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Stochastic Variational Inequalities of Parabolic Type*

U. G. Haussmann[1] and E. Pardoux[2]

[1] Mathematics Department, University of British Columbia, 121-1984 Mathematics Road, Vancouver, B.C., Canada, V6T 1Y4

[2] Mathématiques—case H, Université de Provence, 13331 Marseille Cedex 3, France

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Abstract. Existence and uniqueness of strong solutions of stochastic partial differential equations of parabolic type with reflection (e.g., the solutions are never allowed to be negative) is proved. The problem is formulated as a stochastic variational inequality and then compactness is used to derive the result, but the method requires the space dimension to be one.

1. Introduction

The aim of this paper is to study reflected solutions of stochastic partial differential equations of parabolic type. Specifically we prove existence and uniqueness results for a process $u(t)$ with values in $L^2(0, 1)$ which is such that, roughly speaking, at each point (t, x) where $u(t, x)$ is positive, u obeys a stochastic partial differential equation, and u is reflected at zero, i.e., $u(t, x)$ is nonnegative for all (t, x). Moreover, we require that the force which is applied in order to keep u nonnegative be minimal in a certain sense. In the case of finite-dimensional equations, i.e., stochastic ordinary differential equations, reflected diffusions have been studied notably by Tanaka [17], Lions and Sznitman [8], and Saisho [15], and also by Stroock and Varadhan [16] who used a "submartingale problem" formulation. On the other hand, there is a vast literature concerning reflected solutions of (deterministic) partial differential equations which are studied under the name of variational inequalities. We mention Lions and Stampacchia [7] and

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Brézis [2] among the pioneers, and Bensoussan and Lions [1] as a more recent reference. Research in this field has been motivated by applications to mechanics (see [3]) and to stochastic control theory (see [1]). Menaldi [9] and Bensoussan and Lions [1] have used the framework of variational inequalities to study finite-dimensional reflected diffusions thus introducing the notion of a stochastic variational inequality. We use a similar framework for the case of a stochastic partial differential equation.

Let us explain our results. Let V be an appropriate Sobolev space of functions with V a subspace of $L^2(0,1)$. Let $(\cdot,\cdot)$ denote the inner product in $L^2(0,1)$ and let $\langle\cdot,\cdot\rangle$ denote the pairing between V and its dual V'. A is a bounded linear operator: $V\to V'$ and similarly B_i: $V\to L^2(0,1)$, $i=1,\dots,d$. $\{W_t^1,\dots,W_t^d\}$ is a d-dimensional standard Brownian motion, M_t is a continuous V-valued martingale, $f(t)$ is an $L^2(0,1)$-valued process, and $g_i(t)$ is a V-valued process, $i=1,\dots,d$. u_0 is a suitably measurable V-valued random variable. We are looking for a pair (u,η) such that

(i) u is a V-valued process and $u(\cdot,\cdot)$ is continuous, $u(t,x)\geq 0$ a.s.; η is a finite, nonnegative random measure on $[0,T]\times[0,1]$ a.s.;

(ii) for all t, $0\leq t\leq T$, for all $v\in V$

$$(u(t),v)+\int_0^t \langle Au(s),v\rangle\,ds$$
$$=(u_0,v)+\int_0^t (f(s),v)\,ds+\sum_{i=0}^d\int_0^t (B_iu(s)+g_i(s),v)\,dW_s^i+(M_t,v)$$
$$+\int_0^t\int_0^1 v(x)\eta(ds,dx)\quad\text{a.s.};$$

(iii) for all continuous, nonnegative functions v defined on $[0,T]\times[0,1]$

$$\int_0^T\int_0^1 (v(t,x)-u(t,x))\eta(dt,dx)\geq 0.$$

Heuristically we write (ii) as

$$du(t)+Au(t)\,dt=f(t)\,dt+\sum_i (B_iu(t)+g(t))\,dW_t^i+dM_t+d\eta,$$
$$u(0)=u_0.$$

Intuitively, η represents the amount of pushing upward required to keep the solution nonnegative, and (iii) says that the minimum pushing is performed, i.e., $\eta(dt,dx)\neq 0$, only when $u(t,x)=0$. It is important to note that the measure $\eta(dt,dx)$ is absolutely continuous with respect to dx but *not* with respect to dt. This fact, which is due to the presence of the martingale terms in (ii), is well known in finite dimensions where the analogue of $\int_0^t \eta(ds,\cdot)$ is the local time of the solution $u(t)$ at the boundary $u=0$. On the other hand, if $g_1\equiv g_2\equiv\cdots\equiv g_d\equiv 0$, $M\equiv 0$, and the operators B_i are such that $B_iu>0$ if $u=0$, $i=1,\dots,d$, then η is absolutely continuous with respect to $(dt\times dx)$.

We show under suitable hypotheses on the data that such a pair (u,η) exists, that A maps such u into $L^2(0,1)$, and that there is only one such pair (u,η).

