_t}|Y_t^n|^2 + \int_t^\infty e^{\lambda s+\mu A_s}\left[\left(\bar{\lambda}|Y_s^n|^2 + \bar{\rho}\|Z_s^n\|^2\right) ds + \bar{\mu}|Y_s^n|^2 dA_s\right]$$
$$\leq c\int_t^\infty e^{\lambda s+\mu A_s}\left[|f(s,0,0)|^2 \, ds + |g(s,0,0)|^2 dA_s\right]$$
$$- 2\int_t^\infty e^{\lambda s+\mu A_s}\langle Y_s^n, Z_s^n dB_s\rangle \ ,$$

where $\bar{\lambda} = \lambda - 2\alpha - \rho^{-1}K^2 - \varepsilon$, $\bar{\rho} = 1 - \rho$ and $\bar{\mu} = \mu - 2\beta - \varepsilon$. We assume that ε and ρ have been chosen in such a way that $\bar{\lambda} > 0$, $\bar{\rho} > 0$ and $\bar{\mu} > 0$. (4) then follows from Burkholder's inequality. Let now $m > n$. We have that

$$E\left(e^{\lambda n+\mu A_n}|Y_n^m|^2\right) \leq CE\int_n^m e^{\lambda t+\mu A_t}\left(|f(t,0,0)|^2 \, dt + |g(t,0,0)|^2 \, dA_t\right) \ . \tag{5}$$

We next define $\Delta Y_t = Y_t^m - Y_t^n$, $\Delta Z_t = Z_t^m - Z_t^n$, and note that

$$\Delta Y_t = Y_n^m + \int_t^n \left(f(Y_s^m, Z_s^m) - f\left(Y_s^n, Z_s^n\right)\right) ds \\ + \int_t^n (g(Y_s^m) - g(Y_s^n))\, dA_s - \int_t^n \Delta Z_s\, dB_s,\ 0 \le t \le n\ .$$

We deduce from Itô's formula and a by now standard procedure that

$$\mathrm{e}^{\lambda t+\mu A_t}|\Delta Y_t|^2 + \int_t^n \mathrm{e}^{\lambda s+\mu A_s}\left[\left(\bar{\lambda}|\Delta Y_s|^2 + \bar{\rho}\|\Delta Z_s\|^2\right) ds + \bar{\mu}|\Delta Y_s|^2 dA_s\right] \\ \le \mathrm{e}^{\lambda n+\mu A_n}|Y_n^m|^2 - 2\int_t^n \mathrm{e}^{\lambda s+\mu A_s}\langle \Delta Y_s, \Delta Z_s dB_s\rangle\ .$$

From this and (5), we deduce that

$$E\left(\sup_t \mathrm{e}^{\lambda t+\mu A_t}|\Delta Y_t|^2 + \int_0^\infty \mathrm{e}^{\lambda s+\mu A_s}\left[\left(|\Delta Y_s|^2 + \|\Delta Z_s\|^2\right) ds + |\Delta Y_s|^2 dA_s\right]\right) \\ \le CE\int_n^\infty \mathrm{e}^{\lambda t+\mu A_t}\left(|f(t,0,0)|^2 dt + |g(t,0)|^2 dA_t\right)\ .$$

Taking into account the assumption (ii), we deduce from this inequality that (Y^n, Z^n) is a Cauchy sequence for the norm whose square appears in (2). Hence it converges to a limit (Y, Z), which clearly satisfies (2) and (3).

3. A class of reflected diffusion process

We now introduce a class of reflected diffusion processes. Let G be an open connected bounded subset of $\mathbb{R}^d$, which is such that for a function $\phi \in C_b^2(\mathbb{R}^d)$, $G = \{\phi > 0\}$, $\partial G = \{\phi = 0\}$, and $|\nabla\phi(x)| = 1$, $x \in \partial G$. Note that at any boundary point $x \in \partial G$, $\nabla\phi(x)$ is a unit normal vector to the boundary, pointing towards the interior of G. Let $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ be coefficients satisfying for some $K > 0$, all $x, y \in \mathbb{R}^d$:

