Backward Stochastic Differential Equations and Viscosity Solutions of Systems of Semilinear Parabolic and Elliptic PDEs of Second Order

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Introduction

The aim of this set of lectures is to present the theory of backward stochastic differential equations, in short BSDEs, and its connections with viscosity solutions of systems of semi–linear second order partial differential equations of parabolic and elliptic type, in short PDEs. Linear BSDEs have appeared long time ago, both as the equations for the adjoint process in stochastic control, as well as the model behind the Black & Scholes formula for the pricing and hedging of options in mathematical finance. These linear BSDEs can be solved more or less explicitly (see the proof of theorem 1.6).

However, the first published paper on nonlinear BSDEs, [37], appeared only in 1990. Since then, the interest for BSDEs has increased regularly, due to the connections of this subject with mathematical finance, stochastic control, and partial differential equations. We refer the interested reader to El Karoui, Peng, Quenez [18], El Karoui, Quenez [19], the reference therein and in particular the work of Duffie and his co–authors [11], [12], [13] and [14] for developments on the use of BSDEs as models in mathematical finance, as well as the connection of BSDEs with stochastic control. BSDEs is also an efficient tool for constructing Γ–martingales on manifolds, with rescribed limit, see Darling [9].

The present notes develop the theory of BSDEs, and its connections with PDEs. We have concentrated our presentation on the connection with viscosity solutions of PDEs. Also this approach is appealing, it is not the unique possible presentation. We have developped both the parabolic and the elliptic cases, the latter being presented in the two cases of systems of equations in $\mathbb{R}^d$, and equations in a bounded set with Dirichlet boundary conditions. We have left out the case of equations with Neumann boundary conditions, which is thoroughly exposed in Pardoux, Zhang [44], and the study of coupled forward–backward SDEs and its connections with quasi–linear PDEs, which is a subject of much recent interest, see the works of Ma, Protter, Yong, [33], Hu, Peng [30] and Pardoux, Tang [42] among others.

On the other hand, we present in these notes a sketch of the proof of uniqueness for viscosity solutions of semi–linear PDEs, following Crandall, Ishii, Lions [7].

Let us now motivate the connection between BSDEs and PDEs.

Consider the backward parabolic partial differential equation

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} + (Lu)(t,x) + c(x)u(t,x) &= 0, 0 < t < T,\ x \in \mathbb{R}^d \\ u(T,x) &= g(x),\ x \in \mathbb{R}^d, \end{aligned}$$

where L is the infinitesimal generator of a time–homogeneous diffusion process $\{X_t;\ t \geq 0\}$, and $c, g \in C_b(\mathbb{R}^d)$. Let us denote by $\{T_t,\ t \geq 0\}$ the semi–group generated by L. We want to get a probabilistic formula for $u(t,x)$ (which will of course happen to be the Feynman–Kac formula), where $t \in (0,T)$. Given $h > 0$, such that $n = h^{-1}(T-t)$ is an integer, define the grid

$$t = t_0 < t_1 = t + h < t_2 = t + 2h < \cdots < t_n = T.$$

From the Trotter–Kato formula, we have that for h small,

$$u(t,x) \simeq [T_h \circ\ e^{hc(\cdot)} \circ\ T_h \circ\ e^{hc(\cdot)} \circ \ldots \circ T_h(e^{hc(\cdot)}g)](x),$$

where $e^{hc(\cdot)}$ denotes the multiplication operator by the function $e^{hc(x)}$, and the operator $T_h \circ e^{hc(\cdot)}$is applied n times. We note that for any $s \geq 0$,

$$\begin{aligned} (T_h f)(x) &= E[f(X_{s+h})/X_s = x], \\ (T_h f)(X_s) &= E[f(X_{s+h})/X_s] \\ &= E[f(X_{s+h})/\mathcal{F}_s], \end{aligned}$$

if for instance $\mathcal{F}_s = \mathcal{F}_s^X$, the "natural filtration of X ", due to the Markovian property of X.

Hence the above formula becomes

$$u(t,x) \simeq E^{X_t=x}[e^{hc(X_{t_1})}E^{\mathcal{F}_{t_1}}e^{hc(X_{t_2})}\ldots E^{\mathcal{F}_{t_{n-1}}}(e^{hc(X_T)}g(X_T))].$$

Now, since

$$E^{\mathcal{F}_{t_{i-1}}}[e^{hc(X_{t_i})}E^{\mathcal{F}_{t_i}}(\xi)] = E^{\mathcal{F}_{t_{i-1}}}[e^{hc(X_{t_i})}\xi],$$

we deduce that

$$u(t,x) \simeq E^{X_t=x}[e^{h\sum_{i=1}^{n} c(X_{t_i})}g(X_T)],$$

and taking the limit as $h \to 0$ yields the celebrated Feynman–Kac formula

$$u(t,x) = E^{X_t=x}[e^{\int_t^T c(X_s)ds}g(X_T)].$$

There has been in the past at least three ways of extending the Feynman–Kac formula to nonlinear equations. One is to replace the diffusion $\{X_t\}$ by a controlled diffusion (see Fleming, Soner [20]), the second is to replace it by a branching–diffusion process (or a "superprocess", see e.g. Dynkin [15]), the third is to replace it by a "nonlinear Markov process" in the sense that the evolution of X_t depends not only of X_t, but also on its probability law, see e.g. McKean [34]. What we shall expose in these notes is a fourth such nonlinear generalization of the Feynman–Kac formula, based on BSDEs. Indeed, let us now try to do the same job as above with the semilinear equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + (Lu)(t,x) + f(u(t,x)) &= 0, 0 < t < T, x \in \mathbb{R}^d; \\ u(T,x) &= g(x), x \in \mathbb{R}^d, \end{aligned} \tag{0.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is e.g. a globally Lipschitz function. For $t > 0, u \in \mathbb{R}$, let us denote by $\Phi_t(u)$ the value at time t of the solution of the ODE

$$\frac{dX_t}{dt} = f(X_t), t > 0; X_0 = u$$

We can still apply the Trotter–Kato formula, yielding

$$u(t,x) \simeq E^{X_t=x}[\Phi_h(E^{\mathcal{F}_{t_1}}(\Phi_h(E^{\mathcal{F}_{t_2}} \dots E^{\mathcal{F}_{t_{n-1}}}\Phi_h \circ g(X_T))))].$$

Now we would like to be able to give a meaning to a limiting formula obtained by letting $h \to 0$, that is we would like to obtain a formula for the evolution of the process $u(t, X_t)$, running backward from $t = T$, where its value is $g(X_T)$, which would not rely on the knowledge of the function $\{u(t,x); 0 \le t \le T,\ x \in \mathbb{R}^d\}$. We note that the evolution of $u(t,X_t)$ is a combination of two effects : it follows the ODE with the coefficient $-f$, and it is – continously in time – projected on the σ–algebra $\mathcal{F}_t$ associated to the current time t. (Note that a major difference with the case of the linear equation is that since conditional expectations do not commute with the non–linear mapping Φ_h, we cannot hope for a formula which computes an evolution path by path on the interval $[t,T]$, and then takes an expectation).

In fact it is not hard to define such an evolution. Suppose we can find an adapted process $\{Y_t; 0 \le t \le T\}$ such that

$$Y_t = E^{\mathcal{F}_t}[g(X_T) + \int_t^T f(Y_s)ds], 0 \le t \le T, \tag{0.2}$$

then Y_t is a good candidate for being equal to $u(t,X_t)$. Indeed, suppose $u \in C^{1,2}((0,T) \times \mathbb{R}^d)$ is a classical solution of the above semilinear PDE, then from Itô's formula

$$\begin{aligned} u(t,X_t) &= u(T,X_T) - \int_t^T (\frac{\partial u}{\partial s} + Lu)(s,X_s)ds \\ &- (M_T - M_t), \end{aligned}$$

where $\{M_t, 0 \le t \le T\}$ is a local martingale.

Assuming that this local martingale is a martingale, and exploiting the equation satisfied by u, we deduce that

$$u(t, X_t) = E^{\mathcal{F}_t}[g(X_T) + \int_t^T f(u(s, X_s))ds],$$

which makes sense in particular whenever f and g are bounded. Hence the "backward stochastic differential equation " (0.2) at least has the solution $Y_t = u(t, X_t)$. Uniqueness in the class of processes satisfying $\sup_{0 \le t \le T} E(Y_t^2)$ follows easily e.g. from the Lipschitz property of f.

So we already see that the BSDE (0.2) is likely to possess a unique solution, under appropriate assumptions on the final condition $g(X_T)$ and the coefficient of f. Moreover, we have seen that there is a connection between the PDE (0.1) and the BSDE (0.2) More precisely, if $\{X_s^{t,x}; t \le s \le T\}$ denotes the diffusion process X on the time interval $[t, T]$, starting at time t from the point x, then $u(t, x) = Y_t^{t,x}$, where $\{Y_s^{t,x}, t \le s \le T\}$ solves the BSDE

$$Y_s^{t,x} = E^{\mathcal{F}_s}[g(X_T^{t,x}) + \int_s^T f(Y_r^{t,x})dr], t \le s \le T$$

Let us now rewrite the BSDE (0.2) in the more fancy form which will be used below. Suppose that the diffusion X is constructed as the solution of a (forward) SDE driven by a d–dimensional Brownian motion $\{B_t; t \ge 0\}$. Then the random variable

$$\chi = g(X_T) + \int_0^T f(Y_t)dt$$

is a functional of $\{B_t\}$ which is $\mathcal{F}_T^B$ measurable, and provided it is square integrable, there exists a unique d–dimensional process $\{Z_t\ ; 0 \le t \le T\}$ such that

(i) $E \int_0^T |Z_t|^2 dt < \infty$

(ii) $\chi = E(\chi) + \int_0^T < Z_t, dB_t >$

It is easily seen that $Y_0 = E^{\mathcal{F}_0}(\chi) = E(\chi)$, hence

$$Y_0 = g(X_T) + \int_0^T f(Y_t)dt - \int_0^T < Z_t, dB_t >,$$

and the quantity

$$Y_0 - \int_0^t f(Y_s)ds + \int_0^t < Z_s, dB_s >$$

is $\mathcal{F}_t$ measurable, and it is equal to

$$g(X_T) + \int_t^T f(Y_s)ds - \int_t^T < Z_s, dB_s > .$$

From this and (0.2) follows the fact that

$$Y_t = g(X_T) + \int_t^T f(Y_s)ds - \int_t^T < Z_s, dB_s >$$

Now that we are at that point, why not let f depend on X_t and on Z_t as well, so that we arrive at the following formulation : find a pair $\{(Y_t, Z_t); 0 \leq t \leq T\}$ of *adapted* processes with values in $\mathbb{R} \times \mathbb{R}^d$, such that :

$$\begin{aligned} &(i) \qquad E\int_0^T |Z_t|^2 dt < \infty \\ &(ii) \qquad Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, 0 \leq t \leq T \end{aligned} \tag{0.3}$$

Note that, since the boundary condition for $\{Y_t\}$ is given at the terminal time T, it is not really natural for the solution $\{Y_t\}$ to be adapted at each time t to the past of the Brownian motion $\{B_s\}$ before time t. The price we have to pay for such a severe constraint to be satisfied is to have the coefficient of the Brownian motion - the process $\{Z_t\}$ - to be choosen independently of $\{Y_t\}$, hence the solution of the BSDE is a *pair* of processes.

One may think that the terminology "backward SDE" is misleading, and that what we are really trying to solve is an inverse problem for an SDE, namely we are looking for a point $y \in \mathbb{R}$, and an adapted process $\{Z_t\}$ satisfying (i), such that the solution $\{Y_t\}$ of

$$(ii') \qquad Y_t = y - \int_0^t f(X_s, Y_s, Z_s)ds + \int_0^t Z_s dB_s$$

satisfies $Y_T = g(X_T)$.

We have motivated the notion of BSDE, and its connection to PDEs. We shall start with a study of an abstract version of (0.3).

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1 Backward stochastic differential equations on a fixed finite time interval

Let $\{B_t\,;\, t \geq 0\}$ be a d–dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. For $t \geq 0$, let $\mathcal{F}_t$ denote the σ–algebra $\sigma(B_s\,;\, 0 \leq s \leq t)$, augmented with the P–null sets of $\mathcal{F}$.

We shall denote below by $M^2(0,T)$ the set of $\mathcal{F}_t$–progressively measurable processes $\{X_t;\ 0 \leq t \leq T\}$ which are such that $E\int_0^T |X_t|^2\,dt < \infty$.

We are given :

(a) a final time T,

(b) a final condition $\xi \in L^2(\Omega, \mathcal{F}_T, P\,;\, \mathbb{R}^k)$,

(c) a coefficient $f\,:\,\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$, which is such that for some $\mathbb{R}_+$–valued progressively measurable process $\{\overline{f}_t;\ 0\leq t\leq T\}$, real numbers μ and $K>0$:

  (i) $f(\cdot, y, z)$ is progressively measurable, $\forall\, y, z$;

  (ii) $|f(t,y,z)| \leq \overline{f}_t + K(|y| + \|z\|)$, $\forall t, y, z$, a.s.;

  (iii) $E\displaystyle\int_0^T |\overline{f}_t|^2\,dt < \infty$

  (iv) $|f(t,y,z) - f(t,y,z')| \leq K\|z - z'\|$, $\forall t, y, z, z'$, a.s., where $\|z\| = [Tr(zz^*)]^{1/2}$;

  (v) $< y - y', f(t,y,z) - f(t,y',z) > \leq \mu|y-y'|^2$, $\forall t, y, y', z$, a.s.;

  (vi) $y \to f(t,y,z)$ is continuous, $\forall t, z$, a.s.

A solution of the BSDE (ξ, f) is a pair $\{(Y_t, Z_t)\,;\, 0\leq t\leq T\}$ of *progressively measurable* processes with values in $\mathbb{R}^k\times\mathbb{R}^{k\times d}$ s.t.

(j) $E\displaystyle\int_0^T \|Z_t\|^2\,dt < \infty$ (i.e. $Z \in (M^2(0,T))^{k\times d}$),

(jj) $Y_t = \xi + \displaystyle\int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dB_s,\ 0\leq t\leq T.$

Note that the progressive measurability of $\{Y_t\}$ implies in particular that Y_0 is deterministic.

