BOUNDARY VALUE PROBLEMS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

By D. Nualart and E. Pardoux

Universitat de Barcelona and Université de Provence and INRIA

In this paper, we study stochastic differential equations with boundary conditions at the endpoints of a time interval (instead of the customary initial condition). We present existence and uniqueness results and study the Markov property of the solution. In the one-dimensional case, we prove that the solution is a Markov field iff the drift is affine.

1. Introduction. This paper is concerned with stochastic differential equations of the type:

$$\frac{dX_t}{dt} + f(X_t) = B\frac{dW_t}{dt},$$ (1)

where the time parameter t runs over the interval $[0, 1]$ and $\{W_t, t \in [0, 1]\}$ is a standard k-dimensional Wiener process. Instead of the customary initial condition where the value of X_0 is specified, we impose a boundary condition which involves both X_0 and X_1 of the form

$$h(X_0, X_1) = \overline{h}.$$ (2)

We assume that $\{X_t\}$ takes values in $\mathbb{R}^d$, $f: \mathbb{R}^d \to \mathbb{R}^d$, B is a $d \times k$ matrix, $h: \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\overline{h} \in \mathbb{R}^d$.

Our goal here is twofold. First, we shall give sufficient conditions for existence and uniqueness of a solution. We shall then study the Markov property of the solution to (1)–(2). There are two types of Markov properties relevant to our problem. The solution is said to be a *Markov process* if for any $t \in [0, 1]$, the past and future of $\{X_s\}$ are conditionally independent, given the present state X_t. The solution is said to be a *Markov field* if for any $0 \le s < t \le 1$, the values inside and outside the interval $[s, t]$ are conditionally independent, given X_s and X_t. In the Gaussian case ($f = 0$ and h linear), it is known that the solution is always a Markov field and even a Markov process for certain types of boundary conditions; see Russek [10]. We show that the same is true if f is linear, but usually fails for nonlinear f. More precisely, when $d = 1$, we establish the following dichotomy: the solution is a Markov field if and only if $f'' \equiv 0$. In the case $d > 1$, there are triangular situations where the solution is a Markov process, even with nonlinear f's. However, it is easy to find higher dimensional examples with solutions which are not

---

Markov fields. For a dichotomy result for the problem (X_t one-dimensional)

$$\frac{d^2X_t}{dt} + f\left(X_t, \frac{dX_t}{dt}\right) = \frac{dW_t}{dt}$$

with Dirichlet type boundary conditions, we refer to the companion paper [7], where it is shown that the solution is a Markov process when f is linear and is not a Markov field if f is nonlinear. Analogous results for a similar type of equation in dimension one, but with a linear (nonconstant) diffusion coefficient, are discussed in Donati-Martin [3].

Our tool for the study of the Markov property is the extended Girsanov theorem due to Kusuoka [5], which allows us to compute conditional expectations under a law under which the Markov property is known to hold.

Note that there exists literature concerning Gaussian solutions of linear SDEs with boundary conditions; see, in particular, Russek [10] and Cinlar and Wang [1]. In the case where both the drift and the diffusion coefficient are linear (the latter being nonconstant), such equations have been studied recently by Ocone and Pardoux [9]. For other types of existence results, see Huang [4] and Dembo and Zeitouni [2]. Our negative results concerning the Markov property seem to be new.

The paper is organized as follows. In Section 2, we prove existence and uniqueness theorems, mostly under monotonicity assumptions. In Section 3, we apply Kusuoka's theorem to our problem and compute a Radon–Nikodym derivative. In Section 4, we study the Markov property in the linear case and in the one-dimensional case and in Section 5, we study the Markov property in higher dimension. In this last section, we establish another existence and uniqueness theorem in dimension two.

2. Existence and uniqueness. Let $\{W_t, \ t \in [0,1]\}$ be a standard k-dimensional Wiener process defined on a probability space $(\Omega, \mathscr{F}, P)$. We are looking for a solution $\{X_t, \ t \in [0,1]\}$ of equations (1) and (2) as an $\mathbb{R}^d$-valued process. We shall assume without loss of generality that $k \leq d$ and that the kernel of the $d \times k$ matrix B reduces to $\{0\}$. We suppose moreover that the mapping $f: \mathbb{R}^d \to \mathbb{R}^d$ takes the form

$$f(x) = Ax + B\bar{f}(x),$$

where A is a $d \times d$ matrix and $\bar{f}: \mathbb{R}^d \to \mathbb{R}^k$ is measurable and locally bounded. We are finally given a mapping $h: \mathbb{R}^{2d} \to \mathbb{R}^d$ and a vector $\bar{h} \in \mathbb{R}^d$ and we consider the equations

$$(3) \qquad \frac{dX_t}{dt} + f(X_t) = B\frac{dW_t}{dt},$$

$$h(X_0, X_1) = \bar{h}.$$

In other words, a solution is thought of as an element $X \in C([0,1]; \mathbb{R}^d)$ which

is such that

$$X_t + \int_0^t f(X_s)\,ds = X_0 + BW_t, \qquad 0 \le t \le 1,$$

$$h(X_0, X_1) = \overline{h}.$$

Note that we shall construct the solution $\{X_t\}$ as a function of the input $\{W_t\}$ defined pointwise on the space $C(\mathbb{R}_+; \mathbb{R}^k)$, so that the fact that $\{W_t\}$ is a Wiener process is in fact irrelevant in this section.

Let us first associate to (3) the equation with $\overline{f} \equiv 0$:

(4)
$$\frac{dY_t}{dt} + AY_t = B\frac{dW_t}{dt},$$

$$h(Y_0, Y_1) = \overline{h}.$$

Note that a solution to (4), if any, takes the form

$$Y_t = e^{-At}\left[Y_0 + \int_0^t e^{As}B\,dW_s\right],$$

where the last expression makes sense for any continuous function $\{W_t\}$ by integration by parts. Therefore a solution to (4) must satisfy

$$h\left(Y_0, e^{-A}\left(Y_0 + \int_0^1 e^{As}B\,dW_s\right)\right) = \overline{h}.$$

We define $F \triangleq \{\int_0^1 e^{At}B\,d\varphi(t);\ \varphi \in C([0,1];\ \mathbb{R}^k)\} \subset \mathbb{R}^d$. We now formulate our first assumption, which is assumed to hold throughout the paper.

$\forall\, z \in F$, the equation

(H1)
$$h(y, e^{-A}(y+z)) = \overline{h}$$

has a unique solution $y = g(z)$.

We shall now give examples of triples $(A, h, \overline{h})$ which satisfy (H1). It is clear that under (H1), equation (4) has the unique solution

$$Y_t = e^{-At}\left[g\left(\int_0^1 e^{As}B\,dW_s\right) + \int_0^t e^{As}B\,dW_s\right].$$

We now define the sets of functions

$$C_0([0,1];\mathbb{R}^k) = \{\eta \in C([0,1];\mathbb{R}^k);\ \eta_0 = 0\},$$

$$\Sigma = \left\{\xi \in C([0,1];\mathbb{R}^d);\ \xi_t - \xi_0 + \int_0^t A\xi_s\,ds \in \mathrm{Im}\,B,\ 0 \le t \le 1;\ h(\xi_0, \xi_1) = h\right\}.$$

It is easily seen that there exists a bijection ψ from $C_0([0,1];\mathbb{R}^k)$ into Σ such that

$$Y_t = (\psi(W))_t$$

$$= \psi_t(W).$$

We finally define the mapping T from $C_0([0, 1]; \mathbb{R}^k)$ into itself by

$$T(\eta) = \eta + \int_0^{\cdot} \bar{f}(\psi_s(\eta)) \, ds.$$

We now have:

THEOREM 2.1. *Suppose that T is a bijection. Then equation (3) has the unique solution $X = \psi \circ T^{-1}(W)$.*

PROOF. Let $\eta \in C_0([0, 1]; \mathbb{R}^k)$ satisfy $T(\eta) = W$. Then $X = \psi(\eta)$ solves (3). Indeed,

$$\begin{aligned}
\dot{X}_t &= \dot{\psi}_t(\eta) \\
&= -A\psi_t(\eta) + B\dot{\eta}_t \\
&= -A\psi_t(\eta) - B\bar{f}(\psi_t(\eta)) + B\dot{W}_t \\
&= -f(X_t) + B\dot{W}_t.
\end{aligned}$$

Conversely, if $\{X_t\}$ solves (3), then $X \in \Sigma$. Define $\eta = \psi^{-1}(X)$. Then

$$\begin{aligned}
\dot{T}(\eta)_t &= \dot{\eta}_t + \bar{f}(X_t), \\
B\dot{T}(\eta)_t &= B\dot{\eta}_t + B\bar{f}(X_t) - \dot{X}_t - f(X_t) + B\dot{W}_t \\
&= B\dot{W}_t.
\end{aligned}$$

That is, $T(\eta) = W$ and $X = \psi \circ T^{-1}(W)$. $\square$

We now give sufficient conditions for T to be one-to-one and onto, starting with the one-to-one property. We recall that the mapping f is said to be monotone if

$$\langle f(x) - f(y), x - y \rangle \geq 0, \qquad \forall \, x, y \in \mathbb{R}^d,$$

and to be strictly monotone if the previous inequality is strict for $x \neq y$.

PROPOSITION 2.2. *Each of the following conditions implies that T is one-to-one:*

(H2i) $\exists \, \lambda \in \mathbb{R}$ *s.t.* $f + \lambda I$ *is monotone and*
$e^{\lambda}|g(z) - g(z')| \geq |e^{-A}(z - z' + g(z) - g(z'))|; z, z' \in F$
$\Rightarrow g(z) = g(z')$.

(H2ii) $\exists \, \lambda \in \mathbb{R}$ *s.t.* $f + \lambda I$ *is strictly monotone and*
$e^{\lambda}|g(z) - g(z')| \leq |e^{-A}(z - z' + g(z) - g(z'))|; \forall \, z, z' \in F$.