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| \le K|x - y|$$

It follows from the results in Lions, Sznitman [6] (see also Saisho [10]) that for each $x \in \bar{G}$ there exists a unique pair of progressively measurable continuous processes $\{(X_t^x, K_t^x);\ t \ge 0\}$, with values in $\bar{G} \times \mathbb{R}_+$, such that

$$X_t^x = x + \int_0^t b(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dB_s + \int_0^t \nabla\phi(X_s^x)\,dK_s^x,\ \ t \geq 0;$$
$$K_t^x = \int_0^t \mathbf{1}_{\{X_s^x \in \partial G\}}\,dK_s^x, \quad K_\cdot^x \text{ is increasing.} \tag{6}$$

It is not hard to see that the above assumptions imply that there exists a constant $\alpha > 0$ such that for any $x \in \partial G, x' \in \bar{G}$,

$$|x - x'|^2 + \alpha\langle x' - x, \phi(x)\rangle \geq 0\ . \tag{7}$$

We have the

Proposition 3.1 *For each* $T \geq 0$, *there exists a constant* C_T *such that for all* $x, x' \in \bar{G}$,

$$E\left(\sup_{0\leq t\leq T} |X_t^x - X_t^{x'}|^4\right) \leq C_T|x - x'|^4\ .$$

Proof: As in Lions, Sznitman [6] page 524, we develop using Itô's formula the semimartingale:

$$\exp\left[-\frac{1}{\alpha}\left(\phi(X_t^x) + \phi(X_t^{x'})\right)\right] \times |X_t^x - X_t^{x'}|^2\ ,$$

and exploit the inequality (7). We end up with

$$E\left(\sup_{0\leq s\leq t} |X_s^x - X_s^{x'}|^4\right) \leq C\left(|x - x'|^4 + E\int_0^t \left|X_s^x - X_s^{x'}\right|^4 ds\right)\ .$$

The result then follows from Gronwall's lemma. ◊

It follows from Itô's formula that

$$K_t^x = \phi(X_t^x) - \phi(x) - \int_0^t L\phi(X_s^x)\,ds - \int_0^t \nabla\phi(X_s^x)\sigma(X_s^x)\,dB_s\ ,$$

where L is the second order partial differential operator

$$L = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i}$$

From this identity and proposition 3.1, we deduce easily the

Proposition 3.2 *For each* $T > 0$, *there exists a constant* C_T *such that for all* $x, x' \in G$,

$$E\left(\sup_{0\leq t\leq T} \left|K_t^x - K_t^{x'}\right|^4\right) \leq C_T|x - x'|^4$$

Moreover, for all $p \geq 1$, *there exists a constant* C_p *such that for all* $(t,x) \in \mathbb{R}_+ \times \bar{G}$,

$$E\left(|K_t^x|^p\right) \leq C_p(1+t^p) \ ,$$

and for each μ, $t > 0$, *there exists* $C(\mu, t)$ *such that for all* $x \in \bar{G}$,

$$E\left(e^{\mu K_t^x}\right) \leq C(\mu, t) \ . \qquad \diamondsuit$$

Finally, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,we define $\{(X_s^{t,x} K_s^{t,x}; s \geq 0\}$ as the unique solution of:

$$X_s^{t,x} = x + \int_t^{t\vee s} b\left(X_r^{t,x}\right) ds + \int_t^{t\vee s} \sigma\left(X_r^{t,x}\right) dB_r + \int_t^{t\vee s} \nabla\phi\left(X_r^{t,x}\right) dK_r^{t,x}, s \geq 0;$$

$$K_s^{t,x} = \int_t^{t\vee s} \mathbf{1}_{\{X_r^{t,x} \in \partial G\}} \, dK_r^{t,x}, K_\cdot^{t,x} \text{ is increasing } . \tag{8}$$

The two above propositions extend trivially to this case.