Proposition 1.1 *Under the above conditions, if (Y,Z) is a solution of the BSDE (ξ, f) , then there exists a constant c, which depends only on T, μ and K, such that*

$$E\left(\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T \|Z_t\|^2 dt\right) < cE\left(|\xi|^2 + \int_0^T |f(t,0,0)|^2 dt\right).$$

Proof : If (Y, Z) is a solution, then

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)\, ds + \int_0^t Z_s\, dB_s. \tag{1.1}$$

Define for each $n \in \mathbb{N}$ the stopping time

$$\tau_n = \inf\{0 \leq t \leq T\,;\, |Y_t| \geq n\},$$

and the process

$$Y_t^n = Y_{t \wedge \tau_n} \quad .$$

Then $|Y_t^n| \leq n \vee |Y_0|$ and

$$Y_t^n = Y_0^n - \int_0^t \mathbf{1}_{[0,\,\tau_n]}(s) f(s, Y_s^n, Z_s)\, ds + \int_0^t \mathbf{1}_{[0,\tau_n]} Z_s\, dB_s$$

$$\begin{aligned} E|Y_t^n|^2 &\leq c\left[|Y_0^n|^2 + \int_0^t E(|f(s, Y_s^n, Z_s)|^2)\, ds + \int_0^t E\|Z_s\|^2\, ds\right], \quad 0 \leq t \leq T \\ &\leq C\left(1 + \int_0^t E|Y_s^n|^2\, ds\right), \end{aligned}$$

where we have used the assumptions (ii), (iii) and (v) on f, and condition (j). Now from Gronwall's lemma

$$E|Y_t^n|^2 \leq Ce^{Ct},$$

and hence from Fatou's lemma

$$E|Y_t|^2 \leq Ce^{Ct}.$$

From this, (j), (1.1) and Burkholder's inequality, we deduce that

$$E\left(\sup_{0 \leq t \leq T} |Y_t|^2\right) < \infty.$$

We can now prove the propostion. Indeed, if (Y, Z) is a solution, it follows from the result we have just proved that $E\int_t^T < Y_s, Z_s dB_s >= 0$, since the local martingale

$$\{\int_0^t < Y_s, Z_s dB_s > \quad, 0 \leq t \leq T\}$$

is a uniformly integrable martingale from the Burkholder–Davis–Gundy inequality and the fact that

$$2E\left[\left(\int_0^T |Y_t|^2 \|Z_t\|^2 dt\right)^{1/2}\right] \leq E\left(\sup_{0 \leq t \leq T} |Y_t|^2 + E\int_0^T \|Z_t\|^2 dt\right) < \infty.$$

Now from Itô's formula,

$$\begin{aligned} E|Y_t|^2 + E\int_t^T \|Z_s\|^2 ds &= E|\xi|^2 + 2E\int_t^T < Y_s, f(s, Y_s, Z_s) > ds \\ &\leq E|\xi|^2 + E\int_t^T |f(s,0,0)|^2 ds + cE\int_t^T |Y_s|^2 + 1/2E\int_t^T \|Z_s\|^2 ds, \end{aligned}$$

where the constant c depends only on T, μ and K. The result, but with the sup outside the expectation, follows from Gronwall's lemma. Then the result follows from the Burkholder–Davis–Gundy inequality. ♢

We now prove a first existence and uniqueness result, under an additional assumption. We reinforce conditions (iv), (v) and (vi) by assuming that f is uniformly Lischitz in y :

(iv') $|f(t, y, z) - f(t, y', z)| \leq K|y - y'|$, $\forall t, y, y', z$, a.s.

Theorem 1.2 *Under the assumptions (i), (ii), (iii) and (iv'), the BSDE (j), (jj) has a unique solution* (Y, Z).

Proof : Let $\mathcal{B}^2 := (M^2(0,T))^k \times (M^2(0,T))^{k\times d}$.

We now define a mapping Φ from $\mathcal{B}^2$ into itself such that $(Y, Z) \in \mathcal{B}^2$ is a solution of the BSDE (jj) iff it is a fixed point of Φ.

Given $(U, V) \in \mathcal{B}^2$, we define $(Y, Z) = \Phi(U, V)$ as follows :

$$Y_t = E[\xi + \int_t^T f(s, U_s, V_s)\, ds/\mathcal{F}_t],\ 0 \leq t \leq T,$$

and $\{Z_t\,;\, 0 \leq t \leq T\}$ is given by Itô's martingales representation theorem applied to the square integrable r.v.

$$\xi + \int_0^T f(s, U_s, V_s)\, ds,$$

i.e.

$$\xi + \int_0^T f(s, U_s, V_s)\, ds = E\left[\xi + \int_0^T f(s, U_s, V_s)\, ds\right] + \int_0^T Z_s\, dB_s.$$

Taking $E(\cdot/\mathcal{F}_t)$ of the last identity yields

$$Y_t + \int_0^t f(s, U_s, V_s)\, ds = Y_0 + \int_0^t Z_s\, dB_s,$$

from which we deduce that

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)\, ds - \int_t^T Z_s\, dB_s,$$

and we have shown that $(Y, Z) \in \mathcal{B}^2$ solves (jj) iff it is a fixed point of Φ.

Now it follows from the Davis–Burkholder–Gundy inequality that whenever $(Y,Z)=\Phi(U,V)$, $(U,V)\in\mathcal{B}^2$, then

$$E\left(\sup_{0\le t\le T}|Y_t|^2\right)<\infty.$$

Consequently, $\{\int_0^t(Y_s,Z_s\,dB_s),\ 0\le t\le T\}$ is a martingale, by the same argument as above.

Let (U,V), $(U',V')\in\mathcal{B}^2$, $(Y,Z)=\Phi(U,V)$, $(Y',Z')=\Phi(U',V')$, $(\overline{U},\overline{V})=(U-U',\ V-V')$, $(\overline{Y},\overline{Z})=(Y-Y',\ Z-Z')$. It follows from Itô's formula that for each $\gamma\in\mathbb{R}$,

$$\begin{aligned}
e^{\gamma t}E|\overline{Y}_t|^2&+E\int_t^T e^{\gamma s}(\gamma|\overline{Y}_s|^2+\|\overline{Z}_s\|^2)\,ds\\
&\le 2KE\int_t^T e^{\gamma s}|\overline{Y}_s|(|\overline{U}_s|+\|\overline{V}_s\|)\,ds\\
&\le 4K^2E\int_t^T e^{\gamma s}|\overline{Y}_s|^2\,ds+\frac{1}{2}E\int_t^T e^{\gamma s}(|\overline{U}_s|^2+\|\overline{V}_s\|^2)\,ds.
\end{aligned}$$

We choose $\gamma=1+4K^2$, hence

$$E\int_0^T e^{\gamma t}(|\overline{Y}_t|^2+\|\overline{Z}_t\|^2)\,dt\le\frac{1}{2}E\int_0^T e^{\gamma t}(|\overline{U}_t|^2+\|\overline{V}_t\|^2)\,ds,$$

from which it follows that Φ is a strict contraction on $\mathcal{B}^2$ equipped with the norm :

$$|||(Y,Z)|||_\gamma=\left(E\int_0^T e^{\gamma t}(|Y_t|^2+\|Z_t\|^2)\,ds\right)^{1/2},$$

if $\gamma=1+4K^2$. Then Φ has a unique fixed point, and the theorem is proved. ◇

We shall now prove existence and uniqueness for the BSDE (j), (jj) under the conditions (i), ..., (vi) on the coefficients.

Theorem 1.3 *Under the conditions (i), (ii), (iii), (iv), (v) and (vi), the BSDE (ξ,f) has a unique solution (satisfying (j) and (jj)).*

Proof : *Proof of uniqueness.* Let (Y,Z) and (Y',Z') be two solutions. It follows from Itô's formula that

$$\begin{aligned}
E|Y_t-Y_t'|^2\ &+\ E\int_t^T\|Z_s-Z_s'\|^2ds=2E\int_t^T<Y_s-Y_s',f(Y_s,Z_s)-f(Y_s',Z_s')>ds\\
&\le\ 2E\int_t^T\left[\mu|Y_s-Y_s'|^2+K|Y_s-Y_s'|\|Z_s-Z_s'\|\right]ds\\
&\le\ (2\mu+K^2)E\int_t^T|Y_s-Y_s'|^2ds+E\int_t^T\|Z_s-Z_s'\|^2ds.
\end{aligned}$$

Hence

$$E|Y_t - Y'_t|^2 \le (2\mu + K^2)E\int_t^T |Y_s - Y'_s|^2 ds,$$

and $E|Y_t - Y'_t|^2 = 0$, $0 \le t \le T$, follows from Gronwall's lemma, and then we have also that $E\int_0^T \|Z_t - Z'_t\|^2 dt = 0$.

Proof of existence. We first note that (Y, Z) solves the BSDE(ξ, f) iff

$$(\bar{Y}_t, \bar{Z}_t) := (e^{\lambda t} Y_t, e^{\lambda t} Z_t)$$

solve the BSDE$(e^{\lambda T}\xi, f')$, where

$$f'(t, y, z) := e^{\lambda t} f(t, e^{-\lambda t} y, e^{-\lambda t} z) - \lambda y.$$

If we choose $\lambda = \mu$, we have that f' satisfies the same assumptions as f, but with (v) replaced by

(v') $< y - y', f(t, y, z) - f(t, y', z) > \le 0$.

Hence we shall assume until the end of this proof that f satisfies (i), (ii), (iii), (iv), (v') and (vi).

Let us admit for a moment the

Proposition 1.4 *Given $V \in (M^2(0,T))^{k\times d}$, there exists a unique pair of progressively measurable processes $\{(Y_t, Z_t),\ 0 \le t \le T\}$ with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ satisfying :*

$$\begin{aligned} &E\int_0^T \|Z_t\|^2 dt < \infty \\ Y_t \;&=\; \xi + \int_t^T f(s, Y_s, V_s) ds - \int_t^T Z_s dB_s,\ 0 \le t \le T. \end{aligned}$$

Using proposition 1.4, we can construct a mapping Φ from $\mathcal{B}^2$ into itself as follows. For any $(U, V) \in \mathcal{B}^2$, $(Y, Z) = \Phi(U, V)$ is the solution of the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s) ds - \int_t^T Z_s\, dB_s,\ \ 0 \le t \le T.$$

Let (U, V), $(U', V') \in \mathcal{B}^2$, $(Y, Z) = \Phi(U, V)$ and $(Y', Z') = \Phi(U', V')$. We shall use the notations $(\bar{U}, \bar{V}) = (U - U', V - V')$, $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. It follows from Itô's formula that for each $\gamma \in \mathbb{R}$,

$$\begin{aligned} e^{\gamma t} E|\bar{Y}_t|^2 \;&+\; E\int_t^T e^{\gamma s}\left(\gamma|\bar{Y}_s|^2 + \|\bar{Z}_s\|^2\right) ds \\ &=\; 2E\int_t^T e^{\gamma s} < \bar{Y}_s, f(Y_s, V_s) - f(Y'_s, V'_s) > ds \\ &\le\; 2E\int_t^T e^{\gamma s}|\bar{Y}_s| \times \|\bar{V}_s\| ds \\ &\le\; E\int_t^T e^{\gamma s}\left(2|\bar{Y}_s|^2 + \frac{1}{2}\|\bar{V}_s\|^2\right) ds. \end{aligned}$$

Hence, if we choose $\gamma = 3$, we have that

$$\begin{aligned} E\int_0^T e^{3t}\left(|\bar{Y}_t|^2 + \|\bar{Z}_t\|^2\right)dt &\le \frac{1}{2}E\int_0^T e^{3t}\|\bar{V}_t\|^2 dt \\ &\le \frac{1}{2}E\int_0^T e^{3t}\left(|\bar{U}_t|^2 + \|\bar{V}_t\|^2\right)dt. \end{aligned}$$

Consequently, Φ is a strict contraction on $\mathcal{B}^2$ equipped with the norm

$$\|(Y,Z)\|_3 = \left[E\int_0^T e^{3t}\left(|Y_t|^2 + \|Z_t\|^2\right)dt\right]^{1/2},$$

and it has a unique fixed point, which is the unique solution of our BSDE.

Proof of proposition 1.4 : Uniqueness is proved as in theorem 1.3. We now prove existence. We shall write $f(s,y)$ for $f(s,y,V_s)$. Note that f satifies the following assumptions :

(ii') $|f(s,y)| \le \overline{f}_s + K(\|V_s\| + |y|)$;

(iii') $E\int_0^T |f(s,0)|^2 ds < \infty$;

(v") $< y - y', f(s,y) - f(s,y') > \le 0$;

(vi') $y \to f(s,y)$ is continuous, $\forall s$, a.s.

We first approximate f by $\bar{f}_n$, which satisfies (v") and (vi'), $|\bar{f}_n| \le |f| \wedge [\overline{f}_s + K\|V_s\| + Kn]$ and $\bar{f}_n(t,y) = f(t,y)$, $|y| \le n$, and then define

$$f_n(t,y) := (\rho_n * \bar{f}_n(t,\cdot))(y),$$

where $\rho_n : \mathbb{R}^k \to \mathbb{R}_+$ is a sequence of smooth functions which approximate the Dirac measure at 0. f_n satisfies again (ii'), (iii'), and (v") with the same constant K.

For each n, f_n is Lipschitz in y, uniformly with respect to s and ω, hence the BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n)ds - \int_t^T Z_s^n dB_s$$

has a unique solution (Y^n, Z^n) in the sense of theorem 1.2, which satisfies moreover

$$\begin{aligned} |Y_t^n|^2 + \int_t^T \|Z_s^n\|^2 ds &= |\xi|^2 + 2\int_t^T < Y_s^n, f_n(Y_s^n) > ds - 2\int_t^T < Y_s^n, Z_s^n dB_s > \\ E|Y_t^n|^2 + E\int_t^T \|Z_s^n\|^2 ds &\le E|\xi|^2 + CE\int_t^T (1 + |Y_s^n|^2)ds. \end{aligned}$$

It then follows from standard estimates that :

$$\sup_n E\left(\sup_{0\le t\le T}|Y_t^n|^2 + \int_0^T \|Z_t^n\|^2 dt\right) < \infty.$$

Let $U_t^n := f_n(t, Y_t^n)$. It follows from the last estimate and (ii'), (iii') that

$$\sup_n E \int_0^T |U_t^n|^2 dt < \infty.$$

Hence there exists a subsequence, which we still denote (Y^n, Z^n, U^n), and which converges weakly in the space $L^2(\Omega \times (0,T), dP \times dt; \mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k)$ to a limit (Y, Z, U). We first note that the stochastic integral term converges weakly in $L^2(\Omega)$. Indeed, let $\eta \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^k)$, which can be written as $\eta = E(\eta) + \int_0^T \varphi_t dB_t$. Then

$$\begin{aligned} E < \eta, \int_t^T Z_s^n dB_s > &= E \int_t^T Tr(Z_s^n \varphi_s^*) ds \\ &\to E \int_t^T Tr(Z_s \varphi_s^*) ds \\ &= E < \eta, \int_t^T Z_s dB_s >, \end{aligned}$$

and it is not hard to conclude that $\int_\cdot^T Z_s^n dB_s \to \int_\cdot^T Z_s dB_s$ in $L^2(\Omega \times (0,T), dP \times dt; \mathbb{R}^k)$ weakly. Taking weak limits in the approximating equation yields :

$$Y_t = \xi + \int_t^T U_s ds - \int_t^T Z_s dB_s,\ 0 \le t \le T.$$

It remains to show that $U_t = f(t, Y_t)$. Let $X \in (M^2(0,T))^k$. From (v") for f_n, and the fact that $f_n(\cdot, X)$ converges in mean square to $f(\cdot, X)$, we deduce that

$$\limsup_{n\to\infty} E \int_0^T < Y_t^n - X_t, f_n(t, Y_t^n) - f(t, X_t) > dt \le 0.$$

Moreover

$$2E \int_0^T < Y_t^n, f_n(t, Y_t^n) > dt = |Y_0^n|^2 - E|\xi|^2 + E \int_0^T \|Z_t^n\|^2 dt.$$

But Y_0^n converges in $\mathbb{R}^k$ to Y_0, and since the mapping

$$Z \to E \int_0^T \|Z_t\|^2 dt$$

is convex and continuous for the strong topology of $L^2(\Omega \times (0,T), dP \times dt, \mathbb{R}^{k\times d})$, it is weakly l.s.c., and consequently

$$\begin{aligned} \liminf_{n\to\infty} 2E \int_0^T < Y_t^n, f_n(t, Y_t^n) > dt &\ge |Y_0|^2 - E|\xi|^2 + E \int_0^T \|Z_t\|^2 dt \\ &= 2E \int_t^T < Y_t, U_t > dt. \end{aligned}$$

Combining this with weak convergence and the previous inequality, we deduce that

$$
\begin{aligned}
& E\int_0^T < Y_t - X_t, U_t - f(t,X_t) > dt \\
& \leq \liminf E\int_0^T < Y_t^n - X_t, f_n(t,Y_t^n) - f(t,X_t) > dt \\
& \leq 0.
\end{aligned}
$$

