Spatially dense stochastic epidemic models with infection-age dependent infectivity

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Abstract. We study an individual-based stochastic spatial epidemic model where the number of locations and the number of individuals at each location both grow to infinity. Each individual is associated with a random infectivity function, which depends on the age of infection. Individuals are infected through interactions across the locations with heterogeneous effects. The epidemic dynamics in each location is described by the total force of infection, the number of susceptible individuals, the number of infected individuals that are infected at each time and have been infected for a certain amount of time, as well as the number of recovered individuals. The processes can be described using a time-space representation. We prove a functional law of large numbers for these time-space processes, and in the limit, we obtain a set of time-space integral equations. We then derive the PDE models from the limiting time-space integral equations, in particular, the density (with respect to the infection age) of the time-age-space integral equation for the number of infected individuals tracking the age of infection satisfies a linear PDE in time and age with an integral boundary condition. These integral equation and PDE limits can be regarded as dynamics on graphons under certain conditions.

1. Introduction

In order to capture the geographic heterogeneity, spatial epidemic models have been well developed, both in discrete and continuous spaces. With discrete space, multi-patch epidemic models have been studied in [1, 2, 5, 27, 31] and recently by the authors [22], where each patch represents a geographic location, and infection may occur within each batch and from the distance (for example, due to short travels). See also the multi-patch multi-type epidemic models in [5, 11], as well as relevant models in [4, 17, 18]. With continuous space, various PDE models have been developed (see the surveys in [25, 26, 19, 8]). There are two well–known models: Kendall's PDE model [14, 15] and Diekmann-Thieme's PDE model [9, 10, 28, 29]. Kendall's PDE model has a constant recovery rate while in the Diekmann-Thieme PDE model, the infection rate depends on the age of infection, as in the PDE model proposed by Kermack and McKendrick in their 1932 paper [16]. Kendall's PDE model was proved to be the FLLN limit for the multitype Markovian model by Andersson and Djehiche [3], where both the number of types and the population size go to infinity. However, the Diekmann-Thieme PDE model has not been proved to be the FLLN limit of a stochastic epidemic model.

In this paper, we start with an individual-based stochastic epidemic model at a finite number of locations. Each individual at every location may be infected from his or her own location or from other locations (see the infection rate function in equation (2.4)). Note that individuals do not migrate from one location to another in our model. Each individual is associated with a random infectivity function/process, independent from any other individual but having the same law as all the other individuals (this is reasonable since the model is for the same disease). This random

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infectivity function also determines the law of the infectious duration of each individual, which is i.i.d. for all individuals. For each individual, we track the age of infection, that is, the elapsed time since the individual was infected (for the initially infected individuals, this means we also know their infection times before time zero). To describe the epidemic dynamics at each location, we use the aggregate infectivity process of the population and a two-parameter (equivalently, measure-valued) process tracking the number of individuals that have been infected for less than or equal to a certain amount time as well as the numbers of susceptible and recovered individuals. Such an individual-based stochastic model with only one location has been studied by the authors in [21], where an FLLN is established and the associated PDE model for the limit is derived.

We consider this stochastic epidemic model in a spatially dense setting, where the number of locations increases to infinity while the number of individuals in each location (and the total population) also goes to infinity. This has the same flavor as the asymptotic regime in [3] for the multitype Markovian model where the number of types goes to infinity while the population in each type also go to infinity. This is also in a similar fashion as the asymptotic regime of the Markovian epidemic model on a refining spatial grid in $\mathbb{R}^d$ $(d = 1, 2, 3)$, recently studied in [20], where the mesh of the grid goes to zero and the population size at each site also goes to infinity. However, our model is non-Markovian and the infection process is much more complicated with the infection-age dependent infectivity.

For this model, it is convenient to describe the epidemic dynamics at all locations using a time-space representation of the vector-valued processes (for the number of infected individuals tracking the age of infection, this in fact becomes a time-age-space process). We treat the time-space processes in the functional spaces $\mathbf{D}$ and $\mathbf{D}_{\mathbf{D}}$ given the spatial component, while choosing the L^1 norm on the spatial component. We prove an FLLN (Theorem 2.1) for the scaled time-space processes under a set of regularity conditions on the initial conditions, infection contact rates and and random infectivity functions (Assumptions 2.1, 2.2 and 2.3). The limits in the FLLN are described by a set of time-space integral equations. It is worth highlighting that the heterogeneity of interaction effects between different locations is represented by a function $\beta(x, y)$ for $x, y \in [0, 1]$ (which resembles the kernel function in graphon, see further discussions below).

For the weak convergence of the time-space processes, we introduce new weak convergence criteria for these time-space processes (Theorems 4.1 and 4.2), which involves the L_1 norm for the spatial component. To verify these criteria, we establish moment estimates for the increments of these processes, which is challenging due to the interactions among the individuals at the different locations. In particular, the interactions introduce nontrivial dependence in various components of the time-space processes. We first study the joint time-space dynamics of the susceptible population and total force of infectivity (Section 5). This involves the existence and uniqueness of solution to a set of time-space Volterra-type integral equations (see equations (5.8)-(5.9)), and the moment estimates associated with the increments involving the varying infectivity functions together with their interactions (in order to use Theorem 4.1). Given their convergence, we then establish the convergence of the time-age-space process tracking the infection ages of individuals (Section 6). In order to employ Theorem 4.2, we need to establish the moment estimates for the increments with respect to both time and infection-age parameters, for which the dependence due to interactions also brings additional challenges.

From the limit tracking the rescaled number of infected individuals with a given age of infection, we derive a PDE model with partial derivatives with respect to time and the age of infection (not with respect to the spatial variable, since there is no migration among locations). It is a linear PDE model with an integral boundary condition. It may be seen as an extension of the PDE models in [21], with the addition of a spatial component. We then discuss how the PDE model is related to the well-known Diekmann-Thieme PDE model and how it reduces to Kendall's PDE model in the Markovian case (see Remarks 3.2 and 3.4). Note that our PDE model is much more general since we do not require any condition on the distribution function of the infectious periods.

This work is part of our continuing efforts on the understanding of non-Markovian epidemic models (see the survey in [12]). In our previous work, most models consider a homogeneous population with the two exceptions of a multi-patch (discrete space) model [22, 11]. Our model in this paper starts from a dense discrete space model, while the limit becomes a spatial model in continuous space. We should also mention the spatial models in continuous space in [7] and [30], where the stochastic model starts with a continuous process for the movement of individuals, in particular, it is assumed that individual movements follow an Itô diffusion process, and the epidemic models are Markovian.

Our work also contributes to the recent studies of stochastic dynamics on graphons. Keliger et al. [13] consider a finite-state Markov chain on a discretized graph of a graphon, and then prove a PDE limit for the dynamics as an FLLN. Their model includes an SIS model on graphon. Petit et al. [23] consider a random walk on graphon and prove a PDE limit for the Markovian dynamics. However, we start with a non-Markovian epidemic dynamics, and the limiting integral equations in Theorem 2.1 and the PDE models in Proposition 3.1 and Corollaries 3.1 and 3.2 can be regarded as dynamics on graphons, when the kernel function $\beta(x, y)$ is symmetric and takes values in $[0, 1]$ (see further discussions in Remark 2.2).

1.1. **Organization of the paper.** The paper is organized as follows. In Section 2.1, we provide the detailed model description. We then present the scaled processes and assumptions and state the FLLN result in Section 2.2. We derive the PDE models from the FLLN limits and discuss how they are related to the already known spatial PDE models in Section 3. The proofs of the FLLN are given in Sections 5 and 6 after some technical preliminaries in Section 4.

1.2. **Notation.** All random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{R}^k(\mathbb{R}^k_+)$ denotes the space of k-dimensional vectors with real (nonnegative) coordinates, with $\mathbb{R}(\mathbb{R}_+)$ for $k = 1$. Let $\mathbf{D} = \mathbf{D}(\mathbb{R}_+; \mathbb{R})$ denote the space of $\mathbb{R}$–valued càdlàg functions defined on $\mathbb{R}_+$. Here, convergence in $\mathbf{D}$ means convergence in the Skorohod J_1 topology, see Chapter 3 of [6]. Let $\mathbf{C}$ be the subset of $\mathbf{D}$ consisting of continuous functions. Let $\mathbf{D_D} = \mathbf{D}(\mathbb{R}_+; \mathbf{D}(\mathbb{R}_+; \mathbb{R}))$ be the $\mathbf{D}$-valued $\mathbf{D}$ space, and the convergence in the space $\mathbf{D_D}$ means that both $\mathbf{D}$ spaces are endowed with the Skorohod J_1 topology. For any increasing càdlàg function $F(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, abusing notation, we write $F(dx)$ by treating $F(\cdot)$ as the positive (finite) measure on $\mathbb{R}_+$ whose distribution function is F. For any $\mathbb{R}$–valued càdlàg function $\phi(\cdot)$ on $\mathbb{R}_+$, the integral $\int_a^b \phi(x)F(dx)$ represents $\int_{(a,b]} \phi(x)F(dx)$ for $a < b$. We use $\mathbf{1}_{\{\cdot\}}$ for the indicator function. For $x, y \in \mathbb{R}$, we denote $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

We use $\|\cdot\|_1$ to denote the $L^1([0, 1])$ norm. For time-space processes $Z(t, x)$ and $Z(t, s, x)$, for each x, we regard them in the spaces $\mathbf{D}$ and $\mathbf{D_D}$, respectively. For the weak convergence of the time-space processes $Z^N(t, x)$ to $Z(t, x)$ as $N \to \infty$, we use the Skorohod topology for the processes in $\mathbf{D}$ with the $L^1([0, 1])$ norm with respect to x. Similarly, for the weak convergence of the time-space processes $Z^N(t, s, x)$ to $Z(t, s, x)$ as $N \to \infty$, we use the Skorohod topology for the processes in $\mathbf{D_D}$ with the $L^1([0, 1])$ norm with respect to x. We write these spaces as $\mathbf{D}(\mathbb{R}_+, L^1([0, 1]))$ and $\mathbf{D}(\mathbb{R}_+, \mathbf{D}(\mathbb{R}_+, L^1([0, 1]))$, or $\mathbf{D}(\mathbb{R}_+, L^1)$ and $\mathbf{D}(\mathbb{R}_+, \mathbf{D}(\mathbb{R}_+, L^1)$ for short. See the weak convergence criteria in Theorems 4.1 and 4.2.

2. Model and FLLN

2.1. **Model Description.** We consider a population of fixed size N distributed in K locations in a space $\mathcal{S}$ (for example, $[0, 1]$, $\mathbb{R}^2$ or $\mathbb{R}^3$). For simplicity, we take $\mathcal{S} = [0, 1]$. Also let K depend on N, denoted as K^N. Let the K^N locations be at x^N_k, $k = 1, \dots, K^N$ in $[0, 1]$ such that $0 \le x^N_1 < x^N_2 < \cdots < x^N_{K^N} \le 1$. For notational convenience, let $\mathtt{I}^N_k, k = 1, \dots, K^N$ be a partition of $[0, 1]$ such that $x^N_k \in \mathtt{I}^N_k$ and $|\mathtt{I}^N_k| = (K^N)^{-1}$ for all $1 \le k \le K^N$. In each location, individuals are categorized into three groups: susceptible, infected (possibly including both exposed and infectious) and recovered. We assume that individuals do not move among the different locations, and susceptible individuals in each location can be infected from their own location as well as from other locations (as explained below). Suppose that there are B_k^N individuals at location k, such that $B_1^N + \cdots + B_{K^N}^N = N$. (For example, there is an equal number of individuals in each path, that is, $B_k^N = N/K^N$ for all k.) We assume that

$$\text{both } K^N \to \infty \quad \text{and} \quad \frac{N}{K^N} \to \infty, \quad \text{as} \quad N \to \infty\,. \tag{2.1}$$

Notation: Whenever not causing any confusion, we drop the superscript N in x_k^N, $\mathtt{I}_k^N$, K^N and B_k^N. For any vector $\mathbf{z} = (z_1, \dots, z_K)$, we write $z(x) = \sum_{k=1}^K z_k \mathbf{1}_{\mathtt{I}_k^N}(x)$ where $\mathbf{1}_{\mathtt{I}_k^N}(\cdot)$ denotes the indicator function of the set $\mathtt{I}_k^N$. For a process $\mathbf{Z}(t) = (Z_1(t), \dots, Z_K(t))$, we write $Z(t,x) = \sum_{k=1}^K Z_k(t) \mathbf{1}_{\mathtt{I}_k^N}(x)$ for $t \ge 0, x \in [0,1]$.

Let $S_k^N(t)$, $I_k^N(t)$ and $R_k^N(t)$ be the numbers of susceptible, infected and recovered individuals in location x_k at time t. We can also write the vectors $\mathbf{S}^N(t) = (S_1^N(t), \dots, S_K^N(t))$, $\mathbf{I}^N(t) = (I_1^N(t), \dots, I_K^N(t))$ and $\mathbf{R}^N(t) = (R_1^N(t), \dots, R_K^N(t))$, as the following time-space processes $S^N(t,x) = \sum_{k=1}^K S_k^N(t) \mathbf{1}_{\mathtt{I}_k}(x)$, $I^N(t,x) = \sum_{k=1}^K I_k^N(t) \mathbf{1}_{\mathtt{I}_k}(x)$ and $R^N(t,x) = \sum_{k=1}^K R_k^N(t) \mathbf{1}_{\mathtt{I}_k}(x)$, respectively. Note that $S_k^N(t) = S^N(t, x_k)$, and so on.

To each infected individual is attached a random infectivity function. Individual j in location k has a random infectivity function $\lambda_{j,k}(\cdot)$. The initially infected individual j from location k gets infected at time $\tau_{j,k}^N < 0$, for $j = -I_k^N(0), \dots, -1$, and has at time $t \ge 0$ the infectivity $\lambda_{j,k}(\tilde\tau_{-j,k}^N + t)$ where $\tilde\tau_{-j,k}^N = -\tau_{j,k}^N$. The initially susceptible individual j that gets infected at time $\tau_{j,k}^N$ has the infectivity $\lambda_{j,k}(t - \tau_{j,k}^N)$ at time $t \ge 0$ for each $j \ge 1$. We assume that the sequence $\{\lambda_{j,k} : j \in \mathbb{Z}\backslash\{0\}, k = 1, \dots, K\}$ is i.i.d. (Since we are concerned about the same disease, it is reasonable to require all the individuals at all the locations have the same law of infectivity and recovery, that is, homogeneous over locations.) Also, let $\bar\lambda(t) = \mathbb{E}[\lambda_{j,k}(t)]$ for $j \in \mathbb{Z}\backslash\{0\}$ and $k = 1, \dots, K$, for each $t \ge 0$.

We assume that $\lambda_{j,k}(t) = 0$ a.s. for $t < 0$, for all $j \in \mathbb{Z}\backslash\{0\}$, $k = 1, \dots, K$, and that each $\lambda_{j,k}$ has paths in $\mathbf{D}$. Define $\eta_{j,k} = \sup\{t > 0 : \lambda_{j,k}(t) > 0\}$, which represents the duration of the infected period for individual j. Note that this may include both the exposed and infectious periods. Under the above assumption on $\{\lambda_{j,k}\}$, the variables $\{\eta_{j,k}\}$ are also i.i.d. Let $F(t) = \mathbb{P}(\eta_{j,k} \le t)$ for $j \in \mathbb{Z}\backslash\{0\}$ and $k = 1, \dots, K$, representing the cumulative distribution function for the newly infected individuals.

For each $j \le -1$, let $\eta_{j,k}^0 = \inf\{t > 0 : \lambda_j(\tilde\tau_{j,k}^N + r) = 0, \forall r \ge t\}$ be the remaining infected period, which depends on the elapsed infection time $\tilde\tau_{j,k}^N$, but is independent of the elapsed infection times of other initially infected individuals. In particular, for $j \le -1$, the conditional distribution of $\eta_{j,k}^0$ given that $\tilde\tau_{j,k}^N = s > 0$ is given by

$$\mathbb{P}(\eta_{j,k}^0 > t | \tilde\tau_{j,k}^N = s) = \frac{F^c(t+s)}{F^c(s)}, \quad \text{for} \quad t, s > 0. \tag{2.2}$$

Note that the $\eta_{j,k}^0$'s are independent but not identically distributed.

The total force of infection of the infected individuals in location k is given by

$$\mathfrak{F}_k^N(t) = \sum_{j=1}^{I_k^N(0)} \lambda_{-j,k}(\tilde\tau_{-j,k}^N + t) + \sum_{j=1}^{S_k^N(0)} \lambda_{j,k}(t - \tau_{j,k}^N), \quad t \ge 0. \tag{2.3}$$

We similarly write the time-space process for the total force of infection in the population:

$$\mathfrak{F}^N(t,x) = \sum_{k=1}^{K^N} \mathfrak{F}_k^N(t)\mathbf{1}_{\mathtt{I}_k}(x).$$

The rate of infection for individuals in location k is given by

$$\Upsilon_k^N(t) = \frac{S_k^N(t)}{B_k^N}\frac{1}{K^N}\sum_{k'=1}^{K^N}\beta_{k,k'}^N\mathfrak{F}_{k'}^N(t), \quad t \geq 0. \tag{2.4}$$

Here the factor $\beta_{k,k'}^N$ reflects the effect of infection of individuals from location k' upon those from location k. It also represents the heterogeneity of the effects of the interactions among different locations.

The number of newly infected individuals in location k by time t is given by

$$A_k^N(t) = \int_0^t\int_0^\infty \mathbf{1}_{u\leq \Upsilon_k^N(s)}Q_k(ds,du), \tag{2.5}$$

where $\{Q_k(ds,du),\ 1\leq k\leq K\}$ are mutually independent standard (i.e., with mean measure the Lebesgue measure) Poisson random measures (PRMs) on $\mathbb{R}_+^2$. The counting process A_k^N has the event times $\{\tau_{j,k}^N, j\geq 1\}$.

Let $\mathfrak{I}_k^N(t,\mathfrak{a})$ be the number of infected individuals in location k that are infected at time t and have been infected for less than or equal to $\mathfrak{a}$. Then we can write

$$\mathfrak{I}_k^N(t,\mathfrak{a}) = \sum_{j=1}^{I_k^N(0)}\mathbf{1}_{\eta_{-j,k}^0>t}\mathbf{1}_{\tilde{\tau}_{-j,k}^N\leq(\mathfrak{a}-t)^+} + \sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t)}\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t}\,. \tag{2.6}$$

We suppose that there exists $\bar{\mathfrak{a}}\in[0,\infty)$ such that $I_k^N(0) = \mathfrak{I}_k^N(0,\bar{\mathfrak{a}})$. It is also clear that for all $t\geq 0$,

$$I_k^N(t) = \mathfrak{I}_k^N(t,\infty).$$

To account for the location, we also write the time-age-space process

$$\mathfrak{I}^N(t,\mathfrak{a},x) = \sum_{k=1}^{K^N}\mathfrak{I}_k^N(t,\mathfrak{a})\mathbf{1}_{\mathtt{I}_k}(x).$$

Note that for each x, the process $\mathfrak{I}^N(t,\mathfrak{a},x)$ has paths in $\mathbf{D_D}$.

The dynamics of $S_k^N(t)$, $I_k^N(t)$ and $R_k^N(t)$ can be expressed as

$$\begin{aligned}
S_k^N(t) &= S_k^N(0) - A_k^N(t),\\
I_k^N(t) &= \sum_{j=1}^{I_k^N(0)}\mathbf{1}_{\eta_{-j,k}^0>t} + \sum_{j=1}^{A_k^N(t)}\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t}\,,\\
R_k^N(t) &= R_k^N(0) + \sum_{j=1}^{I_k^N(0)}\mathbf{1}_{\eta_{-j,k}^0\leq t} + \sum_{j=1}^{A_k^N(t)}\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}\leq t}\,.
\end{aligned}$$

2.2. **FLLN.** We recall that

$$N = \sum_{k=1}^{K^N}(S_k^N(t)+I_k^N(t)+R_k^N(t)) = \sum_{k=1}^{K^N}B_k^N\,, \tag{2.7}$$

and observe that

$$\int_0^1 (S^N(t,x)+I^N(t,x)+R^N(t,x))dx = \frac{1}{K^N}\sum_{k=1}^{K^N}(S_k^N(t)+I_k^N(t)+R_k^N(t)) = \frac{N}{K^N}\,. \tag{2.8}$$

It is then reasonable to introduce the scaling of the processes by N/K^N, that is, for any process $Z_k^N = \mathfrak{F}_k^N, \mathfrak{I}_k^N, \Upsilon_k^N, A_k^N, S_k^N, I_k^N, R_k^N$, we define $\bar{Z}_k^N = (N/K^N)^{-1} Z_k^N$. We then define the scaled time-space processes

$$\bar{Z}^N(t,x) = \sum_{k=1}^{K^N} \bar{Z}_k^N(t)\mathbf{1}_{\mathtt{I}_k}(x), \quad Z_k^N = \mathfrak{F}_k^N, \Upsilon_k^N, A_k^N, S_k^N, I_k^N, R_k^N$$

and

$$\bar{\mathfrak{I}}^N(t,\mathfrak{a},x) = \sum_{k=1}^{K^N} \bar{\mathfrak{I}}_k^N(t,\mathfrak{a})\mathbf{1}_{\mathtt{I}_k}(x)\,.$$

In addition, define the scaled population size at each location

$$\bar{B}^N(x) = \sum_{k=1}^{K^N} \bar{B}_k^N \mathbf{1}_{\mathtt{I}_k}(x), \quad \text{with} \quad \bar{B}_k^N = (N/K^N)^{-1}B_k^N.$$

Hence, from (2.8) and the scaling, we obtain

$$\int_0^1 (\bar{S}^N(t,x)+\bar{I}^N(t,x)+\bar{R}^N(t,x))dx = \int_0^1 \bar{B}^N(x)dx = 1\,.$$

We make the following assumption on the initial condition.