In the case where $B_1 \equiv B_2 \equiv \cdots \equiv B_d \equiv 0$, a weak formulation of the above stochastic variational inequality (i.e., where the process η does not appear explicitly) has been considered by Rascanu in [13] and [14] where he establishes the existence of "weak" and "almost weak" solutions and the uniqueness of the latter. This formulation seems somewhat unnatural to us; however, it imposes fewer restrictions on the set where u assumes its values. A major difference in our work is that we assume slightly more regularity (in x) of the data to obtain continuity of the solution u and hence uniqueness. Moreover, our solution is a strong solution.

As far as the organization of this paper is concerned, in Section 2 we define the problem precisely and state and prove the existence and uniqueness result, modulo some technical lemmas which comprise Section 3.

2. Existence and Uniqueness

Notation 2.1. We set $O=(0,1)$, $\bar{O}=[0,1]$, and $Q=(0,T)\times O$ for some fixed finite $T>0$. Set $\Gamma=\{0,1\}$, the boundary of O and let Γ_0, Γ_1 be two possibly empty subsets of Γ such that $\Gamma=\Gamma_0\cup\Gamma_1$. Let $H=L^2(O)$ and

$$V=\{u\in H^1(O)\colon u(x)=0,\, x\in\Gamma_0\}.$$

Recall that

$$H^1(O)=\{u\in H\colon u'_x\in H\},$$

where u'_x is the distributional derivative of u. Hence if u is in V, then it is square integrable, has square integrable derivate, and, depending on Γ_0, u may satisfy some Dirichlet-type boundary condition. We let $\|\cdot\|$ and $((\cdot,\cdot))$ denote the norm and scalar product in V. On H we denote the norm and scalar product by $|\cdot|$ and $(\cdot,\cdot)$. V is a dense subspace of H with induced topology weaker than the $\|\cdot\|$-norm topology so the injection of V into H is continuous. Let V' denote the dual of V (with the $\|\cdot\|$ norm topology) and denote this pairing by $\langle\cdot,\cdot\rangle$. Identifying H with its dual H' we have

$$V\subset H\cong H'\subset V', \tag{2.1}$$

where the two injections are continuous with dense range. From Sobolev's imbedding theorem we have

$$V\subset C(\bar{O}) \tag{2.2}$$

with compact imbedding since O is one-dimensional and bounded.

We also make use of $H^m(O)$, the set of functions in H which have m derivatives in H, and we denote the norm in $H^m(O)$ by $\|\cdot\|_{H^m}$, so that

$$\|\cdot\|_{H^1}=\|\cdot\|.$$

As usual V^d denotes the product of d copies of V and

$$W^{1,\infty}(O)=\{u\in L^\infty(O)\colon u'_x\in L^\infty(O)\}.$$

We write $\mathscr{M}(\bar{Q})$ for the Banach space of signed measures on $\bar{Q}$ with the norm being the total variation $\|\cdot\|_{\bar{Q}}$. Let $\mathscr{M}_+(\bar{Q})$ denote the subset of nonnegative measures on $\bar{Q}$. If $C(\bar{Q})$ denotes the set of continuous functions on $\bar{Q}$, then $\mathscr{M}(\bar{Q})$ can and will be identified with $C(\bar{Q})'$.

We define

$$C_+(\bar{Q})=\{v\in C(\bar{Q}): v(t,x)\geq 0, (t,x)\in\bar{Q}\}.$$

We assume that we are given a stochastic basis $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{0\leq t\leq T})$ such that $\mathscr{F}_0$ contains all the P-null sets of $\mathscr{F}$. Let $M^2(0, T; V)$ denote the set of a.s. continuous V-valued martingales $\{M_t: 0\leq t\leq T\}$ such that $M_0=0$ and

$$E\|M_T\|^2<\infty.$$

If M is in $M^2(0, T; V)$, then $\{\|M_t\|^2\}$ is a real-valued continuous submartingale, so there exists a continuous increasing process $\{\langle M\rangle_t: 0\leq t\leq T\}$ such that $\|M_t\|^2-\langle M\rangle_t$ is a martingale (see p. 132 of [10]). $\langle M\rangle_t$ is called the quadratic variation of M_t. In addition there exists a process $\{Q_t^M: 0\leq t\leq T\}$ assuming values in the set of positive symmetric elements of $\mathscr{L}(V, V)$ of trace class such that

$$\operatorname{tr} Q_t^M=1 \qquad \text{a.s., a.e.}$$

and $M_t\otimes M_t-\int_0^t Q_s^M\, d\langle M\rangle_s$ is a martingale (see [10] again). We call Q_t^M the normalized covariance of M_t.