We finally choose $X_t = Y_t - \varepsilon(U_t - f(t,X_t))$, with $\varepsilon > 0$, divide the resulting inequality by ε and let ε tend to 0. We obtain that :

$$
E\int_0^T |U_t - f(t,Y_t)|^2 dt \leq 0,
$$

which concludes the proof of the proposition. ♢

We now want to estimate the difference between two solutions in terms of the difference between the data. Given two final conditions $\xi, \xi' \in L^2(\Omega, \mathcal{F}_T, P)$, and two coefficients f, f' both satisfying conditions (i), (ii), (iii), (iv), (v) and (vi), let $\{(Y_t, Z_t)\,;\, 0 \leq t \leq T\}$ (resp. $\{(Y_t', Z_t')\,;\, 0 \leq t \leq T\}$) be the solution of the BSDE (ξ, f) (resp. of the BSDE (ξ', f')). We have the

Theorem 1.5 *There exists a constant c, which depends upon the Lipschitz and monotonicity constants of f', such that*

$$
E\left(\sup_{0\leq t\leq T} |Y_t - Y_t'|^2 + \int_0^T \|Z_t - Z_t'\|^2\, dt\right) \leq c\left(E|\xi - \xi'|^2 + E\int_0^T |f(Y_s, Z_s) - f'(Y_s, Z_s)|^2\, ds\right).
$$

Proof : We use Itô's formula to develop the increment of $|Y_s - Y_s'|^2$ between $s = t$ and $s = T$, yielding :

$$
\begin{aligned}
& |Y_t - Y_t'|^2 + \int_t^T \|Z_s - Z_s'\|^2\, ds = |\xi - \xi'|^2 \\
& +2\int_t^T < Y_s - Y_s',\, f(s,Y_s,Z_s) - f'(s,Y_s',Z_s') >\, ds \\
& -2\int_t^T < Y_s - Y_s',\, (Z_s - Z_s')\, dB_s > .
\end{aligned}
$$

We note that

$$
\begin{aligned}
& < Y_s - Y_s', f(Y_s,Z_s) - f'(Y_s',Z_s') > \\
& \quad \leq |Y_s - Y_s'| \times (|f(Y_s,Z_s) - f'(Y_s,Z_s)| + K'\|Z_s - Z_s'\|) + \mu'|Y_s - Y_s'|^2,
\end{aligned}
$$

where K' and μ' are respectively the Lipschitz and the monotonicity constant of f'. Hence taking the expectation in the above identity, we deduce that

$$
\begin{aligned}
&E|Y_t - Y'_t|^2 + \frac{1}{2}E\int_t^T \|Z_s - Z'_s\|^2\, ds \leq E|\xi - \xi'|^2 \\
&+E\int_0^T |f(s, Y_s, Z_s) - f'(s, Y_s, Z_s)|^2 ds + (1 + 2\mu' + 2K'^2)]E\int_t^T |Y_s - Y'_s|^2\, ds.
\end{aligned}
$$

The result, but with the sup outside the expectation, now follows from Gronwall's lemma. We can then conclude using this result, the first identity in this proof and the Burkholder–Davis–Gundy inequality. $\diamondsuit$

We continue with the same set–up, restricting now ourselves to the case $k = 1$, and prove a comparison theorem.

Theorem 1.6 *Suppose that $k = 1$, $\xi \leq \xi'$ a.s., and $f(t, y, z) \leq f'(t, y, z)$ $dt \times dP$ a.e. Then $Y_t \leq Y'_t$, $0 \leq t \leq T$, a.s.*

If moreover $Y_0 = Y'_0$, then $Y_t = Y'_t$, $0 \leq t \leq T$, a.s. In particular, whenever moreover either $P(\xi < \xi') > 0$ or $f(t, y, z) < f'(t, y, z)$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, on a set of positive $dt \times dP$ measure, then $Y_0 < Y'_0$.

Proof : Define

$$
\alpha_t = \begin{cases} (Y'_t - Y_t)^{-1}(f(t, Y'_t, Z_t) - f(t, Y_t, Z_t)) & \text{if } Y_t \neq Y'_t; \\ 0 & \text{if } Y_t = Y'_t; \end{cases}
$$

and the $\mathbb{R}^d$–valued process $\{\beta_t;\ 0 \leq t \leq T\}$ as follows. For $1 \leq i \leq d$, let $Z_t^{(i)}$ denote the d–dimensional vector whose i first components are equal to those of Z'_t, and whose $d - i$ last components are equal to those of Z_t. With this notation, we define for each $1 \leq i \leq d$,

$$
\beta_t^i = \begin{cases} (Z_t'^i - Z_t^i)^{-1}(f(t, Y_t, Z_t^{(i)}) - f(t, Y_t, Z_t^{(i-1)})) & \text{if } Z_t^i \neq Z_t'^i; \\ 0 & \text{if } Z_t^i = Z_t'^i. \end{cases}
$$

We note that $\{\alpha_t;\ 0 \leq t \leq T\}$ and $\{\beta_t;\ 0 \leq t \leq T\}$ are progressively measurable, $\alpha_t \leq \mu$ and $|\beta| \leq K$.

For $0 \leq s \leq t \leq T$, let

$$
\Gamma_{s,t} = \exp[\int_s^t (\alpha_r - 1/2|\beta_r|^2)\, dr + \int_s^t < \beta_r, dB_r >].
$$

Define $(\overline{Y}_t, \overline{Z}_t) = (Y'_t - Y_t, Z'_t - Z_t)$, $\bar{\xi} = \xi' - \xi$, $U_t = f'(t, Y'_t, Z'_t) - f(t, Y'_t, Z'_t)$.

Then $(\overline{Y}, \overline{Z})$ solves the linear BSDE

$$
\overline{Y}_t = \bar{\xi} + \int_t^T (\alpha_s \overline{Y}_s + < \beta_s, \overline{Z}_s >)\, ds + \int_t^T U_s\, ds - \int_t^T \overline{Z}_s dB_s.
$$

It is not hard to see that for $0 \le s \le t \le T$,

$$\begin{aligned}\overline{Y}_s &= \Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r} U_r \, dr - \int_s^t \Gamma_{s,r}(\overline{Z}_r + \overline{Y}_r \beta_r) \, dB_r \\ \overline{Y}_s &= E(\Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r} U_r \, dr / \mathcal{F}_s).\end{aligned}$$

The result follows from this formula and the positivity of $\bar{\xi}$ and U.

Remark 1.7 *Suppose that*

$$\begin{aligned}Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s \\ Y'_t &= \xi' + \int_t^T V_s \, ds - \int_t^T Z'_s dB_s,\end{aligned}$$

and $\xi \le \xi'$, $f(t, Y'_t, Z'_t) \le V_t$. *Then we can apply theorem 1.6, defining*

$$f'(t, y, z) = f(t, y, z) + (V_t - f(t, Y'_t, Z'_t)).$$

If moreover $f(t, Y'_t, Z'_t) < V_t$ *on a set of* $dt \times dP$ *positive measure, then* $Y_0 < Y'_0$.

Proposition 1.8 *Let* $\{(Y_t, Z_t);\ 0 \le t \le T\}$ *be the solution of the BSDE (j), (jj). Assume that for some stopping time* $\tau \le T$,

(a) ξ *is* $\mathcal{F}_\tau$*–measurable;*

(b) $f(t, y, z) = 0$ *on the interval* (τ, T).

Then $Y_t = Y_{t \wedge \tau}$, *and* $Z_t = 0$ *on the interval* (τ, T).

Proof : Since

$$\begin{aligned}Y_\tau &= \xi - \int_\tau^T Z_s dB_s, \\ Y_\tau &= E(\xi / \mathcal{F}_\tau) \\ &= \xi.\end{aligned}$$

On the other hand,

$$|Y_\tau|^2 + \int_\tau^T \|Z_s\|^2 ds = |\xi|^2 - 2 \int_\tau^T < Y_s, Z_s dB_s > .$$

Hence

$$|Y_\tau|^2 + E^{\mathcal{F}_\tau} \int_\tau^T \|Z_s\|^2 ds = |\xi|^2.$$

Consequently $\int_\tau^T \|Z_s\|^2 ds = 0$ a.s.

Remark 1.9 *In particular, if ξ and $f(t,y,z)$ are deterministic, then $Z_t \equiv 0$, and $\{Y_t\}$ is the solution of the ODE*

$$\frac{dY_t}{dt} = -f(t, Y_t, 0); \quad Y_T = \xi.$$

What makes the solution of a BSDE random is the randomness of the final condition and of the coefficient. The role of the stochastic integral term $\int_t^T Z_s dB_s$ is to make the process $\{Y_t\}$ adapted, i.e. to reduce its randomness. Whenever Z is not necessary to make Y adapted, then it is equal to zero.

2 BSDE's and systems of semilinear parabolic PDE's

We need to put our BSDE is a Markovian framework : ξ and f will be functionals of B as "explicit" functions of the solution of a forward SDE driven by $\{B_t\}$.

Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be measurable functions which are globally Lipschitz in x uniformly with respect to t, and locally bounded. Let $\{X_s^{t,x}\,;\, t \le s \le T\}$ denote the solution of the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\, dr + \int_t^s \sigma(r, X_r^{t,x}) dB_r,\ t \le s \le T\,; \tag{2.1}$$

and consider the backward SDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x},\, Z_r^{t,x})\, dr - \int_s^T Z_r^{t,x}\, dB_r,\ t \le s \le T, \tag{2.2}$$

where $g : \mathbb{R}^d \to \mathbb{R}^k$ and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ are continuous and such that for some K, μ, $p > 0$,

$$\begin{aligned}
|g(x)| &\le K(1 + |x|^p), \\
|f(t,x,y,z)| &\le K(1 + |x|^p + |y| + |z|), \\
< y - y', f(t,x,y,z) - f(t,x,y',z) > &\le \mu |y - y'|^2, \\
|f(t,x,y,z) - f(t,x,y,z')| &\le K\|z - z'\|.
\end{aligned}$$

Remark 2.1 *(i) Clearly, for each $t \le s \le T$, $Y_s^{t,x}$ is $\mathcal{F}_s^t = \sigma\{B_r - B_t,\, t \le r \le s\} \vee \mathcal{N}$ measurable, where $\mathcal{N}$ is the class of the P–null sets of $\mathcal{F}$. Hence $Y_t^{t,x}$ is a.s. constant (i.e. deterministic).*

(ii) It is not hard to see, using uniqueness for BSDEs, that $Y_{t+h}^{t,x} = Y_{t+h}^{t+h, X_{t+h}^{t,x}}$, $h > 0$.

Denote by

$$L_t = \frac{1}{2}\sum_{i,j} (\sigma\sigma^*)_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(t,x) \frac{\partial}{\partial x_i}$$

the infinitesimal generator of the Markov process $\{X_s^{t,x}\,;\, t \le s \le T\}$, and consider the following system of backward semilinear parabolic PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial u_i}{\partial t}(t,x) + L_t u_i(t,x) + f_i(t,x,u(t,x),(\nabla u\sigma)(t,x)) & = & 0, \\ (t,x) \in [0,T] \times \mathbb{R}^d,\ 0 \le i \le k; \qquad u(T,x) & = & g(x),\ x \in \mathbb{R}^d. \end{array} \right. \tag{2.3}$$

We can first establish the :

Theorem 2.2 *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^d; \mathbb{R}^k)$ be a classical solution of (2.3), such that for some $c, q > 0$,*

$$|\nabla_x u(t,x)| \le C(1 + |x|^q). \tag{2.4}$$

Then for each $(t,x) \in [0,T] \times \mathbb{R}^d$, $\{(u(s,X_s^{t,x}),(\nabla u\sigma)(s,X_s^{t,x})); t \le s \le T\}$ is the solution of the BSDE (2.2). In particular, $u(t,x) = Y_t^{t,x}$.

Proof : The result follows by applying Itô's formula to $u(s,X_s^{t,x})$, and the fact that (2.4) implies that

$$E\int_t^T \|(\nabla_x u\sigma)(s,X_s^{t,x})\|^2 ds < \infty.$$

$\diamondsuit$

We want now to connect (2.1)–(2.2) with (2.3) in the other direction, i.e. prove that (2.1)–(2.2) provides a solution of (2.3). In order to avoid restrictive assumptions on the coefficients in (2.1)–(2.2), we will consider (2.3) in the viscosity sense. This imposes us one restriction. Indeed for the notion of viscosity solution of the system of PDEs (2.3) to make sense, we need to make the following restriction : *for $0 \le i \le k$, the i–th coordinate of f depends only on the i–th row of the matrix z.* Then the first line in (2.3) reads

$$\frac{\partial u_i}{\partial t}(t,x) + L_t u_i(t,x) + f_i(t,x,u(t,x),(\nabla u_i\sigma)(t,x)) = 0$$

We now define the notion of viscosity solution of (2.3).

Definition 2.3 **a** *$u \in C([0,T]\times\mathbb{R}^d;\mathbb{R}^k)$ is called a viscosity subsolution of (2.3) if $u_i(T,x) \le g_i(x)$, $x \in \mathbb{R}^d$, $0 \le i \le k$, and moreover for any $1 \le i \le k$, $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and $(t,x) \in [0,T) \times \mathbb{R}^d$ which is a local maximum of $u_i - \varphi$,*

$$-\frac{\partial \varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),\,(\nabla\varphi\sigma)(t,x)) \le 0.$$

b *$u \in C([0,T]\times\mathbb{R}^d;\mathbb{R}^k)$ is called a viscosity supersolution of (2.3) if $u_i(T,x) \ge g_i(x)$, $x \in \mathbb{R}^d$, $0 \le i \le k$, and moreover for any $1 \le i \le k$, $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and $(t,x) \in [0,T)\times\mathbb{R}^d$ which is a local minimum of $u_i - \varphi$,*

$$-\frac{\partial \varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),\,(\nabla\varphi\sigma)(t,x)) \ge 0.$$

c $u \in C([0,T]\times \mathbb{R}^d; \mathbb{R}^k)$ *is called a viscosity solution of (2.3) if it is both a viscosity sub– and supersolution.*

We now establish the main result of this section.

Theorem 2.4 *Under the above assumptions,* $u(t,x) \triangleq Y_t^{t,x}$ *is a continuous function of* (t,x) *which grows at most polynomially at infinity, and it is a viscosity solution of (2.3).*

Proof : The continuity follows from the mean–square continuity of $\{Y_s^{t,x},\ x \in \mathbb{R}^d,\ 0 \leq t \leq s \leq T\}$, which is turn follows from the continuity of $X_\cdot^{t,x}$ with respect to t,x and theorem 1.5. The polynomial growth follows from classical moment estimates for $X_\cdot^{t,x}$, the assumptions on the growth of f and g, and proposition 1.1.

To prove that u is a viscosity subsolution, take any $1 \leq i \leq k$, $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and $(t,x) \in [0,T) \times \mathbb{R}^d$ such that (t,x) is a point of local maximum of $u_i - \varphi$. We assume w.l.o.g. that

$$u_i(t,x) = \varphi(t,x).$$

We suppose that

$$\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(t,x) + f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) < 0,$$

and we will find a contradiction.