Assumption 2.1. *There exist nonnegative deterministic functions* $(\bar{S}(0,x), \bar{\mathfrak{I}}(0,\mathfrak{a},x), \bar{R}(0,x))$ *such that for each* x, $\bar{\mathfrak{I}}(0,\cdot,x)$ *is in* $\mathbf{C}$, *and for each* $\mathfrak{a} \in [0,\bar{\mathfrak{a}}]$,

$$\|\bar{S}^N(0,\cdot)-\bar{S}(0,\cdot)\|_1 \to 0, \quad \|\bar{\mathfrak{I}}^N(0,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(0,\mathfrak{a},\cdot)\|_1 \to 0, \quad \|\bar{R}^N(0,\cdot)-\bar{R}(0,\cdot)\|_1 \to 0 \tag{2.9}$$

in probability as $N\to\infty$, *where letting* $\bar{I}(0,x) = \bar{\mathfrak{I}}(0,\infty,x)$, *we have*

$$\int_0^1 (\bar{S}(0,x)+\bar{I}(0,x)+\bar{R}(0,x))dx = 1. \tag{2.10}$$

In addition, there exists $\bar{B}(x)$ *such that*

$$\|\bar{B}^N(\cdot)-\bar{B}(\cdot)\|_\infty = \sup_{x\in[0,1]} |\bar{B}^N(x)-\bar{B}(x)| \to 0, \tag{2.11}$$

where for some constants $0 < c_B < C_B < \infty$,

$$\bar{B}(x) \in [c_B, C_B] \quad \forall x\in[0,1], \tag{2.12}$$

and

$$\int_0^1 \bar{B}(x)dx = 1\,.$$

Note that, thanks to (2.11) and (2.12), we may and do assume that c_B and C_B have been chosen in such a way that

$$\bar{B}^N(x) \in [c_B, C_B] \quad \forall N\ge 1,\ x\in[0,1]. \tag{2.13}$$

Under the assumption in (2.9), it follows that

$$\|\bar{I}^N(0,\cdot)-\bar{I}(0,\cdot)\|_1 \to 0$$

in probability as $N\to\infty$.

We introduce for each $x, x' \in [0,1]$,

$$\beta^N(x,x') = \sum_{k,k'} \beta^N_{k,k'} \mathbf{1}_{\mathtt{I}_k}(x) \mathbf{1}_{\mathtt{I}_{k'}}(x') \,. \tag{2.14}$$

Assumption 2.2. *There exists a constant $C_\beta > 0$ such that for all $N \geq 1$, $x \in [0,1]$,*

$$\int_0^1 \beta^N(x,y)dy \vee \int_0^1 \beta^N(y,x)dy \leq C_\beta \,. \tag{2.15}$$

There exists a function $\beta : [0,1] \times [0,1] \mapsto \mathbb{R}_+$ such that for any bounded measurable function $\phi : [0,1] \mapsto \mathbb{R}$,

$$\left\| \int_0^1 [\beta^N(\cdot,y) - \beta(\cdot,y)]\phi(y)dy \right\|_1 \to 0 \,. \tag{2.16}$$

Remark 2.1. *Concerning condition (2.15), let us first note that, if $\beta^N_{k,k'} = \beta^N_{k',k}$ (symmetric) for all $N \geq 1$, $1 \leq k, k' \leq K$, the boundedness of $\int_0^1 \beta^N(x,y)dy$ is equivalent to that of $\int_0^1 \beta^N(y,x)dy$. Clearly (2.16) implies that (2.15) is satisfied with β^N replaced by β. We note that this assumption allows in particular $\beta(x,y)$ to explode on the diagonal $x = y$, for example, $\beta(x,y) = \frac{c}{\sqrt{|x-y|}}$ for some $c > 0$, meaning that infectious interactions between "close by" individuals are much more frequent than between distant ones. See further discussions in Remark 2.2.*

We make the following assumption on the random function λ.

Assumption 2.3. *Let $\lambda(\cdot)$ be a process having the same law of $\{\lambda_j^0(\cdot)\}_j$ and $\{\lambda_i(\cdot)\}_i$. Assume that there exists a constant λ^* such that for each $0 < T < \infty$, $\sup_{t\in[0,T]} \lambda(t) \leq \lambda^*$ almost surely. Assume that there exist an integer κ, a random sequence $0 = \zeta^0 \leq \zeta^1 \leq \cdots \leq \zeta^\kappa$ and associated random functions $\lambda^\ell \in \mathbf{C}(\mathbb{R}_+; [0,\lambda^*])$, $1 \leq \ell \leq \kappa$, such that*

$$\lambda(t) = \sum_{\ell=1}^{\kappa} \lambda^\ell(t) \mathbf{1}_{[\zeta^{\ell-1},\zeta^\ell)}(t). \tag{2.17}$$

We write F_ℓ for the c.d.f. of ζ^ℓ, $\ell = 1,\dots,\kappa$. In addition, we assume that there exists a deterministic nondecreasing function $\varphi \in \mathbf{C}(\mathbb{R}_+;\mathbb{R}_+)$ with $\varphi(0) = 0$ such that $|\lambda^\ell(t) - \lambda^\ell(s)| \leq \varphi(t-s)$ almost surely for all $t, s \geq 0$ and for all $\ell \geq 1$. Let $\bar\lambda(t) = \mathbb{E}[\lambda_i(t)] = \mathbb{E}[\lambda_j^0(t)]$ and $v(t) = \mathrm{Var}(\lambda(t)) = \mathbb{E}\big[\big(\lambda(t) - \bar\lambda(t)\big)^2\big]$ for $t \geq 0$.

Theorem 2.1. *Under Assumptions 2.1, 2.2 and 2.3,*

$$\begin{aligned} &\|\bar{\mathfrak{F}}^N(t,\cdot) - \bar{\mathfrak{F}}(t,\cdot)\|_1 \to 0, \quad \|\bar S^N(t,\cdot) - \bar S(t,\cdot)\|_1 \to 0, \qquad \|\bar R^N(t,\cdot) - \bar R(t,\cdot)\|_1 \to 0, \\ &\|\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)\|_1 \to 0 \end{aligned} \tag{2.18}$$

in probability as $N \to \infty$, locally uniformly in t and $\mathfrak{a}$, where the limits are given by the unique solution to the following set of integral equations. The limit $(\bar S(t,x), \bar{\mathfrak{F}}(t,x))$ is a unique solution to the system of integral equations: for $t \geq 0$ and $x \in [0,1]$,

$$\bar S(t,x) = \bar S(0,x) - \int_0^t \bar\Upsilon(s,x)ds \,, \tag{2.19}$$

$$\bar{\mathfrak{F}}(t,x) = \int_0^\infty \bar\lambda(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x) + \int_0^t \bar\lambda(t-s)\bar\Upsilon(s,x)ds \,, \tag{2.20}$$

where

$$\bar\Upsilon(t,x) = \frac{\bar S(t,x)}{\bar B(x)} \int_0^1 \beta(x,x')\bar{\mathfrak{F}}(t,x')dx' = \bar{\mathfrak{I}}_{\mathfrak{a}}(t,0,x) \,. \tag{2.21}$$

Given $\bar{S}(t,x)$ and $\bar{\mathfrak{F}}(t,x)$, the limits $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ and $\bar{R}(t,x)$ are given by

$$\bar{\mathfrak{I}}(t,\mathfrak{a},x) = \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_{(t-\mathfrak{a})^+}^t F^c(t-s)\bar{\Upsilon}(s,x)ds\,, \tag{2.22}$$

$$\bar{R}(t,x) = \bar{R}(0,x) + \int_0^\infty \Big(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big)\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_0^t F(t-s)\bar{\Upsilon}(s,x)ds\,. \tag{2.23}$$

In addition,

$$\|\bar{I}^N(t,\cdot)-\bar{I}(t,\cdot)\|_1 \to 0$$

locally uniformly in t in probability as $N\to\infty$, where

$$\bar{I}(t,x) = \int_0^\infty \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_0^t F^c(t-s)\bar{\Upsilon}(s,x)ds\,. \tag{2.24}$$

For each x, the limits $\bar{S}(t,x)$, $\bar{\mathfrak{F}}(t,x)$, $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$, $\bar{I}(t,x)$ and $\bar{R}(t,x)$ are continuous in t and $\mathfrak{a}$.

Remark 2.2. *Our model can be regarded in some sense as non-Markovian population dynamics on graphons. In particular, the function $\beta(x,x')$ can be regarded as the graphon kernel function, representing the inhomogeneity in the connectivity. However, the kernel function is often assumed to take values in $[0,1]$ and to be symmetric in the graphon literature. In our model, $\beta(x,x')$ does not necessarily take values in $[0,1]$ although it can be rescaled to $[0,1]$, and the function $\beta(x,x')$ may not be necessarily symmetric. In the prelimit (the N^{th} system), the locations $\{\mathtt{I}_k^N\}_k$ can be regarded as a discretization of the unit interval $[0,1]$ and the infection rate functions between different locations $\beta^N_{k,k'}$ in (2.14) can then be regarded as the corresponding discretization of the function $\beta(x,x')$. We refer the readers to [13] and [23] for Markov dynamics on graphons and PDE approximations.*

Remark 2.3. *For the spatial SIS model, we have the identity $\sum_{k=1}^{K^N}(S_k^N(t)+I_k^N(t)) = N$ and $\int_0^1(\bar{S}(t,x)+\bar{I}(t,x))dx = 1$. We use two processes $\bar{\mathfrak{F}}^N(t,x)$ and $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$ to describe the epidemic dynamics, and can show that $\|\bar{\mathfrak{F}}^N(t,\cdot)-\bar{\mathfrak{F}}(t,\cdot)\|_1\to 0$ and $\|\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(t,\mathfrak{a},x)\|_1\to 0$ in probability locally uniformly in t and $\mathfrak{a}$ as $N\to\infty$, where*

$$\bar{\mathfrak{F}}(t,x) = \int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x) + \int_0^t \bar{\lambda}(t-s)\bar{S}(s,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(s,x')dx'ds\,, \tag{2.25}$$

and

$$\bar{\mathfrak{I}}(t,\mathfrak{a},x) = \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_{(t-\mathfrak{a})^+}^t F^c(t-s)\bar{S}(s,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(s,x')dx'ds\,, \tag{2.26}$$

with $\bar{S}(t,x)$ satisfying

$$\int_0^1(\bar{S}(t,x)+\bar{\mathfrak{I}}(t,\infty,x))dx = 1\,. \tag{2.27}$$

Using $\bar{I}(t,x) = \bar{\mathfrak{I}}(t,\infty,x)$, we can write the last equation as $\int_0^1(\bar{S}(t,x)+\bar{I}(t,x))dx = 1$, and the limit $\bar{I}(t,x)$ is given by

$$\bar{I}(t,x) = \int_0^\infty \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x) + \int_0^t F^c(t-s)\bar{S}(s,x)\int_0^1 \beta(x,x')\bar{\mathfrak{F}}(s,x')dx'ds\,.$$

3. PDE Models

In this section we derive the PDE models associated with the limits from the FLLN. For each t, the limits $\bar{S}(t,x), \bar{\mathfrak{F}}(t,x), \bar{I}(t,x), \bar{R}(t,x)$ can be regarded as the densities of the quantities, susceptibles, aggregate infectivity, infected and recovered, distributed over the location $x \in [0,1]$, and for each t and $\mathfrak{a}$, the function $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ can be also regarded as the density of the proportion of infected individuals at time t with infection age less than or equal to $\mathfrak{a}$, over the location $x \in [0,1]$. In addition, for each fixed t and x, $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ is increasing in $\mathfrak{a}$, and can be regarded as a "distribution" over the infection ages. If $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ is absolutely continuous in $\mathfrak{a}$, we let $\bar{\mathfrak{i}}(t,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(t,\mathfrak{a},x)$ be the density function of $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ with respect to the infection age $\mathfrak{a}$.

In the following we will consider the dynamics of $\bar{S}(t,x), \bar{\mathfrak{F}}(t,x), \bar{I}(t,x), \bar{R}(t,x), \bar{\mathfrak{I}}(t,\mathfrak{a},x)$ in t and $\mathfrak{a}$, as a PDE model. Since there is no movement of individuals between locations, no derivative with respect to x will appear. However, the interaction among individuals in different locations will be captured in these dynamics, in particular, in the expression of $\bar{\Upsilon}(t,x)$ in (2.21).

We consider any arbitrary distribution F, and for notational convenience, we let $G(t) = F(t^-)$ and $G^c(t) = 1 - G(t) = F^c(t^-)$, which are the left continuous versions of F and F^c. Denote $\nu(\cdot)$ the law of η. Then $\frac{\nu(d\mathfrak{a})}{G^c(\mathfrak{a})}$ can be regarded as a generalized hazard rate function.

Remark 3.1. *Let us explain why we introduce here the left continuous version of F. Note that when F is absolutely continuous, this makes no difference. The simplest example which motivates this choice is the following: $\nu = \delta_{t_0}$. In this case, $G^c(t_0) = 1$, while $F^c(t_0) = 0$. So, with the convention that $\frac{\nu(dt)}{G^c(t)}$ is zero outside the support of ν, whatever the denominator might be, this fraction is well defined, which would not be the case if we replace $G^c(t)$ by $F^c(t)$ in the denominator.*

Proposition 3.1. *Suppose that for each x, $\bar{\mathfrak{I}}(0,\mathfrak{a},x)$ is absolutely continuous with respect to $\mathfrak{a}$ with density $\bar{\mathfrak{i}}(0,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a},x)$. Then for $t, \mathfrak{a} > 0$ and $x \in [0,1]$, the function $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ is absolutely continuous in t and $\mathfrak{a}$, and its density $\bar{\mathfrak{i}}(t,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(t,\mathfrak{a},x)$ with respect to $\mathfrak{a}$ satisfies*

$$\frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial t} + \frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial \mathfrak{a}} = -\frac{\bar{\mathfrak{i}}(t,\mathfrak{a},x)}{G^c(\mathfrak{a})}\nu(d\mathfrak{a}), \tag{3.1}$$

$(t,\mathfrak{a},x)$ in $(0,\infty)^2 \times [0,1]$, with the initial condition $\bar{\mathfrak{i}}(0,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a},x)$ for $(\mathfrak{a},x) \in (0,\infty) \times [0,1]$, and the boundary condition

$$\bar{\mathfrak{i}}(t,0,x) = \frac{\bar{S}(t,x)}{\bar{B}(x)} \int_0^1 \beta(x,x') \left(\int_0^{t+\bar{\mathfrak{a}}} \frac{\bar{\lambda}(\mathfrak{a}')}{\frac{G^c(\mathfrak{a}')}{G^c(\mathfrak{a}'-t)}} \bar{\mathfrak{i}}(t,\mathfrak{a}',x') d\mathfrak{a}' \right) dx', \tag{3.2}$$

where $G^c \equiv 1$ on $\mathbb{R}_-$ and the integrand inside the second integral is set to zero whenever $G^c(\mathfrak{a}) = 0$.

The function $\bar{S}(t,x)$ satisfies

$$\frac{\partial \bar{S}(t,x)}{\partial t} = -\bar{\mathfrak{i}}(t,0,x), \tag{3.3}$$

with $\bar{S}(0,x)$ satisfying (2.10).

Moreover, the PDE (3.1)-(3.2) has a unique non-negative solution which is given as follows: for $\mathfrak{a} \geq t$ and $x \in [0,1]$,

$$\bar{\mathfrak{i}}(t,\mathfrak{a},x) = \frac{G^c(\mathfrak{a})}{G^c(\mathfrak{a}-t)} \bar{\mathfrak{i}}(0,\mathfrak{a}-t,x), \tag{3.4}$$

and for $t > \mathfrak{a}$ and $x \in [0,1]$,

$$\bar{\mathfrak{i}}(t,\mathfrak{a},x) = G^c(\mathfrak{a}) \bar{\mathfrak{i}}(t-\mathfrak{a},0,x), \tag{3.5}$$

and the boundary function is the unique non-negative solution to the integral equation

$$\bar{\mathfrak{i}}(t,0,x) = (\bar{B}(x))^{-1} \Big(\bar{S}(0,x) - \int_0^t \bar{\mathfrak{i}}(s,0,x) ds \Big)$$

$$\times \int_0^1 \beta(x,x') \left(\int_0^\infty \bar\lambda(\mathfrak{a}+t)\,\bar{\mathfrak{i}}(0,\mathfrak{a},x')d\mathfrak{a} + \int_0^t \bar\lambda(t-s)\,\bar{\mathfrak{i}}(s,0,x')ds \right) dx' \,. \tag{3.6}$$

Provided with the PDE solution $\bar{\mathfrak{i}}(t,\mathfrak{a},x)$ and with $\bar\Upsilon(t,x)=\bar{\mathfrak{i}}(t,0,x)$, the functions $\bar I(t,x)$ and $\bar R(t,x)$ are given by

$$\bar I(t,x) = \int_0^\infty \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\,\bar{\mathfrak{i}}(0,\mathfrak{a}',x)d\mathfrak{a}' + \int_0^t F^c(t-s)\,\bar{\mathfrak{i}}(s,0,x)ds\,,$$

$$\bar R(t,x) = \bar R(0,x) + \int_0^\infty \left(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\right)\bar{\mathfrak{i}}(0,\mathfrak{a}',x)d\mathfrak{a}' + \int_0^t F(t-s)\,\bar{\mathfrak{i}}(s,0,x)ds\,.$$

Also, by definition,

$$\bar I(t,x) = \bar{\mathfrak{I}}(t,\infty,x) = \int_0^\infty \bar{\mathfrak{i}}(t,\mathfrak{a},x)d\mathfrak{a}.$$

Proof. Using the expression of $\bar\Upsilon(s,x)=\bar{\mathfrak{I}}_{\mathfrak{a}}(s,0,x)$ in (2.21) and with G^c, we can equivalently rewrite (2.22) as

$$\bar{\mathfrak{I}}(t,\mathfrak{a},x) = \int_0^{(\mathfrak{a}-t)^+} \frac{G^c(\mathfrak{a}'+t)}{G^c(\mathfrak{a}')}\bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a}',x)d\mathfrak{a}' + \int_{(t-\mathfrak{a})^+}^t G^c(t-s)\bar{\mathfrak{I}}_{\mathfrak{a}}(s,0,x)ds\,. \tag{3.7}$$

Exploiting the fact that $\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathfrak{a}}$ of a function of $t-\mathfrak{a}$ vanishes, we deduce from (3.7) that

$$\begin{aligned}
\bar{\mathfrak{I}}_t(t,\mathfrak{a},x) + \bar{\mathfrak{I}}_{\mathfrak{a}}(t,\mathfrak{a},x) &= -\int_0^{(\mathfrak{a}-t)^+} \frac{1}{G^c(\mathfrak{a}')}\bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a}',x)\nu(t+d\mathfrak{a}') \\
&\qquad + \bar{\mathfrak{I}}_{\mathfrak{a}}(t,0,x) - \int_{(t-\mathfrak{a})^+}^t \bar{\mathfrak{I}}_{\mathfrak{a}}(s,0,x)\nu(t-ds) \\
&= -\int_t^{\mathfrak{a}\vee t} \frac{1}{G^c(\mathfrak{a}'-t)}\bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a}'-t,x)\nu(d\mathfrak{a}') \\
&\qquad + \bar{\mathfrak{I}}_{\mathfrak{a}}(t,0,x) - \int_0^{\mathfrak{a}\wedge t} \bar{\mathfrak{I}}_{\mathfrak{a}}(t-s,0,x)\nu(ds)\,.
\end{aligned}$$

Here we consider the derivative with respect to t in the distributional sense and use the measure $\nu(\cdot)$ associated with G since we do not necessarily have differentiability of G^c. We then take derivative with respect to $\mathfrak{a}$ on both sides of this equation (denoting $\bar{\mathfrak{I}}_{t,\mathfrak{a}}(t,\mathfrak{a},x)$ and $\bar{\mathfrak{I}}_{\mathfrak{a},\mathfrak{a}}(t,\mathfrak{a},x)$ as the derivatives of $\bar{\mathfrak{I}}_t(t,\mathfrak{a},x)$ and $\bar{\mathfrak{I}}_{\mathfrak{a}}(t,\mathfrak{a},x)$ with respect to $\mathfrak{a}$) and obtain the following:

$$\bar{\mathfrak{I}}_{t,\mathfrak{a}}(t,\mathfrak{a},x) + \bar{\mathfrak{I}}_{\mathfrak{a},\mathfrak{a}}(t,\mathfrak{a},x) = -\mathbf{1}_{\mathfrak{a}\ge t}\frac{\nu(d\mathfrak{a})}{G^c(\mathfrak{a}-t)}\bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a}-t,x) - \mathbf{1}_{t>\mathfrak{a}}\nu(d\mathfrak{a})\bar{\mathfrak{I}}_{\mathfrak{a}}(t-\mathfrak{a},0,x)\,.$$

Rewriting $\frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial t} = \bar{\mathfrak{I}}_{\mathfrak{a},t}(t,\mathfrak{a},x) = \bar{\mathfrak{I}}_{t,\mathfrak{a}}(t,\mathfrak{a},x)$ and $\frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial \mathfrak{a}} = \bar{\mathfrak{I}}_{\mathfrak{a},\mathfrak{a}}(t,\mathfrak{a},x)$, we obtain the PDE:

$$\frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial t} + \frac{\partial \bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial \mathfrak{a}} = -\mathbf{1}_{\mathfrak{a}\ge t}\frac{\nu(d\mathfrak{a})}{G^c(\mathfrak{a}-t)}\,\bar{\mathfrak{i}}(0,\mathfrak{a}-t,x) - \mathbf{1}_{t>\mathfrak{a}}\nu(d\mathfrak{a})\,\bar{\mathfrak{i}}(t-\mathfrak{a},0,x)\,. \tag{3.8}$$

In order to see that the right hand side coincides with that in (3.1), we first establish (3.4) and (3.5). For $\mathfrak{a}\ge t$, $0\le s\le t$ and $x\in[0,1]$,

$$\frac{\partial \bar{\mathfrak{i}}(s,\mathfrak{a}-t+s,x)}{\partial s} = -\frac{\nu(\mathfrak{a}-t+ds)}{G^c(\mathfrak{a}-t)}\,\bar{\mathfrak{i}}(0,\mathfrak{a}-t,x)\,,$$

and for $t>\mathfrak{a}$, $0\le s\le \mathfrak{a}$ and $x\in[0,1]$,

$$\frac{\partial \bar{\mathfrak{i}}(t-\mathfrak{a}+s,s,x)}{\partial s} = -\nu(ds)\,\bar{\mathfrak{i}}(t-\mathfrak{a},0,x)\,.$$

From these, by integration and simple calculations, we obtain (3.4) and (3.5). Now (3.1) follows from (3.8), (3.4) and (3.5).