On the other hand, if M is in $M^2(0, T; V)$, then it is also in $M^2(0, T; H)$. Let us denote the quadratic variation of M_t in H by $\langle M\rangle_t^H$ and the normalized covariance in H by q_t^M. Observe that

$$0\leq\langle M\rangle_t^H\leq\langle M\rangle_t.$$

If u is in H, then define

$$u^-(x)=\max\{0, -u(x)\}.$$

If u is in $H^1(O)$, then so is u^- and for almost all x

$$(u^-)'_x(x)=-u'_x(x)1_{\{u(x)<0\}}$$

see p. 145 of [4].

Finally, we remark that we use the convention that repeated indices are summed—the summation sign is omitted. For convenience of notation we frequently write

$$u'_x=\nabla u, \qquad u''_{xx}=\Delta u.$$

The Stochastic Variational Inequality 2.2. We define a bilinear form $a(u, v)$ on V by

$$a(u,v)=\int_0^1 a(x)u'_x(x)v'_x(x)\,dx+\int_0^1 b(x)u'_x(x)v(x)\,dx+\int_0^1 c(x)u(x)v(x)\,dx. \tag{2.3}$$

We assume that a, b, c are in $W^{1,\infty}(O)$ so that this bilinear form is continuous; hence

$$\langle Au, v\rangle = a(u, v)$$

defines a continuous linear mapping A of V into V', i.e., A is an element of $\mathscr{L}(V; V')$. Now for $i=1, \ldots, d$ let α_i, β_i be elements of $W^{1,\infty}(O)$ and define

$$(B_i u)(x) = \alpha_i(x)u_x'(x) + \beta_i(x)u(x); \qquad i=1, \ldots, d. \tag{2.4}$$

Then B_i is in $\mathscr{L}(V; H)$. Observe that if α_i is also an element of V, i.e., that $\alpha_i(x)=0$ for x in Γ_0, then B_i is an element of $\mathscr{L}(V \cap H^2(O); V)$.

Let us collect the hypotheses which we impose on the data:

$a, b, c \in W^{1,\infty}(O)$; for $i=1, 2, \ldots, d$, $\alpha_i, \beta_i \in W^{1,\infty}(O)$ and $\alpha_i(x)=0$ for x in Γ_0; there exists $\theta>0$ such that, for all x in $\bar{O}$, $2a(x) - \sum_{i=1}^d \alpha_i^2(x) \geq \theta$. (2.5)

$(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{0\leq t\leq T})$ is a stochastic basis; $\{W_t: 0\leq t\leq T\}$ is a standard d-dimensional Brownian motion on this basis; $\{M_t: 0\leq t\leq T\} \in M^2(0, T; V)$ such that $E\langle M\rangle_T^2 < \infty$ and there exists $m \in L^2(\Omega\times(0, T))$ such that $d\langle M\rangle_t^H = m(t)\,dt$; M and W are independent. (2.6)

$u_0 \in L^4(\Omega; V)$, $u_0(x) \geq 0$, $dx\,dP$ a.e., u_0 is $\mathscr{F}_0$ measurable, independent of M; $f \in L^4(\Omega\times(0, T); H)$, $g \in L^2(\Omega\times(0, T); V^d) \cap L^4(\Omega\times(0, T); H^d)$ and f, g are $\{\mathscr{F}_t\}$ adapted. (2.7)

Observe that (2.2) implies that $u_0(\cdot)$ is continuous a.s. The extension to the case, when A and B are time-dependent, is standard.

We can now define precisely what is meant by a solution of a stochastic variational inequality (S.V.I.).

Definition 2.3. The pair (u, η) solves the S.V.I.

$$\begin{aligned} &du + Au\,dt = (B_i u + g_i)\,dW_t^i + dM_t + f\,dt + d\eta, \\ &u(0) = u_0, \end{aligned} \tag{2.8}$$

if

(i) $u \in L^2(\Omega; L^2((0, T); V) \cap C_+(\bar{Q}))$ and u is $\{\mathscr{F}_t\}$ adapted, $\eta \in L^2(\Omega; \mathscr{M}(\bar{Q}))$,

(ii) $$\begin{aligned}(u(t), v) &+ \int_0^t a(u(s), v)\,ds \\ &= (u_0, v) + \int_0^t (B_i u(s) + g_i(s), v)\,dW_s^i + (M_t, v) + \int_0^t (f(s), v)\,ds \\ &\quad + \int_0^t \int_0^1 v(x)\eta(dt, dx), \forall t, \text{ a.s., } \forall v \in V,\end{aligned}$$

(iii) $$\int_{\bar{Q}} (v-u)\,d\eta \geq 0 \text{ a.s. } \forall v \in C_+(\bar{Q}).$$

Note that the double integral in (ii) makes sense by (2.2), and that from (iii) it follows that a.s. $\eta(\omega) \in \mathcal{M}_+(\bar{Q})$ and has support on $\{(t, x)\colon u(t, x) = 0\}$.