Let $0 < \alpha \leq T - t$ be such that for all $t \leq s \leq t+\alpha$, $|y-x| \leq \alpha$,

$$\begin{aligned} u_i(s,y) &\leq \varphi(s,y), \\ \left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(s,y) + f_i(s,y,u(s,y),(\nabla\varphi\sigma)(s,y)) &< 0, \end{aligned}$$

and define

$$\tau = \inf\{s \geq t;\ |X_s^{t,x} - x| \geq \alpha\} \wedge (t+\alpha).$$

Let now

$$(\overline{Y}_s, \overline{Z}_s) = ((Y_{s\wedge\tau}^{t,x})^i, \mathbf{1}_{[0,\tau]}(s)(Z_s^{t,x})^i),\ t \leq s \leq t+\alpha.$$

$(\overline{Y}, \overline{Z})$ solves the one–dimensional BSDE

$$\overline{Y}_s = u_i(\tau, X_\tau^{t,x}) + \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) f_i(r, X_r^{t,x}, u(r, X_r^{t,x}), \overline{Z}_r)dr - \int_s^{t+\alpha} \overline{Z}_r dB_r,\ t \leq s \leq t+\alpha.$$

On the other hand, from Itô's formula,

$$(\widehat{Y}_s, \widehat{Z}_s) = (\varphi(s, X_{s\wedge\tau}^{t,x}), \mathbf{1}_{[0,\tau]}(s)(\nabla\varphi\sigma)(s, X_s^{t,x})),\ t \leq s \leq t+\alpha$$

solves the BSDE

$$\widehat{Y}_s = \varphi(\tau, X_\tau^{t,x}) - \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r)(\frac{\partial\varphi}{\partial r} + L\varphi)(r, X_r^{t,x})dr - \int_s^{t+\alpha} \widehat{Z}_r dB_r,\ t \leq s \leq t+\alpha.$$

From $u_i \leq \varphi$, and the choice of α and τ, we deduce with the help of the comparison theorem 1.6 (see remark 1.7) that $\overline{Y}_0 < \widehat{Y}_0$, i.e. $u_i(x) < \varphi(x)$, which contradicts our assumptions.

Remark 2.5 *Suppose that $k = 1$ and f has the special form :*

$$f(t, x, r, z) = c(t, x)r + h(t, x).$$

In that case, the BSDE is linear :

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T [c(r, X_r^{t,x})Y_s^{t,x} + h(r, X_r^{t,x})]\, dr - \int_s^T Z_r^{t,x}\, dB_r,$$

hence it has an explicit solution, from an extension of the classical "variation of constants formula"(see the argument in the proof of theorem 1.6) :

$$\begin{aligned} Y_s^{t,x} &= g(X_T^{t,x})e^{\int_s^T c(r, X_r^{t,x})\, dr} + \int_s^T h(r, X_r^{t,x})e^{\int_s^r c(\alpha, X_\alpha^{t,x})\, d\alpha}\, dr \\ &\quad - \int_s^T e^{\int_s^r c(\alpha, X_\alpha^{t,x}) d\alpha} Z_r^{t,x}\, dB_r. \end{aligned}$$

Now $Y_t^{t,x} = E(Y_t^{t,x})$, so that

$$Y_t^{t,x} = E\left[g(X_T^{t,x})e^{\int_t^T c(s, X_s^{t,x})\, ds} + \int_t^T h(s, X_s^{t,x})e^{\int_t^s c(r, X_r^{t,x}) dr} ds \right],$$

which is the well–known Feynman–Kac formula.

Clearly, theorem 2.4 can be considered as a nonlinear extension of the Feynman–Kac formula.

Remark 2.6 *We have proved that a certain function of (t, x), defined through the solution of a probabilistic problem, is the solution of a system of backward parabolic partial differential equations. Suppose that b, σ and f do not depend on t, and let*

$$v(t, x) = u(T - t, x),\ (t, x) \in [0, T] \times \mathbb{R}^d$$

The v solves the system of forward parabolic PDEs :

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) &= Lv(t, x) + f(x, v, (t, x), (\nabla v \sigma)(t, x)),\ t > 0, x \in \mathbb{R}^d; \\ v(0, x) &= h(x),\ x \in \bar{G}. \end{aligned}$$

On the other hand, we have that

$$v(t, x) = Y_{T-t}^{T-t,x} = \bar{Y}_0^{t,x},$$

where $\{(\bar{Y}_s^{t,x}, \bar{Z}_s^{t,x});\, 0 \le s \le t\}$, solves the BSDE

$$\begin{aligned} \bar{Y}_s^{t,x} &= g(X_t^x) + \int_s^t f(X_r^x, \bar{Y}_r^{t,x}, \bar{Z}_r^{t,x}) dr \\ &\quad - \int_s^t \bar{Z}_r^{t,x} dB_r, 0 \le s \le t. \end{aligned}$$

So we have a probabilistic repesentation for a system of forward parabolic PDEs, which is valid on $\mathbb{R}_+ \times \mathbb{R}^d$.

3 BSDE's with random terminal time

We are given :

(a) a final time τ, which is an $\mathcal{F}_t$–stopping time;

(b) a coefficient $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$, which is such that for some $\mathbb{R}_+$–valued progressively measurable process $\{\overline{f}_t\}$, and real numbers μ, λ, K, K' such that $K, K' > 0$ and $2\mu + K^2 < \lambda$:

(3.i) $f(\cdot, y, z)$ is progressively measurable, $\forall\, y, z$;

(3.ii) $|f(t,y,z) - f(t,y,z')| \le K\|z - z'\|$, $\forall t, y, z, z'$, a.s., where $\|z\| = [Tr(zz^*)]^{1/2}$.;

(3.iii) $< y - y', f(t,y,z) - f(t,y',z) > \le \mu|y - y'|^2$, $\forall t, y, y', z$, a.s.;

(3.iv) $|f(t,y,z)| \le \overline{f}_t + K'(|y| + \|z\|)$, $\forall t, y, z$, a.s.;

(3.v) $E\int_0^\tau e^{\lambda t}\,\overline{f}_t^2\,dt < \infty$

(3.vi) $y \to f(t,y,z)$ is continuous, $\forall t, z$, a.s.;

(c) a final condition ξ which is an $\mathcal{F}_\tau$–measurable and k–dimensional r.v. such that $E(e^{\lambda\tau}|\xi|^2) < \infty$, $\xi = 0$ on the set $\{\tau = \infty\}$, and

$$E\int_0^\tau e^{\lambda t}|f(t,\xi_t,\eta_t)|^2\,dt < \infty,$$

where $\xi_t = E(\xi/\mathcal{F}_t)$ and $\eta \in (M^2(\mathbb{R}_+))^{k\times d}$ is such that $\xi = E(\xi) + \int_0^\infty \eta_t dB_t$.

A solution of the BSDE (τ, ξ, f) is a pair $\{(Y_t, Z_t)\,;\, t \ge 0\}$ of progressively measurable processes with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ s.t. $Z_t = 0$, $t > \tau$ and

(3.j) $E\int_0^\tau e^{\lambda t}\|Z_t\|^2\,dt < \infty$,

(3.jj) $Y_t = Y_T + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y_s, Z_s)\,ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s\,dB_s$, for all t, T s.t. $0 \le t \le T$,

(3.jjj) $Y_t = \xi$, on the set $\{t \ge \tau\}$.

Remark 3.1 *Intuitively, we are solving the BSDE*

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\,ds - \int_{t\wedge\tau}^{\tau} Z_s dB_s,\ t \ge 0,$$

but the integrals here may not make sense on the set $\{\tau = +\infty\}$.

We have assumed that $\xi = 0$ on the set $\{\tau = +\infty\}$, but in fact the value of ξ on that set is irrelevant. Our set–up contains the case $\tau \equiv +\infty$ as a particular case, in which the condition (3.jjj) drops out.

Theorem 3.2 *Under the above conditions, there exists a unique solution* (Y,Z) *of the BSDE* (τ,ξ,f) *, which satisfies moreover, for any* $\lambda > 2\mu + K^2$*,*

$$E\left(\sup_{0\le t\le\tau} e^{\lambda t}|Y_t|^2 + \int_0^\tau e^{\lambda t}|Y_t|^2\,dt + \int_0^\tau e^{\lambda t}\|Z_t\|^2\,dt\right) < cE\left(e^{\lambda\tau}|\xi|^2 + \int_0^\tau e^{\lambda t}|f(t,\xi_t,\eta_t)|^2 dt\right). \tag{3.1}$$

Proof : *Proof of uniqueness.* Let (Y,Z) and (Y',Z') be two solutions, which satisfy (3.1), and let $(\bar{Y},\bar{Z}) = (Y-Y', Z-Z')$. It follows from Itô's formula, and the assumptions (3.iii) and (3.iv) that

$$\begin{aligned} & e^{\lambda t\wedge\tau}|\bar{Y}_t|^2 + \int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s}(\lambda|\bar{Y}_s|^2 + \|\bar{Z}_s\|^2)ds \\ \le\, & e^{\lambda T\wedge\tau}|\bar{Y}_T|^2 + 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda t}(\mu|\bar{Y}_s|^2 + K|\bar{Y}_s|\times\|\bar{Z}_s\|)ds - 2\int_{t\wedge\tau}^{T\wedge\tau} e^{\lambda s} < \bar{Y}_s, \bar{Z}_s dB_s > . \end{aligned}$$

Combining the above inequality with

$$2K|\bar{Y}_s|\times\|\bar{Z}_s\| \le \|\bar{Z}_s\|^2 + K^2|\bar{Y}_s|^2,$$

we deduce, since $\lambda > 2\mu + K^2$, that for $t < T$,

$$E\left(e^{\lambda t\wedge\tau}|\bar{Y}_t|^2\right) \le E\left(e^{\lambda T\wedge\tau}|\bar{Y}_T|^2\right).$$

The same result holds with λ replaced by λ', with $2\mu + K^2 < \lambda' < \lambda$. Hence

$$E\left(e^{\lambda' t\wedge\tau}|\bar{Y}_t|^2\right) \le e^{(\lambda'-\lambda)T} E\left(e^{\lambda T\wedge\tau}|\bar{Y}_T|^2 \mathbf{1}_{\{T<\tau\}}\right).$$

The condition (3.1) implies that the second factor of the right hand side remains bounded as $T\to\infty$, while the first factor tends to 0 as $T\to\infty$. Uniqueness is proved.

Proof of existence. For each $n\in\mathbb{N}$, we construct a solution $\{(Y_t^n, Z_t^n);\, t\ge 0\}$ of the BSDE

$$Y_t^n = \xi + \int_{t\wedge\tau}^{n\wedge\tau} f(s,Y_s^n,Z_s^n)ds - \int_{t\wedge\tau}^{\tau} Z_s^n dB_s,\ t\ge 0,$$

as follows. $\{(Y_t^n, Z_t^n);\, 0\le t\le n\}$ is defined as the solution of the following BSDE on the fixed intervall $[0,n]$:

$$Y_t^n = E(\xi|\mathcal{F}_n) + \int_t^n \mathbf{1}_{[0,\tau]}(s) f(s,Y_s^n,Z_s^n)ds - \int_t^n Z_s^n dB_s,\ 0\le t\le n,$$

$\{(Y_t^n, Z_t^n);\, t\ge n\}$ is defined by

$$Y_t^n = \xi_t, \quad Z_t^n = \eta_t, \quad t > n.$$

We note that from proposition 1.8, $Z_t^n = 0$ on the set $t > \tau$.

We will first establish an a–priori estimate for the sequence (Y^n, Z^n). For that sake, we will use the fact that for any arbitrarily small $\varepsilon > 0$, and any $\rho < 1$ arbitrarily close to one, for all $t \geq 0$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{d\times k}$, if $c = \varepsilon^{-1}$,

$$2 < y, f(t, y, z) > \leq (2\mu + \rho^{-1}K^2 + \varepsilon)|y|^2 + \rho\|z\|^2 + c|f(t, 0, 0)|^2.$$

From these and Itô's formula, we deduce that

$$\begin{aligned} e^{\lambda t\wedge\tau}|Y^n_{t\wedge\tau}|^2 \quad &+ \quad \int_{t\wedge\tau}^{\tau} e^{\lambda s}(\bar\lambda|Y^n_s|^2 + \bar\rho\|Z^n_s\|^2)\,ds \\ &\leq \quad e^{\lambda\tau}|\xi|^2 + c\int_{t\wedge\tau}^{\tau} e^{\lambda s}|f(s, 0, 0)|^2\,ds - 2\int_{t\wedge\tau}^{\tau} e^{\lambda s} < Y^n_s, Z^n_s dB_s >, \end{aligned}$$

with $\bar\lambda = \lambda - 2\mu - \rho^{-1}K^2 - \varepsilon > 0$ and $\bar\rho = 1 - \rho > 0$. It then follows from Burkholder's inequality

$$\begin{aligned} E\left(\sup_{t\geq s} e^{\lambda t\wedge\tau}|Y^n_{t\wedge\tau}|^2 + \int_{s\wedge\tau}^{\tau} e^{\lambda r}(|Y^n_r|^2 + \|Z^n_r\|^2)\,dr\right) \\ \leq E(e^{\lambda\tau}|\xi|^2) + cE\int_{s\wedge\tau}^{\tau} e^{\lambda r}|f(r, 0, 0)|^2\,dr. \end{aligned}$$

Let now $m > n$, and define $\Delta Y_t = Y^m_t - Y^n_t$, $\Delta Z_t = Z^m_t - Z^n_t$. We first have that for $n \leq t \leq m$,

$$\Delta Y_t = \int_{t\wedge\tau}^{m\wedge\tau} f(s, Y^m_s, Z^m_s)\,ds - \int_{t\wedge\tau}^{m\wedge\tau} \Delta Z_s dB_s.$$

Consequently, for $n \leq t \leq m$,

$$\begin{aligned} e^{\lambda t}|\Delta Y_t|^2 + \int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}(\lambda|\Delta Y_s|^2 + \|\Delta Z_s\|^2)\,ds \\ = 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s} < \Delta Y_s, f(s, Y^m_s, Z^m_s) > \,ds - 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s} < \Delta Y_s, \Delta Z_s dB_s > \\ \leq 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}(\mu|\Delta Y_s|^2 + K|\Delta Y_s| \times \|\Delta Z_s\|)\,ds + 2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s}|\Delta Y_s| \times |f(s, \xi_s, \eta_s)|\,ds \\ -2\int_{t\wedge\tau}^{m\wedge\tau} e^{\lambda s} < \Delta Y_s, \Delta Z_s dB_s > . \end{aligned}$$

We then deduce, by an argument already used, that

$$\begin{aligned} E\left(\sup_{n\leq t\leq m} e^{\lambda t}|\Delta Y_t|^2 + \int_{n\wedge\tau}^{m\wedge\tau} e^{\lambda s}(|\Delta Y_s|^2 + \|\Delta Z_s\|^2)\,ds\right) \\ \leq cE\int_{n\wedge\tau}^{\tau} e^{\lambda s}|f(s, \xi_s, \eta_s)|^2\,ds, \end{aligned}$$

and this last term tends to zero, as $n \to \infty$. Next, for $t \leq n$,

$$\Delta Y_t = \Delta Y_n + \int_{t\wedge\tau}^{n\wedge\tau} (f(s, Y^m_s, Z^m_s) - f(s, Y^n_s, Z^n_s))\,ds - \int_{t\wedge\tau}^{n\wedge\tau} \Delta Z_s dB_s.$$

It follows from the same argument as in the proof of uniqueness that

$$\begin{aligned} E\left(e^{\lambda t\wedge\tau}|\Delta Y_t|^2\right) &\leq E\left(e^{\lambda n\wedge\tau}|\Delta Y_n|^2\right) \\ &\leq cE\int_{n\wedge\tau}^{\tau} e^{\lambda s}|f(s,\xi_s,\eta_s)|^2\,ds. \end{aligned}$$

It is then easy to show that the sequence (Y^n, Z^n) is Cauchy for the norm whose square appears on the left side of (3.1), and that the limit (Y, Z) is a solution of the BSDE (τ, ξ, f) which satisfies (3.1). The proof is complete.

4 BSDE's and semilinear elliptic PDE's

We will first consider elliptic PDEs in $\mathbb{R}^d$, and then in a bounded open subset of $\mathbb{R}^d$, with Dirichlet boundary condition.