4. Probabilistic formula for the solution of a system of parabolic PDEs with nonlinear Neumann boundary condition

We fix $T > 0$. For each $(t,x) \in [0,T] \times \bar{G}$, let $\{(X_s^{t,x}, K_s^{t,x}); s \geq 0\}$ denote the solution of the reflected SDE (8). Let $h \in C(\bar{G}; \mathbb{R}^k)$, $f \in C([0,T] \times \bar{G} \times \mathbb{R}^k \times \mathbb{R}^{k\times d}; \mathbb{R}^k)$, $g \in C([0,T] \times \bar{G} \times \mathbb{R}^k; \mathbb{R}^k)$ satisfy the assumptions

(4.iii) $\langle y - y', f(t,x,y,z) - f(t,x,y',z)\rangle \leq \alpha|y - y'|^2$
(4.iv) $\langle y - y', g(t,x,y) - g(t,x,y')\rangle \leq \beta|y - y'|^2$
(4.v) $|f(t,x,y,z) - f(t,x,y,z')| \leq K\|z - z'\|$
(4.vi) $|f(t,x,y,z)| + |g(t,y)| \leq K(1 + |y| + \|z\|)$,

and for each $(t,x) \in [0,T] \times \bar{G}$, let $\{(Y_s^{t,x}, Z_s^{t,x}); t <\leq s \leq T\}$ be the unique solution of the BSDE

(4.j) $E(\sup_{t\leq s\leq T} |Y_s^{t,x}|^2 + \int_t^T \|Z_s^{t,x}\|^2 ds) < \infty$
(4.jj) $Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\, dr + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x})\, dK_r^{t,x} - \int_s^T Z_r^{t,x}\, dB_r,\ t \leq s \leq T.$

Existence and uniqueness follow from theorem 1.7. We define

$$u(t,x) \stackrel{\triangle}{=} Y_t^{t,x}, (t,x) \in [0,T] \times \bar{G} \ , \tag{9}$$

which is clearly a deterministic quantity, since $Y_s^{t,x}$ is measurable with respect to the σ-algebra $\sigma\{B_r - B_t, t \leq r \leq s\}$. Tedious but standard estimates permit to deduce from proposition 1.1 and 1.2, 3.1 and 3.2 the

Proposition 4.1

$$u \in C\big([0,T] \times \bar{G}; \mathbb{R}^k\big)$$

Let us now introduce the system of parabolic PDEs, of which u will be a solution. First of all, we shall make one restriction, which is due to the fact that we want to consider viscosity solutions of our system of PDEs. We assume that for each $1 \le i \le k$, f_i, the i-th coordinate of f, depends only on the i-th row of the matrix z, and not on the other rows of z. Consider the system of semi-linear PDE:

$$\begin{aligned}
&\frac{\partial u_i}{\partial t}(t,x) + Lu_i(t,x) + f_i(t,x,u(t,x),(\nabla u_i\sigma)(t,x)) = 0,\\
&\qquad 1 \le i \le k,\ \ 0 < t < T,\ \ x \in G;\\
&\frac{\partial u_i}{\partial n}(t,x) + g_i(t,x,u(t,x)) = 0,\\
&\qquad 1 \le i \le k,\ \ 0 < t < T,\ \ x \in \partial G;\\
&u(T,x) = h(x),\ \ x \in \bar{G},
\end{aligned} \tag{10}$$

where at a point $x \in \partial G$

$$\frac{\partial}{\partial n} = \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i}(x)\frac{\partial}{\partial x_i} \quad .$$

We now explain what we mean by a viscosity solution of (10).

Definition 4.2 (*a*) $u \in C([0,T] \times \bar{G}, \mathbb{R}^k)$ *is called a viscosity subsolution of* (10) *if* $u_i(T,x) \le h_i(x)$, $x \in \bar{G}$, $1 \le i \le k$, *and moreover for any* $1 \le i \le k$, $\varphi \in C^{1,2}[0,T] \times \bar{\mathbb{R}})$, *whenever* $(t,x) \in [0,T] \times \bar{G}$ *is a local maximum of* $u_i - \varphi$, *then*

$$-\frac{\partial \varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \le 0,\ \ if\ x \in G$$

$$\begin{aligned}
\min\Bigg(&-\frac{\partial \varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),\nabla\varphi\sigma(t,x)),\\
&-\frac{\partial \varphi}{\partial n}(t,x) - g_i(t,x,u(t,x))\Bigg) \le 0,\ \ if\ \ x \in \partial G\ \ .
\end{aligned}$$