Theorem 2.4. *Assume* (2.5), (2.6), *and* (2.7). *Then the* S.V.I. (2.8) *has at least one solution.*

Proof. With (2.8) we associate the following penalized equation for $\varepsilon > 0$:

$$du_\varepsilon(t) + Au_\varepsilon(t)\,dt = [B_i u_\varepsilon(t) + g_i(t)]\,dW_t^i + dM_t + f(t)\,dt + \varepsilon^{-1} u_\varepsilon^-(t)\,dt, \qquad u_\varepsilon(0) = u_0. \tag{2.9}$$

According to Pardoux [11, p. 105], (2.9) has a unique adapted solution u_ε in $L^2(\Omega; L^2((0, T); V) \cap C([0, T]; H))$. However, we require more regularity of u_ε in order to pass to the limit as $\varepsilon \to 0$. We organize the rest of the proof into four steps.

Step 1. Uniform (in ε) bounds on u_ε.

There exists a complete orthonormal basis $\{e_i\}$ of H such that e_i is a solution of the eigenvalue problem:

$$e_i \in V, \qquad \lambda_i \geq 1, \qquad (e_i, v) + (\nabla e_i, \nabla v) = \lambda_i (e_i, v), \qquad \forall v \in V. \tag{2.10}$$

Note that each e_i is either a sine or cosine depending on the boundary conditions, hence is in $C^\infty(\bar{O})$. Alternatively, since $\Delta e_i = (1 - \lambda_i) e_i$ then $e_i \in H^m(O)$ implies $e_i \in H^{m+2}(O)$; consequently $e_i \in H^m(O)$ for all m, hence is in $C^\infty(\bar{O})$. Moreover, $\nabla e_i(x) = 0$ for x in Γ_1, so that

$$v(x) \nabla e_i(x) = 0, \qquad \forall x \in \Gamma, \tag{2.11}$$

if $v(x) = 0$ for x in Γ_0.

Let $\{u_\varepsilon^n\colon n = 1, 2, \ldots\}$ be a Galerkin approximation of u_ε constructed as follows:

$$u_\varepsilon^n(t) = \sum_{j=1}^n (u_\varepsilon^n(t), e_j) e_j, \tag{2.12}$$

$$\begin{aligned}(u_\varepsilon^n(t), e_j) &+ \int_0^t a(u_\varepsilon^n(s), e_j)\,ds \\ &= (u_0, e_j) + \int_0^t (B_i u_\varepsilon^n(s) + g_i(s), e_j)\,dW_s^i + (M_t, e_j) \\ &\quad + \int_0^t (f(s), e_j)\,ds + \frac{1}{\varepsilon} \int_0^t (u_\varepsilon^{n-}(s), e_j)\,ds, \qquad j = 1, \ldots, n.\end{aligned} \tag{2.13}$$

Note that we have written u_ε^{n-} for $(u_\varepsilon^n)^-$. After we substitute the right-hand side of (2.12) for u_ε^n in (2.13) we find that (2.13) is a stochastic differential equation with bounded, Lipschitz continuous coefficients for the n dependent variables

$(u_\varepsilon^n, e_j), j=1,\ldots,n$; hence we can solve it uniquely and then use (2.12) to define u_ε^n.

From Lemma 3.4 it follows that, for some constant c depending on the data but not on n or ε,

$$E\left\{\sup_{t\le T}\|u_\varepsilon^n(t)\|^4+\left(\int_0^T\|u_\varepsilon^n(t)\|_{H^2}^2\,dt\right)^2\right\}\le c,$$

$$E\left\{\left(\int_0^T\|u_\varepsilon^{n-}(t)\|^2\,dt\right)^2\right\}\le \varepsilon^2 c.$$

It follows that, for fixed ε, the set $\{u_\varepsilon^n: n=1,2,\ldots\}$ is weakly sequentially compact in $L^4(\Omega; L^2((0,T); H^2))$ and weak-* sequentially compact in $L^4(\Omega; L^\infty((0,T); V))$, and that $\{u_\varepsilon^{n-}: n=1,2,\ldots\}$ is weakly sequentially compact in $L^4(\Omega; L^2((0,T); V))$. Since we also know that $\{u_\varepsilon^n\}$ converges weakly to u_ε in $L^2(\Omega\colon L^2((0,T); V))$ (see p. 113 of [11]) then it must also converge to u_ε in the above stronger sense and

$$E\left\{\sup_{t\le T}\|u_\varepsilon(t)\|^4+\left(\int_0^T\|u_\varepsilon(t)\|_{H^2}^2\,dt\right)^2\right\}\le c. \tag{2.14}$$