PROOF. Let $\eta, \bar{\eta} \in C_0([0, 1]; \mathbb{R}^k)$ satisfy $T(\eta) = T(\bar{\eta})$. We denote $Y = \psi(\eta)$, $\bar{Y} = \psi(\bar{\eta})$.

$$\frac{d}{dt}(Y_t - \bar{Y}_t) + A(Y_t - \bar{Y}_t) = B\frac{d}{dt}(\eta_t - \bar{\eta}_t) = -B[\bar{f}(Y_t) - \bar{f}(\bar{Y}_t)].$$

It follows that

$$\frac{d}{dt}\left(e^{-2\lambda t}|Y_t - \overline{Y}_t|^2\right) + 2e^{-2\lambda t}\left[\langle f(Y_t) - f(\overline{Y}_t), Y_t - \overline{Y}_t\rangle + \lambda|Y_t - \overline{Y}_t|^2\right] = 0.$$

The first part of (H2i) implies that

$$|Y_1 - \overline{Y}_1| \le e^{\lambda}|Y_0 - \overline{Y}_0|.$$

On the other hand, if $z = \int_0^1 e^{At}B\, d\eta_t$, $\overline{z} = \int_0^1 e^{At}B\, d\overline{\eta}_t$, $Y_0 = g(z)$, $\overline{Y}_0 = g(\overline{z})$ and $Y_1 = e^{-A}(z + g(z))$, $\overline{Y}_1 = e^{-A}(\overline{z} + g(\overline{z}))$. It now follows from the second half of (H2i) that $Y_0 = \overline{Y}_0$, hence $Y \equiv \overline{Y}$ and $\eta \equiv \overline{\eta}$.

Similarly, from the first half of (H2ii), unless $Y \equiv \overline{Y}$,

$$|Y_1 - \overline{Y}_1| < e^{\lambda}|Y_0 - \overline{Y}_0|.$$

But the reversed inequality

$$|Y_1 - \overline{Y}_1| \ge e^{\lambda}|Y_0 - \overline{Y}_0|$$

follows from the second half of (H2ii), hence $Y \equiv \overline{Y}$. □

REMARK 2.3. Sufficient conditions for (H2i) and (H2ii) are respectively as follows:

(H2i′)
$$\exists\, \lambda \in \mathbb{R} \text{ s.t. } f + \lambda I \text{ is monotone and}$$
$$(j) + (jj) \Rightarrow x = \overline{x}, \text{ where}$$
$$(j)h(x,y) = h(\overline{x},\overline{y}) = \overline{h}$$
$$(jj)e^{\lambda}|x - \overline{x}| \ge |y - \overline{y}|.$$

(H2ii′)
$$\exists\, \lambda \in \mathbb{R} \text{ s.t. } f + \lambda I \text{ is strictly monotone and}$$
$$(j) \Rightarrow e^{\lambda}|x - \overline{x}| \le |y - \overline{y}|.$$

REMARK 2.4. The following condition also implies that T is one-to-one:

(H2iii) g and $e^{At}B\overline{f}(e^{-At}\cdot)$ are monotone $\forall\, 0 \le t \le 1$.

The proof is similar to that of Proposition 2.2, noting that it implies that

$$\frac{d}{dt}\left|e^{At}(Y_t - \overline{Y}_t)\right|^2 \le 0,$$

that is,

$$|Y_0 - \overline{Y}_0| \ge \left|e^{A}(Y_1 - \overline{Y}_1)\right|$$

or

$$|g(\xi_1) - g(\overline{\xi}_1)| \ge |g(\xi_1) - g(\overline{\xi}_1) + \xi_1 - \overline{\xi}_1|,$$

with $\xi_1 = e^{A}Y_1 - Y_0$, $\overline{\xi}_1 = e^{A}\overline{Y}_1 - \overline{Y}_0$. Then the monotonicity of g implies

$$|g(\xi_1) - g(\overline{\xi}_1)|^2 \ge |g(\xi_1) - g(\overline{\xi}_1)|^2 + |\xi_1 - \overline{\xi}_1|^2.$$

Hence $\xi_1 = \overline{\xi}_1$ and $Y_0 = g(\xi_1) = g(\overline{\xi}_1) = \overline{Y}_0$.

We now give a sufficient condition for T to be onto.

PROPOSITION 2.5. *The following three conditions imply that T is onto*

(H3) $\bar{f}$ *is locally Lipschitz*,

(H4) $\lim_{a \to \infty} \dfrac{1}{a} \sup_{|x| \le a} |\bar{f}(x)| = 0$,

(H5)   *g is continuous and $\exists\, c$ s.t. $|g(x)| \le c(1 + |x|)$, $x \in F$.*

PROOF.   Let $\eta \in C_0([0, 1]; \mathbb{R}^k)$. We need to find $W \in C_0([0, 1]; \mathbb{R}^k)$ s.t.

$$W_t + \int_0^t \bar{f}(\psi_s(W))\,ds = \eta_t,$$

which is equivalent to

$$\int_0^t e^{As}B\,dW_s = \int_0^t e^{As}B\,d\eta_t - \int_0^t e^{As}B\bar{f}(\psi_s(W))\,ds \equiv \xi_t + u_t.$$

Then

$$\dot{u}_t = -e^{At}B\bar{f}\big(e^{-At}g(u_1 + \xi_1) + e^{-At}(u_t + \xi_t)\big).$$

For any $y \in \mathbb{R}^d$, let $\{u_t(y),\ 0 \le t \le 1\}$ denote the unique solution of the differential equation

$$u_t(y) = -\int_0^t e^{As}B\bar{f}\big(e^{-As}g(y + \xi_1) + e^{-As}(u_s(y) + \xi_s)\big)\,ds.$$

Clearly, $y \to u_1(y)$ is continuous. It suffices to show that it has a fixed point. From (H4), for any $\varepsilon > 0$, there exists $a > 0$ s.t.

$$|B\bar{f}(y)| \le \varepsilon(a + |y|),   y \in \mathbb{R}^d.$$

Set $\alpha = \sup_{0 \le t \le 1}(|e^{At}| \vee |e^{-At}|)$. Then (with $c \ge 1$),

$$|u_t(y)| \le \alpha\varepsilon \int_0^t \big[a + \alpha c(1 + 2\|\xi\|_\infty + |y|) + \alpha|u_s(y)|\big]\,ds.$$

We can choose a large enough such that

$$\alpha c(1 + 2\|\xi\|_\infty) \le a.$$

It then follows from Gronwall's lemma that

$$|u_1(y)| \le \alpha^2 \varepsilon e^{\alpha^2 \varepsilon}\left(\frac{2a}{\alpha} + c|y|\right).$$

Thus $|y| \le ha \Rightarrow |u_1(y)| \le \alpha^2 \varepsilon e^{\alpha^2 \varepsilon}((2/\alpha) + cR)a$ and there exists $\varepsilon(\alpha, R)$ s.t.

$$\alpha^2 \varepsilon(\alpha, R)e^{\alpha^2 \varepsilon(\alpha, R)}\left(\frac{2}{\alpha} + cR\right) \le R.$$

Let $\bar{a}$ denote the a corresponding to $\varepsilon(\alpha, R)$. Then $|y| \le R\bar{a} \Rightarrow |u_1(y)| \le R\bar{a}$. The result follows from the Brouwer fixed point theorem. $\square$

REMARK 2.6. We do not really need (H4), but only the fact that $\bar{f}$ at infinity grows at most linearly:

$$|\bar{f}(y)| \le \rho(a + |y|),$$

with ρ small enough.

REMARK 2.7. The bijectivity of T would also follow from the fact that $y \to u_1(y)$ is a strict contraction, which is the case if $\bar{f}$ and g are Lipschitz, with small enough Lipschitz constants.

We now consider examples where our assumptions are satisfied.
Let us consider the case where the boundary condition is linear, that is,

$$h(y, z) = H_0 y + H_1 z,$$

where H_0 and H_1 are $d \times d$ matrices. Then a sufficient condition for (H1) is that the matrix $H_0 + H_1 e^{-A}$ is invertible and moreover:

$$g(z) = \left(H_0 + H_1 e^{-A}\right)^{-1}\left(\bar{h} - H_1 e^{-A} z\right)$$

and (H5) is satisfied. Assume (H3) and (H4). In order to express condition (H2), let us now assume that

$$H_0 \text{ is invertible.}$$

We now have that (H2ii') is satisfied whenever

$$f - \left(\log|H_0^{-1}H_1|\right)I \text{ is strictly monotone.}$$

EXAMPLE 2.8. Periodic boundary condition. This is the case $H_0 = -H_1 = I$, $\bar{h} = 0$. The boundary condition is: $X_0 = X_1$. In this case, $|H_0^{-1}H_1| = 1$, and we need to assume that f is strictly monotone. Moreover, A should not have zero as an eigenvalue, for $I - e^{-A}$ to be invertible. From (H4), we then need A to be positive definite.

EXAMPLE 2.9. Proportional initial and final value. Let $a, b \in \mathbb{R} \setminus \{0\}$ and suppose that the boundary condition is of the form

$$aX_0 = bX_1$$

or in other words $H_0 = aI$, $H_1 = -bI$, $\bar{h} = 0$. Note that we can choose $A = 0$, provided $a \ne b$. We are interested in the case $a \ne 0$ only. $|H_0^{-1}H_1| = |b/a|$. We need for (H2ii') to hold that

$$f - \log\left|\frac{b}{a}\right|I \text{ is strictly monotone.}$$

EXAMPLE 2.10. Nonlinear boundary conditions. Let $d = k = 1$, $A = 0$, $B = 1$, f be monotone and satisfying (H3), (H4), $h(x, y) = e^x + y$. Then the corresponding stochastic boundary value problem has a unique solution.

We note that the two endpoints $t = 0$ and $t = 1$ do not play symmetric roles in our conditions. However, it is easy by time reversal to deduce from our results a result where the initial and final states are exchanged and $f + \lambda$ is required to be monotone decreasing for a certain λ.

Other types of boundary conditions which are not covered by the previous results consist in specifying l components of $X_0 (1 \leq l \leq d - 1)$ and $d - l$ components of X_1. For example, a second order scalar equation

$$\frac{d^2 Z_t}{dt^2} + f\left(Z_t, \frac{dZ_t}{dt}\right) = \frac{dW_t}{dt}$$

with Dirichlet (resp., Neumann) boundary conditions leads to a first order equation in $\mathbb{R}^2$, where the first (resp., second) component of X_0 and X_1 are given. Such an equation is studied in the companion paper [7], using specific methods which differ from the general methods presented here. Another situation of interest is where we fix the first l components of X_0 and the last $d - l$ components of X_1. An example of such a situation (with $d = 2$, $l = 1$) will be treated in the last section of this paper.

3. Computation of a Radon–Nikodym derivative.

From now on, we assume that $k = d$ and $B = I$. In order to study the Markov property of the solution, we shall exploit the extended Girsanov theorem of Kusuoka (Theorem 6.4 of [5]). In this section, we first state that theorem and then apply it to our situation. We now assume that $\Omega = C_0([0, 1]; \mathbb{R}^d)$ equipped with the topology of uniform convergence, $\mathscr{F}$ is the Borel field over Ω, P is standard Wiener measure and $W_t(\omega) = \omega(t)$ is the canonical process.