Then using (3.4) and (3.5), by (2.20) and the second equality in (2.21), we obtain

$$\bar{\mathfrak{F}}(t,x) = \int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}(0,\mathfrak{a},x)d\mathfrak{a} + \int_0^t \bar{\lambda}(t-s)\,\bar{\mathfrak{i}}(s,0,x)ds\,. \tag{3.9}$$

The expression for the boundary condition in (3.6) then follows directly from (2.21) using this expression of $\bar{\mathfrak{F}}(t,x)$. Again, using (3.4) and (3.5), we see that the boundary condition (3.6) is equivalent to (3.2).

We now sketch the proof of existence and uniqueness of a non-negative solution to (3.6). Note that, thanks to (3.4) and (3.5), existence and uniqueness of a non-negative solution to the PDE (3.1)-(3.2) will follow from that result. First of all, let us rewrite that equation as

$$u(t,x) = (\bar{B}(x))^{-1}\left(f(x) - \int_0^t u(s,x)ds\right) \times \int_0^1 \beta(x,x')\left(g(t,x') + \int_0^t \bar{\lambda}(t-s)u(s,x')ds\right)dx',$$

where $0 \le f(x) \le 1$ and $0 \le g(t,x) \le \lambda^*$ are given from the initial conditions. Any nonnegative solution satisfies

$$\begin{aligned} u(t,x) &\le \int_0^1 \beta(x,x')\left(g(t,x') + \int_0^t \bar{\lambda}(t-s)u(s,x')ds\right)dx', \text{ hence} \\ \|u(t,\cdot)\|_\infty &\le C_\beta\lambda^*\left(1 + \int_0^t \|u(s,\cdot)\|_\infty ds\right) \\ &\le C_\beta\lambda^* e^{C_\beta\lambda^* t}\,. \end{aligned}$$

Here $\|u(t,\cdot)\|_\infty = \sup_{x\in[0,1]}|u(t,x)|$. Let now u and v be two non negative solutions. Then,

$$\begin{aligned} |u(t,x)-v(t,x)| &\le (\bar{B}(x))^{-1}\left(\int_0^1 \beta(x,x')\left[g(t,x') + \int_0^t \bar{\lambda}(t-s)u(s,x')ds\right]\right)\int_0^t |u(s,x)-v(s,x)|ds \\ &\quad + (\bar{B}(x))^{-1}\left(f(x) + \int_0^t v(s,x)ds\right)\int_0^1 \beta(x,x')\int_0^t \bar{\lambda}(t-s)|u(s,x')-v(s,x')|dsdx'\,. \end{aligned}$$

Integrating over dx, exploiting the previous a priori estimate and (2.15), we deduce the uniqueness from Gronwall's Lemma. Finally, the existence of a nonnegative $L^1([0,1])$-valued solution can be established using a Picard iteration argument. Note that in the previous lines we have used the two distinct inequalities contained in (2.15). □

If F is absolutely continuous, with density f, we denote by $\mu(\mathfrak{a})$ the hazard function, i.e., $\mu(\mathfrak{a}) = \frac{f(\mathfrak{a})}{F^c(\mathfrak{a})}$ for all $\mathfrak{a} \ge 0$. We obtain the following corollary in this case.

Corollary 3.1. *Under the assumptions of Proposition 3.1, if F is absolutely continuous with density f, then the PDE in (3.1) becomes*

$$\frac{\partial\bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial t} + \frac{\partial\bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial\mathfrak{a}} = -\mu(\mathfrak{a})\,\bar{\mathfrak{i}}(t,\mathfrak{a},x)\,, \tag{3.10}$$

with the initial condition $\bar{\mathfrak{i}}(0,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a},x)$ for $(\mathfrak{a},x) \in (0,\infty)\times[0,1]$ and the boundary condition (3.2). The function $\bar{S}(t,x)$ satisfies (3.3), and the PDE (3.10) has a unique solution which is given by (3.4) and (3.5), and the boundary function is the same as in (3.6).

When the infectious periods are deterministic, we obtain the following corollary.

Corollary 3.2. *Suppose that the infectious periods are deterministic and equal to t_i, that is, $F(t) = \mathbf{1}_{t\ge t_i}$ and $G(t) = \mathbf{1}_{t>t_i}$. Then the PDE in in (3.1) becomes*

$$\frac{\partial\bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial t} + \frac{\partial\bar{\mathfrak{i}}(t,\mathfrak{a},x)}{\partial\mathfrak{a}} = -\delta_{t_i}(\mathfrak{a})\,\bar{\mathfrak{i}}(t,\mathfrak{a},x)\,, \tag{3.11}$$

with $\delta_{t_i}(\mathfrak{a})$ being the Dirac measure at t_i, with the initial condition $\bar{\mathfrak{i}}(0,\mathfrak{a},x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(0,\mathfrak{a},x)$ for $\mathfrak{a} \in (0,t_i)\times[0,1]$, and the boundary condition

$$\bar{\mathfrak{i}}(t,0,x) = \frac{\bar{S}(t,x)}{\bar{B}(x)} \int_0^1 \beta(x,x') \left(\int_0^{t_i} \bar{\lambda}(\mathfrak{a}')\, \bar{\mathfrak{i}}(t,\mathfrak{a}',x') d\mathfrak{a}' \right) dx' \,, \tag{3.12}$$

The PDE (3.11) *has a unique solution which is given as follows: for $t \le \mathfrak{a} < t_i$ and $x \in [0,1]$,*

$$\bar{\mathfrak{i}}(t,\mathfrak{a},x) = \bar{\mathfrak{i}}(0,\mathfrak{a}-t,x), \tag{3.13}$$

and for $\mathfrak{a} < t \wedge t_i$ and $x \in [0,1]$,

$$\bar{\mathfrak{i}}(t,\mathfrak{a},x) = \bar{\mathfrak{i}}(t-\mathfrak{a},0,x), \tag{3.14}$$

and for $\mathfrak{a} \ge t_i$, $\bar{\mathfrak{i}}(t,\mathfrak{a},x) = 0$. The boundary function is the unique solution to the integral equation: for $0 < t < t_i$,

$$\begin{aligned} \bar{\mathfrak{i}}(t,0,x) &= \bar{B}(x)^{-1} \Big(\bar{S}(0,x) - \int_0^t \bar{\mathfrak{i}}(s,0,x) ds \Big) \\ &\quad \times \int_0^1 \beta(x,x') \left(\int_t^{t_i} \bar{\lambda}(\mathfrak{a})\, \bar{\mathfrak{i}}(0,\mathfrak{a}-t,x') d\mathfrak{a} + \int_0^t \bar{\lambda}(t-s)\, \bar{\mathfrak{i}}(s,0,x') ds \right) dx' \,, \end{aligned} \tag{3.15}$$

and for $t \ge t_i$,

$$\bar{\mathfrak{i}}(t,0,x) = \bar{B}(x)^{-1} \Big(\bar{S}(0,x) - \int_0^t \bar{\mathfrak{i}}(s,0,x) ds \Big) \times \int_0^1 \beta(x,x') \int_0^{t_i} \bar{\lambda}(t-s)\, \bar{\mathfrak{i}}(s,0,x') ds dx' \,. \tag{3.16}$$

Remark 3.2. *In the special case when $\lambda_i(t) = \tilde{\lambda}(t) \mathbf{1}_{t<\eta_i}$ for a deterministic function $\tilde{\lambda}(t)$, the boundary condition* (3.2) *becomes*

$$\bar{\mathfrak{i}}(t,0,x) = \frac{\bar{S}(t,x)}{\bar{B}(x)} \int_0^1 \beta(x,x') \left(\int_0^{t+\bar{\mathfrak{a}}} \tilde{\lambda}(\mathfrak{a}')\, \bar{\mathfrak{i}}(t,\mathfrak{a}',x') d\mathfrak{a}' \right) dx' \,. \tag{3.17}$$

This is because $\bar{\lambda}(t) = \tilde{\lambda}(t) F^c(t)$ and $\mathbb{E}[\tilde{\lambda}(t)\mathbf{1}_{t<\eta_0} | \tilde{\tau}_0 = y] = \tilde{\lambda}(t+y) \frac{F^c(t+y)}{F^c(y)}$. This boundary condition resembles that given in the Diekmann PDE model [9] *(without $\bar{B}(x)$ in the denominator). See further discussions in Remark 3.4.*

Remark 3.3. *By using the solution expressions in* (3.4) *and* (3.5) *together with the second identity $\bar{\Upsilon}(t,x) = \bar{\mathfrak{I}}_{\mathfrak{a}}(t,0,x)$ in* (2.21)*, we can rewrite $\bar{\mathfrak{F}}(t,x)$ in* (2.20) *as*

$$\begin{aligned} \bar{\mathfrak{F}}(t,x) &= \int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\, \bar{\mathfrak{i}}(0,\mathfrak{a},x) d\mathfrak{a} + \int_0^t \bar{\lambda}(t-s)\, \bar{\mathfrak{i}}(s,0,x) ds \\ &= \int_0^\infty \bar{\lambda}(\mathfrak{a}+t) \frac{G^c(\mathfrak{a})}{G^c(t+\mathfrak{a})} \bar{\mathfrak{i}}(t,t+\mathfrak{a},x) d\mathfrak{a} + \int_0^t \bar{\lambda}(\mathfrak{a}) \frac{1}{G^c(\mathfrak{a})} \bar{\mathfrak{i}}(t,\mathfrak{a},x) d\mathfrak{a} \\ &= \int_0^{t+\bar{\mathfrak{a}}} \frac{G^c(\mathfrak{a}-t)}{G^c(\mathfrak{a})} \bar{\lambda}(\mathfrak{a})\, \bar{\mathfrak{i}}(t,\mathfrak{a},x) d\mathfrak{a} \,, \end{aligned} \tag{3.18}$$

where $G^c(\mathfrak{a}) = 1$ for $\mathfrak{a} \le 0$. In the special case when $\lambda_i(t) = \tilde{\lambda}(t)\mathbf{1}_{t<\eta_i}$ as described in the previous remark, we obtain

$$\bar{\mathfrak{F}}(t,x) = \int_0^{t+\bar{\mathfrak{a}}} \tilde{\lambda}(\mathfrak{a})\, \bar{\mathfrak{i}}(t,\mathfrak{a},x) d\mathfrak{a} \,, \tag{3.19}$$

which further gives

$$\bar{\Upsilon}(t,x) = \frac{\bar{S}(t,x)}{\bar{B}(x)} \int_0^1 \beta(x,x') \int_0^{t+\bar{\mathfrak{a}}} \tilde{\lambda}(\mathfrak{a})\, \bar{\mathfrak{i}}(t,\mathfrak{a},x') d\mathfrak{a} dx'$$

$$= \frac{\bar{S}(t,x)}{\bar{B}(x)} \int_0^{t+\bar{\mathfrak{a}}} \int_0^1 \beta(x,x')\tilde{\lambda}(\mathfrak{a})\,\bar{\mathfrak{i}}(t,\mathfrak{a},x')dx'd\mathfrak{a}\,. \tag{3.20}$$

Remark 3.4. *In Diekmann [9], the spatial-temporal deterministic model is specified as follows. The function $\bar{I}(t,x)$ is written as an integral of the function $\bar{\mathfrak{i}}(t,\mathfrak{a},x)$:*

$$\bar{I}(t,x) = \int_0^\infty \bar{\mathfrak{i}}(t,\mathfrak{a},x)d\mathfrak{a}\,.$$

The infectivity function is given by

$$\bar{\Upsilon}(t,x) = \bar{S}(t,x)\int_0^\infty \int_0^1 \bar{\mathfrak{i}}(t,\mathfrak{a},x')A(\mathfrak{a},x,x')dx'd\mathfrak{a}\,, \tag{3.21}$$

where $A(\mathfrak{a},x,x')$ is the infectivity at x due to the infected individual with the infection age $\mathfrak{a}$ at x'. (Note the difference of $\bar{\Upsilon}(t,x)$ in (3.21) from our limit $\bar{\Upsilon}(t,x)$ in (3.20) with $\bar{B}(x)$ in the denominator, and abusing notation we use the same symbols in this remark). Therefore, in order to match the model by Diekmann [9], we can take

$$A(\mathfrak{a},x,x') = \beta(x,x')\bar{\lambda}(\mathfrak{a}). \tag{3.22}$$

By (3.3) and (3.6), we obtain

$$\begin{aligned}\frac{\partial\bar{S}(t,x)}{\partial t} &= -\bar{S}(t,x)\int_0^\infty\int_0^1 \beta(x,x')\bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}(0,\mathfrak{a},x')dx'd\mathfrak{a} \\ &\quad - \bar{S}(t,x)\int_0^t\int_0^1 \beta(x,x')\bar{\lambda}(t-s)\,\bar{\mathfrak{i}}(s,0,x')dx'ds \\ &= \bar{S}(t,x)\bigg(\int_0^t\int_0^1 \beta(x,x')\bar{\lambda}(\mathfrak{a})\frac{\partial\bar{S}(t-\mathfrak{a},x')}{\partial t}dx'd\mathfrak{a} - h(t,x)\bigg)\,,\end{aligned} \tag{3.23}$$

where

$$h(t,x) = \int_0^\infty\int_0^1 \beta(x,x')\bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}(0,\mathfrak{a},x')dx'd\mathfrak{a}\,.$$

Then integrating (3.23) with respect to t, we also get

$$u(t,x) = -\ln\frac{\bar{S}(t,x)}{\bar{S}(0,x)} = \int_0^t\int_0^1 (1-e^{-u(t-\mathfrak{a},x')})\bar{S}(0,x')\beta(x,x')\bar{\lambda}(\mathfrak{a})dx'd\mathfrak{a} + \int_0^t h(s,x)ds.$$

By using (3.22), we obtain the specification of $u(t,x)$ in [9].

In the special case $\lambda(t) = \tilde{\lambda}(t)\mathbf{1}_{t<\eta}$ for some deterministic function $\tilde{\lambda}(t)$ as described in Remark 3.2, given the expressions in (3.19) and (3.20), to match the model by Diekmann [9], we can take

$$A(\mathfrak{a},x,x') = \beta(x,x')\tilde{\lambda}(\mathfrak{a}).$$

Moreover, if the infection rate is constant λ and the infectious periods are exponential of rate μ, we have $\bar{\mathfrak{F}}(t,x) = \lambda\bar{I}(t,x)$, and as a result, the infectivity function of Diekmann in (3.21) becomes

$$\bar{\Upsilon}(t,x) = \bar{S}(t,x)\int_0^1 \beta(x,x')\lambda\bar{I}(t,x')dx'\,. \tag{3.24}$$

Because of the memoryless property of exponential periods, it is adequate to use the process $I(t,x)$ to describe the dynamics instead of $\mathfrak{I}(t,\mathfrak{a},x)$. In this case, we obtain the PDE model by Kendall [14, 15], in which given the limit $\bar{\Upsilon}(t,x)$ in (3.24),

$$\frac{\partial\bar{S}(t,x)}{\partial t} = -\bar{\Upsilon}(t,x), \quad \frac{\partial\bar{I}(t,x)}{\partial t} = \bar{\Upsilon}(t,x) - \mu\bar{I}(t,x), \quad \frac{\partial\bar{R}(t,x)}{\partial t} = \mu\bar{I}(t,x)\,.$$

Remark 3.5. *Recall the spatial SIS model in Remark 2.3. We obtain the same PDE in (3.1) with the boundary condition in (3.2), in which $\bar{S}(t,x)$ is the solution to (3.3) with $\bar{S}(0,x)$ satisfying $\int_0^1(\bar{S}(0,x)+\bar{I}(0,x))dx = 1$. The solution to the PDE is also given by (3.4)–(3.5) with the boundary condition in (3.2), in which $\bar{S}(0,x)$ satisfying $\int_0^1(\bar{S}(0,x)+\bar{I}(0,x))dx = 1$. Similarly, we also obtain the expression of $\bar{\mathfrak{F}}(t,x)$ in (3.18).*

Assume that $\lim_{t\to\infty}\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ exists and the limit is denoted as $\bar{\mathfrak{I}}^*(\mathfrak{a},x)$, and let $\bar{I}^*(x) = \lim_{t\to\infty}\bar{I}(t,x) = \bar{\mathfrak{I}}^*(\infty,x)$. Also let $\bar{S}^*(x) = \lim_{t\to\infty}\bar{S}(t,x)$. Note that

$$\int_0^1(\bar{S}^*(x)+\bar{I}^*(x))dx = 1. \tag{3.25}$$

Let $\beta^{-1} = \int_0^\infty F^c(\mathfrak{a})d\mathfrak{a}$ and $F_e(\mathfrak{a}) = \beta\int_0^{\mathfrak{a}} F^c(s)ds$.

By (2.26) and (3.18), we obtain

$$\begin{aligned}\bar{\mathfrak{I}}^*(\mathfrak{a},x) &= \int_0^{\mathfrak{a}} F^c(s)ds\,\bar{S}^*(x)\int_0^1\beta(x,x')\int_0^\infty\frac{1}{G^c(\mathfrak{a}')}\bar{\lambda}(\mathfrak{a}')\bar{\mathfrak{I}}^*(d\mathfrak{a}',x')dx'\\ &= \beta^{-1}F_e(\mathfrak{a})\bar{S}^*(x)\int_0^1\beta(x,x')\int_0^\infty\frac{1}{G^c(\mathfrak{a}')}\bar{\lambda}(\mathfrak{a}')\bar{\mathfrak{I}}^*(d\mathfrak{a}',x')dx'\,.\end{aligned} \tag{3.26}$$

By letting $\mathfrak{a}\to\infty$ on the both sides, we obtain

$$\bar{I}^*(x) = \beta^{-1}\bar{S}^*(x)\int_0^1\beta(x,x')\int_0^\infty\frac{1}{G^c(\mathfrak{a}')}\bar{\lambda}(\mathfrak{a}')\bar{\mathfrak{I}}^*(d\mathfrak{a}',x')dx'\,. \tag{3.27}$$

This implies

$$\bar{\mathfrak{I}}^*(\mathfrak{a},x) = F_e(\mathfrak{a})\bar{I}^*(x)\,,$$

which then gives

$$\frac{\partial}{\partial\mathfrak{a}}\bar{\mathfrak{I}}^*(\mathfrak{a},x) = \beta F^c(\mathfrak{a})\bar{I}^*(x)\,.$$

Thus,

$$\begin{aligned}\bar{\mathfrak{I}}^*(\mathfrak{a},x) &= \beta^{-1}F_e(\mathfrak{a})\bar{S}^*(x)\int_0^1\beta(x,x')\int_0^\infty\frac{1}{G^c(\mathfrak{a}')}\bar{\lambda}(\mathfrak{a}')\beta F^c(\mathfrak{a}')\bar{I}^*(x')d\mathfrak{a}'dx'\\ &= F_e(\mathfrak{a})\Big(\int_0^\infty\bar{\lambda}(\mathfrak{a}')d\mathfrak{a}'\Big)\bar{S}^*(x)\int_0^1\beta(x,x')\bar{I}^*(x')dx'\,.\end{aligned}$$

Recall that $R_0 = \int_0^\infty\bar{\lambda}(t)dt$. By letting $\mathfrak{a}\to\infty$ again on both sides, we obtain

$$\bar{I}^*(x) = R_0\bar{S}^*(x)\int_0^1\beta(x,x')\bar{I}^*(x')dx'\,.$$

This equation together with the identity (3.25) determines the values $\bar{I}^*(x)$ and $\bar{S}^*(x)$.

4. Some technical preliminaries

We will use the following convergence criteria for the processes: a) $X^N(t,x)$ in $\mathbf{D}(\mathbb{R}_+,L^1)$ and b) $X^N(t,s,x)$ in $\mathbf{D}(\mathbb{R}_+,\mathbf{D}(\mathbb{R}_+,L^1))$. They extend the convergence criterion for the processes in $\mathbf{D}$ (the Corollary on page 83 of [6]) and in $\mathbf{D}_{\mathbf{D}}$ ([21, Theorem 4.1]). The proof is a straightforward extension of those results (in [6] it is noted that with very little change, the theory can be extended to functions taking values in metric spaces that are separable and complete). We remark that one may also replace the L_1 norm $\|\cdot\|_1$ by the L_2 norm in the following results.

Theorem 4.1. *Let $\{X^N(t,x) : N\ge 1\}$ be a sequence of random elements such that X^N is in $\mathbf{D}(\mathbb{R}_+,L^1)$. If the following two conditions are satisfied: for any $T>0$,*

(i) *for any* $\epsilon > 0$, $\sup_{t\in[0,T]} \mathbb{P}\big(\|X^N(t,\cdot)\|_1 > \epsilon\big) \to 0$ *as* $N \to \infty$, *and*
(ii) *for any* $\epsilon > 0$, *as* $\delta \to 0$,

$$\limsup_N \sup_{t\in[0,T]} \frac{1}{\delta}\mathbb{P}\bigg(\sup_{u\in[0,\delta]} \|X^N(t+u,\cdot) - X^N(t,\cdot)\|_1 > \epsilon\bigg) \to 0,$$

then $\|X^N(t,\cdot)\|_1 \to 0$ *in probability, locally uniformly in* t*, as* $N \to \infty$.