(*b*) $u \in C([0,T] \times \bar{G}, \mathbb{R}^k)$ *is called a viscosity supersolution of* (10) *if* $u_i(T,x) \ge h_i(x)$, $x \in \bar{G}$, $1 \le i \le k$, *and moreover for any* $1 \le i \le k$, $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$, *whenever* $(t,x) \in [0,T] \times \bar{G}$ *is a local minimum of* $u_i - \varphi$, *then*

$$-\frac{\partial\varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \geq 0,\ \textit{if}\ x \in G,$$

$$\max\bigg(-\frac{\partial\varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)),$$
$$-\frac{\partial\varphi}{\partial n}(t,x) - g_i(t,x,u(t,x))\bigg) \geq 0,\ \textit{if}\ x \in \partial G.$$

(*c*) $u \in C([0,T]\times\bar{G},\mathbb{R}^k)$ *is called a viscosity solution of* (10) *if it is both a viscosity sub- and supersolution.*

It can be deduced from the uniqueness theorem for BSDEs that

$$Y^{t,x}_{t+h} = Y^{t+h,X^{t,x}_{t+h}}_{t+h}, \quad h > 0 \ .$$

This implies that $Y^{t,x}_s = u(s,X^{t,x}_s)$, $t \leq s \leq T$.

We can now prove the

Theorem 4.3 u, *defined by* (9), *is a viscosity solution of the system of parabolic PDEs* (10).

Proof: We note that clearly $u(T,x) = h(x)$. We shall prove that u is a viscosity subsolution. The property of being a supersolution can be proved similarly. Take any $1 \leq i \leq k$, $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$, and $(t,x) \in [0,T]\times\bar{G}$ such that (t,x) is a local maximum of $u_i - \varphi$, and $u_i(t,x) = \varphi(t,x)$.

We first consider the case $x \in G$. We suppose that

$$\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(t,x) + f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) < 0 \ ,$$

and we will find a contradiction.

Let $0 < \alpha \leq T - t$ be such that $\{y;\ |y-x| \leq \alpha\} \subset G$,

$$\sup_{t\leq s\leq t+\alpha,|y-x|\leq\alpha}\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s,y) + f_i(s,y,u(s,y),(\nabla\varphi\sigma)(s,y)) < 0$$
$$u_i(s,y) \leq \varphi(s,y),\ t \leq s \leq t+\alpha, |y-x| \leq \alpha$$

and define

$$\tau = \inf\{s \geq t;\ |X^{t,x}_s - x| \geq \alpha\} \wedge (t+\alpha) \ .$$

Let now

$$(\overline{Y}_s, \overline{Z}_s) = \left(\left(Y^{t,x}_{s\wedge\tau}\right)^i, \mathbf{1}_{[0,\tau]}(s)\left(Z^{t,x}_s\right)^i\right),\ \ t \leq s \leq t+\alpha \ .$$

$(\overline{Y},\overline{Z})$ solves the one-dimensional BSDE

$$\overline{Y}_s = u_i\big(\tau, X_\tau^{t,x}\big) + \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) f_i\big(r, X_r^{t,x}, u(r, X_r^{t,x}), \overline{Z}_r\big)\, dr$$
$$- \int_s^{t+\alpha} \overline{Z}_r \, dB_r, \;\; t \le s \le t+\alpha \;\; .$$

On the other hand, from Itô's formula,

$$\Big(\widehat{Y}_s, \widehat{Z}_s\Big) = (\varphi(s, X_{s\wedge\tau}^{t,x}), \mathbf{1}_{[0,\tau]}(s)(\nabla\varphi\sigma)(s, X_s^{t,x})), \;\; t \le s \le t+\alpha$$

solves the BSDE

$$\widehat{Y}_s = \varphi\big(\tau, X_\tau^{t,x}\big) - \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) \left(\frac{\partial\varphi}{\partial r} + L\varphi\right)\big(r, X_r^{t,x}\big)\, dr$$
$$- \int_s^{t+\alpha} \widehat{Z}_r \, dB_r, \;\; t \le s \le t+\alpha \;\; .$$

From $u_i \le \varphi$, and the choice of α and τ, we deduce with the help of the comparison theorem 1.4 that $\overline{Y}_0 < \widehat{Y}_0$, i.e. $u_i(x) < \varphi(x)$, which is a contradiction.