THEOREM 3.1.   *Let $T = \Omega \to \Omega$ be a mapping of the form*

$$T(\omega) = \omega + \int_0^{\cdot} K_s(\omega) \, ds,$$

where K is a measurable mapping from Ω into $H = L^2(0, 1; \mathbb{R}^d)$ and suppose that the following conditions are satisfied:

(i) *T is bijective.*

(ii) *For all $\omega \in \Omega$, there exists a Hilbert–Schmidt operator $DK(\omega)$ from H into itself such that* (a) *$\|K(\omega + \int_0^{\cdot} h_s \, ds) - K(\omega) - DK(\omega)h\|_H = o(\|h\|_H)$ as $\|h\|_H$ tends to zero;* (b) *$h \to DK(\omega + \int_0^{\cdot} h_s \, ds)$ is continuous from H into $\mathscr{L}^2(H)$, the space of Hilbert–Schmidt operators;* (c) *$I + DK(\omega)$ is invertible.*

Then if Q is the measure on $(\Omega, \mathscr{F})$ s.t. $P = QT^{-1}$, Q is absolutely continuous with respect to P and

$$\frac{dQ}{dP} = |d_c(-DK)|\exp\left(-\delta(K) - \frac{1}{2}\int_0^1 |K_t|^2 \, dt\right),$$

where $d_c(-DK)$ denotes the Carleman–Fredholm determinant of the Hilbert–Schmidt operator $-DK$ and $\delta(K)$ the Skorohod integral of K.

To define the Carleman–Fredholm determinant (see, e.g., [11]), it is sufficient to say that: (i) if A is a linear operator from $\mathbb{R}^N$ into itself,

$$d_c(A) = \prod_j (1 - \lambda_j)\exp \lambda_j,$$

where the λ_j's are the nonzero eigenvalues of A counted with their multiplicities; (ii) $A \to d_c(A)$ is continuous from $\mathscr{L}^2(H)$ into $\mathbb{R}$.

Let us recall the notions of derivation on Wiener space and Skorohod integral. Let **S** denote the subset of $L^2(\Omega)$ consisting of those random variables F of the form

$$F = f\left(\int_0^1 \langle h_1(t), dW_t\rangle, \dots, \int_0^1 \langle h_n(t), dW_t\rangle\right),$$

where $n \in \mathbb{N}$; $h_1, \dots, h_n \in L^2(0, 1; \mathbb{R}^d)$; $f \in C_b^\infty(\mathbb{R}^n)$. For $F \in \mathbf{S}$,

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^1 \langle h_1(t), dW_t\rangle, \dots, \int_0^1 \langle h_n(t), dW_t\rangle\right) h_i(t)$$

and we denote by $\mathbb{D}^{1,2}$ the completion of **S** with respect to the norm $\|\cdot\|_{1,2}$ defined by

$$\|F\|_{1,2}^2 = E(F^2) + E\int_0^1 |D_t F|^2\, dt, \qquad F \in \mathbf{S}.$$

Note that $D_t F$ is a d-dimensional random vector. We shall denote by $D_t^j F$ its jth component.

If a process $u \in L^2(\Omega \times (0,1); \mathbb{R}^d)$ is Skorohod integrable,

$$E(\delta(u)F) = E\int_0^1 \langle u_t, D_t F\rangle\, dt, \qquad F \in \mathbf{S}.$$

It is part of Theorem 3.1 that K is locally Skorohod integrable in the sense that there exists a sequence $\{(\Omega_n, K_n)\}$ such that $\Omega_n \in \mathscr{F}$, $K_n \in L^2(0,1; \mathbb{D}^{1,2})$, $n \in \mathbb{N}$; $\Omega_n \uparrow \Omega$ a.s. as $n \to \infty$, $K = K_n$ on $[0,1] \times \Omega_n$. It then follows that $\delta(K)$ is well-defined by $\delta(K)(\omega) = \delta(K_n)(\omega)$, $\omega \in \Omega_n$, $n \in \mathbb{N}$; see [6]. For more information about D and δ, we refer in particular to Nualart and Zakai [8] and Nualart and Pardoux [6].

We now want to apply Theorem 3.1 to the mapping T from Section 2 and to compute the Radon–Nikodym derivative.

We have $K_t(\omega) = \bar{f}(\psi_t(\omega))$. Assume that $\bar{f}, g \in C^1(\mathbb{R}^d; \mathbb{R}^d)$.

$$\begin{aligned} D_s K_t(\omega) &= \bar{f}'(\psi_t(\omega)) D_s \psi_t(\omega) \\ &= \bar{f}'(\psi_t(\omega)) e^{-At}\big[g'(\xi_1)e^{As} + e^{As}1_{[0,t]}(s)\big], \end{aligned}$$

where $\xi_t = \int_0^t e^{As}\, dW_s$. The operator $DK(\omega) \in \mathscr{L}^2(H)$ is given as

$$(DK(\omega)(h))^i(t) = \sum_{j=1}^d \int_0^1 D_s^j K_t^i(\omega) h_s^j\, ds.$$

Conditions (iia) and (iib) are satisfied here. These properties are easy conse-

quences of the fact that $h \to (\psi_t(\omega + \int_0^\cdot h_s \, ds), \, \xi_1(\omega + \int_0^\cdot h_s \, ds))$ is continuous from H into $\mathbb{R}^{2d}$ for any $\omega \in \Omega$. Recall that everything here is defined for any $\omega \in \Omega$ and not just a.s., by integrating by parts the Wiener integrals.

Let $\{\Phi_t, \, 0 \le t \le 1\}$ denote the $d \times d$ matrix valued solution of

$$\frac{d\Phi_t}{dt} = -f'(\psi_t)\Phi_t,$$

$$\Phi_0 = I.$$

We now have:

PROPOSITION 3.2. *Suppose that $\bar{f} \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, (H1) holds and $g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Assume moreover that T defined in Section 2 is bijective and further that*

(5) $\det\!\big(I - e^A \Phi_1 g'(\xi_1) + g'(\xi_1)\big) \ne 0.$

Then the conditions of Theorem 3.1 are satisfied.

PROOF. It remains to check that $K_t(\omega) = \bar{f}(\psi_t(\omega))$ satisfies condition (iic) of Theorem 3.1. From the Fredholm alternative, it suffices to check that -1 is not an eigenvalue of $DK(\omega), \, \forall \, \omega \in \Omega$. Let $h \in H$ s.t.

$$h + DK(\omega)h = 0.$$

Then

$$h_t + \bar{f}'(\psi_t)e^{-At}\left[g'(\xi_1)\int_0^1 e^{As}h_s \, ds + \int_0^t e^{As}h_s \, ds\right] = 0.$$

Define $H_t = e^{-At}\int_0^t e^{As}h_s \, ds$. We get

$$\frac{dH_t}{dt} + f'(\psi_t)H_t + \bar{f}'(\psi_t)e^{-At}g'(\xi_1)e^A H_1 = 0.$$

Consequently,

$$\left(I + \Phi_1\int_0^1 \Phi_s^{-1}\bar{f}'(\psi_s)e^{-As}g'(\xi_1)e^A \, ds\right)H_1 = 0.$$

Since $H_1 = 0 \Rightarrow H \equiv 0 \Rightarrow h \equiv 0$, it suffices to show that

$$\det\!\left(I + \Phi_1\int_0^1 \Phi_s^{-1}\bar{f}'(\psi_s)e^{-As}g'(\xi_1)e^A \, ds\right) \ne 0.$$

Since

$$\Phi_t^{-1} = I + \int_0^t \Phi_s^{-1}f'(\psi_s) \, ds,$$

$$\Phi_1^{-1}e^{-A} = I + \int_0^1 \Phi_t^{-1}f'(\psi_t)e^{-At} \, dt - \int_0^1 \Phi_t^{-1}Ae^{-At} \, dt$$

$$= I + \int_0^1 \Phi_t^{-1}\bar{f}'(\psi_t)e^{-At} \, dt,$$

the previous condition is equivalent to

$$\det\bigl(I + \Phi_1(\Phi_1^{-1}e^{-A} - I)g'(\xi_1)e^A\bigr) \neq 0,$$

which is clearly equivalent to (5). □

Now we need to give sufficient conditions for (5).

PROPOSITION 3.3. *Suppose that* $\bar{f}, g \in C^1(\mathbb{R}^d; \mathbb{R}^d)$. *Then* (5) *follows from each of the following slightly stronger versions of the hypotheses of Proposition 2.2:*

(H2i″) $\exists\, \lambda \in \mathbb{R}$ *s.t.* $f + \lambda I$ *is monotone and* $\forall\, x, y \in \mathbb{R}^d$:
$$e^\lambda |g'(y)x| \geq |e^{-A}(I + g'(y))x| \Rightarrow x = 0.$$

(H2ii″) $\exists\, \lambda \in \mathbb{R}$ *s.t.* $f'(y) + \lambda I > 0, \forall\, y \in \mathbb{R}^d$ *and*
$$\forall\, x, y \in \mathbb{R}^d, e^\lambda |g'(y)x| \leq |e^{-A}(I + g'(y))x|.$$

PROOF. Suppose first that (H2i″) holds. Then $f'(y) + \lambda I \geq 0, \forall\, y \in \mathbb{R}^d$. Consequently, $\forall\, v \in \mathbb{R}^d$,

$$\frac{d}{dt}\bigl[e^{-2\lambda t}|\Phi_t g'(\xi_1)v|^2\bigr] = -2\langle \Phi_t g'(\xi_1)v, [\,f'(\psi_t) + \lambda I\,]\Phi_t g'(\xi_1)v\rangle \leq 0,$$

$$e^\lambda |g'(\xi_1)v| \geq |\Phi_1 g'(\xi_1)v|.$$

If (5) is not true, then there exists $v \neq 0$ s.t.

$$\Phi_1 g'(\xi_1)v = e^{-A}(I + g'(\xi_1))v.$$

Consequently,

$$e^\lambda |g'(\xi_1)v| \geq |e^{-A}(v + g'(\xi_1)v)|,$$

which, from the second half of (H2i″), contradicts $v \neq 0$.
    Similarly, under (H2ii″), if there exists $v \neq 0$ s.t.

$$\Phi_1 g'(\xi_1)v = e^{-A}(I + g'(\xi_1))v,$$

then $\Phi_1 g'(\xi_1)v \neq 0$ and

$$e^\lambda |g'(\xi_1)v| > |e^{-A}(v + g'(\xi_1)v)|,$$

which contradicts the second half of (H2ii″). □

REMARK 3.4. Again with $\bar{f}, g \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, if T is bijective and (H2iii) holds, then the conditions of Theorem 3.1 are satisfied. Indeed, going back to the proof of Proposition 3.2, if we consider the equation for $G_t = \int_0^t e^{sA}h_s\,ds$, we see that the sufficient condition (5) can be replaced by

$$\det(I - \psi_1 g'(\xi_1) + g'(\xi_1)) \neq 0,$$

where

$$\frac{d\Psi_t}{dt} = -e^{At}\bar{f}'(\psi_t)e^{-At}\Psi_t,$$

$$\Psi_0 = I.$$

But if the previous determinant vanishes, then there exists $v \in \mathbb{R}^d \setminus \{0\}$ s.t.

$$v - \Psi_1 g'(\xi_1)v + g'(\xi_1)v = 0,$$

$$\langle v - \Psi_1 g'(\xi_1)v + g'(\xi_1)v, g'(\xi_1)v \rangle = 0,$$

hence

$$\langle v, g'(\xi_1)v \rangle + \tfrac{1}{2}|\Psi_1 g'(\xi_1)v - g'(\xi_1)v|^2 - \tfrac{1}{2}|\Psi_1 g'(\xi_1)v|^2 + \tfrac{1}{2}|g'(\xi_1)v|^2 = 0.$$

But $\langle v, g'(\xi_1)v \rangle \geq 0$ and the monotonicity of $e^{At}f'(\psi_t)e^{-At}$ implies that $|g'(\xi_1)v|^2 \geq |\Psi_1 g'(\xi_1)v|^2$. Consequently, $\Psi_1 g'(\xi_1)v = g'(\xi_1)v$, hence $v = 0$, which contradicts the previous assumption.