Theorem 4.2. *Let* $\{X^N : N \geq 1\}$ *be a sequence of random elements such that* X^N *is in* $\mathbf{D}(\mathbb{R}_+, \mathbf{D}(\mathbb{R}_+, L^1))$. *If the following two conditions are satisfied: for any* $T, S > 0$,

(i) *for any* $\epsilon > 0$, $\sup_{t\in[0,T]} \sup_{s\in[0,S]} \mathbb{P}\big(\|X^N(t,s,\cdot)\|_1 > \epsilon\big) \to 0$ *as* $N \to \infty$, *and*
(ii) *for any* $\epsilon > 0$, *as* $\delta \to 0$,

$$\limsup_N \sup_{t\in[0,T]} \frac{1}{\delta}\mathbb{P}\bigg(\sup_{u\in[0,\delta]} \sup_{s\in[0,S]} \|X^N(t+u,s,\cdot) - X^N(t,s,\cdot)\|_1 > \epsilon\bigg) \to 0,$$

$$\limsup_N \sup_{s\in[0,S]} \frac{1}{\delta}\mathbb{P}\bigg(\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \|X^N(t,s+v,\cdot) - X^N(t,s,\cdot)\|_1 > \epsilon\bigg) \to 0,$$

then $\|X^N(t,s,\cdot)\|_1 \to 0$ *in probability, locally uniformly in* t *and* s*, as* $N \to \infty$.

We shall also need the following Lemma.

Lemma 4.1. *For each* $N \geq 1$*, let* $f_N : \mathbb{R}_+ \times [0,1] \mapsto \mathbb{R}_+$ *be measurable and such that* $t \mapsto f_N(t,x)$ *is non–decreasing for each* $x \in [0,1]$. *Assume that there exists* $f : \mathbb{R}_+ \times [0,1] \mapsto \mathbb{R}_+$ *such that* $t \mapsto f(t,x)$ *is continuous for each* $x \in [0,1]$*, and for all* $t \geq 0$*, as* $N \to \infty$,

$$\|f_N(t,\cdot) - f(t,\cdot)\|_1 \to 0\,. \tag{4.1}$$

Let $g \in \mathbf{D}(\mathbb{R};\mathbb{R}_+)$ *be such that there exists* $C > 0$ *with* $g(t) \leq C$ *for all* $t \geq 0$. *Define*

$$h_N(t,x) = \int_0^t g(s) f_N(ds,x), \quad h(t,x) = \int_0^t g(s) f(ds,x)\,.$$

Then for any $t > 0$, $\|h_N(t,\cdot) - h(t,\cdot)\|_1 \to 0$ *as* $N \to \infty$. *In addition,* $\int_0^1 h_N(t,x)dx \to \int_0^1 h(t,x)dx$ *locally uniformly in* t*, as* $N \to \infty$.

Moreover, if for each $N \geq 1$*,* f_N *is random and the convergence* (4.1) *holds in probability, then the conclusion holds in probability as well.*

Proof. Let $\{s_n,\ n \geq 1\}$ be a countable dense subset of $[0,1]$. By successive extraction of subsequences we can extract a subsequence from the original sequence $\{f_N,\ N \geq 1\}$, which by an abuse of notation we still denote as the original sequence, and which is such that there exists a subset $\mathcal{N} \subset [0,1]$ with zero Lebesgue measure, such that for all $n \geq 1$ and $x \in [0,1]\backslash\mathcal{N}$, $f_N(s_n,x) \to f(s_n,x)$. Since for all N and x, $s \mapsto f_N(s,x)$ is nondecreasing and $s \mapsto f(s,x)$ is continuous, we deduce that for all $s \in [0,T]$ and $x \in [0,1]\backslash\mathcal{N}$, $f_N(s,x) \to f(s,x)$. Consequently, for all $x \in [0,1]\backslash\mathcal{N}$, the sequence of measures $f_N(ds,x)$ on $[0,T]$ converges weakly to the measure $f(ds,x)$. Since the set of points of discontinuity of g on $[0,T]$ is at most countable and $s \mapsto f(s,x)$ is continuous, that set is of zero $f(ds,x)$ measure. Hence a slight extension of the Portmanteau theorem (see Theorem 1.2.1 in [6]) yields that for all $x \in [0,1]\backslash\mathcal{N}$, $h_N(t,x) \to h(t,x)$. Moreover, $0 \leq h_N(t,x) \leq C f_N(t,x)$, and the upper bound converges in $L^1([0,1])$, hence the sequence $h_N(t,\cdot)$ is uniformly integrable and converges in $L^1([0,1])$ towards $h(t,x)$. Now all converging subsequences have the same limit, so the the whole sequence converges.

The "locally uniform in t" convergence of the integrals follows from the second Dini theorem (see, e..g, Problem 127 on pages 81 and 270 in [24]). Indeed the convergence $\int_0^1 h_N(t,x)dx \to \int_0^1 h(t,x)dx$ for each t follows from the above arguments, for each $N \geq 1$, $t \mapsto \int_0^1 h_N(t,x)dx$ is non–decreasing and the limit $t \mapsto \int_0^1 h(t,x)dx$ is continuous.

The case of random f_N is treated similarly. The extraction of subsequences is done in such a way that for each n, $f_N(s_n,x)$ converges as $N\to\infty$ on a subset of $\Omega\times[0,1]$ of full $d\mathbb{P}\otimes dx$ measure. We conclude that from any subsequence of the original sequence $\{h_N(t,\cdot),\ N\geq 1\}$, we can extract a further subsequence which converges a.s. in $L^1([0,1])$, hence the convergence in probability in $L^1([0,1])$, as claimed. □

5. Proof of the Convergence of $\bar S^N(t,x)$ and $\bar{\mathfrak{F}}^N(t,x)$

In this section we prove the convergence of $\bar S^N(t,x)$ and $\bar{\mathfrak{F}}^N(t,x)$ to $\bar S(t,x)$ and $\bar{\mathfrak{F}}(t,x)$ given by the set of equations (2.19) and (2.20) together with (2.21). We first write $S_k^N(t)=S_k^N(0)-A_k^N(t)$ as follows by (2.5):

$$S_k^N(t)=S_k^N(0)-\int_0^t\int_0^\infty \mathbf{1}_{u\leq\Upsilon_k^N(s)}Q_k(ds,du),$$

and recall $\mathfrak{F}_k^N(t)$ in (2.3). Then, we have

$$\begin{aligned}\bar S^N(t,x)&=\bar S^N(0,x)-\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\int_0^\infty \mathbf{1}_{u\leq\Upsilon_k^N(s)}Q_k(ds,du)\,\mathbf{1}_{\mathtt{I}_k}(x)\\&=\bar S^N(0,x)-\int_0^t\bar\Upsilon^N(s,x)ds\,-\bar M_A^N(t,x)\,,\end{aligned}\tag{5.1}$$

where $\overline{Q}_k(ds,du)=Q_k(ds,du)-dsdu$ and

$$\bar M_A^N(t,x):=\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\int_0^\infty \mathbf{1}_{u\leq\Upsilon_k^N(s)}\overline{Q}_k(ds,du)\,\mathbf{1}_{\mathtt{I}_k}(x)\,.\tag{5.2}$$

We then write

$$\bar{\mathfrak{F}}^N(t,x)=\bar{\mathfrak{F}}_0^N(t,x)+\int_0^t\bar\lambda(t-s)\bar\Upsilon^N(s,x)ds+\Delta_{1,1}^N(t,x)+\Delta_{1,2}^N(t,x)\,,\tag{5.3}$$

where

$$\bar{\mathfrak{F}}_0^N(t,x)=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\lambda_{-j,k}(\tilde\tau_{-j,k}^N+t)\mathbf{1}_{\mathtt{I}_k}(x),\tag{5.4}$$

$$\Delta_{1,1}^N(t,x)=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A_k^N(t)}\Big(\lambda_{j,k}(t-\tau_{j,k}^N)-\bar\lambda(t-\tau_{j,k}^N)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\,,\tag{5.5}$$

and

$$\Delta_{1,2}^N(t,x)=\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\bar\lambda(t-s)\int_0^\infty \mathbf{1}_{u\leq\Upsilon_k^N(s)}\overline{Q}_k(ds,du)\,\mathbf{1}_{\mathtt{I}_k}(x)\,.\tag{5.6}$$

Observe that

$$\begin{aligned}\bar\Upsilon^N(s,x)&=\sum_{k=1}^{K^N}\frac{K^N}{N}\frac{S_k^N(s)}{B_k^N}\frac{1}{K^N}\sum_{k'=1}^{K^N}\beta_{k,k'}^N\mathfrak{F}_{k'}^N(s)\mathbf{1}_{\mathtt{I}_k}(x)\\&=\sum_{k=1}^{K^N}\frac{\bar S_k^N(s)}{\bar B_k^N}\mathbf{1}_{\mathtt{I}_k}(x)\int_0^1\sum_{k'=1}^{K^N}\beta_{k,k'}^N\bar{\mathfrak{F}}_{k'}^N(s)\mathbf{1}_{\mathtt{I}_{k'}}(x')dx'\\&=\frac{\bar S^N(s,x)}{\bar B^N(x)}\int_0^1\beta^N(x,x')\bar{\mathfrak{F}}^N(s,x')dx'\,,\end{aligned}\tag{5.7}$$

where $\beta^N(x,x')$ is defined in (2.14).

Before proceeding to prove the convergence of $\bar{S}^N(t,x)$ and $\bar{\mathfrak{F}}^N(t,x)$, we describe the proof strategy as follows. In the expressions of $\bar{S}^N(t,x)$ and $\bar{\mathfrak{F}}^N(t,x)$ in (5.1) and (5.3), the stochastic terms $\bar{M}_A^N(t,x)$, $\Delta_{1,1}^N(t,x)$ and $\Delta_{1,2}^N(t,x)$ will converge to zero in probability as $N\to\infty$, which are proved in Lemmas 5.5 and 5.6. The term $\bar{\mathfrak{F}}_0^N(t,\cdot)$ will converge to a limit $\bar{\mathfrak{F}}_0(t,\cdot)$ (in the $\|\cdot\|_1$ norm in probability), which is proved in Lemma 5.4. Thus, the proof for the convergence of $\bar{S}(t,x)$ and $\bar{\mathfrak{F}}^N(t,x)$ can be carried out by studying the set of integral equations (5.1) and (5.3) together with the expression of $\bar{\Upsilon}^N(s,x)$ above, given the convergence of the terms $\bar{S}^N(0,\cdot)$, $\bar{\mathfrak{F}}_0^N(t,\cdot)$, $\bar{M}_A^N(t,x)$, $\Delta_{1,1}^N(t,x)$ and $\Delta_{1,2}^N(t,x)$. In the following we will first provide this argument in Proposition 5.1 and then provide the proofs for the convergence of the required individual terms.

The following Lemma follows readily from (2.8) and (2.3), and the conditions on $\bar{B}^N(x)$ in (2.13).

Lemma 5.1. *The processes $\bar{S}^N(t,x)$ and $\bar{\mathfrak{F}}^N(t,x)$ are nonnegative and satisfy the following a priori bounds:*

$$\sup_N \sup_{t\ge 0,\ x\in[0,1]} \bar{S}^N(t,x) \le C_B \quad \textit{and} \quad \sup_N \sup_{t\ge 0,\ x\in[0,1]} \bar{\mathfrak{F}}^N(t,x) \le \lambda^* C_B \quad a.s.$$

Next, recall the set of the limiting equations:

$$\begin{aligned} \bar{S}(t,x) &= \bar{S}(0,x) - \int_0^t \frac{\bar{S}(s,x)}{\bar{B}(x)} \int_0^1 \beta(x,y)\bar{\mathfrak{F}}(s,y)dyds, \\ \bar{\mathfrak{F}}(t,x) &= \bar{\mathfrak{F}}_0(t,x) + \int_0^t \bar{\lambda}(t-s)\frac{\bar{S}(s,x)}{\bar{B}(x)} \int_0^1 \beta(x,y)\bar{\mathfrak{F}}(s,y)dyds\,, \end{aligned} \tag{5.8}$$

where

$$\bar{\mathfrak{F}}_0(t,x) := \int_0^\infty \bar{\lambda}(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x). \tag{5.9}$$

We have the following lemmas on the solution properties to this set of equations, and also the existence and uniqueness of its solution.

Lemma 5.2. *Under Assumptions 2.1 and 2.3, any $(L^\infty([0,1]))^2$–valued solution $(\bar{S}(t,x),\bar{\mathfrak{F}}(t,x))$ of equation (5.8) is nonnegative, and satisfies $\sup_{t\ge0}\bar{S}(t,x)\le\bar{S}(0,x)\le C_B$ and for any $T>0$, there exists $C_T>0$ such that*

$$\sup_{0\le t\le T, x\in[0,1]} \bar{\mathfrak{F}}(t,x) \le C_T\,.$$

Proof. The non–negativity of $\bar{S}$ follows from that of the initial condition and the linearity of the equation. For the second statement, we first note that $\bar{\mathfrak{I}}(0,\infty,x)\le C_B$, hence from (5.9) and Assumption 2.3, $0\le\bar{\mathfrak{F}}_0(t,x)\le\lambda^*C_B$. Hence from the second line of (5.8) and (2.15) and from the assumption that $\bar{B}(x)\ge c_B>0$ for each $x\in[0,1]$ in (2.12), we obtain

$$\|\bar{\mathfrak{F}}(t,\cdot)\|_\infty \le \lambda^*C_B + \frac{C_\beta}{c_B}\lambda^*C_B\int_0^t \|\bar{\mathfrak{F}}(s,\cdot)\|_\infty ds.$$

Thus, the second statement with $C_T=\lambda^*C_B\exp\left(\frac{C_\beta}{c_B}\lambda^*C_BT\right)$ follows from Gronwall's lemma. We next show that $\bar{\mathfrak{F}}(t,x)\ge0$. Suppose that $\bar{\mathfrak{F}}(t,x)=\bar{\mathfrak{F}}_+(t,x)-\bar{\mathfrak{F}}_-(t,x)$. Then we have

$$\bar{\mathfrak{F}}_-(t,x) \le \int_0^t \bar{\lambda}(t-s)\frac{\bar{S}(s,x)}{\bar{B}(x)}\int_0^1 \beta(x,y)\bar{\mathfrak{F}}_-(s,y)dyds,$$

and by a similar argument as above using Gronwall's Lemma, we deduce that $\|\bar{\mathfrak{F}}_-(t,\cdot)\|_\infty=0$, hence the result. Finally it follows readily from Assumption 2.1 that $\bar{S}(0,x)\le\sup_N\bar{S}^N(0,x)\le C_B$ for all x. From the first line of (5.8), since $\bar{S}$ and $\bar{\mathfrak{F}}$ are nonnegative, $\bar{S}(t,x)\le\bar{S}(0,x)$, hence the first statement. □

Lemma 5.3. *Under Assumptions 2.1 and 2.3, equation (5.8) has a unique $(L^\infty([0,1]))^2$–valued solution.*

Proof. We already know that any solution is nonnegative and locally bounded. Uniqueness is then easy to deduce from the following estimate. Consider two solutions $(\bar{S},\bar{\mathfrak{F}})$ and $(\bar{S}',\bar{\mathfrak{F}}')$, and define $\bar{\Upsilon}(t,x)=\frac{\bar{S}(t,x)}{\bar{B}(x)}\int_0^1\beta(x,y)\bar{\mathfrak{F}}(t,y)dy$, $\bar{\Upsilon}'(t,x)$ similarly, replacing $(\bar{S},\bar{\mathfrak{F}})$ by $(\bar{S}',\bar{\mathfrak{F}}')$.

Since from (2.12) $\bar{B}(x)\ge c_B$, and from Lemma 5.2 $\bar{S}(t,x)\le C_B$ and for $0\le t\le T, x\in[0,1]$, $\bar{\mathfrak{F}}(t,x)\le C_T$, we obtain

$$\begin{aligned}\|\bar{\Upsilon}(t,\cdot)-\bar{\Upsilon}'(t,\cdot)\|_\infty &\le \sup_{x\in[0,1]}\left|\frac{\bar{S}(t,x)}{\bar{B}(x)}-\frac{\bar{S}'(t,x)}{\bar{B}(x)}\right|\int_0^1\beta(x,y)\bar{\mathfrak{F}}(t,y)dy\\ &\quad+\sup_{x\in[0,1]}\frac{\bar{S}'(t,x)}{\bar{B}(x)}\int_0^1\beta(x,y)|\bar{\mathfrak{F}}(t,y)-\bar{\mathfrak{F}}'(t,y)|dy\\ &\le\frac{1}{c_B}\|\bar{S}(t,\cdot)-\bar{S}'(t,\cdot)\|_\infty\sup_{x\in[0,1]}\int_0^1\beta(x,y)\bar{\mathfrak{F}}(t,y)dy\\ &\quad+\frac{C_\beta C_B}{c_B}\|\bar{\mathfrak{F}}(t,\cdot)-\bar{\mathfrak{F}}'(t,\cdot)\|_\infty\\ &\le\frac{C_\beta}{c_B}C_T\|\bar{S}(t,\cdot)-\bar{S}'(t,\cdot)\|_\infty+\frac{C_\beta C_B}{c_B}\|\bar{\mathfrak{F}}(t,\cdot)-\bar{\mathfrak{F}}'(t,\cdot)\|_\infty.\end{aligned}$$

From this inequality, we see that uniqueness follows from Gronwall's Lemma. The same estimate can be used repeatedly for proving convergence in $L^\infty([0,1])$ of the Picard iteration procedure, which establishes existence. □

We can now prove the main result of this section. Let us first introduce a notation. We let $\mathcal{E}^N_{\mathfrak{F}}(t,x)=\Delta^N_{1,1}(t,x)+\Delta^N_{1,2}(t,x)$ and

$$\Psi^N(t):=\int_0^1|\bar{\mathfrak{F}}^N_0(t,x)-\bar{\mathfrak{F}}_0(t,x)|dx+\int_0^1|\bar{M}^N_A(t,x)|dx+\int_0^1|\mathcal{E}^N_{\mathfrak{F}}(t,x)|dx\,.$$

Proposition 5.1. *Let $T>0$ be arbitrary. Given that $\int_0^1|\bar{S}^N(0,x)-\bar{S}(0,x)|dx\to0$ in Assumption 2.1, and assuming that $\sup_{0\le t\le T}\Psi^N(t)\to0$ in probability as $N\to\infty$, we have*

$$\sup_{0\le t\le T}\big(\|\bar{S}^N(t,\cdot)-\bar{S}(t,\cdot)\|_1+\|\bar{\mathfrak{F}}^N(t,\cdot)-\bar{\mathfrak{F}}(t,\cdot)\|_1\big)\to0$$

in probability as $N\to\infty$.

Proof. Referring to the notations in Lemmas 5.1 and 5.2, let us assume that $\lambda^*\le C_T$. We first upper bound the following difference

$$\begin{aligned}&\frac{\bar{S}(t,x)}{\bar{B}(x)}\int_0^1\beta(x,y)\bar{\mathfrak{F}}(t,y)dy-\frac{\bar{S}^N(t,x)}{\bar{B}^N(x)}\int_0^1\beta^N(x,y)\bar{\mathfrak{F}}^N(t,y)dy\\ &=\left(\frac{\bar{S}(t,x)}{\bar{B}(x)}-\frac{\bar{S}^N(t,x)}{\bar{B}^N(x)}\right)\int_0^1\beta^N(x,y)\bar{\mathfrak{F}}^N(t,y)dy\\ &\quad+\frac{\bar{S}(t,x)}{\bar{B}(x)}\left(\int_0^1\beta(x,y)\bar{\mathfrak{F}}(t,y)dy-\int_0^1\beta^N(x,y)\bar{\mathfrak{F}}^N(t,y)dy\right)\\ &\le C_\beta C_T\left|\frac{\bar{S}(t,x)}{\bar{B}(x)}-\frac{\bar{S}^N(t,x)}{\bar{B}^N(x)}\right|+\int_0^1\beta^N(x,y)(\bar{\mathfrak{F}}(t,y)-\bar{\mathfrak{F}}^N(t,y))dy\\ &\qquad+\int_0^1(\beta(x,y)-\beta^N(x,y))\bar{\mathfrak{F}}(t,y)dy\,.\end{aligned}$$

Note that by (2.12) and (2.13),

$$\left|\frac{\bar S(t,x)}{\bar B(x)}-\frac{\bar S^N(t,x)}{\bar B^N(x)}\right| = \left|\frac{\bar S(t,x)-\bar S^N(t,x)}{\bar B(x)}+\bar S^N(t,x)\left(\frac{1}{\bar B(x)}-\frac{1}{\bar B^N(x)}\right)\right|$$
$$\le c_B^{-1}|\bar S(t,x)-\bar S^N(t,x)|+c_B^{-2}C_B|\bar B^N(x)-\bar B(x)|\,.$$

Consequently,

$$\begin{aligned}\left\|\frac{\bar S(t,\cdot)}{\bar B(\cdot)}\int_0^1\beta(\cdot,y)\bar{\mathfrak{F}}(t,y)dy-\frac{\bar S^N(t,\cdot)}{\bar B^N(\cdot)}\int_0^1\beta^N(\cdot,y)\bar{\mathfrak{F}}^N(t,y)dy\right\|_1 \\ \le C_\beta C_T c_B^{-1}\|\bar S(t,\cdot)-\bar S^N(t,\cdot)\|_1+C_\beta C_T c_B^{-2}C_B\|\bar B^N(\cdot)-\bar B(\cdot)\|_1 \\ +\left(\sup_{N,y}\int_0^1\beta^N(x,y)dx\right)\|\bar{\mathfrak{F}}(t,\cdot)-\bar{\mathfrak{F}}^N(t,\cdot)\|_1 \\ +\int_0^1\left|\int_0^1(\beta(x,y)-\beta^N(x,y))\bar{\mathfrak{F}}(t,y)dx\right|dy\,.\end{aligned}$$

We can now estimate the norm $\|\bar S(t,\cdot)-\bar S^N(t,\cdot)\|_1$ and $\|\bar{\mathfrak{F}}(t,\cdot)-\bar{\mathfrak{F}}^N(t,\cdot)\|_1$. Let $\bar C:=\max\{C_\beta, C_\beta C_T c_B^{-1}, C_\beta C_T c_B^{-2}C_B\}$. We now deduce from (5.1), (5.8) and the last computation that

$$\begin{aligned}\|\bar S(t,\cdot)-\bar S^N(t,\cdot)\|_1 \le \|\bar S(0,\cdot)-\bar S^N(0,\cdot)\|_1+\|\bar M_A^N(t,\cdot)\|_1 \\ +\int_0^t\int_0^1\left|\int_0^1(\beta(x,y)-\beta^N(x,y))\bar{\mathfrak{F}}(s,y)dx\right|dyds \\ +\bar C\int_0^t\|\bar S(s,\cdot)-\bar S^N(s,\cdot)\|_1ds+\bar C\|\bar B^N(\cdot)-\bar B(\cdot)\|_1 \\ +\bar C\int_0^t\|\bar{\mathfrak{F}}(s,\cdot)-\bar{\mathfrak{F}}^N(s,\cdot)\|_1ds.\end{aligned}$$

Next from (5.3) and (5.8), we get

$$\begin{aligned}\|\bar{\mathfrak{F}}(t,\cdot)-\bar{\mathfrak{F}}^N(t,\cdot)\|_1 \le \|\bar{\mathfrak{F}}_0(t,\cdot)-\bar{\mathfrak{F}}_0^N(t,\cdot)\|_1+\|\mathcal{E}_{\mathfrak{F}}^N(t,\cdot)\|_1 \\ +\int_0^t\int_0^1\left|\int_0^1(\beta(x,y)-\beta^N(x,y))\bar{\mathfrak{F}}(s,y)dx\right|dyds \\ +\bar C\int_0^t\|\bar S(s,\cdot)-\bar S^N(s,\cdot)\|_1ds+\bar C\|\bar B^N(\cdot)-\bar B(\cdot)\|_1 \\ +\bar C\int_0^t\|\bar{\mathfrak{F}}(s,\cdot)-\bar{\mathfrak{F}}^N(s,\cdot)\|_1ds\,.\end{aligned}$$

Adding those two inequalities, the result follows from our assumptions, the fact that (2.16) in Assumption 2.2 implies that

$$\int_0^t\int_0^1\left|\int_0^1(\beta(x,y)-\beta^N(x,y))\bar{\mathfrak{F}}(s,y)dx\right|dyds\to 0 \quad \text{as} \quad N\to\infty,$$

and the following variant of Gronwall's Lemma: if $f(t)$ and $g(t)$ are nonnegative real-valued functions of t and satisfy $f(t)\le g(t)+c\int_0^t f(s)ds$ for all $0\le t\le T$ and for some $c>0$, then for those t, $f(t)\le g(t)+c\int_0^t e^{c(t-s)}g(s)ds$. ☐

It remains to show that $\sup_{0\le t\le T}\Upsilon^N(t)\to 0$ in probability, which follows from the next three lemmas, where we establish the convergence of $\bar{\mathfrak{F}}_0^N(t,\cdot)$ to $\bar{\mathfrak{F}}_0(t,x)$, and that the stochastic terms $\bar M_A^N(t,x)$, $\Delta_{1,1}^N(t,x)$ and $\Delta_{1,2}^N(t,x)$ of (5.2), (5.5) and (5.6) tend to 0 in probability, as $N\to\infty$.