Since the operator $\tilde{A}$ from V into V' defined by

$$\tilde{A}u=Au-\varepsilon^{-1}u^-,\qquad u\in V,$$

is monotone in the sense that

$$\langle\tilde{A}u-\tilde{A}v, u-v\rangle\ge 0,\qquad u, v\in V,$$

it follows from the proof on pp. 116–119 of [11] that $\{\tilde{A}u_\varepsilon^n\}$ converges weakly to $\tilde{A}u_\varepsilon$ in $L^2(\Omega; L^2((0,T); V'))$. It now follows from the above estimate that $\{u_\varepsilon^{n-}\}$ converges weakly to u_ε^- in $L^4(\Omega; L^2((0,T); V))$ and that moreover

$$E\left\{\left(\int_0^T\|u_\varepsilon^-(t)\|^2\,dt\right)^2\right\}\le \varepsilon^2 c. \tag{2.15}$$

This result tells us that $u_\varepsilon^-/\sqrt{\varepsilon}$ lies in a bounded set in $L^2(\Omega\times(0,T); V)$ but we require more: we will show that $u_\varepsilon^-/\varepsilon$ lies in a bounded set of $L^2(\Omega; L^1(Q))$. Define

$$\mathcal{V}=\{u\in H^2(O)\colon u(x)=0 \text{ if } x\in\Gamma_0, \nabla u(x)=0 \text{ if } x\in\Gamma_1\}.$$

Although $A\in\mathcal{L}(V; V')$ it is easy to see that $A\in\mathcal{L}(\mathcal{V}; H)$ also. Hence if u_ε is $\mathcal{V}$-valued then $Au_\varepsilon\in H$. In view of (2.14) to show that u_ε is $\mathcal{V}$-valued all we need to show is that $\nabla u_\varepsilon(t,x)=0$ for x in Γ_1. According to (2.9)

$$d(u_\varepsilon(t), v)+a(u_\varepsilon(t), v)\,dt=d\Phi_t^v \tag{2.16}$$

for any $v \in V$, where $d\Phi_t^v$ represents the inner product of the right-hand side of (2.9) and v. Since $u_\varepsilon \in H^2(O)$, then for $v \in H_0^1(O)$ we have

$$d(u_\varepsilon(t), v) + (-\nabla(a\nabla u_\varepsilon) + b\nabla u_\varepsilon + cu_\varepsilon, v)\, dt = d\Phi_t^v. \tag{2.17}$$

But $H_0^1(O)$ is dense in $H = L^2(O)$ so (2.17) holds for all $v \in H$. On the other hand, from (2.16) for $v \in V$

$$d(u_\varepsilon(t), v) + (-\nabla(a\nabla u_\varepsilon) + b\nabla u_\varepsilon + cu_\varepsilon, v)\, dt + \int_\Gamma a\nabla u_\varepsilon v\, dx\, dt = d\Phi_t^v. \tag{2.18}$$

Comparing (2.17) and (2.18), noting that $v = O$ on Γ_0 but is arbitrary on Γ_1 and that $a(x) > 0$ in view of (2.5), we see that $u_\varepsilon \in L^4(\Omega; L^2((0, T); \mathscr{V}))$.

As noted above, it now follows that Au_ε is H-valued, so (2.9) implies that u_ε is an H-valued semimartingale. Then Itô's formula (see p. 19 of [11]) implies that

$$\begin{aligned}
&|u_\varepsilon(t) - 1|^2 + 2\int_0^t (Au_\varepsilon(s), u_\varepsilon(s) - 1)\, ds \\
&= |u_0 - 1|^2 + 2\int_0^t (B_i u_\varepsilon(s) + g_i(s), u_\varepsilon(s) - 1)\, dW_s^i \\
&\quad + 2\int_0^t (u_\varepsilon(s) - 1, dM_s) + 2\int_0^t (f(s), u_\varepsilon(s) - 1)\, ds \\
&\quad + 2\varepsilon^{-1}\int_0^t (u_\varepsilon^-(s), u_\varepsilon(s) - 1)\, ds \\
&\quad + \sum_{i=1}^d \int_0^t |B_i u_\varepsilon(s) + g_i(s)|^2\, ds + \langle M\rangle_t^H.
\end{aligned} \tag{2.19}$$

Since A is in $\mathscr{L}(\mathscr{V}; H)$ and since for any real number r we have

$$r^-(r-1) = rr^- - r^- \leq -r^-,$$

then there exist constants c_0, c_1 such that

$$\begin{aligned}
&|u_\varepsilon(t) - 1|^2 + 2\varepsilon^{-1}\int_0^t \|u_\varepsilon^-(s)\|_{L^1(O)}\, ds \\
&\leq |u_0 - 1|^2 + c_0\int_0^t \|u_\varepsilon(s)\|_{H^2} |u_\varepsilon(s) - 1|\, ds \\
&\quad + 2\int_0^t (B_i u_\varepsilon(s) + g_i(s), u_\varepsilon(s) - 1)\, dW_s^i + 2\int_0^t (u_\varepsilon(s) - 1, dM_s) \\
&\quad + 2\int_0^t (f(s), u_\varepsilon(s) - 1)\, ds + \sum_{i=1}^d \int_0^t |B_i u_\varepsilon(s) + g_i(s)|^2\, ds + \langle M\rangle_t^H,
\end{aligned}$$