It now remains to compute the Radon–Nikodym derivative $J = dQ/dP$ of Theorem 3.1. The main step is the computation of the Carleman–Fredholm determinant $d_c(-DK)$, which is given by:

LEMMA 3.5. *Under the assumptions of Proposition 3.2, if $K_t = \bar{f}(\psi_t)$,*

$$d_c(-DK) = \det(I - e^A \Phi_1 g'(\xi_1) + g'(\xi_1))$$

$$\times \exp\left(-\int_0^1 \mathrm{Tr}\left[\bar{f}'(\psi_t)e^{-tA}g'(\xi_1)e^{tA}\right]dt\right).$$

The proof of Lemma 3.5, which is a little long and technical, is given in the Appendix.

THEOREM 3.6. *Under the assumptions of Proposition 3.2, if Q is defined as in Theorem 3.1 with $K_t = \bar{f}(\psi_t)$, then*

$$J = |\det(I - e^A \Phi_1 g'(\xi_1) + g'(\xi_1))|$$

$$\times \exp\left[\tfrac{1}{2}\int_0^1 \mathrm{Tr}\,\bar{f}'(\psi_t)\,dt - \int_0^1 \bar{f}(\psi_t) \circ dW_t - \tfrac{1}{2}\int_0^1 |\bar{f}(\psi_t)|^2\,dt\right],$$

where $\int_0^1 \bar{f}(\psi_t) \circ dW_t$ denotes the generalized Stratonovich integral (see Nualart and Pardoux [6]).

PROOF. The result follows from Theorem 3.1, Lemma 3.5 and the following expression of the correction term between the Skorohod and the Stratonovich

integrals (see Nualart and Pardoux [6]):

$$\delta\big(\bar{f}(\psi)\big) = \int_0^1 \bar{f}(\psi_t)\, dW_t$$

$$= \int_0^1 \bar{f}(\psi_t)\circ dW_t - \tfrac{1}{2}\int_0^1 \operatorname{Tr}\big[(D^+\bar{f}(\psi))_t + (D^-\bar{f}(\psi))_t\big]\, dt$$

$$= \int_0^1 \bar{f}(\psi_t)\circ dW_t - \tfrac{1}{2}\int_0^1 \operatorname{Tr}\bar{f}'(\psi_t)\, dt - \int_0^1 \operatorname{Tr}\big[\bar{f}'(\psi_t)e^{-tA}g(\xi_1)e^{tA}\big]\, dt.$$

$\square$

4. The Markov property.

In this section, we want to study the Markov properties of the solution $\{X_t\}$ of equation (3). We first need to specify the two Markov properties which are relevant in our framework.

DEFINITION 4.1. We say that $\{X_t;\ 0 \le 1\}$ is a *Markov process* if for any $t \in [0,1]$, $\sigma\{X_s;\ 0 \le s \le t\}$ and $\sigma\{X_\tau;\ t \le \tau \le 1\}$ are conditionally independent given X_t, i.e., past and future are conditionally independent given the present.

DEFINITION 4.2. We say that $\{X_t;\ 0 \le t \le 1\}$ is a *Markov field* if for any $0 \le s < t \le 1$, $\sigma(X_\tau;\ \tau \in [s,t])$ and $\sigma(X_u;\ u \in [0,1]\setminus(s,t))$ are conditionally independent, given $\sigma(X_s, X_t)$.

It is easily seen that any Markov process is a Markov field, but the converse is not true.

Clearly, in the case of periodic boundary condition $X_1 = X_0$, we cannot expect $\{X_t\}$ to be a Markov process, but at most a Markov field. It is known (see Russek [10], Ocone and Pardoux [9]) that in the Gaussian case (f affine, h linear) the solution is always a Markov field and is moreover a Markov process if $h(x,y) = H_0 x + H_1 y$ is such that $\operatorname{Im} H_0 \cap \operatorname{Im} H_1 = \{0\}$.

We shall see that the solution is again a Markov field whenever f is affine (even for nonlinear h's), but that it is not always a Markov field if f is nonlinear. More precisely, we shall prove that in case $d = 1$ the solution is a Markov field iff f is affine. As we shall see, the situation in higher dimension is more complex.

Let us first study the case of an affine f. Consider the equation

(6)
$$\frac{dX_t}{dt} + AX_t + c = B\frac{dW_t}{dt},$$

$$h(X_0, X_1) = \bar{h},$$

where A, B, h and $\bar{h}$ are as in Section 2 and $c \in \mathbb{R}^d$. We assume that (H1) holds with $E = \mathbb{R}^d$, which implies that (6) has the unique solution

(7) $$X_t = e^{-At}g\left(\int_0^1 e^{As}(B\,dW_s - c\,ds)\right) + e^{-At}\int_0^t e^{As}(B\,dW_s - c\,ds).$$

THEOREM 4.3. *The process* $\{X_t; \, t \in [0,1]\}$ *given by* (7) *is a Markov field*.

PROOF. Define $\xi_t = \int_0^t e^{As}(B\,dW_s - c\,ds)$.
Let $\varphi \in C_b(\mathbb{R}^d)$ and $0 \le s < r < t \le 1$.

$$E\big[\varphi(X_r)\,|\,X_u;\, u \in [0,1] \setminus (s,t)\big]$$

$$= E\big[\varphi\big(e^{-rA}(g(\xi_1) + \xi_r)\big)\,|\,\xi_u;\, u \in [0,1] \setminus (s,t)\big]$$

$$= E\big[\varphi\big(e^{-rA}(y + \xi_r)\big)\,|\,\xi_u;\, u \in [0,1] \setminus (s,t)\big]\big|_{y = g(\xi_1)}.$$

$\{\xi_t\}$ is a Gauss–Markov process, hence also a Markov field. Therefore, the conditional law of ξ_r given $\sigma(\xi_u;\, u \in [0,1] \setminus (s,t))$ is Gaussian with mean $c_0 + C_1\xi_s + C_2\xi_t$ and constant covariance, where $c_0 \in \mathbb{R}^d$ and C_1, C_2 are $d \times d$ matrices satisfying $C_1 + C_2 = I$. Therefore, the quantity

$$E\big[\varphi\big(e^{-rA}(y + \xi_r)\big)\,|\,\xi_u;\, u \in [0,1] \setminus (s,t)\big]$$

is a function of $y + C_1\xi_s + C_2\xi_t = C_1(y + \xi_s) + C_2(y + \xi_t)$.
  Consequently,

$$E\big[\varphi(X_r)\,|\,X_u;\, u \in [0,1] \setminus (s,t)\big]$$

is a function of X_s, X_t, hence equals

$$E\big(\varphi(X_r)\,|\,X_s, X_t\big). \qquad\qquad \square$$

We now come back to our original equation, but with $k = d$ and $B = I$:

$$(8) \qquad \frac{dX_t}{dt} + f(X_t) = \frac{dW_t}{dt},$$

$$h(X_0, X_1) = \overline{h}.$$

Recall that $\Omega = C_0([0,1]; \mathbb{R}^d)$, $\mathscr{F}$ is its Borel field, P its Wiener measure, $P = QT^{-1}$, $J = dQ/dP$ and $W_t(\omega) = \omega(t)$.
  Note that whenever f is affine, (H1) is satisfied and T is bijective, we are in the situation of Theorem 4.3 and the solution $\{X_t\}$ is a Markov field. We now prove a converse to that property, in the one-dimensional case.

THEOREM 4.4. *Suppose that* $d = 1$, (H1) *holds*, f *and* g *are of* C^2 *class*, T *is bijective and either* (H2i″) *or* (H2ii″) *holds. Assume moreover that* $g'(x) > -1$, $\forall\, x \in \mathbb{R}$ *and* $g' \not\equiv 0$. *Then if* $\{X_t, \, t \in [0,1]\}$ *is a Markov field*, f *is affine* (*i.e.*, $f'' \equiv 0$).

PROOF. Let f be any nonnegative measurable functional defined on Ω. Since

$$X(\omega) = \psi \circ T^{-1}(\omega),$$

$$Y(\omega) = \psi(\omega),$$

we have that

$$\int_\Omega f(X(\omega))\,dP(\omega) = \int_\Omega f(Y(\omega))\,dQ(\omega),$$

that is, the law of $\{X_t;\ y \in [0,1]\}$ under P is the same as the law of $\{Y_t;\ t \in [0,1]\}$ under Q. Therefore we are going to assume that $\{Y_t\}$ is a Markov field under Q, or in other words that for any $0 \le r < t \le 1$ and any nonnegative r.v. χ which is $\mathscr{F}_{r,t}^i = \sigma\{Y_s;\ s \in [r,t]\}$ measurable,

$$\Lambda_\chi = E_Q(\chi | \mathscr{F}_{r,t}^e) = \frac{E_P(\chi J | \mathscr{F}_{r,t}^e)}{E_P(J | \mathscr{F}_{r,t}^e)}$$

(where $\mathscr{F}_{r,t}^e = \sigma(Y_s;\ s \in [0,1] \setminus (r,t))$) is $\mathscr{F}_{r,t} = \sigma(Y_r, Y_t)$ measurable. Note that the superscript i stands for interior and e for exterior. Recall that

$$J = \big[(1 + g'(\xi_1))H^i H^e - g'(\xi_1)\big]K^i K^e,$$

where $\xi_1 = \int_0^1 e^{At}\,dW_t$ and

$$H^i = \exp\left(\int_{[r,t]} \bar{f}'(Y_s)\,ds\right), \qquad H^e = \exp\left(\int_{[r,t]^c} \bar{f}'(Y_s)\,ds\right),$$

$$K^i = \exp\left(-\int_{[r,t]} \bar{f}(Y_s) \circ dW_s - \tfrac{1}{2}\int_{[r,t]} \bar{f}'(Y_s)\,ds - \tfrac{1}{2}\int_{[r,t]} \bar{f}(Y_s)^2\,ds\right),$$

$$K^e = \exp\left(-\int_{[r,t]^c} \bar{f}(Y_s) \circ dW_s - \tfrac{1}{2}\int_{[r,t]^c} \bar{f}'(Y_s)\,ds - \tfrac{1}{2}\int_{[r,t]^c} \bar{f}(Y_s)^2\,ds\right).$$

Indeed, $1 + g'(\xi_1) - e^A \Phi_1 g'(\xi_1) > 0$ a.s. This is clear from $1 + g'(\xi_1) \ge 0$ if $g'(\xi_1) < 0$ and follows from both (H2i″) and (H2ii″) (see the proof of Proposition 3.3), if $g'(\xi_1) \ge 0$.