Let $\{X_t^x;\, t \geq 0\}$ denote the solution of the forward SDE :

$$X_t^x = x + \int_0^t b(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dB_s,\ t \geq 0, \tag{4.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are globally Lipschitz coefficients, and consider the backward SDE

$$Y_t^x = Y_T^x + \int_t^T f(X_s^x, Y_s^x, Z_s^x)\,ds - \int_t^T Z_s^x\,dB_s,\ \text{for all } t, T \text{ s.t. } 0 \leq t \leq T, \tag{4.2}$$

where $f : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ is continuous and such that for some K, K', $\mu < 0$, $p > 0$,

$$\begin{aligned} |f(x,y,z)| &\leq K'(1 + |x|^p + |y| + |z|), \\ < y - y', f(x,y,z) - f(x,y',z) > &\leq \mu|y - y'|^2, \\ |f(x,y,z) - f(x,y,z')| &\leq K\|z - z'\|. \end{aligned} \tag{4.3}$$

We assume moreover that for some $\lambda > 2\mu + K^2$, and all $x \in \mathbb{R}^d$,

$$E\int_0^\infty e^{\lambda t}|f(X_t^x, 0, 0)|^2 dt < \infty, \tag{4.4}$$

which essentially implies that $\lambda < 0$.

Under these assumptions, the BSDE (4.2) has a unique solution, in the sense of theorem 3.2.

It is not hard to see, using uniqueness for BSDEs, that

$$Y_t^x = Y_0^{X_t^x}, \quad t > 0. \tag{4.5}$$

Denote by

$$L = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i}$$

the infinitesimal generator of the Markov process $\{X_t^x\,;\,t\geq 0\}$, and consider the following system of backward semilinear elliptic PDEs in $\mathbb{R}^d$

$$Lu_i(x)+f_i(x,u(x),(\nabla u\sigma)(x))=0, x\in\mathbb{R}^d,\ 0\leq i\leq k. \tag{4.6}$$

As in section 2, one easily establishes the

Theorem 4.1 *Let $u\in C^2(\mathbb{R}^d;\mathbb{R}^k)$ be a classical solution of (4.6) such that*

$$E\int_0^\infty e^{\lambda t}\|(\nabla u\sigma)(X_t^x)\|^2dt<\infty,\ x\in\mathbb{R}^d.$$

Then for each $x\in\mathbb{R}^d$, $\{(u(X_t^x),(\nabla u\sigma)(X_t^x));\ t\geq 0\}$ is the solution of the BSDE (4.3). In particular $u(x)=Y_0^x$.

We want now to prove that (4.1)–(4.2) provides a viscosity solution to (4.6)

Again, for the notion of viscosity solution of the system of PDEs we have (4.6) to make sense, we need to make the following restriction : *for $0\leq i\leq k$, the i–th coordinate of f depends only on the i–th row of the matrix z.*

Then the system (4.6) reads

$$Lu_i(x)+f_i(x,u(x),\nabla u_i\sigma(x))=0,\ x\in\mathbb{R}^d,\ 0\leq i\leq k$$

The notion of viscosity solution of (4.6) is defined similarly as for parabolic systems. Let us just state the :

Definition 4.2 *$u\in C(\mathbb{R}^d;\mathbb{R}^k)$ is called a viscosity subsolution of (4.6) if for any $1\leq i\leq k$, $\varphi\in C^2(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$ which is a local maximum of $u_i-\varphi$,*

$$-L\varphi(x)-f_i(x,u(x),\,(\nabla\varphi\sigma)(x))\leq 0.$$

We can now prove the

Theorem 4.3 *Under the above assumptions, $u(x)\triangleq Y_0^x$ is a continuous function which satisfies*

$$|Y_0^x|\leq c\sqrt{E\int_0^\infty e^{\lambda t}|f(X_t^x,0,0)|^2dt}, \tag{4.7}$$

for any $\lambda>2\mu+K^2$, and it is a viscosity solution of (4.6).

Proof : The continuity follows from the mean–square continuity of $\{Y^x_\cdot,\ x \in \mathbb{R}^d\}$. The inequality (4.7) follows from (3.1).

To prove that u is a viscosity subsolution, take any $1 \le i \le k$, $\varphi \in C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that x is a point of local maximum of $u_i - \varphi$. We assume w.l.o.g. that

$$u_i(x) = \varphi(x).$$

We suppose that

$$(L\varphi)(x) + f_i(x, u(x), (\nabla\varphi\sigma)(x)) < 0,$$

and we will find a contradiction.

Let $\alpha > 0$ be such that whenever $|y - x| \le \alpha$,

$$\begin{aligned} u_i(y) &\le \varphi(y), \\ (L\varphi)(y) + f_i(y, u(y), (\nabla\varphi\sigma)(y)) &< 0, \end{aligned}$$

and define, for some $T > 0$,

$$\tau = \inf\{t > 0;\ |X^x_t - x| \ge \alpha\} \wedge T.$$

Let now

$$(\overline{Y}_t, \overline{Z}_t) = ((Y^x_{t\wedge\tau})^i, \mathbf{1}_{[0,\tau]}(t)(Z^x_t)^i),\ 0 \le t \le T.$$

$(\overline{Y}, \overline{Z})$ solves the one–dimensional BSDE

$$\overline{Y}_t = u_i(X^x_\tau) + \int_t^T \mathbf{1}_{[0,\tau]}(s) f_i(X^x_s, u(X^x_s), \overline{Z}_s)ds - \int_t^T \overline{Z}_s dB_s,\ 0 \le t \le T.$$

On the other hand, from Itô's formula,

$$(\widehat{Y}_t, \widehat{Z}_t) = (\varphi(X^x_{t\wedge\tau}), \mathbf{1}_{[0,\tau]}(t)(\nabla\varphi\sigma)(X^x_t)),\ 0 \le t \le T$$

solves the BSDE

$$\widehat{Y}_t = \varphi(X^x_\tau) - \int_t^T \mathbf{1}_{[0,\tau]}(s) L\varphi(X^x_s)ds - \int_t^T \widehat{Z}_s dB_s,\ 0 \le t \le T.$$

From $u_i \le \varphi$, and the choice of α and τ, we deduce with the help of the comparison theorem 1.6 that $\overline{Y}_t < \widehat{Y}_t$, i.e. $u_i(x) < \varphi(x)$, which is a contradiction. $\diamondsuit$

We now give a similar result, for a system of elliptic PDEs in an open bounded subset of $\mathbb{R}^d$, with Dirichlet boundary condition. The process $\{X^x_t;\ t \ge 0\}$ is defined as above. Let G be an open bounded subset of $\mathbb{R}^d$, whose boundary is of class C^1. For each $x \in \bar{G}$, we define the stopping time

$$\tau_x = \inf\{t \ge 0;\ X^x_t \notin \bar{G}\}.$$

We assume that $P(\tau_x < \infty) = 1$, for all $x \in \bar{G}$, that the set

$$\Gamma = \{x \in \partial G;\ P(\tau_x > 0) = 0\} \quad \text{is closed,} \tag{4.8}$$

and that for some $\lambda > 2\mu + K^2$, and all $x \in \bar{G}$,

$$Ee^{\lambda \tau_x} < \infty.$$

We are finally given a function $g \in C(\mathbb{R}^d)$. Let $\{(Y_t^x, Z_t^x);\ 0 \le t \le \tau_x\}$ be the solution, in the sense of theorem 3.2, of the BSDE

$$Y_t^x = g(X_{\tau_x}^x) + \int_{t\wedge\tau_x}^{\tau_x} f(X_s^x, Y_s^x, Z_s^x)\, ds - \int_{t\wedge\tau_x}^{\tau_x} Z_s^x dB_s,\ t \ge 0.$$

We again define $u(x) = Y_0^x$. The continuity of u relies, besides some arguments which we have already used, on the

Proposition 4.4 *Under the condition (4.8), the mapping $x \to \tau_x$ is a.s. continuous on $\bar{G}$.*

Proof : Let $\{x_n,\ n \in \mathbb{N}\}$ be a sequence in $\bar{G}$ such that $x_n \to x$, as $n \to \infty$.

We first show that

$$\limsup_{n\to\infty} \tau_{x_n} \le \tau_x \quad \text{a.s..} \tag{4.9}$$

Suppose that (4.9) is false. Then

$$P(\tau_x < \limsup_{n\to\infty} \tau_{x_n}) > 0. \tag{4.10}$$

For each $\varepsilon > 0$, let

$$\tau_x^\varepsilon = \inf\{t \ge 0;\ d(X_t^x, G) \ge \varepsilon\}.$$

From (4.10), there exists ε and T such that

$$P(\tau_x^\varepsilon < \limsup_{n\to\infty} \tau_{x_n} \le T) > 0.$$

But since $X_\cdot^{x_n} \to X_\cdot^x$ uniformly on $[0, T]$ a.s., it implies that

$$P(\limsup_{n\to\infty} \tau_{x_n}^{\varepsilon/2} \le \tau_x^\varepsilon < \limsup_{n\to\infty} \tau_{x_n} \le T) > 0,$$

which would mean that for some n, X^{x_n} exits the $\varepsilon/2$–neighbourhood of G before exiting G, which is impossible.

We next prove that

$$\liminf_{n\to\infty} \tau_{x_n} \ge \tau_x \quad \text{a.s.} \tag{4.11}$$

For this part of the proof, we will need the assumption (4.8) that Γ is closed.

It suffices to prove that (4.11) holds a.s. on $\Omega_M = \{\tau_x \le M\}$, with M arbitrary. From the result of the first step, for almost all $\omega \in \Omega_M$, there exists $n(\omega)$ such that $n \ge n(\omega)$

implies $\tau_{x_n} \leq M+1$. From the a.s. (on Ω_M) uniform convergence of $X_\cdot^{x_n} \to X_\cdot^x$ on the intervall $[0, M+1]$, $X_\cdot^x$ hits the set

$$\overline{\{X_{\tau_{x_n}}^{x_n};\ n \in \mathbb{N}\}} \subset \bar{\Gamma} = \Gamma$$

on the random interval $[0, \liminf_n \tau_{x_n}]$ a.s. on Ω_M. The result follows, since $X_\cdot^x$ exits $\bar{G}$ when it hits Γ. $\diamondsuit$

We now state the system of elliptic PDEs , of which u is a viscosity solution.

$$\begin{aligned} Lu_i(x) + f_i(x, u(x), (\nabla u_i \sigma)(x)) = 0,\ 1 \leq i \leq k, x \in G; \\ u_i(x) = g_i(x),\ 1 \leq i \leq k, x \in \bar{G}. \end{aligned} \tag{4.12}$$

We now define the notion of viscosity solution of (4.12).

Definition 4.5 **a** *$u \in C(\bar{G}; \mathbb{R}^k)$ is called a viscosity subsolution of (4.12) if for all $1 \leq i \leq k$, all $\varphi \in C^2(\mathbb{R}^d)$, whenever $x \in \bar{G}$ is a point of local maximum of $u_i - \varphi$,*

$$\begin{aligned} -L\varphi(x) - f_i(x, u(x), (\nabla\varphi\sigma)(x)) \leq 0,\ if\ x \in G; \\ \min\left(-L\varphi(x) - f_i(x, u(x), (\nabla\varphi\sigma)(x)), u_i(x) - g_i(x)\right) \leq 0,\ if\ x \in \partial G. \end{aligned}$$

b *$u \in C(\bar{G}; \mathbb{R}^k)$ is called a viscosity supersolution of (4.12) if for all $1 \leq i \leq k$, all $\varphi \in C^2(\mathbb{R}^d)$, whenever $x \in \bar{G}$ is a point of local minimum of $u_i - \varphi$,*

$$\begin{aligned} -L\varphi(x) - f_i(x, u(x), (\nabla\varphi\sigma)(x)) \geq 0,\ if\ x \in G; \\ \max\left(-L\varphi(x) - f_i(x, u(x), (\nabla\varphi\sigma)(x)), u_i(x) - g_i(x)\right) \geq 0,\ if\ x \in \partial G. \end{aligned}$$

c *$u \in C(\bar{G}; \mathbb{R}^k)$ is called a viscosity solution of (4.12) if it is both a viscosity sub– and supersolution.*

Theorem 4.6 *Under the assumptions of theorem 4.3, the above conditions on G and the condition (4.8), $u(x) \stackrel{\triangle}{=} Y_0^x$ is continuous on $\bar{G}$ and it is a viscosity solution of the system of equations (4.12).*

Proof : We only prove that u is a subsolution. Let $1 \leq i \leq k$, $\varphi \in C^2(\mathbb{R}^d)$ and $x \in \bar{G}$ be a point of local maximum of $u_i - \varphi$ in $\bar{G}$, such that $u_i(x) = \varphi(x)$. If $x \in \Gamma$, then $\tau_x = 0$, and hence $u(x) = g(x)$. We now consider the case $x \notin \Gamma$. Then $\tau_x > 0$ a.s.

We suppose that

$$(L\varphi)(x) + f_i(x, u(x), (\nabla\varphi\sigma)(x)) < 0,$$

and we find a contradiction as in the proof of theorem 4.3, if we choose $\alpha > 0$ such that whenever $|y - x| \le \alpha$,

$$\begin{aligned} u_i(y) &\le \varphi(y), \\ (L\varphi)(y) + f_i(y, u(y), (\nabla\varphi\sigma)(y)) &< 0, \end{aligned}$$

and

$$\bar{\tau} = \inf\{t > 0;\ |X_t^x - x| \ge \alpha\} \wedge \tau_x \wedge T.$$

5 BSDE's and systems of semilinear elliptic PDE's

In this section, we want to consider a system of elliptic PDE's of the form

$$L_i u_i(x) + f_i(x, u(x), (\nabla u_i \sigma)(x)) = 0,\ 1 \le i \le k,\ x \in \mathbb{R}^d, \tag{5.1}$$

where, and this is the novelty with respect to the results of section 4, the second order operator

$$L_i = \frac{1}{2} Tr[\sigma_i \sigma_i^*(x) D^2.] + < b_i(x), D. >,\ 1 \le i \le k,$$

depends on the index i, and for each $1 \le i \le k$,

$$b_i : \mathbb{R}^d \to \mathbb{R}^d,\ \sigma_i : \mathbb{R}^d \to \mathbb{R}^{d \times d}$$

are uniformly Lipschitz continuous. Let $B_t = (B_t^1, \cdots, B_t^d)$ be a d–dimensional standard Brownian motion and $P_t = (P_t^1, \cdots, P_t^{k-1})$ a $k-1$–dimensional standard Poisson process, defined on the probability space $(\Omega, \mathcal{F}, P)$, such that $\{B_t, t \ge 0\}$ and $\{P_t, t \ge 0\}$ are mutually independent. We denote by $\{\mathcal{F}_t,\ t \ge 0\}$ the smallest filtration with respect to which $\{B_t\}$ and $\{P_t\}$ are adapted, and such that $\mathcal{F}_0$ contains all P–null sets of $\mathcal{F}$.

We define

$$M_t = (M_t^1, \cdots, M_t^{k-1}) = (P_t^1 - t, \cdots, P_t^{k-1} - t),$$

and let for each $1 \le i \le k$, $x \in \mathbb{R}^d$, $\{(N_t^i, X_t^{x,i}),\ t \ge 0\}$ denote the Marlov "transmutation–diffusion" process solution of the system of equations :

$$\begin{aligned} N_t^i &= i \overset{k}{+} \sum_{\ell=1}^{k-1} \ell P_t^\ell, \\ X_t^{x,i} &= x + \int_0^t b_{N_s^i}(X_s^{x,i}) ds + \int_0^t \sigma_{N_s^i}(X_s^{x,i}) dB_s,\ t \ge 0, \end{aligned} \tag{5.2}$$

where "$\overset{k}{+}$ "denotes addition modulo k, in other words N_t^i takes values in the set $\{1, 2, \cdots, k\}$. This approach of introducing a system of the type (5.2) in order to get probabilistic formulas for systems of PDEs is originally due to Milstein [35], in the case of linear PDEs.