Lemma 5.4. *Under Assumptions 2.1 and 2.3,*

$$\|\bar{\mathfrak{F}}_0^N(t,\cdot) - \bar{\mathfrak{F}}_0(t,\cdot)\|_1 \to 0 \tag{5.10}$$

in probability, locally uniformly in t*, as* $N\to\infty$*, where* $\bar{\mathfrak{F}}_0(t,x)$ *is defined in* (5.9)*.*

Proof. We apply Theorem 4.1. First, we have

$$\bar{\mathfrak{F}}_0^N(t,x) - \bar{\mathfrak{F}}_0(t,x) = \Delta_{0,1}^N(t,x) + \Delta_{0,2}^N(t,x),$$

where

$$\Delta_{0,1}^N(t,x) = \sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big(\lambda_{-j,k}(\tilde\tau_{-j,k}^N+t) - \bar\lambda(\tilde\tau_{-j,k}^N+t)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\,,$$

$$\begin{aligned}\Delta_{0,2}^N(t,x) &= \sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\bar\lambda(\tilde\tau_{-j,k}^N+t)\mathbf{1}_{\mathtt{I}_k}(x) - \int_0^{\bar{\mathfrak{a}}}\bar\lambda(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x)\\ &= \int_0^{\bar{\mathfrak{a}}}\bar\lambda(\mathfrak{a}+t)[\bar{\mathfrak{I}}^N(0,d\mathfrak{a},x) - \bar{\mathfrak{I}}(0,d\mathfrak{a},x)]\,.\end{aligned}$$

We now verify condition (i) of Theorem 4.1. For the first term $\Delta_{0,1}^N(t,x)$, we have

$$\|\Delta_{0,1}^N(t,\cdot)\|_1 \le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\left|\sum_{j=1}^{I_k^N(0)}\Big(\lambda_{-j,k}(\tilde\tau_{-j,k}^N+t) - \bar\lambda(\tilde\tau_{-j,k}^N+t)\Big)\right|\,.$$

Here the summands over k are independent, and for each k, conditional on $\{\tilde\tau_{-j,k}^N\}_j$, the summands over j are also independent and centered. Using Jensen's inequality for the sum over k, and the conditional independence for the sum over j, we deduce

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\left|\sum_{j=1}^{I_k^N(0)}\Big(\lambda_{-j,k}(\tilde\tau_{-j,k}^N+t) - \bar\lambda(\tilde\tau_{-j,k}^N+t)\Big)\right|\right)^2\right]\\ &\le \mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{\bar{\mathfrak{a}}}v(\mathfrak{a}+t)\bar{\mathfrak{I}}_k^N(0,d\mathfrak{a})\right] \to 0 \quad \text{as} \quad N\to\infty,\end{aligned}$$

since under Assumption 2.1, thanks to Lemma 4.1,

$$\frac{1}{K^N}\sum_{k=1}^{K^N}\int_0^{\bar{\mathfrak{a}}}v(\mathfrak{a}+t)\bar{\mathfrak{I}}_k^N(0,d\mathfrak{a}) \to \int_0^1\int_0^{\bar{\mathfrak{a}}}v(\mathfrak{a}+t)\bar{\mathfrak{I}}(0,d\mathfrak{a},x)dx$$

in probability and $\frac{K^N}{N}\to 0$ as $N\to\infty$. Recall that $v(t)$ is the variance of the random function $\lambda(t)$ in Assumption 2.3, which is bounded.

The fact that $\|\Delta_{0,2}^N\|_1\to 0$ in probability follows again from Lemma 4.1 and Assumption 2.1.

Now to check condition (ii) of Theorem 4.1, we first have for $t,u>0$,

$$\begin{aligned}&\Delta_{0,1}^N(t+u,x) - \Delta_{0,1}^N(t,x)\\ &= \sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big(\lambda_{-j,k}(\tilde\tau_{-j,k}^N+t+u) - \lambda_{-j,k}(\tilde\tau_{-j,k}^N+t)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\ &\quad - \sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big(\bar\lambda(\tilde\tau_{-j,k}^N+t+u) - \bar\lambda(\tilde\tau_{-j,k}^N+t)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\,.\end{aligned}$$

Observe that

$$\left\|\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big(\lambda_{-j,k}(\tilde\tau^N_{-j,k}+t+u)-\lambda_{-j,k}(\tilde\tau^N_{-j,k}+t)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\right\|_1$$
$$\le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big|\lambda_{-j,k}(\tilde\tau^N_{-j,k}+t+u)-\lambda_{-j,k}(\tilde\tau^N_{-j,k}+t)\Big|\,,$$

and similarly for the second term. Thus,

$$\begin{aligned}\|\Delta^N_{0,1}(t+u,x)-\Delta^N_{0,1}(t,x)\|_1 &\le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big|\lambda_{-j,k}(\tilde\tau^N_{-j,k}+t+u)-\lambda_{-j,k}(\tilde\tau^N_{-j,k}+t)\Big|\\ &\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big|\bar\lambda(\tilde\tau^N_{-j,k}+t+u)-\bar\lambda(\tilde\tau^N_{-j,k}+t)\Big|\\ &=: \Delta^{N,(1)}_{0,1}(t,u)+\Delta^{N,(2)}_{0,1}(t,u)\,.\end{aligned}$$

By Assumption 2.3, using the expression of $\lambda(t)$ in (2.17), that is, $\lambda_{-j,k}(t)=\sum_{\ell=1}^{\kappa}\lambda^\ell_{-j,k}(t)\mathbf{1}_{[\zeta^{\ell-1}_{-j,k},\zeta^\ell_{-j,k})}(t)$, we obtain

$$\begin{aligned}\Delta^{N,(1)}_{0,1}(t,u)&=\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\Big|\sum_{\ell=1}^{\kappa}\lambda^\ell_{-j,k}(\tilde\tau^N_{-j,k}+t+u)\mathbf{1}_{[\zeta^{\ell-1}_{-j,k},\zeta^\ell_{-j,k})}(\tilde\tau^N_{-j,k}+t+u)\\ &\qquad-\sum_{\ell=1}^{\kappa}\lambda^\ell_{-j,k}(\tilde\tau^N_{-j,k}+t)\mathbf{1}_{[\zeta^{\ell-1}_{-j,k},\zeta^\ell_{-j,k})}(\tilde\tau^N_{-j,k}+t)\Big|\\ &\le\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\sum_{\ell=1}^{\kappa}\Big|\lambda^\ell_{-j,k}(\tilde\tau^N_{-j,k}+t+u)-\lambda^\ell_{-j,k}(\tilde\tau^N_{-j,k}+t)\Big|\mathbf{1}_{\zeta^{\ell-1}_{-j,k}\le\tilde\tau^N_{-j,k}+t\le\tilde\tau^N_{-j,k}+t+u\le\zeta^\ell_{-j,k}}\\ &\quad+\lambda^*\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\sum_{\ell=1}^{\kappa}\mathbf{1}_{\tilde\tau^N_{-j,k}+t\le\zeta^\ell_{-j,k}\le\tilde\tau^N_{-j,k}+t+u}\\ &\le\varphi(u)\frac{1}{K^N}\sum_{k=1}^{K^N}\bar I_k^N(0)+\lambda^*\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\sum_{\ell=1}^{\kappa}\mathbf{1}_{\tilde\tau^N_{-j,k}+t\le\zeta^\ell_{-j,k}\le\tilde\tau^N_{-j,k}+t+u}\,.\end{aligned}\tag{5.11}$$

Since both terms in the right hand side are increasing in u, we obtain

$$\sup_{u\in[0,\delta]}\Delta^{N,(1)}_{0,1}(t,u)\le\varphi(\delta)\frac{1}{K^N}\sum_{k=1}^{K^N}\bar I_k^N(0)+\lambda^*\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\mathbf{1}_{\tilde\tau^N_{-j,k}+t\le\zeta^\ell_{-j,k}\le\tilde\tau^N_{-j,k}+t+\delta}\,.\tag{5.12}$$

Note that

$$\frac{1}{K^N}\sum_{k=1}^{K^N}\bar I_k^N(0)\to\int_0^1\bar I(0,x)dx\quad\text{as}\quad N\to\infty$$

under Assumption 2.1. For the second term in (5.12), we have

$$\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{I_k^N(0)}\mathbf{1}_{\tilde\tau^N_{-j,k}+t\le\zeta^\ell_{-j,k}\le\tilde\tau^N_{-j,k}+t+\delta}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{I_k^N(0)} \Big[\mathbf{1}_{\tilde{\tau}^N_{-j,k}+t \le \zeta^\ell_{-j,k} \le \tilde{\tau}^N_{-j,k}+t+\delta} - \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \Big] \\
&\quad + \sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{I_k^N(0)} \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) . \qquad (5.13)
\end{aligned}
$$

In both expressions, the summands over k are independent, and in the first, for each k, conditional on $\{\tilde{\tau}^N_{-j,k}\}_j$, the summands over j are also independent. We have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{I_k^N(0)} \Big[\mathbf{1}_{\tilde{\tau}^N_{-j,k}+t \le \zeta^\ell_{-j,k} \le \tilde{\tau}^N_{-j,k}+t+\delta} - \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \Big] \right)^2 \right] \\
&\le \mathbb{E}\left[\frac{1}{K^N} \sum_{k=1}^{K^N} \Big(\frac{K^N}{N}\Big)^2 \left(\sum_{j=1}^{I_k^N(0)} \Big[\mathbf{1}_{\tilde{\tau}^N_{-j,k}+t \le \zeta^\ell_{-j,k} \le \tilde{\tau}^N_{-j,k}+t+\delta} - \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \Big] \right)^2 \right] \\
&= \mathbb{E}\left[\frac{1}{K^N} \sum_{k=1}^{K^N} \Big(\frac{K^N}{N}\Big)^2 \sum_{j=1}^{I_k^N(0)} \Big[\mathbf{1}_{\tilde{\tau}^N_{-j,k}+t \le \zeta^\ell_{-j,k} \le \tilde{\tau}^N_{-j,k}+t+\delta} - \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \Big]^2 \right] \\
&= \mathbb{E}\Bigg[\frac{1}{K^N} \sum_{k=1}^{K^N} \Big(\frac{K^N}{N}\Big)^2 \sum_{j=1}^{I_k^N(0)} \Big[\Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \\
&\qquad\qquad \times \Big(1 - \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \Big) \Big] \Bigg] \\
&= \mathbb{E}\left[\frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \int_0^{\bar{\mathfrak{a}}} \Big[\Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \Big(1 - \Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \Big) \Big] \bar{\mathfrak{I}}_k^N(0, d\mathfrak{a}) \right] \\
&\to 0 \quad \text{as} \quad N \to \infty ,
\end{aligned}
$$

since under Assumption 2.1,

$$
\begin{aligned}
&\frac{1}{K^N} \sum_{k=1}^{K^N} \int_0^{\bar{\mathfrak{a}}} \Big[\Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \Big(1 - \Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \Big) \Big] \bar{\mathfrak{I}}_k^N(0, d\mathfrak{a}) \\
&\to \int_0^1 \int_0^{\bar{\mathfrak{a}}} \Big[\Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \Big(1 - \Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \Big) \Big] \bar{\mathfrak{I}}(0, d\mathfrak{a}, x) dx ,
\end{aligned}
$$

and $\frac{K^N}{N} \to 0$ as $N \to \infty$. Hence, the first term in (5.13) converges to zero in probability as $N \to \infty$. For the second term in (5.13), we have

$$
\begin{aligned}
&\sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{I_k^N(0)} \Big(F_\ell(\tilde{\tau}^N_{-j,k}+t+\delta) - F_\ell(\tilde{\tau}^N_{-j,k}+t) \Big) \\
&= \sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \int_0^{\bar{\mathfrak{a}}} \Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \bar{\mathfrak{I}}_k^N(0, d\mathfrak{a}) \\
&\to \sum_{\ell=1}^{\kappa} \int_0^1 \int_0^{\bar{\mathfrak{a}}} \Big(F_\ell(\mathfrak{a}+t+\delta) - F_\ell(\mathfrak{a}+t) \Big) \bar{\mathfrak{I}}(0, d\mathfrak{a}, x) dx ,
\end{aligned}
$$

in probability as $N \to \infty$. For each $\ell = 1, \dots, \kappa$, the function $\delta \to \int_0^1 \int_0^{\bar{x}} \Big(F_\ell(\mathfrak{a} + t + \delta) - F_\ell(\mathfrak{a} + t) \Big) \bar{\mathfrak{I}}(0, d\mathfrak{a}, x)dx$ is continuous and equal to zero at $\delta = 0$. Thus we have shown that for any small enough $\delta > 0$,

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\le u\le\delta} \Delta_{0,1}^{N,(1)}(t,u) > \epsilon/2 \right) = 0. \tag{5.14}$$

Note that

$$\Delta_{0,1}^{N,(2)}(t,u) = \int_0^1 \int_0^{\bar{\mathfrak{a}}} \left|\bar{\lambda}(\mathfrak{a}+t+u) - \bar{\lambda}(\mathfrak{a}+t)\right| \bar{\mathfrak{I}}^N(0,d\mathfrak{a},x)dx\,. \tag{5.15}$$

By similar calculations leading to (5.12), we obtain for any small enough $\delta > 0$,

$$\sup_{u\in[0,\delta]} \Delta_{0,1}^{N,(2)}(t,u) \le \varphi(\delta) \frac{1}{K^N} \sum_{k=1}^{K^N} \bar{I}_k^N(0) + \lambda^* \sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{I_k^N(0)} \Big(F_\ell(\tilde{\tau}_{-j,k}^N + t + \delta) - F_\ell(\tilde{\tau}_{-j,k}^N + t) \Big)\,.$$

Thus, by the same arguments for these two terms as in the proof for (5.14), we obtain that (5.14) holds for $\Delta_{0,1}^{N,(2)}(t,u)$. Thus, combining these two results, we obtain that for any $\epsilon > 0$, for $\delta > 0$ small enough,

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\le u\le\delta} \|\Delta_{0,1}^N(t+u,x) - \Delta_{0,1}^N(t,x)\|_1 > \epsilon \right) = 0\,. \tag{5.16}$$

Now for $\Delta_{0,2}^N(t,x)$, we have for $t, u > 0$,

$$\begin{aligned} &\|\Delta_{0,2}^N(t+u,x) - \Delta_{0,2}^N(t,x)\|_1 \\ &\le \int_0^1 \int_0^{\bar{\mathfrak{a}}} |\bar{\lambda}(\mathfrak{a}+t+u) - \bar{\lambda}(\mathfrak{a}+t)| [\bar{\mathfrak{I}}^N(0,d\mathfrak{a},x) + \bar{\mathfrak{I}}(0,d\mathfrak{a},x)]dx, \end{aligned}$$

which is treated exactly as $\Delta_{0,1}^{N,(2)}(t,u)$, see formula (5.15). This completes the proof of the lemma. □

Lemma 5.5. *Under Assumptions 2.1, 2.2 and 2.3, for all $T > 0$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\bar{M}_A^N(t,\cdot)\|_1^2 \right] \to 0\,, \tag{5.17}$$

and thus,

$$\left\| \bar{A}^N(t,\cdot) - \int_0^t \bar{\Upsilon}^N(s,\cdot)ds \right\|_1 \to 0 \tag{5.18}$$

in probability, locally uniformly in t.

In addition, there exists $C_T > 0$ such that for all $N \ge 1$,

$$\mathbb{E}\left[\sup_{t\le T} \|\bar{A}^N(t,\cdot)\|_1 \right] \le C_T\,. \tag{5.19}$$

Proof. Recall the expressions of $A_k^N(t)$ in (2.5) and $\Upsilon_k^N(t)$ in (2.4). By (2.3), under Assumption 2.3 that $\lambda(t) \le \lambda^*$, under the condition on $\bar{B}(x)$ in (2.12), and (2.13), we have $\bar{\mathfrak{F}}^N(t,x) \le \lambda^* C_B$ and thus, under Assumption 2.2, $\bar{\Upsilon}^N(t,x) \le \lambda^* C_B C_\beta$, where we have used (2.15). Hence $\|\bar{\Upsilon}^N(t,\cdot)\|_1 \le \lambda^* C_B C_\beta$, and

$$\Big\| \int_0^t \bar{\Upsilon}^N(r,\cdot)dr - \int_0^s \bar{\Upsilon}^N(r,\cdot)dr \Big\|_1 \le \lambda^* C_B C_\beta (t-s)\,. \tag{5.20}$$

For each k, we can write

$$\bar{A}_k^N(t) = \int_0^t \bar{\Upsilon}_k^N(s)ds + \bar{M}_{A,k}^N(t)$$

where

$$\bar{M}_{A,k}^N(t) = \frac{K^N}{N}\int_0^t \int_0^\infty \mathbf{1}_{u \le \Upsilon_k^N(s^-)} \bar{Q}_k(ds,du)$$

with $\bar{Q}_k(ds,du) = Q_k(ds,du) - dsdu$ being the compensated PRM associated with Q_k. Let $\bar{M}_A^N(t,x) = \sum_{k=1}^{K^N} \bar{M}_{A,k}^N(t)\mathbf{1}_{\mathtt{I}_k}(x)$. Then we have the time-space representation:

$$\bar{A}^N(t,x) = \int_0^t \bar{\Upsilon}^N(r,x)dr + \bar{M}_A^N(t,x)\,. \tag{5.21}$$

It is clear that for each k, $\{\bar{M}_{A,k}^N(t) : t \ge 0\}$ is a square-integrable martingale with respect to the filtration $\mathcal{F}_A^N = \{\mathcal{F}_A^N(t) : t \ge 0\}$ where

$$\mathcal{F}_A^N(t) := \sigma\big\{I_k^N(0), \tilde{\tau}_{-j,k}^N : j = 1,\dots,I_k^N(0), k = 1,\dots,K\big\} \vee \sigma\big\{\lambda_{j,k}(\cdot),\ j \in \mathbb{Z}\setminus\{0\}, k = 1,\dots,K\big\}$$
$$\vee\, \sigma\bigg\{ \int_0^{t'}\int_0^\infty \mathbf{1}_{u \le \Upsilon_k^N(s^-)} Q_k(ds,du) : 0 \le t' \le t,\ k = 1,\dots,K \bigg\}.$$

and has the quadratic variation

$$\langle \bar{M}_{A,k}^N \rangle(t) = \frac{K^N}{N}\int_0^t \bar{\Upsilon}_k^N(s)ds, \quad t \ge 0.$$

Then,

$$\|\bar{M}_A^N(t,\cdot)\|_1 \le \int_0^1 \Big| \sum_{k=1}^{K^N} \bar{M}_{A,k}^N(t)\mathbf{1}_{\mathtt{I}_k}(x) \Big| dx \le \frac{1}{K^N}\sum_{k=1}^{K^N} \big|\bar{M}_{A,k}^N(t)\big|\,. \tag{5.22}$$

By Doob's inequality for submartingales,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|\bar{M}_{A,k}^N(t)\big|^2\Big] \le \mathbb{E}\big[\big|\bar{M}_{A,k}^N(T)\big|^2\big] = \mathbb{E}\left[\frac{K^N}{N}\int_0^T \bar{\Upsilon}_k^N(s)ds\right] \le \lambda^* C_B C_\beta T \frac{K^N}{N}\,.$$

Since $\frac{K^N}{N} \to 0$ as $N \to \infty$, the last inequality entails that as $N \to \infty$,

$$\sup_{1\le k\le K} \mathbb{E}\Big[\sup_{t\in[0,T]} \big|\bar{M}_{A,k}^N(t)\big|^2\Big] \to 0.$$

This combined with (5.22) implies that (5.17) holds.