Since the increments of $\{W_t\}$ in any interval I are $\sigma(Y_t;\ t \in I)$ measurable, we conclude that K^i and H^i are $\mathscr{F}_{r,t}^i$ measurable, K^e and H^e are $\mathscr{F}_{r,t}^e$ measurable. Also, ξ_1 is $\mathscr{F}_{r,t}^e$ measurable. We shall next use the following notation:

$$\bar{Z} = E_p(Z | F_{r,t})$$

for any nonnegative (or P-integrable) r.v. Z.

Since $\{Y_t\}$ is a Markov field P, we have

$$\Lambda_\chi = \frac{(1 + g'(\xi_1))H^e \overline{\xi H^i K^i} - g'(\xi_1)\overline{\xi K^i}}{(1 + g'(\xi_1))H^e \overline{H^i K^i} - g'(\xi_1)\overline{K^i}}.$$

Consequently,

(9) $$H^e(1 + g'(\xi_1))\big[\overline{\chi H^i K^i} - \Lambda_\chi \overline{H^i K^i}\big] = g'(\xi_1)\big(\overline{\chi K^i} - \Lambda_\chi \overline{K^i}\big).$$

We now choose two particular χ's:

$$\chi_1 = (H^i K^i)^{-1}, \qquad \chi_2 = (K^i)^{-1}.$$

and write Λ_j for Λ_{χ_j}, $j = 1, 2$. Define the set

$$G = \left\{ \overline{\chi_1 K^i} = \Lambda_1 \overline{K^i} \right\} \cap \left\{ \overline{\chi_2 K^i} = \Lambda_2 \overline{K^i} \right\}.$$

$G \in \mathscr{F}_{r,t}$, and on G, we have

$$\Lambda_1 = \frac{\overline{\chi_1 K^i}}{\overline{K^i}} = \frac{\overline{\chi_1 H^i K^i}}{\overline{H^i K^i}},$$

$$\Lambda_2 = \frac{\overline{\chi_2 K^i}}{\overline{K^i}} = \frac{\overline{\chi_2 H^i K^i}}{\overline{H^i K^i}}.$$

Consequently,

$$\overline{(H^i)^{-1} H^i K^i} = \overline{K^i},$$

$$\overline{H^i K^i} = \overline{H^i}\, \overline{K^i},$$

which imply that

$$\left(\overline{H^i} \right)^{-1} = \overline{\left(H^i \right)^{-1}}.$$

The strict Jensen inequality now implies that $1_G \int_r^t \bar{f}'(Y_s)\, ds$ is $\mathscr{F}_{r,t}$ measurable. Now the following set is also in $\mathscr{F}_{r,t}$:

$$\overline{G} = G^c \cap \{ g'(\xi_1) \neq 0 \}$$

$$= \left(G^c \cap \left\{ \overline{\chi_1 H^i K^i} \neq \Lambda_1 \overline{H^i K^i} \right\} \right) \cup \left(G^c \cap \left\{ \overline{\chi_2 H^i K^i} \neq \Lambda_2 \overline{H^i K^i} \right\} \right).$$

So from (9), $1_{\overline{G}}[\log(|1 + g'(\xi_1)^{-1}|) + \int_{[r,t]^c} \bar{f}'(Y_s)\, ds]$ is $\mathscr{F}_{r,t}$ measurable.

We now want to deduce from the two previous measurability properties that

(10) $$f''(Y_s(\omega)) = 0,$$
$$\forall\, s \in [r, t] \text{ a.s. on } G \text{ and } \forall\, s \in [0, 1] \setminus (r, t) \text{ a.s. on } \overline{G}.$$

First observe that $\mathscr{F}_{r,t}$ is generated by the random variables $\xi_r + g(\xi_1)$ and $\xi_t + g(\xi_1)$, where the process $\{ \xi_s = \int_0^s e^{Au}\, dW_u;\ 0 \le s \le 1 \}$ is a Gaussian process with covariance function given by

$$E(\xi_s \xi_u) = \langle 1_{[0,s]}, 1_{[0,u]} \rangle_{\overline{H}},$$

where $\overline{H} = L^2(0, 1;\ e^{2uA}\, du)$. We denote by D the derivation operator with respect to this Gaussian process, which is defined as follows. We denote here by $\mathbf{S}$ the subset of $L^2(\Omega)$ consisting of those random variables F of the form

$$F = f(\xi(h_1), \ldots, \xi(h_n)),$$

where $n \in \mathbb{N}$; $h_1, \ldots, h_n \in L^2(0, 1)$; $f \in C_b^\infty(\mathbb{R}^n)$ and $\xi(h) \triangleq \int_0^1 h(t)\, d\xi_t$. For $F \in \mathbf{S}$,

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (\xi(h_1), \ldots, \xi(h_n)) h_i(t)$$

and we define $\mathbb{D}^{1,2}$ (resp., $\mathbb{D}^{h,2}$) as the closure of $\mathbf{S}$ with respect to the norm

$$\|F\|_{1,2} = \left(E\left[F^2 + \|DF\|_H^2 \right] \right)^{1/2}$$

(resp., $\|F\|_{h,2} = (E[F^2 + (D_h F)^2])^{1/2}$, where $D_h F = \langle h, DF \rangle_H$).

Since D (resp., D_h) is closable, DF (resp., $D_h F$) is well defined for $F \in \mathbb{D}^{1,2}$ (resp., $\mathbb{D}^{h,2}$). Moreover, it is proved in Nualart and Pardoux [6] that D (resp., D_h) is a local operator and DF (resp., $D_h F$) is well-defined for $F \in \mathbb{D}_{\text{loc}}^{1,2}$ (resp., $\mathbb{D}_{\text{loc}}^{h,2}$), where $\mathbb{D}_{\text{loc}}^{1,2}$ (resp., $\mathbb{D}_{\text{loc}}^{h,2}$) is the set of r.v. F such that there exists $\{(\Omega_n, F_n)\} \subset \mathscr{F} \times \mathbb{D}^{1,2}$ (resp., $\mathbb{D}^{h,2}$) with (i) $\Omega_n \uparrow \Omega$ a.s. (ii) $F = F_n$ a.s. on Ω_n.

Note that $\mathscr{F}_{r,t} \subset \mathscr{G} = \sigma(\xi_r, \xi_t, \xi_1) = \sigma(\xi(h_1), \xi(h_2), \xi(h_3))$, with $h_1 = 1_{[0,r]}$, $h_2 = 1_{[r,t]}$, $h_3 = 1_{[t,1]}$. Let $K = sp\{h_1, h_2, h_3\}$. We shall use the following lemma, whose proof will be given at the end of the section.

LEMMA 4.5.   Let $F \in \mathbb{D}_{\text{loc}}^{1,2}$, $A \in \mathscr{G}$ and $1_A F$ be $\mathscr{G}$ measurable. Then $DF \in K$ a.s. on A.

We shall choose succesively (G, Z) and $(\overline{G}, U)$ for (A, F), where ·

$$Z = \int_{[r,t]} \bar{f}'(Y_s) \, ds, \qquad U = \log\left(|1 + g'(\xi_1)^{-1}|\right) + \int_{[r,t]^c} \bar{f}'(Y_s) \, ds,$$

$$D_\theta Z = \int_r^t \bar{f}''(Y_s) \, D_\theta Y_s \, ds$$

$$= \left(g'(\xi_1) \int_r^t \bar{f}''(Y_s) e^{-As} \, ds + \int_\theta^t \bar{f}''(Y_s) e^{-As} \, ds \right) 1_{[r,t]}(\theta)$$

$$+ \left(\int_r^t \bar{f}''(Y_s) e^{-As} \, ds \right) \left[(1 + g'(\xi_1)) 1_{[0,r]}(\theta) + g'(\xi_1) 1_{[t,1]}(\theta) \right].$$

Hence $DZ \in K$ a.s. on G implies that $\int_\theta^t f''(Y_s) e^{-As} \, ds$ is constant for $\theta \in [r,t]$, that is, that

$$f''(Y_s) = 0; \qquad s \in [r,t] \quad \text{a.s. on } G,$$

$$D_\theta U = -g''(\xi_1) \left[g'(\xi_1)^2 + g'(\xi_1) \right]^{-1} + \int_{[r,t]^c} \bar{f}''(Y_s) e^{-As} \left[g'(\xi_1) + 1_{[0,s]}(\theta) \right] \, ds$$

$$= -g''(\xi_1) g'(\xi_1)^{-1} (1 + g'(\xi_1))^{-1} + g'(\xi_1) \int_{[r,t]^c} \bar{f}''(Y_s) e^{-As} \, ds$$

$$+ \int_{\theta \wedge r}^r \bar{f}''(Y_s) e^{-As} \, ds + \int_{t \vee \theta}^1 \bar{f}''(Y_s) e^{-As} \, ds.$$

Again $DU \in K$ a.s. on $\overline{G}$ implies in particular that

$$f''(Y_s) = 0; \qquad s \in [0,1] \setminus (r,t) \quad \text{a.s. on } \overline{G}.$$

Hence (10) is established. Let us finally conclude that $f'' \equiv 0$. Note that this implies that $P(G) = 1$ so that $DU \in K$ a.s. on $\overline{G}$ will be automatically satisfied, without g'' having to vanish.