Suppose now that for each $1 \leq i \leq k$, we are given a mapping :

$$\bar{f}_i \in C(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^d)$$

which is such that for some K, K', $\mu < 0$, $p > 0$, all $1 \leq i \leq k$, $(x, y, u, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^d$,

$$\begin{aligned} |f_i(x,y,0,0)| &\leq K'(1+|x|^p+|y|), \\ < y-y',\ f_i(x,y,u,z)-f_i(x,y',u,z) > &\leq \mu|y-y'|^2, \\ |f(x,y,u,z)-f(x,y,u',z')| &\leq K(|u-u'|+\|z-z'\|). \end{aligned}$$

We assume moreover that for some $\lambda > 2\mu + K^2$, and all $1 \leq i \leq k$, $x \in \mathbb{R}^d$,

$$E\int_0^\infty e^{\lambda t}|f_{N_t^i}(X_t^{x,i},0,0,0)|^2 dt < \infty$$

It then follows from theorem 0.2 in Pardoux [36] that the BSDE

$$\begin{aligned} Y_t^{x,i} &= Y_T^{x,i} + \int_t^T \bar{f}^i_{N_s}(X_s^{x,i}, Y_s^{x,i}, V_s^{x,i}, Z_s^{x,i})ds \\ &\quad - \int_t^T < Z_s^{x,i} dB_s > - \int_t^T < V_s^{x,i} dP_s >,\ \forall t, T\ s.t.\ 0 \leq t < T, \end{aligned} \tag{5.3}$$

has a unique adapted solution $\{(Y_t^{x,i}, V_t^{x,i}, Z_t^{x,i});\ t \geq 0\}$ with values in $\mathbb{R} \times \mathbb{R}^{k-1} \times \mathbb{R}^d$ such that $\{V_t^{x,i}\}$ is predictable and

$$E\int_0^\infty e^{\lambda t}(\|Z_t^{x,i}\|^2 + |V_t^{x,i}|^2)dt < \infty.$$

For each $1 \leq i \leq k$, we define the mapping

$$f_i : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}$$

as follows :

$$f_i(x, u_1, \cdots, u_k, z) = \bar{f}_i(x, u_i, u_{i+1}-u_i, \cdots, u_k-u_i, u_1-u_i, \cdots, u_{i-1}-u_i, z).$$

f_i is the function which appears in the systems of elliptic PDEs (5.1). We now prove the :

Theorem 5.1 *$u(x) \triangleq (Y_0^{x,1}, \cdots, Y_0^{x,k})$ is a continous function of $x \in \mathbb{R}^d$, and it is a viscosity solution of the system of PDEs (5.1).*

Proof : Let $1 \leq i \leq k$, $\varphi \in C^2(\mathbb{R})$, and $x \in \mathbb{R}^d$ such that $u_i - \varphi$ has a local maximum at x. We assume w.l.o.g. that $u_i(x) = \varphi(x)$. We now assume that

$$L_i\varphi(x) + f_i(x, u(x), (\nabla\varphi\sigma_i)(x)) < 0,$$

and we will find a contradiction. This will show that u is a subsolution.

It follows from the above assumptions that there exists $\alpha > 0$ such that whenever $|y-x| \leq \alpha$,

$$\begin{aligned} u_i(y) &\leq \varphi(y) \\ L_i\varphi(y) + f_i(y, u(y), (\nabla\varphi\sigma_i)(y)) &< 0. \end{aligned}$$

Define, for some $T > 0$,

$$\tau = \inf\{t > 0;\ |X_t^{x,i} - x| \geq \alpha\} \wedge \inf\{t > 0;\ N_t^i \neq i\} \wedge T.$$

We have, by the same argument as in the previous section, that

$$u_{N_t^i}(X_t^{x,i}) = Y_t^{x,i},\ t \geq 0.$$

Moreover, identifying the jumps of these two processes, we have that for $1 \leq \ell \leq k-1$,

$$(u_{N_{t-}^i+\ell} - u_{N_{t-}^i})(X_t^{x,i}) = (V_t^{x,i})^\ell, dP_t^\ell \text{ a.e.}$$

Hence

$$E\int_0^T |(u_{N_{t-}^i+\ell} - u_{N_{t-}^i})(X_t^{x,i}) - (V_t^{x,i})^\ell|^2 dP_t^\ell = 0,$$

and then also

$$E\int_0^T |(u_{N_{t-}^i+\ell} - u_{N_{t-}^i})(X_t^{x,i}) - (V_t^{x,i})^\ell|^2 dt = 0,$$

from which we deduce that

$$(u_{N_{t-}^i+\ell} - u_{N_{t-}^i})(X_t^{x,i}) = (V_t^{x,i})^\ell,\ dP \times dt \,\text{a.e.}$$

Consequently, on the interval $[0, \tau]$,

$$\bar{f}_i(X_s^{x,i}, Y_s^{x,i}, V_s^{x,i}, Z_s^{x,i}) = f_i(X_s^{x,i}, u(X_s^{x,i}), Z_s^{x,i}),\ dP \times dt \,\text{a.e.},$$

and $(\bar{Y}_t, \bar{Z}_t) \triangleq (Y_{t\wedge\tau}^{x,i}, \mathbf{1}_{[0,\tau]}(t) Z_t^{x,i}),\ 0 \leq t \leq T$ solves the BSDE

$$\begin{aligned} \bar{Y}_t &= u_i(X_\tau^{x,i}) + \int_t^T \mathbf{1}_{[0,\tau]}(s) f_i(X_s^{x,i}, u(X_s^{x,i}), \bar{Z}_s) ds \\ &\quad - \int_t^T \bar{Z}_s dB_s,\ 0 \leq t \leq T. \end{aligned}$$

On the other hand, from Itô's formula, $(\hat{Y}_t, \hat{Z}_t) \triangleq (\varphi(X_{t\wedge\tau}^{x,i}), \mathbf{1}_{[0,\tau]}(t)\nabla\varphi\sigma_i(X_t^{x,i})),\ 0 \leq t \leq T$ solves the BSDE

$$\hat{Y}_t = \varphi(X_\tau^{x,i}) - \int_t^T \mathbf{1}_{[0,\tau]}(s) L\varphi(X_s^{x,i}) ds - \int_t^T \hat{Z}_s dB_s,\ 0 \leq t \leq T.$$

We conclude as in the proof of theorem 4.3

6 Viscosity solutions of PDEs–Uniqueness

6.1 Motivation and definition

Consider the following nonlinear partial differential equation in IR^d

$$F(x, u(x), Du(x), D^2u(x)) = 0, x \in \mathrm{IR}^d, \tag{6.1}$$

where $F : \mathrm{IR}^d \times \mathrm{IR} \times \mathrm{IR}^d \times S_d \to \mathrm{IR}$, S_d denoting the set of $d \times d$ symmetric non negative matrices. We want to motivate the notion of viscosity solution of equation (6.1).

The standard theory of viscosity solution of (6.1) requires that F be *proper*, i.e. that it satisfies the two following conditions

$$F(x, r, p, X) \leq F(x, s, p, X), \text{whenever } r \leq s; \tag{6.2}$$

$$F(x, r, p, X) \leq F(x, r, p, Y), \text{whenever } Y \leq X. \tag{6.3}$$

Condition (6.3) is called “degenerate ellipticity ”. We note that in our semilinear case, with the notation $a(x) = \sigma(x)\sigma^*(x)$,

$$F(x, r, p, X) = -\frac{1}{2}Tr[a(x)X] - < b(x), p > -f(x, r, p^*\sigma(x)).$$

Degenerate ellipticity follows immediately from the fact that $a(x) \geq 0$, and the monotonicity condition (6.2) follows from the condition that μ, the monotonicity constant of f, is non positive.

In order to introduce the notion of viscosity solution, we first need to split the notion of classical solution of (6.1) into the notion of subsolution :

$$u \in C^2(\mathrm{IR}^d) \text{ s.t. } F(x, u(x), Du(x), D^2u(x)) \leq 0 \ , x \in \mathrm{IR}^d;$$

and the notion of supersolution :

$$u \in C^2(\mathrm{IR}^d) \text{ s.t. } F(x, u(x), Du(x), D^2u(x)) \geq 0, \ x \in \mathrm{IR}^d.$$

Suppose now that u is a classical subsolution, $\varphi \in C^2(\mathrm{IR}^d)$, and x is a point of local maximum of $u - \varphi$. Then $Du(x) = D\varphi(x)$, $D^2u(x) \leq D^2\varphi(x)$, hence if F is denegerate elliptic :

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0.$$

This and an analogous remark concerning supersolutions motivate the following definition

Definition 6.1 *Let $G \subset \mathrm{IR}^d$.*

a) $u \in C(G)$ is called a viscosity subsolution of equation (6.1) in G, if for any $\varphi \in C^2(\mathrm{IR}^d)$, any $x \in G$ which is a point of local maximum of $u - \varphi$ in G,

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0.$$

b) $u \in C(G)$ is called a viscosity supersolution of equation (6.1) in G if for any $\varphi \in C^2(\mathbb{R}^d)$, any $x \in G$ which is a point of local minimum of $u - \varphi$ in G,

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \geq 0.$$

c) $u \in C(G)$ is called a viscosity solution of equation (6.1) in G if it is both a viscosity sub– and supersolution.

Remark 6.2 *The classical formulation of the notion of viscosity solution requires that the subsolution be just upper semicontinuous (and not necessarily continuous), and that the supersolution be just lower semi continuous. Then a viscosity solution is either a continuous function which is both a sub– and a supersolution, or a (possibly discontinuous) function whose upper semicontinous envelope is a subsolution, and whose lower semicontinuous envelope is a supersolution. For the simplicity of the exposition, and since it is sufficient for our purpose, we will restrict ourselves to continuous solutions, as well to continuous sub– and supersolutions.*

It will be useful to exploit alternative definitions of sub– and supersolutions, using the notions of second–order super– and subjets.

Definition 6.3 *Let G be a subset of $\mathbb{R}^d$, u a mapping from G into $\mathbb{R}$. The second order superjet of u at $\hat{x} \in G$, relative to G, is the set*

$$\begin{aligned} J_G^{2,+}u(\hat{x}) &= \{(p, X) \in \mathbb{R}^d \times S_d; u(x) \leq u(\hat{x}) + < p, x - \hat{x} > \\ &\quad + \frac{1}{2} < X(x - \hat{x}), x - \hat{x} > + o|x - \hat{x}|^2, x \in G\}. \end{aligned}$$

The second order subjet of u at point $\hat{x}$, relative to G, is the set

$$\begin{aligned} J_G^{2,-}u(\hat{x}) &= \{(p, X) \in \mathbb{R}^d \times S_d; u(x) \geq u(\hat{x}) + < p, x - \hat{x} > \\ &\quad + \frac{1}{2} < X(x - \hat{x} >, x - \hat{x} > + o|x - \hat{x}|^2, x \in G\}. \end{aligned}$$

The following lemma follows easily from the definitions :

Lemma 6.4 *Let $\hat{x} \in G \subset \mathbb{R}^d$, $u \in C(G)$. Then*

$J_G^{2,+}u(\hat{x}) = \{(D\varphi(\hat{x}), D^2\varphi(\hat{x})); \varphi \in C^2(\mathbb{R}^d), \hat{x}$ *is a local maximum of* $u - \varphi$ *in* $G\}$.
$J_G^{2,-}u(\hat{x}) = \{(D\varphi(\hat{x}), D^2\varphi(\hat{x})); \varphi \in C^2(\mathbb{R}^d), \hat{x}$ *is a local minimum of* $u - \varphi$ *in* $G\}$.

Another formulation of Definition 6.1 is obvious from this lemma, namely :

Definition 6.5 *Let $G \subset \mathbb{R}^d$.*
a) $u \in C(G)$ is a viscosity subsolution of (6.1) in G if for all $x \in G$, all $(p,X) \in J_G^{2,+}u(x)$,

$$F(x,u(x),p,X) \leq 0.$$

b) $u \in C(G)$ is a viscosity supersolution of (6.1) in G if for all $x \in G$, all $(p,X) \in J_G^{2,-}u(x)$,

$$F(x,u(x),p,X) \geq 0.$$

Note that in the case where u is very irregular, $J_G^{2,+}u(x)$ and $J_G^{2,-}u(x)$ can very well be empty at many points $x \in G$. In this sense, the notion of viscosity solution is really weaker than the notion of classical solution. The strength of that theory (in other words, the fact that the notion of solution has not been weakened too much) comes from the fact that one has a uniqueness result under quite reasonable conditions (essentially properness plus some continuity in the variable x). The main argument in uniqueness results is the maximum principle, which we will present in the next subsection.

Let us now introduce some further notations, which will be useful below. For $x \in G$, let

$$\begin{aligned} J_G^2u(x) &\triangleq J_G^{2,+}u(x) \cap J_G^{2,-}u(x), \text{ and} \\ \bar{J}_G^{2,+}u(x) &= \{(p,X) \in \mathbb{R}^d \times S_d \; s.t. \; \exists \{(x_n,p_n,X_n)\} \subset G \times \mathbb{R}^d \times S_d \\ &\quad \text{such that } (p_n,X_n) \in J_G^{2,+}u(x_n), n \in \mathbb{N}, \\ &\quad \text{and } (x_n,u(x_n),p_n,X_n) \to (x,u(x),p,X) \text{ as } n \to \infty\} \end{aligned}$$

We define similarly $\bar{J}_G^{2,-}u(x)$ and $\bar{J}_G^2u(x)$. Finally we shall omit the index G whenever $G = \mathbb{R}^d$.

6.2 The maximum principle

The aim of this section is to sketch the proof that under conditions to be specified below, the following holds (in this section 6.2, G denotes an open and bounded subset of $\mathbb{R}^d$).

Maximum principle *Let $u, v \in C(\bar{G})$ be respectively a sub– and a supersolution of equation (6.1) in G. Then, if $u \leq v$ on ∂G, $u \leq v$ in G.*

One important consequence of the maximum principle is the following corollary.

Corollary 6.6 *Suppose that the maximum principle holds, and F satisfies (6.2). Then if $u, v \in C(\bar{G})$ are respectively a sub– and a supersolution,*

$$\sup_{x \in \bar{G}} (u(x) - v(x))^+ \leq \sup_{x \in \partial G} (u(x) - v(x))^+$$

Proof : It suffices to note that, from (6.2), if v is a supersolution,

$$\bar{v} = v + \sup_{x\in\partial G} (u - v)^+$$

is also a supersolution, and apply the maximum principle. ♢

In order to motivate the ideas in the proof, we shall first prove the maximum principle for classical solutions, then in the case of first order equations, and finally in the general case.

In all what follows, we shall need to assume that F satisfies the following reinforced version of condition (6.2)

$$\begin{aligned}&\text{For each } R > 0, \text{ there exists } \gamma_R > 0 \text{ such that}\\ &\gamma_R(s-r) \le F(x,s,p,X) - F(x,r,p,X),\\ &\text{for all } |x| \le R, -R \le r \le s \le R, p \in \mathbb{R}^d, X \in S_d.\end{aligned} \tag{6.2'}$$

Let us start with the classical case.