We now consider the case $x \in \partial G$. We suppose that

$$\max\left(\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(t,x) + f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)),\right.$$
$$\left.\frac{\partial\varphi}{\partial n}(t,x) + g_i(t,x,u(t,x))\right) < 0 \;\; ,$$

and we will find a contradiction.

Let $0 < \alpha \le T - t$ be such that

$$\sup_{t \le s \le t+\alpha, |y-x| \le \alpha} \max\left(\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s,y) + f_i(s,y,u(s,y),(\nabla\varphi\sigma)(s,y)),\right.$$
$$\left.\frac{\partial\varphi}{\partial n}(s,y) + g_i(s,y,u(s,y))\right) < 0 \;\; ,$$

and define

$$\tau = \inf\{s \ge t; \; |X_s^{t,x} - x| \ge \alpha\} \wedge (t+\alpha) \;\; .$$

Let now

$$(\overline{Y}_s, \overline{Z}_s) = ((Y_{s\wedge\tau}^{t,x})^i, \mathbf{1}_{[0,\tau]}(s)(Z_s^{t,x})^i), t \le s \le t+\alpha \;\; .$$

$(\overline{Y}, \overline{Z})$ solves the one-dimensional BSDE

$$\overline{Y}_s = u_i\big(\tau, X_\tau^{t,x}\big) + \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) f_i\big(r, X_r^{t,x}, u\big(r, X_r^{t,x}\big), \overline{Z}_r\big)\, dr$$
$$+ \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) g_i\big(r, X_r^{t,x}, u(r, X_r^{t,x})\big)\, dK_r^{t,x}$$
$$- \int_s^{t+\alpha} \overline{Z}_r\, dB_r, \quad t \le s \le t+\alpha \ .$$

On the other hand, from Itô's formula,

$$\Big(\widehat{Y}_s, \widehat{Z}_s\Big) = \big(\varphi(s, X_{s\wedge\tau}^{t,x}), \mathbf{1}_{[0,\tau]}(s)(\nabla\varphi\sigma)(s, X_s^{t,x})\big), \quad t \le s \le t+\alpha$$

solves the BSDE

$$\widehat{Y}_s = \varphi\big(\tau, X_\tau^{t,x}\big) - \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r)\left(\frac{\partial\varphi}{\partial r} + L\varphi\right)\big(r, X_r^{t,x}\big)\, dr$$
$$+ \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) \frac{\partial\varphi}{\partial n}\big(r, X_r^{t,x}\big)\, dK_r^{t,x} - \int_s^{t+\alpha} \widehat{Z}_r\, dB_r, t \le s \le t+\alpha \ .$$

From $u_i \le \varphi$, and the choice of α and τ, we deduce with the help of the comparison theorem 1.4 that $\overline{Y}_0 < \widehat{Y}_0$, i.e. $u_i(x) < \varphi(x)$, which is a contradiction. $\diamondsuit$

We have proved that a certain function of (t, x), defined through the solution of a probabilistic problem, is the solution of a system of backward parabolic partial differential equations. Suppose that f and g do not depend on t, and let

$$v(t,x) = u(T-t, x),\ (t,x) \in [0,T] \times \bar{G}$$

The v solves the system of forward parabolic PDEs:

$$\frac{\partial v}{\partial t}(t,x) = Lv(t,x) + f(x, v, (t,x), (\nabla v\sigma)(t,x)), \quad t > 0, x \in G;$$
$$\frac{\partial v}{\partial n}(t,x) + g(x, v(t,x)) = 0, \qquad t > 0, \quad x \in \partial G;$$
$$v(0,x) = h(x), \quad x \in \bar{G} \ .$$

On the other hand, we have that

$$v(t,x) = Y_{T-t}^{T-t,x} = \bar{Y}_0^{t,x} \ ,$$

where $\{(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x}); 0 \le s \le t\}$, solves the BSDE

$$\bar{Y}_s^{t,x} = h(X_t^x) + \int_s^t f\big(X_r^x, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}\big)\, dr$$
$$+ \int_s^t g\big(X_r^x, \bar{Y}_r^{t,x}\big)\, dK_r^x - \int_s^t \bar{Z}_r^{t,x}\, dB_r,\ 0 \le s \le t \ .$$

So we have a probabilistic representation for a system of forward parabolic PDEs, which is valid on $\mathbb{R}_+ \times \bar{G}$.