Suppose that f'' is not identically zero. Then by continuity there exists an interval $I \subset \mathbb{R}$ with positive Lebesgue measure s.t. $f''(y) \neq 0$, $y \in I$. Choose $\theta \in (r, t)$ and $\tau \in (0, 1) \setminus [r, t]$. It follows from (10) that

$$(11) \qquad\qquad P(G \cap \{Y_\theta \in I\}) = 0,$$

$$(12) \qquad\qquad P(\overline{G} \cap \{Y_\tau \in I\}) = 0.$$

However, $Y_s = e^{-As}(g(\xi_1) + \xi_s)$ and $G \in \mathscr{F}_{r,t}$, $\overline{G} \in \mathscr{G}$. But the conditional law of Y_θ given $\mathscr{F}_{r,t}$ and the conditional law of Y_τ given $\mathscr{G}$ both have $\mathbb{R}$ as their support. Consequently (11) and (12) imply that

$$P(G) = P(\overline{G}) = 0,$$

which contradicts the assumption $g' \not\equiv 0$, since the law of ξ_1 is equivalent to Lebesgue measure.

Note that if $g' \equiv 0$, then X_0 is deterministic and $\{X_t; t \in [0, 1]\}$ is a Markov process and that the case X_1 deterministic corresponds to $g' \equiv -1$.

PROOF OF LEMMA 4.5. Since we can approximate F by $\varphi_n(F)$ with $\varphi_n \in C_b^\infty(\mathbb{R})$, $\varphi_n(x) = x$ for $|x| \leq n$, it is sufficient to prove the result for $F \in \mathbb{D}_{\text{loc}}^{1,2} \cap L^2(\Omega)$. Let $h \in \overline{H}$ with $h \perp K$. Then since $E^{\mathscr{G}}(F)$ is a function of $\xi(h_1)$, $\xi(h_2)$ and $\xi(h_3)$, it is easily seen that $E^{\mathscr{G}}(F) \in \mathbb{D}^{h,2}$ and that $D_h[E^{\mathscr{G}}(F)] = 0$. However, $F \in \mathbb{D}_{\text{loc}}^{h,2}$ and $F = E^{\mathscr{G}}(F)$ a.s. on A. It then follows from the local property of D_h that

$$D_h F = \int_0^1 h(t) D_t F \, dt = 0 \quad \text{a.s. on } A,$$

and this holds for any $h \perp K$. It remains to choose a countable set $\{h_n, n \in \mathbb{N}\} \subset \overline{H}$ s.t.

$$\overline{h} \in K \quad \Leftrightarrow \quad \langle \overline{h}, h_n \rangle_H = 0 \quad \forall n$$

and to remark that

$$D_{h_n} F = 0 \quad \forall n \quad \text{a.s. on } A. \qquad\qquad \square$$

REMARK 4.6. One may wonder whether, in case $f'' \not\equiv 0$, the solution $\{X_t\}$ might be a germ Markov field, that is, for any $0 \leq s < t \leq 1$, $\sigma(X_u; u \in [s, t])$ and $\sigma(X_u; u \in [0, 1] \setminus [s, t])$ are conditionally independent, given $\cap_{\epsilon > 0} \sigma(X_u; u \in [s - \epsilon, s + \epsilon] \cup [t - \epsilon, t + \epsilon])$. We have not been able to answer this question in general. However, in the case where h is linear, hence $\{Y_t\}$ is a Gaussian process, one can show that $f'' \not\equiv 0$ implies that $\{X_t\}$ is not a germ Markov field. The argument is identical to that in the companion paper [7] and we do not develop it here.

5. The Markov property for nonlinear f in higher dimension.

We want now to show via both an example and a counterexample that in higher dimension the solution to a nonlinear stochastic differential equation with boundary conditions may be a Markov process and may also not be a Markov

field in the case $d > 1$. However, the examples which we are going to consider are not covered by the existence and uniqueness result of Section 2. Indeed, we want to consider boundary conditions of the type (here $1 \le l < d$):

$$
\begin{aligned}
X_0^{i_k} &= a_k, & 1 \le k \le l, \\
X_1^{j_k} &= b_k, & 1 \le k \le d - l,
\end{aligned}
\tag{13}
$$

where $a_1, \ldots, a_l,\ b_1, \ldots, b_{d-l}$ are arbitrary real numbers: that is, we fix l coordinates of X_0 and $d - l$ coordinates of X_1. With this type of boundary condition, the second half of (H2i) or (H2ii) cannot possibly be satisfied.

Again we are studying the equation

$$
\frac{dX_t}{dt} + f(X_t) = \frac{dW_t}{dt},
\tag{14}
$$

together with the boundary conditions (13). Note that in case f is affine and (13)–(14) has a unique solution, it is a Gauss–Markov process; see Russek [10] and Ocone and Pardoux [9].

We now give an example with a nonlinear f, where there is a unique solution which is a Markov process.

Let l be the integer part of $(d + 1)/2$; for $1 \le k \le l$, let $i_k = 2k - 1$ and for $1 \le k \le d - l$, let $j_k = 2k$. Suppose now that f has the following triangular form: for $1 \le k \le d$, $f^k(x)$ is a function of $x^1, \ldots, x^k$ only. Suppose moreover that each f^k is a measurable function of $(x^1, \ldots, x^k)$ and satisfies

$$
|f(x^1, \ldots, x^{k-1}, x)| \le L_k(x^1, \ldots, x^{k-1})(1 + |x|),
$$

$$
|f(x^1, \ldots, x^{k-1}, x) - f(x^1, \ldots, x^{k-1}, y)| \le L_k(x^1, \ldots, x^{k-1})|x - y|,
$$

where $L_k \colon \mathbb{R}^{k-1} \to \mathbb{R}_+$ is measurable and locally bounded.

It is easy to show by induction on k that under these conditions the equation (14) together with the boundary condition (13) has a unique solution. Moreover, again by induction on k, we show easily that $\{X_t\}$ is a Markov process. Clearly $\{X_t^1,\ t \in [0, 1]\}$ is a Markov process and the induction is carried over with the help of the following standard lemma, whose proof is left to the reader.

LEMMA 5.1. *Let $\{Y_t,\ t \in [0, 1]\}$ be an m-dimensional Markov process and let $\{V_t;\ t \in [0, 1]\}$ be an n-dimensional standard Wiener process independent of $\{Y_t\}$. Suppose we are given a measurable and locally bounded mapping $g \colon \mathbb{R}^{m \times n} \to \mathbb{R}^n,\ L \colon \mathbb{R}^m \to \mathbb{R}_+$ such that*

$$
|g(y, x)| \le L(y)(1 + |x|),
$$

$$
|g(y, x) - g(y, x')| \le L(y)|x - x'|.
$$

Let $\{X_t;\ t \in [0, 1]\}$ denote the unique solution of the SDE

$$
\frac{dX_t}{dt} = g(Y_t, X_t) + \frac{dV_t}{dt}.
$$

Then the $m + n$-dimensional process $\{(Y_t, X_t);\ t \in [0, 1]\}$ is a Markov process.

Let us now give an example in dimension 2 where the solution is not a Markov field. In fact we are going to prove a partial dichotomy.

Let $d = 2$ and consider the equation

$$\frac{dX_t}{dt} + f(X_t) = \frac{dW_t}{dt},$$

$$X_1^1 = X_0^2 = 0,$$

where $f(x^1, x^2) = \begin{pmatrix} x^1 - x^2 \\ -f_2(x^1) \end{pmatrix}$ and f_2 is measurable and locally bounded and satisfies assumptions to be specified later. The equation for $\{Y_t\}$ is simply

$$\frac{dY_t}{dt} = \frac{dW_t}{dt},$$

$$Y_1^1 = Y_0^2 = 0.$$

Hence

$$Y_t = \psi_t(W) = \begin{pmatrix} W_t^1 - W_1^1 \\ W_t^2 \end{pmatrix}.$$

The mapping g defined by $h(y, y + g(z)) = \overline{h}$ is given by $g(x^1, x^2) = (-x^1, 0)$ and its derivative is the matrix $g' = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. We now assume

(H6) f_2 is of C^2 class,

(H7) $0 \leq f_2'(x) \leq K$, for some $K > 0$ and all $x \in \mathbb{R}$.

We again define $T: C_0([0, 1]; \mathbb{R}^2) \to C_0([0, 1]; \mathbb{R}^2)$ by

$$T_t(\eta) = \eta_t + \int_0^t f(\psi_s(\eta)) \, ds.$$

We have:

PROPOSITION 5.2. *Under the assumptions* (H6) *and* (H7), T *is a bijection.*

PROOF. We need to show that $\forall\ W \in C_0([0, 1]; \mathbb{R}^2)$, there exists a unique $\eta \in C_0([0, 1]; \mathbb{R}^2)$ s.t.

$$\dot{\eta}_t^1 = \eta_1^1 - \eta_t^1 + \eta_t^2 + \dot{W}_t^1,$$

$$\dot{\eta}_t^2 = f_2(\eta_t^1 - \eta_1^1) + \dot{W}_t^2.$$

Let $v = \eta - W$. Then v satisfies

$$\dot{v}_t^1 = v_1^1 - v_t^1 + v_t^2 + W_1^1 - W_t^1 + W_t^2,$$

(15) $$\dot{v}_t^2 = f_2(v_t^1 + W_t^1 - v_1^1 - W_1^1),$$

$$v_0^1 = v_0^2 = 0.$$

For each $y \in \mathbb{R}$, let $\{u_t(y), 0 \le t \le 1\}$ denote the unique solution of

$$\dot{u}_t^1(y) = y - u_t^1(y) + u_t^2(y) + \overline{W}_t,$$

$$\dot{u}_t^2(y) = f_2\big(u_t^1(y) + W_t^1 - y - W_1^1\big),$$

$$u_0^1(y) = u_0^2(y) = 0,$$

where $\overline{W}_t = W_1^1 - W_t^1 + W_t^2$.

The first equation gives us an explicit expression for $u_t^1(y)$ in terms of y, $\{u_s^2(y), \overline{W}_s; 0 \le s \le t\}$.