$$\begin{aligned}
&E\{(\varepsilon^{-1}\|u_\varepsilon^-\|_{L^1(Q)})^2\} \\
&\leq c_1 E\left\{1 + |u_0|^4 + \left(\int_0^T \|u_\varepsilon(s)\|_{H^2}^2\, ds\right)^2 + \sup_{0 \leq t \leq T} |u_\varepsilon(t)|^4 \right. \\
&\quad \left. + \sum_i \left(\int_0^T |g_i(s)|^2\, ds\right)^2 + \left(\int_0^T |f(s)|^2\, ds\right)^2 + (\langle M\rangle_T^H)^2\right\}.
\end{aligned}$$

The bounds of (2.14) now imply that for some $c<\infty$

$$E\{(\varepsilon^{-1}\|u_\varepsilon^-(s)\|_{L^1(Q)})^2\}\le c. \tag{2.20}$$

We now have the required bounds: (2.14), (2.15), and (2.20).

Step 2. $u_\varepsilon \in L^2(\Omega; C(\bar{Q}))$.

Let $L^0(\Omega; X)$ denote the measurable maps: $\Omega\to X$. If $u_\varepsilon\in L^0(\Omega; C([0,T]; H^1(O)))$, i.e., $u_\varepsilon\in L^0(\Omega; C([0,T]; C(\bar{O})))$, see (2.2), then $u_\varepsilon\in L^0(\Omega; C(\bar{Q}))$. This follows from the Sobolev inequality:

$$\sup_x|u(t,x)|+\sup_{x\neq y}\frac{|u(t,x)-u(t,y)|}{|x-y|^{1/2}}\le\|u(t)\|.$$

Then (2.14) implies that $u_\varepsilon\in L^4(\Omega; C(\bar{Q}))$ and the desired result follows. Hence it suffices to show that $u_\varepsilon\in L^0(\Omega, C([0,T]; H^1(O)))$.

Let us for the moment take V as the basic Hilbert space, so for convenience we set $\mathcal{H}=V$. Then identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have

$$\mathcal{V}\subset\mathcal{H}\simeq\mathcal{H}'\subset\mathcal{V}',$$

the injections being continuous with dense range. Note that the norm on $\mathcal{V}$ is $\|\cdot\|_{H^2}$. The following result will be useful.

Lemma 2.5. *H (under an equivalent norm) can be identified with $\mathcal{V}'$.*

Proof. It is convenient to distinguish $\mathcal{H}$ and $\mathcal{H}'$ for the moment. For $v\in\mathcal{H}$, $u\in\mathcal{H}$ let

$$L_v u=((u,v)).$$

Then $L_v\in\mathcal{H}'$ with $\|L_v\|_{\mathcal{H}'}=\|v\|_{\mathcal{H}}=\|v\|$. But $\mathcal{H}'\subset\mathcal{V}'$ so $L_v\in\mathcal{V}'$. For $u\in\mathcal{V}$ we have $v\nabla u=0$ on Γ so

$$L_v u=((u,v))=(u-\Delta u, v).$$

Hence $\|L_v\|_{\mathcal{V}'}\le|v|$, so the mapping $L: v\to L_v$ of $\mathcal{H}$ into $\mathcal{V}'$ is continuous and linear if we use the H-norm on $\mathcal{H}$, i.e., L is a densely defined continuous linear map of H into $\mathcal{V}'$. Thus it can be extended to all of H. If we define a new norm on H by

$$|v|'=\sup_{\|u\|_{\mathcal{V}}=1}|(u-\Delta u, v)|, \tag{$*$}$$

then $|v|'\le|v|$. The a priori estimate $\|u\|_{\mathcal{V}}\le\kappa|u-\Delta u|$ yields that $|v|\le\kappa|v|'$ (in $(*)$ take $u=u/\|u\|_{\mathcal{V}}$ where $u-\Delta u=v$) so that $|\cdot|$ and $|\cdot|'$ are equivalent norms on H. Moreover, $\|L_v\|_{\mathcal{V}'}=|v|'$, i.e., $L: H\to\mathcal{V}'$ is an isometry. Since $L\mathcal{H}=\mathcal{H}'$ is dense in $\mathcal{V}'$ the result follows. □

We can rewrite (2.9) as

$$u_\varepsilon(t)=u_0(t)+\int_0^t v(s)\,ds+N_t,$$

where $u_0\in L^2(\Omega;\mathscr{H})$ and

$$v(s)=-Au_\varepsilon(s)+f(s)+\varepsilon^{-1}u_\varepsilon^-(s)$$

is in $L^0(\Omega;L^2((0,T);H))\subset L^0(\Omega;L^2((0,T);\mathscr{V}'))$, see Lemma 2.5. Moreover,

$$N_t=\int_0^t [B_iu_\varepsilon(s)+g_i(s)]\,dW_s^i+M_t$$

is in $M^2(0,T;V)=M^2(0,T;\mathscr{H})$. Now a result of Pardoux [11, pp. 57-59, Theorems 3.1 and 3.3] implies that $u_\varepsilon\in L^0(\Omega;C([0,T];\mathscr{H}))$.