Theorem 6.7 *Suppose that F satisfies (6.2') and (6.3). Let $u, v \in C^2(G) \cap C(\bar{G})$ be respectively a sub– and a supersolution of equation (6.1) in G. Then the pair (u, v) satisfies the maximum principle.*

Proof : Assume that

$$M = \sup_{x\in\bar{G}} u(x) - v(x) > 0.$$

Since $u - v$ is continuous and $\bar{G}$ is compact, the maximum is achieved at a point $\hat{x}$. The assumption implies that $\hat{x} \in G$. Then

$$F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \le 0 \le F(\hat{x}, v(\hat{x}), D\,v(\hat{x}), D^2v(\hat{x})). \tag{6.4}$$

Since $\hat{x}$ is a maximum of the C^2 function $u - v$ in the open set G,

$$\begin{aligned} Du(\hat{x}) &= D\,v(\hat{x}),\\ D^2u(\hat{x}) &\le D^2v(\hat{x}) \end{aligned}$$

Then, exploiting successively (6.2'), degenerate ellipticity and (6.4) we obtain

$$\begin{aligned}\gamma_R M &\le F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) - F(\hat{x}, v(\hat{x}), Du(\hat{x}), D^2u(\hat{x}))\\ &\le F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x}), -F(\hat{x}, v(\hat{x}), Dv(\hat{x})), D^2v(\hat{x}))\\ &\le 0,\end{aligned}$$

where $R = \sup_{x\in\bar{G}}(|x| \vee |u(x)| \vee |v(x)|)$. We have obtained a contradiction. $\diamondsuit$

We note that we could have replaced in theorem 6.7 the assumption (6.2') by a strict inequality in (6.2), whenever $r < s$. We now turn to the case of viscosity solutions, i.e. from now on a subsolution (resp. a supersolution, resp. a solution) will always be understood in the viscosity sense. We first prove the maximum principle for first order equations, i.e. we consider the equation

$$F(x, u(x), Du(x)) = 0 \text{ in } G, \tag{6.5}$$

where $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. We have the :

Theorem 6.8 *Suppose $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies (6.2') and*

$$\begin{gathered}\text{for each } R, \text{ there exists } \omega_R \in C(\mathbb{R}_+, \mathbb{R}_+) \text{ such that } \omega(0) = 0 \text{ and}\\ F(x,r,p) - F(y,r,p) \le \omega_R(|x-y|(1+|p|)), |x|, |y|, |r| \le R, p \in \mathbb{R}^d.\end{gathered} \tag{6.6}$$

Let $u, v \in C(\bar{G})$ be respectively a sub– and a supersolution of (6.5) in G, then the pair (u, v) satisfies the maximum principle.

Proof : We again assume that

$$M = \sup_{x\in\bar{G}} u(x) - v(x) > 0,$$

and we will obtain a contradiction.

For $\alpha > 0$, we define $\psi_\alpha : \bar{G} \times \bar{G} \to \mathbb{R}$ as

$$\psi_\alpha(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x-y|^2,$$

and let

$$M_\alpha = \sup_{(x,y)\in\bar{G}\times\bar{G}} \psi_\alpha(x, y).$$

Let us admit for a moment the

Lemma 6.9

(i) $\quad M_\alpha \to M$, *as* $\alpha \to \infty$.

If $(\hat{x}, \hat{y})$ denotes a point where the maximum of ψ_α is achieved, then :

(ii) $\quad \alpha|\hat{x} - \hat{y}|^2 \to 0$, *and* $u(\hat{x}) - v(\hat{y}) \to M$, *as* $\alpha \to \infty$;

(iii) $\hat{x}, \hat{y} \in G$, *for* α *large enough.*

Since $u(x) - v(\hat{y}) - \frac{\alpha}{2}|x - \hat{y}|^2$ has a maximum at $\hat{x}$, and u is a subsolution of equation (6.5),

$$F(\hat{x}, u(\hat{x}), \alpha(\hat{x} - \hat{y})) \leq 0.$$

Similarly since v is a supersolution,

$$F(\hat{y}, v(\hat{y}), \alpha(\hat{x} - \hat{y})) \geq 0.$$

Consequently

$$F(\hat{x}, u(\hat{x}), \alpha(\hat{x} - \hat{y})) \leq F(\hat{y}, v(\hat{y}), \alpha(\hat{x} - \hat{y})) \tag{6.7}$$

Using successively (6.2'), (6.7) and (6.6), we deduce that

$$\begin{aligned} \gamma_R(u(\hat{x}) - v(\hat{y})) &\leq F(\hat{x}, u(\hat{x}), \alpha(\hat{x} - \hat{y})) - F(\hat{x}, v(\hat{y}), \alpha(\hat{x} - \hat{y})) \\ &\leq F(\hat{y}, v(\hat{y}), \alpha(\hat{x} - \hat{y})) - F(\hat{x}, v(\hat{y}), \alpha(\hat{x} - \hat{y})) \\ &\leq \omega_R(|\hat{x} - \hat{y}| + \alpha|\hat{x} - \hat{y}|^2), \end{aligned}$$

where R is defined as in the proof of theorem 6.7.

But, from lemma 6.9, the left hand side of this inequality is bounded from below by $\gamma_R M/2$ for α large enough, while the right hand side tends to zero, as $\alpha \to \infty$. Hence the assumption $M > 0$ cannot be true.

Proof of lemma 6.9 : We first note that

$$M = \sup_{x \in G} \psi_\alpha(x, x),$$

and consequently

$$0 < M \leq M_\alpha \leq \sup_{(x,y) \in \bar{G} \times \bar{G}} u(x) - v(y) = K < \infty$$

Since ψ_α is continuous and $\bar{G}$ is compact, there exists $(\hat{x}, \hat{y}) \in \bar{G} \times \bar{G}$ such that

$$\psi_\alpha(\hat{x}, \hat{y}) = \sup_{\bar{G} \times \bar{G}} \psi_\alpha(x, y).$$

Since $M_\alpha \geq 0$,

$$\frac{\alpha}{2}|\hat{x} - \hat{y}|^2 \leq u(\hat{x}) - v(\hat{y}) \leq K,$$

and consequently $\hat{x} - \hat{y} \to 0$, as $\alpha \to \infty$. Define, for $\delta > 0$,

$$K_\delta = \max_{|x-y| \leq \delta} u(x) - v(y)$$

From the uniform continuity of u and v on $\bar{G}$, we deduce that $K_\delta \downarrow M$, as $\delta \downarrow 0$. But

$$\theta(\alpha) \stackrel{\triangle}{=} |\hat{x} - \hat{y}| \to 0, \text{as } \alpha \to \infty,$$

and $M \le M_\alpha \le K_{\theta(\alpha)}$. (i) is proved. (ii) follows from

$$\frac{\alpha}{2}|\hat{x} - \hat{y}|^2 = u(\hat{x}) - v(\hat{y}) - M_\alpha \le K_{\theta(\alpha)} - M.$$

Finally from these and the uniform continuity of u and v on $\bar{G}$, $u(\hat{x}) - v(\hat{x})$ and $u(\hat{y}) - v(\hat{y})$ are strictly positive for α large enough, which proves (iii). $\diamondsuit$

We now turn finally to the general case of second order equations, which is technically more difficult.

Theorem 6.10 *Suppose that F, in addition to the condition (6.2'), satisfies*

$$\begin{gathered}\textit{for each } R > 0, \textit{there exists } \omega_R \in C(\mathbb{R}_+, \mathbb{R}_+) \textit{ such that } \omega_R(0) = 0 \textit{ and}\\ F(y, r, \alpha(x-y), Y) - F(x, r, \alpha(x-y), X) \le \omega(|x-y| + \alpha|x-y|^2), \textit{for each}\\ \alpha > 0, |x|, |y| \le R, r \in [-R, R], X, Y \in S_d \textit{ such that}\\ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\end{gathered} \tag{6.8}$$

Let $u, v \in C(\bar{G})$ be respectively a sub– and a supersolution of (6.1) in G. Then the pair (u, v) satisfies the maximum principle.

The proof of this result will rely on lemma 6.9 and on the

Proposition 6.11 *Given $u, v \in C(\bar{G})$, $\alpha > 0$, we define again*

$$\psi_\alpha(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2$$

Let $(\hat{x}, \hat{y})$ be a local maximum in $G \times G$ of ψ_α. Then there exists $X, Y \in S_d$ such that

(j) $(\alpha(\hat{x} - \hat{y}), X) \in \bar{J}_G^{2,+} u(\hat{x})$

(jj) $(\alpha(\hat{x} - \hat{y}), Y) \in \bar{J}_G^{2,-} v(\hat{y})$

(jjj) $\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$

Proof of theorem 6.10 : Since u and v are respectively a sub– and a supersolution, it follows from proposition 6.11 that

$$F(\hat{x}, u(\hat{x}), \alpha(\hat{x}-\hat{y}), X) \leq 0 \leq F(\hat{y}, v(\hat{y}), \alpha(\hat{x}-\hat{y}), Y), \tag{6.9}$$

$$\text{and} \qquad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

We now use successively (5.2'), (6.9) and (6.8), yielding

$$\begin{aligned} \gamma_R(u(\hat{x}) - v(\hat{y})) &\leq F(\hat{x}, u(\hat{x}), \alpha(\hat{x}-\hat{y}), X) - F(\hat{x}, v(\hat{y}), \alpha(\hat{x}-\hat{y}), X) \\ &\leq F(\hat{y}, v(\hat{y}), \alpha(\hat{x}-\hat{y}), Y) - F(\hat{x}, v(\hat{y}), \alpha(\hat{x}-\hat{y}), X) \\ &\leq \omega_R(|\hat{x}-\hat{y}| + \alpha|\hat{x}-\hat{y}|^2), \end{aligned}$$

where R is defined as the proofs of the previous theorems. We conclude as in the proof of theorem 6.8 ♢

We finally sketch the :

Proof of proposition 6.11 : We shall use the notation

$$A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

It is sufficient to prove the proposition in case $G = \mathbb{R}^d$, $\hat{x} = \hat{y} = 0$, $u(0) = v(0) = 0$, $(0,0)$ is a global maximum of ψ_α, u and $-v$ are bounded from above. Hence we may assume that for all $x, y \in \mathbb{R}^d$,

$$u(x) - v(y) \leq \frac{1}{2} < A \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} >, \tag{6.10}$$

and we need to show that there exist $X, Y \in S_d$ such that

(j') $\quad (0, X) \in \bar{J}^{2,+}u(0)$,

(jj') $\quad (0, Y) \in \bar{J}^{2,+}v(0)$,

(jjj') $\quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3A.$

With the notations $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\bar{\xi} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, we deduce from Schwarz's inequality that (with the notation $\|A\| \triangleq \sup\{| < A\bar{\xi}, \bar{\xi} > |; |\bar{\xi}| \leq 1\}$) :

$$< A\bar{x}, \bar{x} > \; = \; < A\bar{\xi}, \bar{\xi} > + < A(\bar{x}-\bar{\xi}), \bar{x}-\bar{\xi} > +2 < \bar{x}-\bar{\xi}, A\bar{\xi} >$$

$$\begin{aligned} &\leq \; < A\bar{\xi}, \bar{\xi} > + \frac{1}{\alpha}|A\bar{\xi}|^2 + (\alpha + \|A\|)|\bar{x} - \bar{\xi}|^2 \\ &\leq \; < (A + \frac{1}{\alpha}A^2)\bar{\xi}, \bar{\xi} > + (\alpha + \|A\|)|\bar{x} - \bar{\xi}|^2 \end{aligned}$$

Hence if $B \triangleq 3A = A + \frac{1}{\alpha}A^2$, $\lambda \triangleq \alpha + \|A\|$, and $w(\bar{x}) \triangleq u(x) - v(y)$, (6.10) implies

$$w(\bar{x}) - \frac{\lambda}{2}|\bar{x} - \bar{\xi}|^2 \leq \frac{1}{2} < B\bar{\xi}, \bar{\xi} > . \tag{6.11}$$

We now introduce inf– and sup–convolutions. Let

$$\begin{aligned} \hat{w}(\bar{\xi}) &\triangleq \sup_{\bar{x}}(w(\bar{x}) - \frac{\lambda}{2}|\bar{x} - \bar{\xi}|^2) \\ &= \hat{u}(\xi) - \hat{v}(\eta), \end{aligned}$$

where

$$\begin{aligned} \hat{u}(\xi) &= \sup_{x}(u(x) - \frac{\lambda}{2}|x - \xi|^2), \\ \hat{v}(\eta) &= \inf_{y}(v(y) + \frac{\lambda}{2}|y - \eta|^2). \end{aligned}$$

Since a supremum (resp. an infimum) of convex (resp. concave) functions is convex (resp. concave), the mappings

$$\begin{aligned} \bar{\xi} &\rightarrow \hat{w}(\bar{\xi}) + \frac{\lambda}{2}|\bar{\xi}|^2 \\ \text{and } \xi &\rightarrow \hat{u}(\xi) + \frac{\lambda}{2}|\xi|^2 \end{aligned}$$

are convex, while

$$\eta \rightarrow \hat{v}(\eta) - \frac{\lambda}{2}|\eta|^2$$

is concave. Hence $\hat{w}$, $\hat{u}$ and $-\hat{v}$ are "semi–convex ", i.e. they are the sum of a convex function and a function of class C^2.

Moreover :

$$\hat{w}(0) \leq w(0) = 0,$$

and from (6.11)

$$\hat{w}(\bar{\xi}) \leq \frac{1}{2} < B\bar{\xi}, \bar{\xi} >,$$

hence

$$\hat{w}(0) \leq 0$$

and consequently

$$\hat{w}(0) = \max_{\bar{\xi}} \left(\hat{w}(\bar{\xi}) - \frac{1}{2} < B\bar{\xi}, \bar{\xi} > \right).$$

If $\hat{w}$ were smooth, we could deduce that there exists $\mathcal{X} \in S_{2d}$ such that $(0, \mathcal{X}) \in J^2\hat{w}(0)$, and $\mathcal{X} \le B$. Since $\hat{w}$ is semiconvex, it is possible to show, using Alexandrov's theorem (which says that a semiconvex function is a.e. twice differentiable), and a lemma due to R. Jensen, that the above is essentially true in the sense that it is true, provided the first condition is changed into $(0, \mathcal{X}) \in \bar{J}^2\hat{w}(0)$. Now, since $\hat{w}(\bar{\xi}) = \hat{u}(\xi) - \hat{v}(\eta)$, it is not hard to deduce that $\mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}$, and $(0, X) \in \bar{J}^2\hat{u}(0)$, $(0, Y) \in \bar{J}^2\hat{v}(0)$.

The magical property of sup–convolution is that this is enough to conclude that $(0, X) \in \bar{J}^{2,+}u(0)$ and $(0, Y) \in \bar{J}^{2,-}v(0)$. It is a consequence of the

Lemma 6.12 *Let $\lambda > 0$, $u \in C(\mathbb{R}^d)$ bounded from above, and*

$$\hat{u}(\zeta) = \sup_{x \in \mathbb{R}^d} (u(x) - \frac{\lambda}{2}|x - \zeta|^2)$$

If $\eta, q \in \mathbb{R}^d$, $X \in S_d$ and $(\eta, X) \in J^{2,+}\hat{u}(\eta)$, then $(q, X) \in J^{2,+}u(\eta + q/\lambda)$.

Proof : We assume that $(q, X) \in J^{2,+}\hat{u}(\eta)$. Let $y \in \mathbb{R}^d$ be such that

$$\hat{u}(\eta) = u(y) - \frac{\lambda}{2}|y - \eta|^2.$$

Then for any $x, \zeta \in \mathbb{R}^d$,

$$\begin{aligned} u(x) - \frac{\lambda}{2}|x - \zeta|^2 &\le \hat{u}(\zeta) \\ &\le \hat{u}(\eta) + < q, \zeta - \eta > + \frac{1}{2} < X(\zeta - \eta), \zeta - \eta > + \circ\,(|\zeta - \eta|^2) \\ &= u(y) - \frac{\lambda}{2}|y - \eta|^2 + < q, \zeta - \eta > \\ &\quad + \frac{1}{2} < X(\zeta - \eta), \zeta - \eta > + \circ\,(|\zeta - \eta|^2) \\ &= u(y) - \frac{\lambda}{2}|y - \eta|^2 + < q, \zeta - \eta > + O(|\zeta - \eta|^2) \end{aligned}$$

If we choose $\zeta = x - y + \eta$, then we deduce from the above that

$$u(x) \le u(y) + < q, x - y > + \frac{1}{2} < X(x - y), x - y > + \circ\,(|x - y|^2).$$

On the other hand, choosing $x = y$ and $\zeta = \eta + \alpha(\lambda(\eta - y) + q)$, we obtain that

$$0 \le \alpha|\lambda(\eta - y) + q|^2 + O(\alpha^2).$$

The first inequality says that $(q, X) \in J^{2,+}u(y)$, while the second, with $\alpha < 0$ small enough in absolute value implies that $y = \eta + \frac{q}{\lambda}$. The result is proved.