Consequently, the equation can be transformed into a closed equation for $\{u_t^2\}$ and we obtain the following expression for $u_1^1(y)$:

$$u_1^1(y) = (1 - e^{-1})y + \int_0^1 e^{(s-1)} \overline{W}_s\, ds$$

$$+ \int_0^1 e^{s-1} \int_0^s f_2\Bigg(W_t^1 - W_1^1 + \int_0^t e^{\theta - t} \overline{W}_\theta\, d\theta - e^{-t}y$$

$$+ \int_0^t e^{\theta - t} u_\theta^2(y)\, d\theta\Bigg) dt\, ds,$$

where $u_t^2(y)$ is the unique solution of

$$u_t^2(y) = \int_0^t f_2\Bigg(W_t^1 - W_1^1 + \int_0^s e^{\theta - s} \overline{W}_\theta\, d\theta - e^{-s}y + \int_0^s e^{\theta - s} u_\theta^2(y)\, d\theta\Bigg) ds.$$

From (H7) the mapping $y \to u_1^2(y)$ is decreasing. From this and again (H7), we obtain

$$\frac{\partial u_1^1(y)}{\partial y} \le 1 - e^{-1}.$$

Consequently, the equation $u_1^1(y) = y$ has a unique solution, hence (15) has a unique solution for any $\{W_t\}$, and T is a bijection. □

We now want to apply Theorem 3.1, with $A = 0$ and $\bar{f} = f$. The assumptions of that Theorem are satisfied under (H6) and (H7). Indeed, those hypotheses imply (5), since

$$-f'(x) = \begin{pmatrix} -1 & 1 \\ f_2'(x^1) & 0 \end{pmatrix},$$

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} \Phi_{21}(t) - \Phi_{11}(t) & \Phi_{22}(t) - \Phi_{12}(t) \\ f_2'(Y_t^1)\Phi_{11}(t) & f_2'(Y_t^1)\Phi_{12}(t) \end{pmatrix},$$

$$\det[I - \Phi_1 g'(W_1) + g'(W_1)] = \det\begin{pmatrix} \Phi_{11}(1) & 0 \\ \Phi_{21}(1) & 1 \end{pmatrix}$$

$$= \Phi_{11}(1).$$

But

$$\dot{\Phi}_{11}(t) = -\Phi_{11}(t) + \Phi_{21}(t); \quad \Phi_{11}(0) = 1,$$

$$\dot{\Phi}_{21}(t) = f_2'(Y_t^1)\Phi_{11}(t); \quad \Phi_{21}(0) = 0.$$

Consequently, $\Phi_{21}(t) > 0$, $\forall\ t \in [0, 1]$ and $\Phi_{11}(1) > 0$.

Note that whenever f is affine, $\{X_t;\ t \in [0, 1]\}$ is a Markov process (see Russek [10], Ocone and Pardoux [9]). We now have a converse to that result:

THEOREM 5.3. *Suppose that* (H6) *and* (H7) *hold and that* $\{X_t;\ t \in [0, 1]\}$ *is a Markov field. Then f is affine (i.e., $f_2'' \equiv 0$).*

PROOF. We proceed as in the proof of Theorem 4.4 and use the same notation as there. Let us fix $t \in (0, 1)$ and let $\mathscr{G}_t = \sigma\{Y_s;\ 0 \le s \le t\}$, $\mathscr{G}^t = \sigma\{Y_0, Y_s;\ t \le s \le 1\}$. Let χ be a nonnegative $\mathscr{G}_t$ measurable random variable. Since $\{Y_t\}$ is a Markov field under Q,

$$\Lambda_\chi = E_Q(\chi|\mathscr{G}^t) = \frac{E_P(\chi J|\mathscr{G}^t)}{E_P(J|\mathscr{G}^t)}$$

is $\sigma(Y_0, Y_t)$ measurable, where

$$J = \Phi_{11}(1)\exp\left[\tfrac{1}{2} - \int_0^1 f(Y_t) \circ dW_t - \tfrac{1}{2}\int_0^1 |f(Y_t)|^2\, dt\right].$$

Let $H_t = \exp(-\int_0^t f(Y_s) \circ dW_s - \tfrac{1}{2}\int_0^t |f(Y_s)|^2\, ds)$; $0 \le t \le 1$ and $\Phi(1, t) = \Phi(1)\Phi(t)^{-1}$. Then

$$\Lambda_\chi = \frac{E_P\big[\chi(\Phi_{12}(1, t)\Phi_{21}(t) + \Phi_{11}(1, t)\Phi_{11}(t))\exp(H_t)|\mathscr{G}^t\big]}{E_P\big[(\Phi_{12}(1, t)\Phi_{21}(t) + \Phi_{11}(1, t)\Phi_{11}(t))\exp(H_t)|\mathscr{G}^t\big]}.$$

Hence, with the notation $\overline{F} = E_P(F|Y_0, Y_t)$,

$$\big[\overline{\chi\Phi_{21}(t)\exp(H_t)} - \Lambda_\chi\overline{\Phi_{21}(t)\exp(H_t)}\big]\Phi_{12}(1, t)$$

$$+ \big[\overline{\chi\Phi_{11}(t)\exp(H_t)} - \Lambda_\chi\overline{\Phi_{11}(t)\exp(H_t)}\big]\Phi_{11}(1, t) = 0.$$

(H7) implies that $\Phi_{12}(1, t)$, $\Phi_{21}(t)$, $\Phi_{11}(1, t)$ and $\Phi_{11}(t)$ are strictly positive a.s. for $0 < t < 1$. Choose

$$\chi_1 = \Phi_{21}(t)^{-1}\exp(-H_t),$$

$$\chi_2 = \Phi_{11}(t)^{-1}\exp(-H_t).$$

Let

$$G = \left\{\Lambda_1 = \big(\overline{\Phi_{21}(t)\exp(H_t)}\big)^{-1}\right\} \cap \left\{\Lambda_2 = \big(\overline{\Phi_{11}(t)\exp(H_t)}\big)^{-1}\right\}.$$

It follows from the strict Jensen's inequality that on G, $y(t) = \Phi_{21}(t)/\Phi_{11}(t)$ is $\sigma(Y_0, Y_t)$ measurable and on G^c, $z(t) = \Phi_{12}(1, t)/\Phi_{11}(1, t)$ is $\sigma(Y_0, Y_t)$ measurable.

Moreover, $G \in \sigma(Y_0, Y_t)$. But $y(t)$ and $z(t)$ are $\{W_t^1; 0 \leq t \leq 1\}$ measurable. Therefore, from Lemma 4.5 applied to $\mathscr{G} = \sigma(Y_0, Y_1)$, $A = G(\text{resp.}, G^c)$ and $F = y(t)$ [resp. $z(t)$], the previous statement implies that on G (since $\{W_t\}$ is two-dimensional, $\{D_t F\}$ is two-dimensional; $\{D_t^1 F\}$ denotes its first component, that is, the derivative in the direction of $\{W_t^1\}$),

$$D_\theta^1 y(t) = k(\omega) D_\theta^1 Y_t^1 + \bar{k}(\omega) D_\theta^1 Y_0^1$$

and on G^c,

$$D_\theta^1 z(t) = l(\omega) D_\theta^1 Y_t^1 + \bar{l}(\omega) D_\theta^1 Y_0^1.$$

However, $D_\theta^1 Y_t^1$ and $D_\theta^1 Y_0^1$ are constant both for $\theta \in [0, t)$ and for $\theta \in (t, 1]$. On the other hand,

$$\dot{y}_t = f_2'(Y_t^1) + y_t - y_t^2; \qquad y_0 = 0$$

$$D_\theta^1 y_t = -\int_0^{t \wedge \theta} \exp\left(\int_s^t (1 - 2y_u) \, du\right) f_2''(Y_s^1) \, ds,$$

and $D_\theta^1 y_t$ constant for $\theta \in [0, t)$ implies that $f_2''(Y_s^1) = 0$, $s \in [0, t)$ a.s. on G. If $P(G) > 0$, this implies $f_2'' \equiv 0$.

Suppose now that $P(G) = 0$. Then $P(G^c) = 1$ and

$$\dot{z}_t = -1 - z_t + f_2'(Y_t^1) z_t^2, \qquad z_1 = 0,$$

$$D_\theta^1 z_t = \int_t^{\theta \vee t} \exp\left(\int_t^s 1 - 2Z_u f_2'(Y_u^1) \, du\right) f_2''(Y_s^1) Z_s^2 \, ds.$$

But $D_\theta^1 z_t$ constant for $\theta \in (t, 1]$ implies that $f_2''(Y_s^1) = 0$, $s \in (t, 1]$ a.s. on G^c, hence $f_2'' \equiv 0$. $\square$

APPENDIX

PROOF OF LEMMA 3.5. We are going to approximate the process $K_t = \bar{f}(\psi_t)$ by a sequence of elementary processes. For each $n \geq 1$ we introduce the family of orthonormal functions $e_i(t) = \sqrt{n} \, \mathbf{I}_{[t_{i-1}, t_i]}$, $1 \leq i \leq n$, with $t_i = i/n$.
Define

$$\begin{aligned}
K_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^n \bar{f}\Bigg(& e^{-At_{i-1}} \Bigg[\sum_{j=1}^{i-1} e^{t_{j-1}A} \frac{1}{\sqrt{n}} W(e_j) \\
(16) \\
& + g\Bigg(\sum_{j=1}^n e^{t_{j-1}A} \frac{1}{\sqrt{n}} W(e_j) \Bigg) \Bigg] \Bigg) e_i(t),
\end{aligned}$$

where $W(e_j) = \sqrt{n}\,(W(t_j) - W(t_{j-1}))$.
We have, by taking a subsequence if necessary, that

$$\lim_n \left\{ \int_0^1 |K_t - K_t^n|^2 \, dt + \int_0^1 \int_0^1 |D_s K_t - D_s K_t^n|^2 \, ds \, dt \right\} = 0$$

for almost every W. Therefore, using the continuity with respect to the

Hilbert–Schmidt norm of the Carleman–Fredholm determinant, we deduce that $d_c(-DK^n)$ converges to $d_c(-DK)$ as n tends to infinity for almost all W.