5. Probabilistic interpretation of the solution of a system of elliptic PDEs with nonlinear Neumann boundary condition

Let $\{(X_t^x, K_t^x)\ ;\ t \geq 0\}$ be the solution of the reflected BSDE (6), and $h \in C(\bar{G}; \mathbb{R}^k)$, $f \in C_b(\bar{G} \times \mathbb{R}^k \times \mathbb{R}^{k\times d}; \mathbb{R}^k)$, $g \in C(\bar{G} \times \mathbb{R}^k; \mathbb{R}^k)$ satisfy (4.iii), (4.iv), (4.v) and (4.vi), with $2\alpha + K^2 < 0$, $\beta < 0$. Assume that for all $x \in \bar{G}$, some λ, μ such that $2\alpha + K^2 < \lambda < 0$, $2\beta < \mu < 0$,

$$E\int_0^\infty e^{\lambda t + \mu K_t^x}(|f(X_t^x, 0, 0)|^2\, dt + |g(X_t^x, 0)|^2 dK_t^x) < \infty \ .$$

Consequently, from theorem 2.1, there exists a unique pair $\{(Y_t^x, Z_t^x),\ t \geq 0\}$, progressively mesurable and $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ valued, such that
(5.j) $E(\sup_t e^{\lambda t + \mu K_t^x}|Y_t^x|^2 + \int_0^\infty e^{\lambda t + \mu A_t}[(|Y_t^x|^2 + \|K_t^x\|^2)dt + |Y_t^x|^2 dK_t^x]) < \infty$,
(5.jj) $Y_t^x = Y_T^x + \int_t^T f(X_s^x, Y_s^x, Z_s^x)\, ds + \int_t^T g(X_s^x, Y_s^x)\, dK_s^x - \int_t^T Z_s^x dB_s$, $0 \leq t \leq T$.
We define

$$u(x) \triangleq Y_0^x \ . \tag{11}$$

One can prove the:

Theorem 5.1 $u \in C(\bar{G}; \mathbb{R}^k)$.

We again assume that for each $1 \leq i \leq k$, f_i, the i-th coordinate of f, depends only on the i-th row of the matrix z, and not on the other rows of z.

Consider the system of semi-linear elliptic PDEs:

$$\begin{aligned} Lu_i(x) + f_i(x, u(x), (\nabla u_i \sigma)(x)) &= 0, \quad 1 \leq i \leq k,\ \ x \in G \\ \frac{\partial u_i}{\partial n}(x) + g_i(x, u(x)) &= 0, \quad 1 \leq i \leq k,\ \ x \in \partial G \end{aligned} \tag{12}$$

We will not prove that u, defined by (11), is a viscosity solution of (12). The notion of viscosity solution of (12) is analogous to that for the parabolic case. Let us just state the

Definition 5.2 $u \in C(\bar{G};\mathbb{R}^k)$ *is called a viscosity subsolution of* (12) *if for all* $1 \leq i \leq k$, $\varphi \in C^2(\mathbb{R}^d)$, *whenever* $x \in \bar{G}$ *is a local maximum of* $u - \varphi$,

$$-L\varphi(x) - f_i(x, u(x), (\nabla\varphi\sigma)(x)) \leq 0, \quad if\ \ x \in G,$$
$$\min\left(-L\varphi(x) - f_i(x, u(x), (\nabla\varphi\sigma)(x)), -\frac{\partial\varphi}{\partial n}(x) - g_i(x, u(x))\right)$$
$$\leq 0 \ \ if x \in \partial G. \qquad \diamondsuit$$

The following theorem is proved exactly as in the parabolic case.

Theorem 5.3 *u, given by the formula* (11), *is a viscosity solution of the system of elliptic PDEs* (12).

Remark 4 Uniqueness results of viscosity solutions of elliptic equations with nonlinear Neumann boundary condition can be found in Barles [1] and in Crandall, Ishii, Lions [3], section 7B. Uniqueness results for viscosity solutions of systems of parabolic PDEs in the whole space can be found in Barles, Buckdahn, Pardoux [2].

Acknowledgement. The authors want to thank an anonymous Referee, whose comments have led to an improvement of this paper.

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