Note that the above computations, combined with (5.21) and (5.20), yield (5.19).

Finally (5.18) follows directly from (5.21) and (5.17). □

We finally show that $\Delta_{1,1}^N(t,\cdot)$ and $\Delta_{1,2}^N(t,\cdot)$ tend to 0.

Lemma 5.6. *Under Assumptions 2.1, 2.2 and 2.3, as $N \to \infty$, both $\Delta_{1,1}^N(t,\cdot)$ and $\Delta_{1,2}^N(t,\cdot)$ defined in (5.5) and (5.6) converge to zero in $L^1([0,1])$ in probability, locally uniformly in t.*

Proof. We apply Theorem 4.1. We first consider $\Delta^N_{1,1}(t,x)$. To verify condition (i) of Theorem 4.1, we have

$$\|\Delta^N_{1,1}(t,\cdot)\|_1 \le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\left|\sum_{j=1}^{A^N_k(t)}\left(\lambda_{j,k}(t-\tau^N_{j,k})-\bar\lambda(t-\tau^N_{j,k})\right)\right|.$$

Recall the expression of $A^N_k(t)$ in (2.5) and the associated $\Upsilon^N_k(t)$ in (2.4). It is clear that the summands over k are not independent due to the interactions among individuals in different locations in the infection process. Using first Jensen's inequality, and then the fact that for each k, conditional on the arrivals $\{\tau^N_{j,k}\}_j$, the summands over j are independent and centered, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\left|\sum_{j=1}^{A^N_k(t)}\Big(\lambda_{j,k}(t-\tau^N_{j,k})-\bar\lambda(t-\tau^N_{j,k})\Big)\right|\right)^2\right]\\
&\le \mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\sum_{j=1}^{A^N_k(t)}\Big(\lambda_{j,k}(t-\tau^N_{j,k})-\bar\lambda(t-\tau^N_{j,k})\Big)\right)^2\right]\\
&= \mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\frac{K^N}{N}\Big)^2\sum_{j=1}^{A^N_k(t)}\left|\lambda_{j,k}(t-\tau^N_{j,k})-\bar\lambda(t-\tau^N_{j,k})\right|^2\right]\\
&= \mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\frac{K^N}{N}\Big)^2\int_0^t v(t-s)dA^N_k(s)\right]\\
&\le (\lambda^*)^2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\bar A^N_k(t)\right]\\
&= (\lambda^*)^2\frac{K^N}{N}\mathbb{E}\left[\|\bar A^N(t,\cdot)\|_1\right]\to 0 \quad \text{as} \quad N\to\infty\,,
\end{aligned}$$

where we used $v(t)\le(\lambda^*)^2$ under Assumption 2.3, and the convergence follows from the assumption that $\frac{K^N}{N}\to 0$ as $N\to\infty$, and (5.19) in Lemma 5.5.

We next check condition (ii) in Theorem 4.1 for $\Delta^N_{1,1}(t,x)$. We have

$$\begin{aligned}
\Delta^N_{1,1}(t+u,x)-\Delta^N_{1,1}(t,x) &= \sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A^N_k(t)}\Big(\lambda_{j,k}(t+u-\tau^N_{j,k})-\lambda_{j,k}(t-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A^N_k(t)}\Big(\bar\lambda(t+u-\tau^N_{j,k})-\bar\lambda(t-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad+\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k(t)+1}^{A^N_k(t+u)}\Big(\lambda_{j,k}(t+u-\tau^N_{j,k})-\bar\lambda(t+u-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\,,
\end{aligned}$$

and

$$\|\Delta^N_{1,1}(t+u,x)-\Delta^N_{1,1}(t,x)\|_1 \le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A^N_k(t)}\left|\lambda_{j,k}(t+u-\tau^N_{j,k})-\lambda_{j,k}(t-\tau^N_{j,k})\right|$$

$$
\begin{aligned}
&+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A_k^N(t)}\Big|\bar\lambda(t+u-\tau_{j,k}^N)-\bar\lambda(t-\tau_{j,k}^N)\Big|\\
&+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N(t)+1}^{A_k^N(t+u)}\Big|\lambda_{j,k}(t+u-\tau_{j,k}^N)-\bar\lambda(t+u-\tau_{j,k}^N)\Big|\\
=:\;&\Delta_{1,1}^{N,(1)}(t,u)+\Delta_{1,1}^{N,(2)}(t,u)+\Delta_{1,1}^{N,(3)}(t,u)\,.
\end{aligned}
$$

Similar to $\Delta_{0,1}^{N,(1)}(t,u)$ in (5.11), we have

$$
\sup_{u\in[0,\delta]}\Delta_{1,1}^{N,(1)}(t,u)\le\varphi(\delta)\int_0^1\bar A^N(t,x)dx+\lambda^*\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A_k^N(t)}\sum_{\ell=1}^{\kappa}\mathbf{1}_{t-\tau_{j,k}^N\le\zeta_{j,k}^\ell\le t+\delta-\tau_{j,k}^N}\,.
$$

We note that

$$
\begin{aligned}
\int_0^1\bar A^N(t,x)dx&=\int_0^1\int_0^t\bar\Upsilon^N(s,x)dsdx+\int_0^1\bar M_A^N(t,x)dx\\
&\le\lambda^*C_BC_\beta t+\int_0^1\bar M_A^N(t,x)dx\,.
\end{aligned}
$$

Hence, we deduce from (5.17) that as soon as $\delta>0$ is small enough such that $\varphi(\delta)\lambda^*C_BC_\beta t<\epsilon/6$,

$$
\limsup_N\frac{1}{\delta}\mathbb{P}\left(\varphi(\delta)\int_0^1\bar A^N(t,x)dx>\epsilon/6\right)=0\,. \tag{5.23}
$$

For the second term, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{A_k^N(t)}\mathbf{1}_{t-\tau_{j,k}^N\le\zeta_{j,k}^\ell\le t+\delta-\tau_{j,k}^N}\right)^2\right]\\
&\le 2\mathbb{E}\left[\left(\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s}\mathbf{1}_{r\le\Upsilon_k^N(s^-)}\overline{Q}_{k,\ell}(ds,dr,d\zeta)\right)^2\right]\\
&\quad+2\mathbb{E}\left[\left(\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Upsilon_k^N(s)ds\right)^2\right]
\end{aligned}
$$

where $Q_{k,\ell}(ds,dr,d\zeta)$ is a PRM on $\mathbb{R}_+^3$ with mean measure $dsdrF_\ell(d\zeta)$ whose projection on the first two coordinates is Q_k, and $\overline{Q}_{k,\ell}(ds,dr,d\zeta)$ is the corresponding compensated PRM. Observe that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s}\mathbf{1}_{r\le\Upsilon_k^N(s^-)}\overline{Q}_{k,\ell}(ds,dr,d\zeta)\right)^2\right]\\
&\le\kappa\sum_{\ell=1}^{\kappa}\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s}\mathbf{1}_{r\le\Upsilon_k^N(s^-)}\overline{Q}_{k,\ell}(ds,dr,d\zeta)\right)^2\right]\\
&=\kappa\sum_{\ell=1}^{\kappa}\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\frac{K^N}{N}\Big)^2\mathbb{E}\left[\int_0^t\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Upsilon_k^N(s)ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \lambda^* C_B C_\beta \kappa \sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \int_0^t \Big(F_\ell(t+\delta-s) - F_\ell(t-s) \Big) ds \\
&\le \lambda^* C_B C_\beta \kappa^2 \delta \frac{K^N}{N} \to 0 \quad \text{as} \quad N \to \infty,
\end{aligned}
$$

where we have used the inequality

$$
0 \le \int_0^t [F_\ell(s+\delta) - F_\ell(s)] ds \le \int_0^{t+\delta} F_\ell(s) ds - \int_0^t F_\ell(s) ds \le \delta\,, \tag{5.24}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \int_0^t \Big(F_\ell(t+\delta-s) - F_\ell(t-s) \Big) \Upsilon_k^N(s) ds \right)^2 \right] \\
&\le \kappa \sum_{\ell=1}^{\kappa} \frac{1}{K^N} \sum_{k=1}^{K^N} \mathbb{E}\left[\left(\frac{K^N}{N} \int_0^t \Big(F_\ell(t+\delta-s) - F_\ell(t-s) \Big) \Upsilon_k^N(s) ds \right)^2 \right] \\
&\le \kappa (\lambda^* C_B C_\beta)^2 \sum_{\ell=1}^{\kappa} \left(\int_0^t [F_\ell(s+\delta) - F_\ell(s)] ds \right)^2 \\
&\le (\kappa \lambda^* C_B C_\beta \delta)^2 .
\end{aligned}
$$

This combined with (5.23) shows that

$$
\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\le u\le\delta} \Delta_{1,1}^{N,(1)}(t,u) > \epsilon/3 \right) \to 0 \qquad \text{as} \qquad \delta \to 0 . \tag{5.25}
$$

Next, similar to $\Delta_{0,1}^{N,(1)}(t,u)$ in (5.11), we have

$$
\sup_{u\in[0,\delta]} \Delta_{1,1}^{N,(2)}(t,u) \le \varphi(\delta) \frac{1}{K^N} \sum_{k=1}^{K^N} \bar{A}_k^N(t) + \lambda^* \frac{1}{K^N} \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{A_k^N(t)} \sum_{\ell=1}^{\kappa} \Big(F_\ell(t+\delta-\tau_{j,k}^N) - F_\ell(t-\tau_{j,k}^N) \Big) .
$$

Then using the same arguments leading to (5.25), we obtain that (5.25) holds for $\Delta_{1,1}^{N,(2)}(t,u)$.

Finally, for $\Delta_{1,1}^{N,(3)}(t,u)$, we have

$$
\begin{aligned}
\sup_{0\le u\le\delta} \Delta_{1,1}^{N,(3)}(t,u) &\le \lambda^* \frac{1}{K^N} \sum_{k=1}^{K^N} (\bar{A}_k^N(t+\delta) - \bar{A}_k^N(t)) \\
&= \lambda^* \int_0^1 \int_t^{t+\delta} \bar{A}^N(ds,x) dx .
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{P}\left(\sup_{0\le u\le\delta} \Delta_{1,1}^{N,(3)}(t,u) > \epsilon/3 \right) \le \frac{18(\lambda^*)^2}{\epsilon^2} \Bigg\{ &\mathbb{E}\left[\left(\int_0^1 \int_t^{t+\delta} \bar{\Upsilon}^N(s,x) ds dx \right)^2 \right] \\
&+ \mathbb{E}\left[\|\bar{M}_A^N(t+\delta,\cdot) - \bar{M}_A^N(t,\cdot)\|_1^2 \right] \Bigg\},
\end{aligned}
$$

and from (5.17) and (5.20),

$$
\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\le u\le\delta} \Delta_{1,1}^{N,(3)}(t,u) > \epsilon/3 \right) \le \frac{18(\lambda^*)^4 (C_B)^2 C_\beta^2}{\epsilon^2} \delta
$$

$$\to 0, \quad \text{as} \quad \delta \to 0\,.$$

Consequently (5.25) holds for $\Delta_{1,1}^{N,(3)}(t,u)$.

Thus combining the three last results, we obtain

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta}\mathbb{P}\left(\sup_{0\le u\le\delta} \|\Delta_{1,1}^N(t+u,x) - \Delta_{1,1}^N(t,x)\|_1 > \epsilon\right) \to 0, \quad \text{as} \quad \delta \to 0. \tag{5.26}$$

Thus we have shown that $\Delta_{1,1}^N(t,\cdot) \to 0$ in $L^1([0,1])$ in probability, locally uniformly in t, as $N \to \infty$.

We now consider $\Delta_{1,2}^N(t,x)$. To check condition (i) in Theorem 4.1, we have for each $t \le T$,

$$\begin{aligned}
\mathbb{E}\big[\|\Delta_{1,2}^N(t,\cdot)\|_1^2\big] &\le \mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^t\int_0^\infty \bar\lambda(t-s)\mathbf{1}_{u\le\Upsilon_k^N(s)}\overline{Q}_k(ds,du)\right)^2\right] \\
&\le \mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\frac{K^N}{N}\Big)^2\Big(\int_0^t\int_0^\infty \bar\lambda(t-s)\mathbf{1}_{u\le\Upsilon_k^N(s)}\overline{Q}_k(ds,du)\Big)^2\right] \\
&= \mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\frac{K^N}{N}\Big)^2\int_0^t \bar\lambda(t-s)^2\Upsilon_k^N(s)ds\right] \\
&\le (\lambda^*)^2\frac{K^N}{N}\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\int_0^t \bar\Upsilon_k^N(s)ds\right] \\
&\le (\lambda^*)^3 C_B C_\beta T\frac{K^N}{N} \to 0
\end{aligned}$$

as $N\to\infty$. To check condition (ii) in Theorem 4.1, we have

$$\begin{aligned}
&\Delta_{1,2}^N(t+u,x) - \Delta_{1,2}^N(t,x) \\
&= \sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+u}\int_0^\infty \big(\bar\lambda(t+u-s)-\bar\lambda(t-s)\big)\mathbf{1}_{r\le\Upsilon_k^N(s)}\overline{Q}_k(ds,dr)\,\mathbf{1}_{\mathtt{I}_k}(x) \\
&\quad + \sum_{k=1}^{K^N}\frac{K^N}{N}\int_t^{t+u}\int_0^\infty \bar\lambda(t-s)\mathbf{1}_{r\le\Upsilon_k^N(s)}\overline{Q}_k(ds,dr)\,\mathbf{1}_{\mathtt{I}_k}(x)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\Delta_{1,2}^N(t+u,\cdot) - \Delta_{1,2}^N(t,\cdot)\|_1 \\
&\le \frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\int_0^{t+u}\int_0^\infty \big(\bar\lambda(t+u-s)-\bar\lambda(t-s)\big)\mathbf{1}_{r\le\Upsilon_k^N(s)}\overline{Q}_k(ds,dr)\right| \\
&\quad + \frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\int_t^{t+u}\int_0^\infty \bar\lambda(t-s)\mathbf{1}_{r\le\Upsilon_k^N(s)}\overline{Q}_k(ds,dr)\right| \\
&\le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+u}\int_0^\infty \big|\bar\lambda(t+u-s)-\bar\lambda(t-s)\big|\mathbf{1}_{r\le\Upsilon_k^N(s)}Q_k(ds,dr) \\
&\quad + \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+u} \big|\bar\lambda(t+u-s)-\bar\lambda(t-s)\big|\Upsilon_k^N(s)ds
\end{aligned}$$

$$
+ \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_t^{t+u}\int_0^\infty \bar\lambda(t-s)\mathbf{1}_{r\le\Upsilon_k^N(s)}Q_k(ds,dr)
+ \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_t^{t+u}\bar\lambda(t-s)\Upsilon_k^N(s)ds\,,
$$

from which we obtain

$$
\begin{aligned}
&\sup_{0\le u\le\delta}\|\Delta_{1,2}^N(t+u,\cdot)-\Delta_{1,2}^N(t,\cdot)\|_1\\
&\le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+\delta}\int_0^\infty\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\mathbf{1}_{r\le\Upsilon_k^N(s)}Q_k(ds,dr)\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+\delta}\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\Upsilon_k^N(s)ds\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_t^{t+\delta}\int_0^\infty\bar\lambda(t-s)\mathbf{1}_{r\le\Upsilon_k^N(s)}Q_k(ds,dr)\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_t^{t+\delta}\bar\lambda(t-s)\Upsilon_k^N(s)ds\,.
\end{aligned}
$$

For the first term, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+\delta}\int_0^\infty\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\mathbf{1}_{r\le\Upsilon_k^N(s)}Q_k(ds,dr)\right)^2\right]\\
&\le 2\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+\delta}\int_0^\infty\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\mathbf{1}_{r\le\Upsilon_k^N(s)}\overline{Q}_k(ds,dr)\right)^2\right]\\
&\quad+2\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+\delta}\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\Upsilon_k^N(s)ds\right)^2\right]\\
&\le 2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\int_0^{t+\delta}\int_0^\infty\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\mathbf{1}_{r\le\Upsilon_k^N(s)}\overline{Q}_k(ds,dr)\right)^2\right]\\
&\quad+2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\int_0^{t+\delta}\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\Upsilon_k^N(s)ds\right)^2\right]\\
&\le 2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\int_0^{t+\delta}\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]^2\bar\Upsilon_k^N(s)ds\right]\\
&\quad+2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\int_0^{t+\delta}\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]\bar\Upsilon_k^N(s)ds\right)^2\right]\\
&\le 2\frac{K^N}{N}\lambda^*C_BC_\beta\int_0^{t+\delta}\Big[\varphi(\delta)+\lambda^*\sum_{\ell=1}^{\kappa}\Big(F_\ell(t+\delta-s)-F_\ell(t-s)\Big)\Big]^2ds
\end{aligned}
$$

$$+ 2(\lambda^* C_B C_\beta)^2 \left(\int_0^{t+\delta} \Big[\varphi(\delta) + \lambda^* \sum_{\ell=1}^{\kappa} \Big(F_\ell(t+\delta-s) - F_\ell(t-s) \Big) \Big] ds \right)^2 .$$

Since the integral terms can be made arbitrarily small by choosing $\delta > 0$ small enough, we have that

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{P}\left(\sup_{0\le u\le\delta} \Delta_{1,2}^{N,(1)}(t,u) > \epsilon/4 \right) = 0$$

for $\delta > 0$ small enough. The second term is already treated above as the second component in the upper bound. The other two terms can be treated in a similar but simpler way. Thus we have shown that

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{0\le u\le\delta} \|\Delta_{1,2}^N(t+u,x) - \Delta_{1,2}^N(t,x)\|_1 > \epsilon \right) \to 0, \quad \text{as} \quad \delta \to 0. \tag{5.27}$$

Thus we have shown that $\Delta_{1,2}^N(t,\cdot) \to 0$ in $L^1([0,1])$ in probability, locally uniformly in t, as $N \to \infty$. The proof for the lemma is complete. □

We now deduce the following Corollary from the results in Proposition 5.1 and Lemmas 5.5, 5.4 and 5.6.

Corollary 5.1. *Under Assumptions 2.1, 2.2 and 2.3, we have that* $\|\bar{\Upsilon}^N(t,\cdot) - \bar{\Upsilon}(t,\cdot)\|_1 \to 0$ *in probability, locally uniformly in* t*, as* $N \to \infty$ *where* $\bar{\Upsilon}(t,x)$ *is given in* (2.21)*, and thus,* $\|\bar{A}^N(t,\cdot) - \bar{A}(t,\cdot)\|_1 \to 0$ *in probability, locally uniformly in* t*, as* $N \to \infty$*, where*

$$\bar{A}(t,x) = \int_0^t \frac{\bar{S}(s,x)}{\bar{B}(x)} \int_0^1 \beta(x,x') \bar{\mathfrak{F}}(s,x') dx' ds = \int_0^t \bar{\Upsilon}(s,x) ds . \tag{5.28}$$

Proof. Combining the results in Lemmas 5.5, 5.4 and 5.6 we have shown that $\sup_{0\le t\le T} \Psi^N(t) \to 0$ in probability as $N \to \infty$. Thus by Proposition 5.1, we can conclude the convergence of $\bar{S}^N(t,\cdot)$ and $\bar{\mathfrak{F}}^N(t,\cdot)$ in $L^1([0,1])$ in probability, locally uniformly in t. By the expression of $\bar{\Upsilon}^N(t,x)$ in (5.7), we immediately obtain the convergence of $\bar{\Upsilon}^N(t,\cdot)$. Then by the expression of $\bar{A}^N(t,x)$ in (5.21), we obtain the convergence in probability of $\bar{A}^N(t,\cdot)$ to $\bar{A}(t,\cdot)$ given in (5.28), as announced. The uniformity in t follows from the second Dini theorem. □

6. Proof for the Convergence of $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$

In this section, we prove the convergence of $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$ to $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ as stated in Proposition 6.1 below. Recall $\mathfrak{I}_k^N(t,\mathfrak{a})$ in (2.6). We write the two decomposed processes:

$$\bar{\mathfrak{I}}_0^N(t,\mathfrak{a},x) = \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{I_k^N(0)} \mathbf{1}_{\eta_{-j,k}^0 > t} \mathbf{1}_{\tilde{\tau}_{-j,k}^N \le (\mathfrak{a}-t)^+} \mathbf{1}_{\mathtt{I}_k}(x) = \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_{-j,k}^0 > t} \mathbf{1}_{\mathtt{I}_k}(x) \ , \tag{6.1}$$

and

$$\bar{\mathfrak{I}}_1^N(t,\mathfrak{a},x) = \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t)} \mathbf{1}_{\tau_{j,k}^N + \eta_{j,k} > t} \mathbf{1}_{\mathtt{I}_k}(x) \ . \tag{6.2}$$

Lemma 6.1. *Under Assumptions 2.1 and 2.3,*

$$\|\bar{\mathfrak{I}}_0^N(t,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}_0(t,\mathfrak{a},\cdot)\|_1 \to 0 \tag{6.3}$$

in probability, locally uniformly in t *and* $\mathfrak{a}$*, as* $N \to \infty$*, where*

$$\bar{\mathfrak{I}}_0(t,\mathfrak{a},x) := \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}(0,d\mathfrak{a}',x) . \tag{6.4}$$

Proof. We first write

$$\bar{\mathfrak{I}}^N_0(t,\mathfrak{a},x) = \bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x) + \bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},x)$$

where

$$\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x) = \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{\mathfrak{I}^N_k(0,(\mathfrak{a}-t)^+)} \frac{F^c(\tilde{\tau}^N_{-j,k}+t)}{F^c(\tilde{\tau}^N_{-j,k})} \mathbf{1}_{\mathtt{I}_k}(x) = \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)\,, \tag{6.5}$$

$$\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},x) = \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{\mathfrak{I}^N_k(0,(\mathfrak{a}-t)^+)} \left(\mathbf{1}_{\eta^0_{-j,k}>t} - \frac{F^c(\tilde{\tau}^N_{-j,k}+t)}{F^c(\tilde{\tau}^N_{-j,k})} \right) \mathbf{1}_{\mathtt{I}_k}(x)\,. \tag{6.6}$$

We apply Theorem 4.2. We first consider the process $\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x)$ and show that

$$\|\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}_0(t,\mathfrak{a},\cdot)\|_1 \to 0, \quad \text{in probability, locally uniformly in } t \text{ and } \mathfrak{a}, \tag{6.7}$$

as $N\to\infty$. We first check condition (i) of Theorem 4.2. we have

$$\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x) - \bar{\mathfrak{I}}_0(t,\mathfrak{a},x) = \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} [\bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x) - \bar{\mathfrak{I}}(0,d\mathfrak{a}',x)]\,.$$

Condition (i) of Theorem 4.2 follows from Lemma 4.1 and Assumption 2.1.