The processes K^n are elementary in the sense that they can be expressed as

$$(17) \qquad K_t^n = \sum_{i=1}^n \psi_i^n(W(e_1), \ldots, W(e_n)) e_i(t),$$

where the functions $\psi^n: (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$ are defined by

$$(18) \quad \psi_i^n(x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \bar{f}\left(e^{-t_{i-1}A}\left[\frac{1}{\sqrt{n}} \sum_{j=1}^{i-1} e^{t_{j-1}A} x_j + g\left(\frac{1}{\sqrt{n}} \sum_{j=1}^n e^{t_{j-1}A} x_j\right)\right]\right), \quad$$

$$1 \le i \le n.$$

From (17) we deduce

$$D_s^l K_t^{n,j} = \sum_{i,h=1}^n \frac{\partial \psi_i^{n,j}}{\partial x_h^l}(W(e_1), \ldots, W(e_n)) e_i(t) e_h(s), \qquad 1 \le l, j \le d.$$

That means the Carleman–Fredholm determinant of $-DK^n$ is equal to that of the Jacobian matrix of ψ^n composed with the vector $(W(e_1), \ldots, W(e_n))$. We denote by $J\psi^n(W)$ the $n \times d$-dimensional square matrix defined by $(\partial \psi_i^{n,j}/\partial x_h^l)(W(e_1), \ldots, W(e_n))$, $1 \le l, j \le d, 1 \le i, h \le n$. Then we have

$$(19) \qquad d_c(-DK^n) = \det(I + J\psi^n)\exp(-\operatorname{Tr} J\psi^n).$$

We are going to use the following notation:

$$\bar{f}_{i-1}' = \bar{f}'\left(e^{-At_{i-1}}\left[\sum_{j=1}^{i-1} e^{At_{j-1}} \frac{1}{\sqrt{n}} W(e_j) + g\left(\frac{1}{\sqrt{n}} \sum_{j=1}^n e^{At_{j-1}} W(e_j)\right)\right]\right)$$

for $1 \le i \le n$ and

$$g' = g'\left(\frac{1}{\sqrt{n}} \sum_{j=1}^n e^{At_{j-1}} W(e_j)\right).$$

We can decompose the matrix $J\psi^n$ into n^2 square matrices of dimension d. The (i, h)-block, $1 \le i, h \le n$, is given by

$$(20) \qquad \frac{\partial \psi_i^n}{\partial x_h} = \begin{cases} \dfrac{1}{n} \bar{f}_{i-1}' e^{-At_{i-1}} g' e^{t_{h-1}A}, & \text{if } i \le h, \\[2mm] \dfrac{1}{n} \bar{f}_{i-1}' e^{-At_{i-1}}(I + g') e^{-At_{h-1}}, & \text{if } i > h. \end{cases}$$

Therefore, the trace of the matrix $J\psi^n$ is equal to

$$\frac{1}{n} \sum_{i=1}^n \operatorname{Tr}\left[\bar{f}_{i-1}' e^{-At_{i-1}} g' e^{At_{i-1}}\right],$$

which converges as n tends to infinity to

$$\int_0^1 \mathrm{Tr}\big[\, \bar{f}'(\psi_t)e^{-tA}g'(\xi_1)e^{At}\,\big]\, dt.$$

Then it remains to compute the following determinant:

$$\det \begin{bmatrix} I + \dfrac{1}{n}\bar{f}_0'g' & \dfrac{1}{n}\bar{f}_0'g'e^{At_1} & \dfrac{1}{n}\bar{f}_0'g'e^{At_2} & \cdots \\[2mm] \dfrac{1}{n}\bar{f}_1'e^{-At_1}(I+g') & I+\dfrac{1}{n}\bar{f}_1'e^{-At_1}g'e^{At_1} & \dfrac{1}{n}\bar{f}_1'e^{-At_1}g'e^{At_2} & \cdots \\[2mm] \dfrac{1}{n}\bar{f}_2'e^{-At_2}(I+g') & \dfrac{1}{n}\bar{f}_2'e^{-At_2}(I+g')e^{At_1} & I+\dfrac{1}{n}\bar{f}_2'e^{-At_2}(I+g')e^{At_2} & \cdots \\[2mm] \vdots & \vdots & \vdots & \end{bmatrix}$$

$$= \det\!\bigg[e^{A\sum_{i=1}^{n-1}t_i}\bigg] \det \begin{bmatrix} I+\dfrac{1}{n}\bar{f}_0'g' & \dfrac{1}{n}\bar{f}_0'g' & \dfrac{1}{n}\bar{f}_0'g' & \cdots \\[2mm] \dfrac{1}{n}\bar{f}_1'e^{-At_1}(I+g') & e^{-At_1}+\dfrac{1}{n}\bar{f}_1'e^{-At_1}g' & \dfrac{1}{n}\bar{f}_1'e^{-At_1}g' & \cdots \\[2mm] \dfrac{1}{n}\bar{f}_2'e^{-At_2}(I+g') & \dfrac{1}{n}\bar{f}_2'e^{-At_2}(I+g') & e^{-At_2}+\dfrac{1}{n}\bar{f}_2'e^{-At_2}(I+g') & \cdots \\[2mm] \vdots & \vdots & \vdots & \end{bmatrix}.$$

Subtracting each column from the preceding one we get

$$\bigg[e^{A\sum_{i=1}^{n-1}t_i}\bigg] \det \begin{bmatrix} I & 0 & 0 & \cdots & \dfrac{1}{n}\bar{f}_{t_0}'g' \\[2mm] -e^{-At_1}+\dfrac{1}{n}\bar{f}_1'e^{-At_1} & e^{-At_1} & 0 & \cdots & \dfrac{1}{n}\bar{f}_{t_i}'e^{-At_1}g' \\[2mm] 0 & -e^{-At_2}+\dfrac{1}{n}\bar{f}_2'e^{-At_2} & e^{-At_2} & \cdots & \dfrac{1}{n}\bar{f}_{t_2}'e^{-At_2}g' \\[2mm] \vdots & 0 & & & \vdots \\[1mm] & \vdots & & & \\[2mm] 0 & 0 & & e^{-At_{n-1}}+\dfrac{1}{n}\bar{f}_{n-1}'e^{-At_{n-1}}g' \end{bmatrix}$$

$$= \det \begin{bmatrix} I & 0 & 0 & \cdots & \dfrac{1}{n}\bar{f}_{t_0}'g' \\[2mm] -I+\dfrac{1}{n}e^{At_1}\bar{f}_1'e^{-At_1} & I & 0 & \cdots & \dfrac{1}{n}e^{At_1}\bar{f}_1'e^{-At_1}g' \\[2mm] 0 & -I+\dfrac{1}{n}e^{At_2}\bar{f}_2'e^{-At_2} & I & \cdots & \dfrac{1}{n}e^{At_2}\bar{f}_2'e^{-At_2}g' \\[2mm] \vdots & \vdots & & & \\[2mm] & & & I & \dfrac{1}{n}e^{t_{n-2}A}\bar{f}_{n-2}'e^{-At_{n-2}}g' \\[2mm] 0 & 0 & & -I+\dfrac{1}{n}e^{t_{n-1}A}\bar{f}_{n-1}'e^{-t_{n-1}A} & I+\dfrac{1}{n}e^{t_{n-1}A}\bar{f}_{n-1}'e^{-t_{n-1}A}g' \end{bmatrix}$$

To simplify the notation, set $e^{tA}\bar{f}'(\psi_t)e^{-tA} = R_t$. Then we obtain

$$\det\left[I + \frac{1}{n}R_{t_{n-1}}g' + \left(I - \frac{1}{n}R_{t_{n-1}}\right)\frac{1}{n}R_{t_{n-2}}g' + \left(I - \frac{1}{n}R_{t_{n-1}}\right) \right.$$

$$\left. \times \left(I - \frac{1}{n}R_{t_{n-2}}\right)\frac{1}{n}R_{t_{n-3}} + \cdots \right]$$

$$= \det\left[I + \frac{1}{n}\left\{R_{t_{n-1}} + \sum_{i=1}^{n-1}\prod_{j=1}^{i}\left(I - \frac{1}{n}R_{t_{n-j}}\right)R_{t_{n-i-1}}\right\}g' \right].$$

Consider the solution Ψ_t of the linear system

$$\dot{\Psi}_t = -R_t\Psi_t,$$

$$\Psi_0 = I.$$

Then we can approximate each term $I - (1/n)R_{t_{n-j}}$ by $\Psi_{t_{n-j+1}}\Psi_{t_{n-j}}^{-1}$. Indeed, we have

$$\Psi_{t_{n-j+1}}\Psi_{t_{n-j}}^{-1} = I - \int_{t_{n-j}}^{t_{n-j+1}} R_s\Psi_s\Psi_{t_{n-j}}^{-1}\,ds$$

$$= I - \frac{1}{n}R_{t_{n-j}} + \frac{a_{j,n}}{n},$$

where $\sup_j|a_{j,n}|/n \to 0$ as $n \to \infty$.

Consequently, we get

$$\det\left(I + \frac{1}{n}\sum_{i=1}^{n-1}\Psi_{t_{n-1}}\Psi_{t_{n-i}}^{-1}R_{t_{n-i-1}}g'\right)$$

and this converges to

(21) $$\det\left(I + \Psi_1\int_0^1 \Psi_t^{-1}e^{tA}\bar{f}'(\psi_t)e^{-tA}g'(\xi_1)\,dt\right).$$

We have

$$e^{-tA}\Psi_t = I - \int_0^t Ae^{-sA}\Psi_s - \int_0^t \bar{f}'(\psi_s)e^{-sA}\Psi_s\,ds = I - \int_0^t \bar{f}'(\psi_s)e^{-sA}\psi_s\,ds;$$

compare this with the equation for Φ_t in Section 3. We get $e^{-tA}\Psi_t = \Phi_t$. On the other hand, $\Psi_1^{-1} = I + \int_0^1 \psi_t^{-1}e^{tA}\bar{f}'(\psi_t)e^{-tA}\,dt$.

Therefore, the determinant (21) is equal to

$$\det\left(I + \Psi_1\left(-I + \Psi_1^{-1}\right)g'(\xi_1)\right)$$

$$= \det\left(I - \Psi_1 g'(\xi_1) + g'(\xi_1)\right)$$

$$= \det\left(I - e^A\Phi_1 g'(\xi_1) + g'(\xi_1)\right).$$

This completes the proof of the lemma. $\square$

REFERENCES

[1] CINLAR, E. and WANG, J. G. (1987). Random circles and fields on circles. Unpublished manuscript.

[2] DEMBO, A. and ZEITOUNI, O. (1987). Existence of solutions for two point stochastic boundary value problems. Unpublished manuscript.

[3] DONATI-MARTIN, C. (1990). Equations différentielles stochastiques dans $\mathbb{R}$ avec conditions au bord. *Stochastics*. To appear.

[4] HUANG, Z. (1984). On the generalized sample solutions of stochastic boundary value problems. *Stochastics* **11** 237–248.

[5] KUSUOKA, S. (1982). The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity, I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **7** 567–597.

[6] NUALART, D. and PARDOUX, E. (1988). Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields* **78** 535–581.

[7] NUALART, D. and PARDOUX, E. (1989). Second order stochastic differential equations with Dirichlet boundary conditions. *Stochastic Process. Appl.* To appear.

[8] NUALART, D. and ZAKAI, M. (1986). Generalized stochastic integrals and the Malliavin Calculus. *Probab. Theory Related Fields* **73** 255–280.

[9] OCONE, D. and PARDOUX, E. (1989). Linear stochastic differential equations with boundary conditions. *Probab. Theory Related Fields* **82** 439–526.

[10] RUSSEK, A. (1980). Gaussian n-Markovian processes and stochastic boundary value problems. *Z. Wahrsch. Verw. Gebiete* **53** 117–122.

[11] SIMON, B. (1979). *Trace Ideals and Their Applications*. Cambridge Univ. Press.

FACULTAT DE MATEMÀTIQUES   MATHÉMATIQUES, URA 225
UNIVERSITAT DE BARCELONA   UNIVERSITÉ DE PROVENCE
GRAN VIA 585   13331 MARSEILLE CEDEX 3
08007 BARCELONA   FRANCE
SPAIN