6.3 Application to our semi–linear equations

In this subsection, we want to show what condition is needed upon the coefficient f, in order that

$$F(x,r,p,X) \triangleq -\frac{1}{2}Tr[a(x)X] - < b(x), p > -f(x,r,p^*\sigma(x))$$

(we assume that $k = 1$) satisfies conditions (6.2') and (6.8) in theorem 6.10. (6.2') is equivalent to the existence, for each $R > 0$, f of a constant $\gamma_R > 0$ such that

$$\gamma_R(s-r) \leq f(x,r,p^*\sigma(x)) - f(x,s,p^*\sigma(x)),$$

for all $|x| \leq R$,$-R \leq r \leq s \leq R$, $p \in \mathbb{R}^d$. This is implied by the fact that $\mu < 0$ in the second condition following equation (4.10) of the previous section. We now turn to condition (6.8). Let us consider successively the three terms in the expression for F. If $\{e_1, \ldots, e_d\}$ denotes an orthonormal basis of $\mathbb{R}^d$,

$$\begin{aligned}
Tr[a(x)X - a(y)Y] &= Tr[X\sigma(x)\sigma^*(x) - Y\sigma(y)\sigma^*(y)] \\
&= Tr[\sigma^*(x)X\sigma(x) - \sigma^*(y)Y\sigma(y)] \\
&= \sum_{i=1}^{d}[< X\sigma(x)e_i, \sigma(x)e_i > - < Y\sigma(y)e_i, \sigma(y)e_i >] \\
&\leq 3\alpha\sum_i |\sigma(x)e_i - \sigma(y)e_i|^2 \\
&\leq c_R\alpha|x-y|^2,
\end{aligned}$$

where we have used successively the assumption

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

and the local Lipschitz property of σ.

Consider next

$$\begin{aligned}
- < b(y), \alpha(x-y) > &+ < b(x), \alpha(x-y) > \\
&= \alpha < x-y, b(x) - b(y) > \\
&\leq c_R\alpha|x-y|^2,
\end{aligned}$$

provided b is locally lipschitz (or just locally “ semimonotone ”).

Finally

$$\begin{aligned}
&- f(y,r,\alpha(x-y)^*\sigma(y)) + f(x,r,\alpha(x-y)^*\sigma(x)) \\
&= f(x,r,\alpha(x-y)^*\sigma(x)) - f(y,r,\alpha(x-y)^*\sigma(x)) \\
&+ f(y,r,\alpha(x-y)^*\sigma(x)) - f(y,r,\alpha(x-y)^*\sigma(y)).
\end{aligned}$$

From the Lipschitz property of f with respect to its last variable, and the local Lipschitz property of σ, the second term of the above right hand side is dominated by

$$c_R\alpha|x-y|^2.$$

We need to formulate an additional assumption, in order to dominate the first term. The following is a sufficient condition for our F to satisfy condition (6.8).

$$\begin{aligned}&\text{For each } R>0, \text{there exists } m_R\in C(\mathbb{R}_+;\mathbb{R}_+) \text{ such that}\\ &m_R(0)=0, \text{and } |f(x,r,p)-f(y,r,p)|\le m_R(|x-y|(1+|p|))\\ &\qquad \text{for all } |x|,|y|\le R, -R\le r\le R, p\in\mathbb{R}^d.\end{aligned} \tag{6.12}$$

6.4 Uniqueness for an elliptic equation in $\mathbb{R}^d$.

We now formulate a condition which is closely related to condition (4.4) in section 4, i.e. which relates the the constants μ and K attached to f, to the generator L.

Let $c,p>0$ be such that

$$|f(x,0,0)|\le\chi(x),$$

where

$$\chi(x)\triangleq c(r+|x|^2)^{p/2}.$$

Let

$$\bar{K}\triangleq K\times\sup_{x\in\mathbb{R}^d}\frac{|x^*\sigma(x)|}{1+|x|^2},$$

where K denotes the Lipschitz constant of f with respect to z.

We assume that there exists $\lambda>\mu+p\bar{K}$ such that

$$L\chi(x)+\lambda\chi(x)\le 0,\ x\in\mathbb{R}^d; \tag{6.13}$$

and $\bar{\lambda}>2\mu+K^2$ such that for some $c>0$

$$E\int_0^\infty e^{\bar{\lambda}t}|f(X_t^x,0,0)|^2dt\le c(1+|x|^2)^p,\ x\in\mathbb{R}^d. \tag{6.14}$$

Remark 6.13 Note that (6.13) with $\lambda>2\mu+K^2$ is a sufficient condition for (6.14). Indeed, (6.13) implies that

$$\frac{d}{dt}E[e^{\lambda t}\chi(X_t)]\le 0.$$

So if this is true for some $\lambda'>2\mu+K^2$, then for any $\lambda\in(2\mu+K^2,\lambda')$,

$$E\int_0^\infty e^{\lambda t}f(X_t,0,0)dt\le E\int_0^\infty e^{\lambda t}\chi(X_t)dt<\infty.$$

We now give a uniqueness result for the equation

$$-Lu(x)-f(x,u(x),(D^*u\sigma)(x))=0,\ x\in\mathbb{R}^d. \tag{6.15}$$

Theorem 6.14 *Assume that f satisfies condition (4.3) with $\mu < 0$, (6.12), (6.13) and (6.14). Then $u(x) \triangleq Y_0^x$ is the unique continuous viscosity solution of equation (6.15), among those functions whose absolute value grows at most like $c|x|^p$ at infinity, for some $c > 0$.*

Proof : The fact that there exists $c > 0$ such that

$$|Y_0^x| \leq c(1 + |x|^p)$$

follows from (6.14) and (4.7).

Let now $u, v \in C(\mathbb{R}^d)$ be such that

$$\limsup_{|x|\to\infty} \frac{|u(x)| \vee |v(x)|}{1 + |x|^p} < \infty,$$

and u (resp. v) is a subsolution (resp. a supersolution) of (6.15). We only need to show that $u \leq v$.

Let us admit for a moment the :

Lemma 6.15 *$(u - v)^+$ is a viscosity subsolution of the equation :*

$$-Lw(x) + |\mu|w(x) - K|(Dw)^*\sigma|(x) = 0, \ x \in \mathbb{R}^d. \tag{6.16}$$

It easily follows from condition (6.13) that there exists $p' > p$, such that

$$L\chi'(x) + (\mu + p'\bar{K})\chi'(x) \leq 0, \ x \in \mathbb{R}^d;$$

where $\chi'(x) = (1 + |x|^2)^{p'/2}$. Consequently, since $D\chi'(x) = p'\chi'(x)\frac{x}{1+|x|^2}$, it follows from the definition of $\bar{K}$ that χ' is a supersolution of (6.16). For any $\beta > 0$, the same is true for $\beta\chi'$. But

$$\lim_{|x|\to\infty} \frac{(u(x) - v(x))^+}{\beta\chi'(x)} = 0,$$

hence there exists R (which depends on β) such that

$$(u(x) - v(x))^+ \leq \beta\chi'(x), |x| \geq R.$$

Consequently, it follows from theorem 6.10 that

$$(u(x) - v(x))^+ \leq \beta\chi'(x), \ x \in \mathbb{R}^d.$$

Since this is true for all $\beta > 0$, the result follows.

Proof of lemma 6.15 : Let first $\bar{x} \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ be such that $u(\bar{x}) \leq v(\bar{x})$, and $\bar{x}$ is a local maximum of $(u - v)^+ - \varphi$. Then $(u(\bar{x}) - v(\bar{x}))^+ = 0$, and $\bar{x}$ is a local minimum of φ, hence $D\varphi(\bar{x}) = 0$ and $D^2\varphi(\bar{x}) \geq 0$. It is then easy to deduce that

$$-L\varphi(\bar{x}) - \mu(u(\bar{x}) - v(\bar{x})^+ - K|(D\varphi)^*\sigma|(\bar{x}) \leq 0.$$

Next we choose $\bar{x} \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u(\bar{x}) > v(\bar{x})$, and $\bar{x}$ is a local maximum of $u-v-\varphi$. It is sufficient to consider the case where $\bar{x}$ is a strict global maximum of $u-v-\varphi$, and $u-v-\varphi$ is bounded from above. For $\alpha > 0$, we introduce the function :

$$\psi_\alpha(x,y) = u(x) - v(y) - \varphi(x) - \frac{\alpha}{2}|x-y|^2.$$

It is clear that for α large enough, ψ_α has a unique maximum $(\hat{x}, \hat{y})$, and one can show, as in lemma 6.9, that $\hat{x}, \hat{y} \to \bar{x}$ and $\alpha|\hat{x}-\hat{y}|^2 \to 0$, as $\alpha \to \infty$. Moreover, one can deduce from proposition 6.11 that there exist $X, Y \in S_d$ such that

$$\begin{aligned} (\alpha(\hat{x}-\hat{y}) + D\varphi(\hat{x}), X) &\in \bar{J}^{2,+}u(\hat{x}), \\ (\alpha(\hat{x}-\hat{y}), Y) &\in \bar{J}^{2,-}v(\hat{y}), \end{aligned}$$
$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} D^2\varphi(\hat{x}) & 0 \\ 0 & 0 \end{pmatrix}$$

We now use these three statements and the fact that u and v are respectively a sub– and a supersolution, and argue similarly as in the proof of theorem 6.10, yielding

$$\begin{aligned} -\frac{1}{2}Tr[a(\hat{x})X - a(\hat{y})Y] - &< b(\hat{x}) - b(\hat{y}), \alpha(\hat{x}-\hat{y}) > \\ - < b(\hat{x}), D\varphi(\hat{x}) > &- f(\hat{x}, u(\hat{x}), (\alpha(\hat{x}-\hat{y}) + D\varphi(\hat{x}))^*\sigma(\hat{x})) \\ &+ f(\hat{y}, v(\hat{y}), \alpha(\hat{x}-\hat{y})^*\sigma(\hat{y})) \le 0, \\ -L\varphi(\hat{x}) - \mu(u(\hat{x}) - v(\hat{x})) &- K|(D\varphi(\hat{x}))^*\sigma(\hat{x})| \\ &\le c|\hat{x}-\hat{y}|(1+\alpha|\hat{x}-\hat{y}|). \end{aligned}$$

Letting $\alpha \to \infty$, we deduce that

$$-L\varphi(\bar{x}) - \mu(u(\bar{x}) - v(\bar{x})) - K|(D\varphi(\bar{x}))^*\sigma(\bar{x})| \le 0.$$

The result follows.

6.5 Uniqueness for the Dirichlet problem

We now consider the Dirichlet problem (4.12) in the case $k = 1$, where again G is a bounded open subset of $\mathbb{R}^d$, whose boundary is of class C^1.

We need to formulate a new assumption

$$\begin{aligned} &\text{There exists } \epsilon > 0 \text{ and } \omega \in C(\mathbb{R}_+; \mathbb{R}_+) \text{ such that } \omega(0) = 0 \text{ and} \\ &\qquad \text{for all } x \in G \text{ such that } d(x, G^c) \le \epsilon,\ r \in \mathbb{R},\ p, q \in \mathbb{R}^d, \\ &\qquad\qquad |f(x,r,p) - f(x,r,q)| \le \omega(|p-q|). \end{aligned} \tag{6.17}$$

Uniqueness for our Dirichlet problem follows from the

Theorem 6.16 *We assume that f satisfies the conditions of theorem 4.6, (6.12) and (6.17). Then, if $u, v \in C(\bar{G})$, u is a subsolution of (4.12), and v is a supersolution of (4.12), $u \le v$.*

Proof : We assume that $\max_{\bar{G}} u - v > 0$, and will find a contradiction. From theorem 6.10 and corollary 6.6, there exists $z \in \partial G$ such that

$$u(z) > v(z).$$

There are two cases

Case 1 $v(z) < g(z)$. Given $\alpha > 1$, $0 < \varepsilon < 1$, we define

$$\Phi_{\alpha,\varepsilon}(x, y) = u(x) - v(y) - |\alpha(x - y) + \varepsilon n(z)|^2 - \varepsilon |y - z|^2,$$

where $n(z)$ is a unit vector normal at z to ∂G, pointing towards the exterior of G. Let $(\hat{x}, \hat{y})$ be a point of maximum of $\Phi_{\alpha,\varepsilon}$. For $0 < \varepsilon < 1$ fixed, we may assume that α is big enough so that $z - \frac{\varepsilon}{\alpha} n(z) \in G$. Let $\Phi_{\alpha,\varepsilon}(\hat{x}, \hat{y}) = \sup_{\bar{G}\times\bar{G}} \Phi_{\alpha,\varepsilon}(x, y)$. From

$$\Phi_{\alpha,\varepsilon}(\hat{x}, \hat{y}) \ge \Phi_{\alpha,\varepsilon}(z - \frac{\varepsilon}{\alpha} n(z), z)$$

follows that

$$|\alpha(\hat{x} - \hat{y}) + \varepsilon n(z)|^2 + \varepsilon |\hat{y} - z|^2 \le u(\hat{x}) - v(\hat{y}) - u(z - \frac{\varepsilon}{\alpha} n(z)) + v(z).$$

Hence, again with ε fixed, as $\alpha \to \infty$, $\hat{x}, \hat{y} \to z$, and $\alpha(\hat{x} - \hat{y}) + \varepsilon n(z) \to 0$, consequently

$$\hat{x} = \hat{y} - \frac{\varepsilon n(z) + \circ(1)}{\alpha},$$

and $\hat{x} \in G$ for α large enough. Hence we have both (the second statement follows from $v(z) < g(z)$, hence also $v(\hat{y}) < g(\hat{y})$ for α large enough)

$$F(\hat{x}, u(\hat{x}), p, X) \le 0, \ \forall (p, X) \in \bar{J}^{2,+}_{\bar{G}} u(\hat{x}),$$
$$F(\hat{y}, v(\hat{y}), q, Y) \ge 0, \ \forall (q, Y) \in \bar{J}^{2,-}_{\bar{G}} v(\hat{y}),$$

where $F(x, r, p, X) = -\frac{1}{2} Tr(a(x)X) - < b(x), p > - f(x, r, p^* \sigma(x))$. Note that with

$$\varphi_{\alpha,\varepsilon}(x, y) = |\alpha(x - y) + \varepsilon n(z)|^2 + \varepsilon |y - z|^2,$$
$$D_x \varphi_{\alpha,\varepsilon}(x, y) = 2\alpha(\alpha(\alpha(x - y) + \varepsilon n(z)),$$
$$-D_y \varphi_{\alpha,\varepsilon}(x, y) = 2\alpha(\alpha(x - y) + \varepsilon n(z)) - 2\varepsilon(y - z),$$

$$D^2 \varphi_{\alpha,\varepsilon}(x, y) = 2\alpha^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Adapting proposition 6.11 to this situation, arguing as in the proof of theorem 6.10, letting first $\alpha \to \infty$ and then $\varepsilon \to 0$, we obtain a contradiction.

Case 2 $v(z) \geq g(z)$. Then the assumption implies that $u(z) > g(z)$.

We define

$$\Phi_{\alpha,\varepsilon}(x,y) = u(x) - v(y) - |\alpha(x-y) - \varepsilon n(z)|^2 + \varepsilon|x-z|^2,$$

argue as above and obtain a contradiction.

Remark 6.17 *Under appropriate conditions, one can prove uniqueness results for viscosity solutions of systems of second order PDEs, see Ishii, Koike [31]. For a uniqueness result for a system of semilinear parabolic PDEs, see e.g. Pardoux, Pradeilles, Rao [40].*

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