Next, we check condition (ii) of Theorem 4.2 for the processes $\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x) - \bar{\mathfrak{I}}_0(t,\mathfrak{a},x)$. We verify the condition for $\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x)$ in detail below, since the similar calculations can be done for $\bar{\mathfrak{I}}_0(t,\mathfrak{a},x)$. Namely, we show that for any $\epsilon>0$, and for any $T,\bar{\mathfrak{a}}'>0$, as $\delta\to 0$,

$$\limsup_N \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{u\in[0,\delta]} \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \|\bar{\mathfrak{I}}^N_{0,1}(t+u,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},\cdot)\|_1 > \epsilon \right) \to 0\,, \tag{6.8}$$

$$\limsup_N \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \frac{1}{\delta} \mathbb{P}\left(\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \|\bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},\cdot)\|_1 > \epsilon \right) \to 0\,. \tag{6.9}$$

To prove (6.8), we have

$$\begin{aligned}
&\bar{\mathfrak{I}}^N_{0,1}(t+u,\mathfrak{a},x) - \bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},x)\\
&= \int_0^{(\mathfrak{a}-t-u)^+} \frac{F^c(\mathfrak{a}'+t+u)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x) - \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)\,,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\bar{\mathfrak{I}}^N_{0,1}(t+u,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},\cdot)\right\|_1 &\le \int_0^1 \int_0^{(\mathfrak{a}-t-u)^+} \frac{F^c(\mathfrak{a}'+t) - F^c(\mathfrak{a}'+t+u)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\\
&\quad + \int_0^1 \int_{(\mathfrak{a}-t-u)^+}^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{u\in[0,\delta]} \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \left\|\bar{\mathfrak{I}}^N_{0,1}(t+u,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N_{0,1}(t,\mathfrak{a},\cdot)\right\|_1 &\le \int_0^1 \int_0^{(\bar{\mathfrak{a}}'-t)^+} \frac{F^c(\mathfrak{a}'+t) - F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\\
&\quad + \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \int_0^1 \int_{(\mathfrak{a}-t-\delta)^+}^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\,.
\end{aligned}$$

Thanks to Lemma 4.1 and Assumption 2.1, the first term on the right converges in probability as $N\to\infty$ to

$$\int_0^1 \int_0^{(\bar{\mathfrak{a}}'-t)^+} \frac{F^c(\mathfrak{a}'+t) - F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx\,,$$

which converges to zero as $\delta \to 0$. It follows from the uniform convergence established in Lemma 4.1 that the second term on the right converges in probability as $N \to \infty$, to

$$\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \int_0^1 \int_{(\mathfrak{a}-t-\delta)^+}^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx \le \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \int_0^1 \int_{(\mathfrak{a}-t-\delta)^+}^{(\mathfrak{a}-t)^+} \bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx\,.$$

Under Assumption 2.1, it is clear that the upper bound converges to zero at $\delta \to 0$. Thus we have shown that for $\epsilon > 0$, if $\delta > 0$ is small enough,

$$\limsup_N \sup_{t\in[0,T]} \mathbb{P}\Big(\sup_{u\in[0,\delta]} \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}]} \|\bar{\mathfrak{I}}_{0,1}^N(t+u,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},\cdot)\|_1 > \epsilon \Big) = 0\,.$$

To prove (6.9), we have

$$\bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a}+v,x) - \bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},x) = \int_0^1 \int_{(\mathfrak{a}-t)^+}^{(\mathfrak{a}+v-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\,,$$

and

$$\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \big\|\bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},\cdot)\|_1 \le \sup_{t\in[0,T]} \int_0^1 \int_{(\mathfrak{a}-t)^+}^{(\mathfrak{a}+\delta-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\,.$$

In order to show that the $\sup_t$ on the above right hand side converges in probability, as $N \to \infty$, to

$$\sup_{t\in[0,T]} \int_0^1 \int_{(\mathfrak{a}-t)^+}^{(\mathfrak{a}+v-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx \le \sup_{t\in[0,T]} \int_0^1 \int_{(\mathfrak{a}-t)^+}^{(\mathfrak{a}+v-t)^+} \bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx\,, \tag{6.10}$$

it suffices to show that the convergence of $\int_0^1 \int_{(\mathfrak{a}-t)^+}^{(\mathfrak{a}+\delta-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx$ is uniform in t. Indeed, we note that

$$\begin{aligned}
&\int_0^1 \int_{(\mathfrak{a}-t)^+}^{(\mathfrak{a}+\delta-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx \\
&\qquad = \int_0^1 \int_0^{(\mathfrak{a}+\delta-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx - \int_0^{(\mathfrak{a}-t)^+} \frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\,.
\end{aligned}$$

This right hand side is the difference of two non–increasing functions of t which converge pointwise to their limit in probability, as $N \to \infty$, and both limits are continuous in t. Hence the uniform convergence follows from the second Dini theorem, exactly as in the proof of Lemma 4.1. Going back to (6.10), we note that, under Assumption 2.1, the right hand side converges to zero at $\delta \to 0$. Thus we have shown that for $\epsilon > 0$, if $\delta > 0$ is small enough,

$$\limsup_N \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}]} \mathbb{P}\Big(\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \|\bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},\cdot)\|_1 > \epsilon \Big) = 0\,.$$

Thus we have verified condition (ii) of Theorem 4.2 for the processes $\bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},x)$, and with a similar argument for $\bar{\mathfrak{I}}_0(t,\mathfrak{a},x)$, and thus, for the difference $\bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},x) - \bar{\mathfrak{I}}_0(t,\mathfrak{a},x)$. Therefore, the claim on the convergence of $\bar{\mathfrak{I}}_{0,1}^N(t,\mathfrak{a},x)$ in (6.7) is proved.

We next prove the convergence of $\bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a},x)$:

$$\|\bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a},\cdot)\|_1 \to 0, \quad \text{in probability, locally uniformly in } t \text{ and } \mathfrak{a}, \text{ as } N \to \infty. \tag{6.11}$$

To check condition (i) of Theorem 4.2, we have

$$\|\bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a},\cdot)\|_1 \le \frac{1}{K^N} \sum_{k=1}^{K^N} \left| \frac{K^N}{N} \sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)} \left(\mathbf{1}_{\eta_{-j,k}^0 > t} - \frac{F^c(\tilde{\tau}_{-j,k}^N + t)}{F^c(\tilde{\tau}_{-j,k}^N)} \right) \right|.$$

We deduce from Jensen's inequality that

$$\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\left|\sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)}\left(\mathbf{1}_{\eta^0_{-j,k}>t}-\frac{F^c(\tilde{\tau}^N_{-j,k}+t)}{F^c(\tilde{\tau}^N_{-j,k})}\right)\right|\right)^2\right]$$
$$\leq \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\mathbb{E}\left[\int_0^{(\mathfrak{a}-t)^+}\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big)\bar{\mathfrak{I}}_k^N(0,d\mathfrak{a}')\right], \tag{6.12}$$

where we have used the fact that the $\eta^0_{-j,k}$'s are conditionally independent, given the $\tilde{\tau}^N_{-j,k}$'s. Note that under Assumption 2.1, thanks to Lemma 4.1, as $N\to\infty$, in probability,

$$\frac{1}{K^N}\sum_{k=1}^{K^N}\int_0^{(\mathfrak{a}-t)^+}\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big)\bar{\mathfrak{I}}_k^N(0,d\mathfrak{a}')$$
$$=\int_0^1\int_0^{(\mathfrak{a}-t)^+}\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big)\bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx$$
$$\to\int_0^1\int_0^{(\mathfrak{a}-t)^+}\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big(1-\frac{F^c(\mathfrak{a}'+t)}{F^c(\mathfrak{a}')}\Big)\bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx\,.$$

Thus, the upper bound in (6.12) converges to zero as $N\to\infty$. This implies that for any $\epsilon>0$,

$$\sup_{t\in[0,T]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}]}\mathbb{P}\big(\|\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},\cdot)\|_1>\epsilon\big)\to 0\quad\text{as}\quad N\to\infty.$$

Next, to check condition (ii) of Theorem 4.2, we show that for any $\epsilon>0$, and for any $T,\bar{\mathfrak{a}}'>0$, as $\delta\to 0$,

$$\limsup_N\sup_{t\in[0,T]}\frac{1}{\delta}\mathbb{P}\bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\big\|\bar{\mathfrak{I}}^N_{0,2}(t+u,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},\cdot)\big\|_1>\epsilon\bigg)\to 0\,, \tag{6.13}$$
$$\limsup_N\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}]}\frac{1}{\delta}\mathbb{P}\bigg(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\big\|\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a}+v,\cdot)-\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},\cdot)\big\|_1>\epsilon\bigg)\to 0\,. \tag{6.14}$$

To prove (6.13), we have

$$\bar{\mathfrak{I}}^N_{0,2}(t+u,\mathfrak{a},x)-\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},x)$$
$$=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)}\left(\mathbf{1}_{t<\eta^0_{-j,k}\leq t+u}-\frac{F^c(\tilde{\tau}^N_{-j,k}+t)-F^c(\tilde{\tau}^N_{-j,k}+t+u)}{F^c(\tilde{\tau}^N_{-j,k})}\right)\mathbf{1}_{\mathtt{I}_k}(x)$$
$$-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)+1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)}\left(\mathbf{1}_{\eta^0_{-j,k}>t}-\frac{F^c(\tilde{\tau}^N_{-j,k}+t)}{F^c(\tilde{\tau}^N_{-j,k})}\right)\mathbf{1}_{\mathtt{I}_k}(x)\,,$$

and

$$\big\|\bar{\mathfrak{I}}^N_{0,2}(t+u,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}^N_{0,2}(t,\mathfrak{a},\cdot)\big\|_1$$
$$\leq\frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)}\left(\mathbf{1}_{t<\eta^0_{-j,k}\leq t+u}-\frac{F^c(\tilde{\tau}^N_{-j,k}+t)-F^c(\tilde{\tau}^N_{-j,k}+t+u)}{F^c(\tilde{\tau}^N_{-j,k})}\right)\right|$$
$$+\frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\sum_{j=\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)+1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)}\left(\mathbf{1}_{\eta^0_{-j,k}>t}-\frac{F^c(\tilde{\tau}^N_{-j,k}+t)}{F^c(\tilde{\tau}^N_{-j,k})}\right)\right|$$

$$
\begin{aligned}
&\le \frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)}\mathbf{1}_{t<\eta^0_{-j,k}\le t+u}\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)}\frac{F^c(\tilde\tau^N_{-j,k}+t)-F^c(\tilde\tau^N_{-j,k}+t+u)}{F^c(\tilde\tau^N_{-j,k})}\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\bar{\mathfrak{I}}_k^N(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}_k^N(0,(\mathfrak{a}-t-u)^+)\Big). \qquad (6.15)
\end{aligned}
$$

For the first term on the right, we have

$$
\begin{aligned}
&\mathbb{P}\Bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)}\mathbf{1}_{t<\eta^0_{-j,k}\le t+u}>\epsilon\Bigg)\\
&\le \mathbb{P}\Bigg(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\bar{\mathfrak{a}}'-t)^+)}\mathbf{1}_{t<\eta^0_{-j,k}\le t+\delta}>\epsilon\Bigg)\\
&\le \mathbb{P}\Bigg(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\bar{\mathfrak{a}}-t)^+)}\Bigg(\mathbf{1}_{t<\eta^0_{-j,k}\le t+\delta}-\frac{F^c(\tilde\tau^N_{-j,k}+t)-F^c(\tilde\tau^N_{-j,k}+t+\delta)}{F^c(\tilde\tau^N_{-j,k})}\Bigg)>\epsilon/2\Bigg)\\
&\quad+\mathbb{P}\Bigg(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\bar{\mathfrak{a}}-t)^+)}\frac{F^c(\tilde\tau^N_{-j,k}+t)-F^c(\tilde\tau^N_{-j,k}+t+\delta)}{F^c(\tilde\tau^N_{-j,k})}>\epsilon/2\Bigg). \qquad (6.16)
\end{aligned}
$$

Here using Jensen's inequality and the fact that the summands over j are independent, conditionally upon the $\tilde\tau^N_{-j,k}$'s, the first probability is bounded by

$$
\begin{aligned}
&\frac{4}{\epsilon^2}\mathbb{E}\Bigg[\Bigg(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\bar{\mathfrak{a}}-t)^+)}\Bigg(\mathbf{1}_{t<\eta^0_{-j,k}\le t+\delta}-\frac{F^c(\tilde\tau^N_{-j,k}+t)-F^c(\tilde\tau^N_{-j,k}+t+\delta)}{F^c(\tilde\tau^N_{-j,k})}\Bigg)\Bigg)^2\Bigg]\\
&\le \frac{K^N}{N}\frac{4}{\epsilon^2}\mathbb{E}\int_0^1\int_0^{(\bar{\mathfrak{a}}-t)^+}\frac{F^c(\mathfrak{a}'+t)-F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx. \qquad (6.17)
\end{aligned}
$$

Now under Assumption 2.1, it follows from Lemma 4.1 that

$$
\begin{aligned}
&\int_0^1\int_0^{(\bar{\mathfrak{a}}-t)^+}\frac{F^c(\mathfrak{a}'+t)-F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\\
&\quad\to\int_0^1\int_0^{(\bar{\mathfrak{a}}-t)^+}\frac{F^c(\mathfrak{a}'+t)-F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx
\end{aligned}
$$

in probability as $N\to\infty$. Hence the upper bound in (6.17) converges to zero, as $N\to\infty$. Inside the second probability in (6.16), we have

$$
\begin{aligned}
&\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=1}^{\mathfrak{I}_k^N(0,(\bar{\mathfrak{a}}-t)^+)}\frac{F^c(\tilde\tau^N_{-j,k}+t)-F^c(\tilde\tau^N_{-j,k}+t+\delta)}{F^c(\tilde\tau^N_{-j,k})}\\
&=\int_0^1\int_0^{(\bar{\mathfrak{a}}-t)^+}\frac{F^c(\mathfrak{a}'+t)-F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}^N(0,d\mathfrak{a}',x)dx\\
&\to\int_0^1\int_0^{(\bar{\mathfrak{a}}-t)^+}\frac{F^c(\mathfrak{a}'+t)-F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')}\bar{\mathfrak{I}}(0,d\mathfrak{a}',x)dx
\end{aligned}
$$

in probability as $N \to \infty$, again from Lemma 4.1, and the limit converges to zero as $\delta \to 0$. Hence for any $\epsilon > 0$, if $\delta > 0$ is small enough, $\limsup_N$ of the second term in the right hand side of (6.16) is zero.

For the second term on the right of (6.15), we have

$$\sup_{u\in[0,\delta]} \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \frac{1}{K^N}\sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}-t-u)^+)} \frac{F^c(\tilde{\tau}_{-j,k}^N + t) - F^c(\tilde{\tau}_{-j,k}^N + t + u)}{F^c(\tilde{\tau}_{-j,k}^N)}$$
$$\le \int_0^1 \int_0^{(\bar{\mathfrak{a}}'-t)^+} \frac{F^c(\mathfrak{a}'+t) - F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}^N(0, d\mathfrak{a}', x)dx$$

which, thanks to Lemma 4.1 and Assumption 2.1, converges in probability as $N \to \infty$, to

$$\int_0^1 \int_0^{(\bar{\mathfrak{a}}-t)^+} \frac{F^c(\mathfrak{a}'+t) - F^c(\mathfrak{a}'+t+\delta)}{F^c(\mathfrak{a}')} \bar{\mathfrak{I}}(0, d\mathfrak{a}', x)dx.$$

This expression will also converge to zero as $\delta \to 0$. For the third term on the right of (6.15), we have

$$\sup_{u\in[0,\delta]} \int_0^1 \Big(\bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t)^+,x) - \bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t-u)^+,x)\Big)dx$$
$$\le \int_0^1 \Big(\bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t)^+,x) - \bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t-\delta)^+,x)\Big)dx$$

which converges in probability to

$$\int_0^1 \Big(\bar{\mathfrak{I}}(0,(\mathfrak{a}-t)^+,x) - \bar{\mathfrak{I}}(0,(\mathfrak{a}-t-\delta)^+,x)\Big)dx$$

as $N \to \infty$. Since $\bar{\mathfrak{I}}^N(0,\cdot,x)$ and $\bar{\mathfrak{I}}(0,\cdot,x)$ are nondecreasing and the limit is continuous, the convergence also holds uniformly over $\mathfrak{a} \in [0, \bar{\mathfrak{a}}']$. Moreover, we also have that

$$\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}]} \int_0^1 \Big(\bar{\mathfrak{I}}(0,(\mathfrak{a}-t)^+,x) - \bar{\mathfrak{I}}(0,(\mathfrak{a}-t-\delta)^+,x)\Big)dx \to 0,$$

as $\delta \to 0$. Combining the results on the three terms on the right of (6.15), we have shown that (6.13) holds.

We next prove (6.14). We have

$$\bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a}+v,x) - \bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a},x) = \sum_{k=1}^{K^N} \frac{K^N}{N} \sum_{j=\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)+1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}+v-t)^+)} \left(\mathbf{1}_{\eta_{-j,k}^0 > t+u} - \frac{F^c(\tilde{\tau}_{-j,k}^N + t)}{F^c(\tilde{\tau}_{-j,k}^N)}\right)\mathbf{1}_{\mathtt{I}_k}(x)\,,$$

and

$$\|\bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a},\cdot)\|_1 \le \frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N} \sum_{j=\mathfrak{I}_k^N(0,(\mathfrak{a}-t)^+)+1}^{\mathfrak{I}_k^N(0,(\mathfrak{a}+v-t)^+)} \left(\mathbf{1}_{\eta_{-j,k}^0 > t+u} - \frac{F^c(\tilde{\tau}_{-j,k}^N + t)}{F^c(\tilde{\tau}_{-j,k}^N)}\right)\right|$$
$$\le \frac{1}{K^N}\sum_{k=1}^{K^N} \left|\bar{\mathfrak{I}}_k^N(0,(\mathfrak{a}+v-t)^+) - \bar{\mathfrak{I}}_k^N(0,(\mathfrak{a}-t)^+\right|.$$

Thus,

$$\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \|\bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}_{0,2}^N(t,\mathfrak{a},\cdot)\|_1$$

$$
\begin{aligned}
&\le \sup_{t\in[0,T]} \frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\bar{\mathfrak{I}}_k^N(0,(\mathfrak{a}+\delta-t)^+)-\bar{\mathfrak{I}}_k^N(0,(\mathfrak{a}-t)^+)\Big)\\
&=\sup_{t\in[0,T]}\int_0^1\Big(\bar{\mathfrak{I}}^N(0,(\mathfrak{a}+\delta-t)^+,x)-\bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t)^+,x)\Big)dx
\end{aligned}
$$

and we claim that the right hand side converges in probability as $N\to\infty$, to

$$
\sup_{t\in[0,T]}\int_0^1\Big(\bar{\mathfrak{I}}(0,(\mathfrak{a}+\delta-t)^+,x)-\bar{\mathfrak{I}}(0,(\mathfrak{a}-t)^+,x)\Big)dx\,.
$$

Indeed, the convergence without the $\sup_t$ follows from Assumption 2.1, and both $t\mapsto\int_0^1\bar{\mathfrak{I}}^N(0,(\mathfrak{a}+\delta-t)^+,x)dx$ and $t\mapsto\int_0^1\bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t)^+,x)dx$ are non–increasing, while the limits are continuous. Hence again an application of the second Dini theorem implies that the convergence is locally uniform in t, hence the claim. The limit then converges to zero as $\delta\to0$. Thus we have shown (6.14). This completes the proof of the lemma. □

Lemma 6.2. *Under Assumptions 2.1, 2.2 and 2.3,*

$$
\|\bar{\mathfrak{I}}_1^N(t,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}_1(t,\mathfrak{a},\cdot)\|_1\to0 \tag{6.18}
$$

in probability, locally uniformly in t and $\mathfrak{a}$, as $N\to\infty$, where

$$
\bar{\mathfrak{I}}_1(t,\mathfrak{a},x):=\int_{(t-\mathfrak{a})^+}^t F^c(t-s)\bar{A}(ds,x)\,, \tag{6.19}
$$

where $\bar{A}(t,x)$ is given in (5.28).