Step 3. $\{u_\varepsilon\}$ is Cauchy in $L^2(\Omega;C(\bar{Q}))$.
Again a preliminary lemma is helpful.

Lemma 2.6. *If $\{u_\varepsilon\}$ is Cauchy in $L^2(\Omega;C([0,T];H))$, then it is Cauchy in $L^2(\Omega;C(\bar{Q}))$.*

Proof. From (2.2) it follows that for any $\eta>0$ there exists a constant C_η such for all $u\in V$

$$\|u\|_{C(\bar{O})}\le\eta\|u\|+C_\eta|u|$$

(see p. 58 of [6]). If u is in $L^2(\Omega;C([0,T]);V))$ it now follows that

$$\|u\|_{L^2(\Omega;C(\bar{Q}))}\le\sqrt{2}\eta\|u\|_{L^2(\Omega;C([0,T];V))}+\sqrt{2}C_\eta|u|_{L^2(\Omega;C([0,T];H))}. \tag{2.21}$$

Consequently for any $\eta>0$ there exists C_η such that for all $\varepsilon,\delta>0$

$$\|u_\varepsilon-u_\delta\|_{L^2(\Omega;C(\bar{Q}))}\le\sqrt{2}\eta\|u_\varepsilon-u_\delta\|_{L^2(\Omega;C[0,T];V))}+\sqrt{2}C_\eta\|u_\varepsilon-u_\delta\|_{L^2(\Omega;C([0,T];H))}.$$

The result now follows from (2.14). □

To show that $\{u_\varepsilon\}$ is Cauchy in $L^2(\Omega;C([0,T];H))$, let $v=u_\varepsilon-u_\delta$. Then

$$dv(t)+Av(t)\,dt=B_iv(t)\,dW_t^i+(\varepsilon^{-1}u_\varepsilon^--\delta^{-1}u_\delta^-)\,dt,$$

$$v(0)=0.$$

Hence

$$|v(t)|^2+2\int_0^t a(v(s),v(s))\,ds=2\int_0^t (B_iv(s),v(s))\,dW_s^i+\sum_i\int_0^t |B_iv(s)|^2\,ds+2\int_0^t (\varepsilon^{-1}u_\varepsilon^-(s)-\delta^{-1}u_\delta^-(s),v(s))\,ds.$$

But

$$(u_\varepsilon^-, u_\varepsilon - u_\delta) \le -(u_\varepsilon^-, u_\delta) \le (u_\varepsilon^-, u_\delta^-),$$
$$-(u_\delta^-, u_\varepsilon - u_\delta) \le -(u_\delta^-, u_\varepsilon) \le (u_\delta^-, u_\varepsilon^-)$$

so that

$$|v(t)|^2 + 2\int_0^t a(v(s), v(s))\, ds \le 2\int_0^t (B_i v(s), v(s))\, dW_s^i + \sum_i \int_0^t |B_i v(s)|^2\, ds$$
$$+ 2(\varepsilon^{-1} + \delta^{-1}) \int_0^T (u_\varepsilon^-(s), u_\delta^-(s))\, ds.$$

By Lemma 3.5 it now follows that for some constant c

$$E \sup_{t \le T} |u_\varepsilon(t) - u_\delta(t)|^2 \le c(\varepsilon^{-1} + \delta^{-1}) E \int_0^T (u_\varepsilon^-(t), u_\delta^-(t))\, dt$$
$$\le c\left[E\left\{ \left(\frac{1}{\varepsilon} \|u_\varepsilon^-\|_{L^1(Q)} \right)^2 \right\} \right]^{1/2} [E\{\|u_\delta^-\|^2_{L^\infty(Q)}\}]^{1/2}$$
$$+ c\left[E\left\{ \left(\frac{1}{\delta} \|u_\delta^-\|_{L^1(Q)} \right)^2 \right\} \right]^{1/2} [E\{\|u_\varepsilon^-\|^2_{L^\infty(Q)}\}]^{1/2}$$
$$\le c_1(\|u_\varepsilon^-\|_{L^2(\Omega; C(\bar{Q}))} + \|u_\delta^-\|_{L^2(\Omega; C(\bar{Q}))}),$$

where the last inequality follows from (2.20).

But by (2.14) $\{u_\varepsilon\}$, hence $\{u_\varepsilon^-\}$, is bounded in $L^2(\Omega; C([0, T]; V))$ and by Lemma 3.6, $u_\varepsilon^- \to 0$ in $L^2(\Omega; C([0, T]; H))$. Hence by (2.21) $u_\varepsilon^- \to 0$ in $L^2(\Omega; C(\bar{Q}))$ and $\{u_\varepsilon\}$ is Cauchy.