Proof. We first write

$$
\bar{\mathfrak{I}}_1^N(t,\mathfrak{a},x)=\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},x)+\bar{\mathfrak{I}}_{1,2}^N(t,\mathfrak{a},x)
$$

where

$$
\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},x)=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t)}F^c(t-\tau_{j,k}^N)\mathbf{1}_{\mathtt{I}_k}(x)=\int_{(t-\mathfrak{a})^+}^t F^c(t-s)\bar{A}^N(ds,x)\;, \tag{6.20}
$$

$$
\bar{\mathfrak{I}}_{1,2}^N(t,\mathfrak{a},x)=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t)}\Big(\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t}-F^c(t-\tau_{j,k}^N)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\;. \tag{6.21}
$$

We apply Theorem 4.2. We start with the process $\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},x)$ and show that

$$
\big\|\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}_1(t,\mathfrak{a},\cdot)\big\|_1\to0,\quad\text{in probability, locally uniformly in } t \text{ and } \mathfrak{a}, \tag{6.22}
$$

as $N\to\infty$. Since

$$
\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},x)-\bar{\mathfrak{I}}_1(t,\mathfrak{a},x)=\int_{(t-\mathfrak{a})^+}^t F^c(t-s)\Big(\bar{A}^N(ds,x)-\bar{A}(ds,x)\Big),
$$

condition (i) of Theorem 4.2 follows from Lemma 4.1 and Corollary 5.1. In other words, we have that for each t and $\mathfrak{a}$, and for any $\epsilon>0$,

$$
\mathbb{P}(\|\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}_1(t,\mathfrak{a},\cdot)\|_1>\epsilon)\to0\quad\text{as}\quad N\to\infty.
$$

We next want to check (ii) of Theorem 4.2 for the processes $\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},x)-\bar{\mathfrak{I}}_1(t,\mathfrak{a},x)$. We will verify the following conditions for $\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},x)$: for any $\epsilon>0$, and for any $T,\bar{\mathfrak{a}}'>0$, as $\delta\to0$,

$$
\limsup_N\sup_{t\in[0,T]}\frac{1}{\delta}\mathbb{P}\bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\|\bar{\mathfrak{I}}_{1,1}^N(t+u,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}_{1,1}^N(t,\mathfrak{a},\cdot)\|_1>\epsilon\bigg)\to0\,, \tag{6.23}
$$

$$\limsup_N \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \frac{1}{\delta}\mathbb{P}\bigg(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\|\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a}+v,\cdot)-\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},\cdot)\|_1>\epsilon\bigg)\to 0\,. \tag{6.24}$$

It will be clear that the same results hold (and are simpler to prove) for $\bar{\mathfrak{I}}_1(t,\mathfrak{a},\cdot)$. To prove (6.23), we have

$$\begin{aligned}
&\bar{\mathfrak{I}}^N_{1,1}(t+u,\mathfrak{a},x)-\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},x)\\
&=\int_{(t+u-\mathfrak{a})^+}^{t+u}F^c(t+u-s)\bar{A}^N(ds,x)-\int_{(t-\mathfrak{a})^+}^{t}F^c(t-s)\bar{A}^N(ds,x)\\
&=\int_{(t-\mathfrak{a})^+}^{t+u}\Big(F^c(t+u-s)-F^c(t-s)\Big)\bar{A}^N(ds,x)\\
&\quad-\int_{(t-\mathfrak{a})^+}^{t+u-\mathfrak{a})^+}F^c(t+u-s)\bar{A}^N(ds,x)+\int_t^{t+u}F^c(t-s)\bar{A}^N(ds,x)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\bar{\mathfrak{I}}^N_{1,1}(t+u,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},\cdot)\big\|_1\\
&\le\int_0^1\int_{(t-\mathfrak{a})^+}^{t+u}\Big(F^c(t-s)-F^c(t+u-s)\Big)\bar{A}^N(ds,x)dx\\
&\quad+\int_{(t-\mathfrak{a})^+}^{(t+u-\mathfrak{a})^+}F^c(t+u-s)\bar{A}^N(ds,x)dx+\int_t^{t+u}F^c(t-s)\bar{A}^N(ds,x)dx\,.
\end{aligned}\tag{6.25}$$

Here the first term on the right satisfies

$$\begin{aligned}
&\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\int_0^1\int_{(t-\mathfrak{a})^+}^{t+u}\Big(F^c(t-s)-F^c(t+u-s)\Big)\bar{A}^N(ds,x)dx\\
&\qquad\le\int_0^1\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\Big(F^c(t-s)-F^c(t+\delta-s)\Big)\bar{A}^N(ds,x)dx\\
&\qquad\to\int_0^1\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\Big(F^c(t-s)-F^c(t+\delta-s)\Big)\bar{A}(ds,x)dx
\end{aligned}$$

in probability as $N\to\infty$ by Lemma 5.5 and Corollary 5.1, and the limit converges to zero as $\delta\to0$. The second term on the right side of (6.25) satisfies

$$\begin{aligned}
&\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\int_0^1\int_{(t-\mathfrak{a})^+}^{t+u-\mathfrak{a})^+}F^c(t+u-s)\bar{A}^N(ds,x)dx\\
&\le\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\int_0^1\Big(\bar{A}^N((t+\delta-\mathfrak{a})^+,x)-\bar{A}^N((t-\mathfrak{a})^+,x)\Big)dx\\
&\to\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\int_0^1\Big(\bar{A}((t+\delta-\mathfrak{a})^+,x)-\bar{A}((t-\mathfrak{a})^+,x)\Big)dx
\end{aligned}$$

in probability as $N\to\infty$ by Corollary 5.1 and the second Dini theorem, and the limit converges to zero as $\delta\to0$. The third term on the right side of (6.25) does not depend on $\mathfrak{a}$ and satisfies

$$\begin{aligned}
&\sup_{u\in[0,\delta]}\int_0^1\int_t^{t+u}F^c(t-s)\bar{A}^N(ds,x)dx\\
&\le\int_0^1\Big(\bar{A}^N(t+\delta,x)-\bar{A}^N(t,x)\Big)dx\to\int_0^1\Big(\bar{A}(t+\delta,x)-\bar{A}(t,x)\Big)dx
\end{aligned}$$

in probability as $N \to \infty$ by Corollary 5.1, and the limit converges to zero as $\delta \to 0$. Thus we have shown that for small enough $\delta > 0$, for any $\epsilon > 0$, and for any $T, \bar{\mathfrak{a}}' > 0$,

$$\limsup_N \sup_{t\in[0,T]} \mathbb{P}\bigg(\sup_{u\in[0,\delta]} \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \|\bar{\mathfrak{I}}^N_{1,1}(t+u,\mathfrak{a},\cdot) - \bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},\cdot)\|_1 > \epsilon \bigg) = 0\,.$$

To prove (6.24), we have

$$\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a}+v,x) - \bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},x) = \int_{(t-\mathfrak{a}-v)^+}^{(t-\mathfrak{a})^+} F^c(t-s)\bar{A}^N(ds,x)\,,$$

and

$$\big\|\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},\cdot)\big\|_1 = \int_0^1 \int_{(t-\mathfrak{a}-v)^+}^{(t-\mathfrak{a})^+} F^c(t-s)\bar{A}^N(ds,x)dx\,.$$

Hence,

$$\begin{aligned}
&\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \big\|\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},\cdot)\big\|_1 \\
&\le \sup_{t\in[0,T]} \int_0^1 \Big(\bar{A}^N((t-\mathfrak{a})^+,x) - \bar{A}^N((t-\mathfrak{a}-\delta)^+,x)\Big)dx \\
&\to \sup_{t\in[0,T]} \int_0^1 \Big(\bar{A}((t-\mathfrak{a})^+,x) - \bar{A}((t-\mathfrak{a}-\delta)^+,x)\Big)dx
\end{aligned}$$

in probability as $N \to \infty$ by Corollary 5.1 and again the second Dini theorem. Moreover, the limit converges to zero as $\delta \to 0$. Thus we have shown that for small enough $\delta > 0$, for any $\epsilon > 0$, and for any $T, \bar{\mathfrak{a}}' > 0$,

$$\limsup_N \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']} \mathbb{P}\bigg(\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} \|\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a}+v,\cdot) - \bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},\cdot)\|_1 > \epsilon \bigg) = 0\,.$$

Therefore, combining the above, we have proved the convergence of $\bar{\mathfrak{I}}^N_{1,1}(t,\mathfrak{a},x)$ as stated in (6.22).

We next consider the process $\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},x)$ and show that

$$\big\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\big\|_1 \to 0, \quad \text{in probability, locally uniformly in } t \text{ and } \mathfrak{a}, \text{ as } N \to \infty. \tag{6.26}$$

To check condition (i) of Theorem 4.2, we have

$$\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1 = \frac{1}{K^N}\sum_{k=1}^{K^N} \Bigg| \frac{K^N}{N} \sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t)} \Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t} - F^c(t-\tau^N_{j,k})\Big)\Bigg|,$$

and

$$\begin{aligned}
\mathbb{E}\big[\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1^2\big] &= \mathbb{E}\Bigg[\Bigg(\frac{1}{K^N}\sum_{k=1}^{K^N} \Bigg| \frac{K^N}{N} \sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t)} \Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t} - F^c(t-\tau^N_{j,k})\Big)\Bigg|\Bigg)^2\Bigg] \\
&\le \mathbb{E}\Bigg[\frac{1}{K^N}\sum_{k=1}^{K^N} \Bigg(\frac{K^N}{N} \sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t)} \Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t} - F^c(t-\tau^N_{j,k})\Big)\Bigg)^2\Bigg] \\
&= \mathbb{E}\Bigg[\frac{1}{K^N}\sum_{k=1}^{K^N} \Big(\frac{K^N}{N}\Big)^2 \sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t)} F(t-\tau^N_{j,k})F^c(t-\tau^N_{j,k})\Bigg]
\end{aligned}$$

$$\le \frac{K^N}{N}\mathbb{E}\left[\int_0^1\int_{(t-\mathfrak{a})^+}^t F(t-s)F^c(t-s)\bar{A}^N(ds,x)dx\right].$$

By Corollary 5.1 and Lemma 4.1, we obtain the convergence

$$\int_0^1\int_{(t-\mathfrak{a})^+}^t F(t-s)F^c(t-s)\bar{A}^N(ds,x)dx \to \int_0^1\int_{(t-\mathfrak{a})^+}^t F(t-s)F^c(t-s)\bar{A}(ds,x)dx$$

in probability as $N\to\infty$. This implies that for any $\epsilon>0$,

$$\sup_{t\in[0,T]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\mathbb{P}\big(\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1>\epsilon\big)\to 0 \quad \text{as} \quad N\to\infty.$$

Next, to check condition (ii) of Theorem 4.2, we need to show that for any $\epsilon>0$, and for any $T,\bar{\mathfrak{a}}'>0$, as $\delta\to 0$,

$$\limsup_N \sup_{t\in[0,T]}\frac{1}{\delta}\mathbb{P}\bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\|\bar{\mathfrak{I}}^N_{1,2}(t+u,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1>\epsilon\bigg)\to 0\,, \tag{6.27}$$

$$\limsup_N \sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\frac{1}{\delta}\mathbb{P}\bigg(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a}+v,\cdot)-\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1>\epsilon\bigg)\to 0\,. \tag{6.28}$$

To prove (6.27), we have

$$\begin{aligned}
&\bar{\mathfrak{I}}^N_{1,2}(t+u,\mathfrak{a},x)-\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},x)\\
&=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t+u-\mathfrak{a})^+)+1}^{A^N_k(t+u)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t+u}-F^c(t+u-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t}-F^c(t-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t+u)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t+u}-F^c(t+u-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k((t+u-\mathfrak{a})^+)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t+u}-F^c(t+u-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t+u)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t}-F^c(t-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad+\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k(t)+1}^{A^N_k(t+u)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t}-F^c(t-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&=-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k(t+u)}\Big(\mathbf{1}_{t<\tau^N_{j,k}+\eta_{j,k}\le t+u}-\big(F^c(t-\tau^N_{j,k})-F^c(t+u-\tau^N_{j,k})\big)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\\
&\quad-\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a})^+)+1}^{A^N_k((t+u-\mathfrak{a})^+\wedge t)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t+u}-F^c(t+u-\tau^N_{j,k})\Big)\mathbf{1}_{\mathtt{I}_k}(x)
\end{aligned}$$

$$+\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N(t)+1}^{A_k^N(t+u)}\Big(\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t}-F^c(t-\tau_{j,k}^N)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\,.$$

Thus we obtain

$$\begin{aligned}
&\big\|\bar{\mathfrak{I}}_{1,2}^N(t+u,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}_{1,2}^N(t,\mathfrak{a},\cdot)\big\|_1\\
&\le\frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t+u)}\Big(\mathbf{1}_{t<\tau_{j,k}^N+\eta_{j,k}\le t+u}-\Big(F^c(t-\tau_{j,k}^N)-F^c(t+u-\tau_{j,k}^N)\Big)\Big)\right|\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+\wedge t)+1}^{A_k^N((t+u-\mathfrak{a})^+}\Big(\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t+u}-F^c(t+u-\tau_{j,k}^N)\Big)\right|\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\sum_{j=A_k^N(t)+1}^{A_k^N((t+u)^+\wedge t)}\Big(\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t}-F^c(t-\tau_{j,k}^N)\Big)\right|\\
&\le\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t+u)}\mathbf{1}_{t<\tau_{j,k}^N+\eta_{j,k}\le t+u}\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t+u)}\Big(F^c(t-\tau_{j,k}^N)-F^c(t+u-\tau_{j,k}^N)\Big)\\
&\quad+\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\bar{A}_k^N(t+u)-\bar{A}_k^N(t)\Big)+\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\bar{A}_k^N((t+u-\mathfrak{a})^+)-\bar{A}_k^N((t-\mathfrak{a})^+)\Big)\,. \qquad (6.29)
\end{aligned}$$

For the first term on the right, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t+u)}\mathbf{1}_{t<\tau_{j,k}^N+\eta_{j,k}\le t+u}\right)^2\right]\\
&\le\mathbb{E}\left[\left(\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\bar{\mathfrak{a}}')^+)+1}^{A_k^N(t+\delta)}\mathbf{1}_{t<\tau_{j,k}^N+\eta_{j,k}\le t+\delta}\right)^2\right]\\
&\le\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\sum_{j=A_k^N((t-\bar{\mathfrak{a}}')^+)+1}^{A_k^N(t+\delta)}\mathbf{1}_{t<\tau_{j,k}^N+\eta_{j,k}\le t+\delta}\right)^2\right]\\
&\le 2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\int_0^\infty\int_{t-s}^{t+\delta-s}\mathbf{1}_{r\le\Upsilon^N(s^-)}\overline{Q}_{k,\ell}(ds,dr,dz)\right)^2\right]\\
&\quad+2\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\left(\frac{K^N}{N}\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}(F(t+\delta-s)-F(t-s))\Upsilon_k^N(s)ds\right)^2\right]\\
&=2\frac{K^N}{N}\mathbb{E}\left[\frac{1}{K^N}\sum_{k=1}^{K^N}\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}(F(t+\delta-s)-F(t-s))\bar{\Upsilon}_k^N(s)ds\right]
\end{aligned}$$

$$
\begin{aligned}
&+ 2\mathbb{E}\Bigg[\frac{1}{K^N}\sum_{k=1}^{K^N}\bigg(\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}(F(t+\delta-s)-F(t-s))\bar{\Upsilon}_k^N(s)ds\bigg)^2\Bigg]\\
&\le 2\lambda^* C_B C_\beta \frac{K^N}{N}\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}(F(t+\delta-s)-F(t-s))ds\\
&\quad + 2(\lambda^* C_B C_\beta)^2\bigg(\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}(F(t+\delta-s)-F(t-s))ds\bigg)^2,
\end{aligned}
$$

where $Q_{k,\ell}(ds,dr,dz)$ is the PRM on $\mathbb{R}_+^3$ with mean measure $dsdrF(dz)$ already introduced in the proof of Lemma 5.6, and $\overline{Q}_{k,\ell}(ds,dr,dz)$ is the corresponding compensated PRM, and we have used the bound $\bar{\Upsilon}_k^N(t)\le\lambda^* C_B C_\beta$. The first term on the right goes to zero as $N\to\infty$, and the integral in the second is bounded from above by

$$
\int_0^t \big(F(s+\delta)-F(s)\big)ds \le \delta\,,
$$

as in (5.24) above. Thus we obtain that for any $\epsilon>0$, as $\delta\to 0$,

$$
\limsup_N \sup_{t\in[0,T]}\frac{1}{\delta}\mathbb{P}\Bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t+u)}\mathbf{1}_{t<\tau_{j,k}^N+\eta_{j,k}\le t+u}>\epsilon\Bigg)\to 0\,.
$$

For the second term on the right side of (6.29), we have

$$
\begin{aligned}
&\mathbb{E}\Bigg[\bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,\bar{\mathfrak{a}}']}\frac{1}{K^N}\sum_{k=1}^{K^N}\frac{1}{B_k^N}\sum_{j=A_k^N((t-\mathfrak{a})^+)+1}^{A_k^N(t+u)}\big(F^c(t-\tau_{j,k}^N)-F^c(t+u-\tau_{j,k}^N)\big)\bigg)^2\Bigg]\\
&\le \mathbb{E}\Bigg[\frac{1}{K^N}\sum_{k=1}^{K^N}\bigg(\frac{K^N}{N}\sum_{j=A_k^N((t-\bar{\mathfrak{a}}')^+)+1}^{A_k^N(t+\delta)}\big(F^c(t-\tau_{j,k}^N)-F^c(t+\delta-\tau_{j,k}^N)\big)\bigg)^2\Bigg]\\
&\le 2\mathbb{E}\Bigg[\frac{1}{K^N}\sum_{k=1}^{K^N}\Big(\frac{K^N}{N}\Big)^2\bigg(\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\big(F^c(t-s)-F^c(t+\delta-s)\big)dM_{A,k}^N(s)\bigg)^2\Bigg]\\
&\quad + 2\mathbb{E}\Bigg[\frac{1}{K^N}\sum_{k=1}^{K^N}\bigg(\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\big(F^c(t-s)-F^c(t+\delta-s)\big)\bar{\Upsilon}_k^N(s)ds\bigg)^2\Bigg]\\
&= 2\lambda^* C_B C_\beta\frac{K^N}{N}\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\big(F^c(t-s)-F^c(t+\delta-s)\big)^2ds\\
&\quad + 2(\lambda^* C_B C_\beta)^2\bigg(\int_{(t-\bar{\mathfrak{a}}')^+}^{t+\delta}\big(F^c(t-s)-F^c(t+\delta-s)\big)ds\bigg)^2.
\end{aligned}
$$

It is clear that the first term converge to zero locally uniformly in t, and the second term can be treated in the same way above. The third and fourth terms on the right side of (6.29) can be also treated similarly as the last two terms in (6.25). Thus, we have shown that (6.27) holds.

To prove (6.28), we have

$$
\bar{\mathfrak{I}}_{1,2}^N(t,\mathfrak{a}+v,x)-\bar{\mathfrak{I}}_{1,2}^N(t,\mathfrak{a},x)=\sum_{k=1}^{K^N}\frac{K^N}{N}\sum_{j=A_k^N((t-\mathfrak{a}-v)^+)+1}^{A_k^N((t-\mathfrak{a})^+)}\Big(\mathbf{1}_{\tau_{j,k}^N+\eta_{j,k}>t}-F^c(t-\tau_{j,k}^N)\Big)\mathbf{1}_{\mathtt{I}_k}(x)\,,
$$

and

$$\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a}+v,\cdot)-\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1 \le \frac{1}{K^N}\sum_{k=1}^{K^N}\left|\frac{K^N}{N}\sum_{j=A^N_k((t-\mathfrak{a}-v)^+)+1}^{A^N_k((t-\mathfrak{a})^+)}\Big(\mathbf{1}_{\tau^N_{j,k}+\eta_{j,k}>t}-F^c(t-\tau^N_{j,k})\Big)\right|$$
$$\le \int_0^1\Big(\bar{A}^N((t-\mathfrak{a})^+,x)-\bar{A}^N((t-\mathfrak{a}-v)^+,x)\Big)dx\,.$$

Then, we obtain

$$\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\|\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a}+v,\cdot)-\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},\cdot)\|_1$$
$$\le \sup_{t\in[0,T]}\int_0^1\Big(\bar{A}^N((t-\mathfrak{a})^+,x)-\bar{A}^N((t-\mathfrak{a}-\delta)^+,x)\Big)dx\,.$$

Here the upper bound converges in probability to

$$\sup_{t\in[0,T]}\int_0^1\Big(\bar{A}((t-\mathfrak{a})^+,x)-\bar{A}((t-\mathfrak{a}-\delta)^+,x)\Big)dx$$

which converges to zero as $\delta\to 0$, uniformly in $\mathfrak{a}$. Indeed, the convergence of the $\sup_t$ follows from the fact that the convergence in probability $\int_0^1\bar{A}^N(t,x)dx\to\int_0^1\bar{A}(t,x)dx$ is locally uniform in t, thanks to Corollary 5.1. Thus we have proved (6.28) holds, and hence, the convergence of $\bar{\mathfrak{I}}^N_{1,2}(t,\mathfrak{a},x)$ in (6.26). This completes the proof of the lemma. □

By the two lemmas above, we can conclude the convergence of $\bar{\mathfrak{I}}^N(t,\mathfrak{a},x)$ to $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$.

Proposition 6.1. *Under Assumptions 2.1, 2.2 and 2.3,*

$$\|\bar{\mathfrak{I}}^N(t,\mathfrak{a},\cdot)-\bar{\mathfrak{I}}(t,\mathfrak{a},\cdot)\|_1\to 0 \tag{6.30}$$

in probability, locally uniformly in t and $\mathfrak{a}$, as $N\to\infty$, where $\bar{\mathfrak{I}}(t,\mathfrak{a},x)=\bar{\mathfrak{I}}_0(t,\mathfrak{a},x)+\bar{\mathfrak{I}}_1(t,\mathfrak{a},x)$, $\bar{\mathfrak{I}}_0$ and $\bar{\mathfrak{I}}_1$ being given respectively by (6.4) *and* (6.19)*.*

Completing the proof of Theorem 2.1. Given the results in Propositions 5.1 and 6.1 and Corollary 5.1, the convergence of $\bar{R}^N(t,x)$ and $\bar{I}^N(t,x)$ can be easily established and their limits $\bar{R}(t,x)$ and $\bar{I}(t,x)$ follows directly. The second expression of $\bar{\Upsilon}(t,x)$ in (2.21) is obtained from $\bar{\mathfrak{I}}(t,\mathfrak{a},x)$ in (2.22), by noting that $\bar{\mathfrak{I}}_a(t,0,x)=\lim_{\mathfrak{a}\to 0}\frac{\bar{\mathfrak{I}}(t,\mathfrak{a},x)-\bar{\mathfrak{I}}(t,0,x)}{\mathfrak{a}}$.

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