Step 4. Convergence of u_ε and $u_\varepsilon^-/\varepsilon$.

If we define η_ε in $\mathcal{M}_+(\bar{Q})$ by

$$\eta_\varepsilon(dt, dx) = \varepsilon^{-1} u_\varepsilon^-(t, x)\, dt\, dx,$$

then (2.20) says that $\{\eta_\varepsilon\}$ is bounded in $L^2(\Omega; \mathcal{M}(\bar{Q}))$, and since $\mathcal{M}(\bar{Q})$ can be identified with the dual of $C(\bar{Q})$ then we can extract a subsequence, again called $\{\eta_\varepsilon\}$, which converges weak-* to an element η in $L^2(\Omega; \mathcal{M}(\bar{Q}))$. Moreover, the bound (2.14) allows us to assume (by taking a further subsequence if necessary) that $\{u_\varepsilon\}$ converges weakly in $L^4(\Omega; L^2((0, T); \mathcal{V}))$ and weak-* in $L^4(\Omega; L^\infty((0, T); V))$ to an element u, which is again adapted.

To relate u and η we observe, see (2.9), that for any $v \in V$

$$(u_\varepsilon(t), v) + \int_0^t a(u_\varepsilon(s), v)\, ds = (u_0, v) + \int_0^t (B_i u_\varepsilon(s) + g_i(s), v)\, dW_s^i + (M_t, v)$$
$$+ \int_0^t (f(s), v)\, ds + \int_0^t \int_0 v(x) \eta_\varepsilon(ds, dx). \tag{2.22}$$

For $t<T$ fixed, $0<\delta\leq T-t$, define

$$\varphi_\delta(s)=\begin{cases}1 & \text{if } s\leq t,\\ (t+\delta-s)/\delta & \text{if } t\leq s\leq t+\delta,\\ 0 & \text{if } s\geq t+\delta.\end{cases}$$

Then φ_δ is continuous and

$$\left|\int_{\bar{Q}}\varphi_\delta v\,d\eta_\varepsilon-\int_0^t\int_{\bar{O}}v(x)\,d\eta_\varepsilon\right|\leq\int_t^{t+\delta}\int_{\bar{O}}|v(x)|\,d\eta_\varepsilon.$$

Since $|v(\cdot)|\in V$ if $v\in V$, then (2.22) implies that

$$\begin{aligned}E&\left\{\left(\int_t^{t+\delta}\int_{\bar{O}}|v(x)|\,d\eta_\varepsilon\right)^2\right\}\\ &\leq o(1)+cE\left\{\left(\int_t^{t+\delta}\|u_\varepsilon(s)\|^2\,ds\right)^2+|v|^2\int_t^{t+\delta}\|u_\varepsilon(s)\|^2\,ds\right.\\ &\qquad\left.+|u_\varepsilon(t+\delta)-u_\varepsilon(t)|^2\right\}\end{aligned}\tag{2.23}$$

for some constant c where $o(1)\to 0$ as $\delta\to 0$ and is independent of ε (but not of v). The bound (2.14) implies that the two integral terms on the right-hand side of (2.13) are also $o(1)$ uniformly in ε. Since $\{u_\varepsilon\}$ is Cauchy in $L^2(\Omega;\,C([0,T];\,H))$ then, for a subsequence $\varepsilon_n\to 0$, $E\{|u_{\varepsilon_n}(t+\delta)-u_{\varepsilon_n}(t)|^2\}$ is also $o(1)$ as $\delta\to 0$ uniformly in n.

We can now replace the last term in (2.22) by $\int_{\bar{Q}}\varphi_\delta v\,d\eta_\varepsilon+\tilde{o}(1)$, and pass to the limit along $\varepsilon_n\to 0$, and then let $\delta\to 0$. Note that $\lim_\delta\sup_n E|\tilde{o}(1)|^2=0$. This shows that (u,η) satisfies Definition 2.3(ii).

Since $\{u_\varepsilon\}$ is Cauchy in $L^2(\Omega;\,C(\bar{Q}))$ then u lies in $L^2(Q;\,L^2((0,T);\,V)\cap C(\bar{Q}))$. We shall now show that $u\geq 0$ a.s., i.e., (u,η) satisfies Definition 2.3(i). Since

$$-E\int_0^T((u_\varepsilon^-,u))\,dt\to E\int_0^T\|u^-\|^2\,dt,$$

and since (2.15) implies that

$$\left|-E\int_0^T((u_\varepsilon^-,u))\,dt\right|\leq\kappa\sqrt{\varepsilon},$$

where κ depends on u but not ε, then $\|u^-\|^2=0$ a.e. $dt\,dP$ and so $u(t,x)\geq 0$ a.s.

Finally, the strong convergence of u_ε to u in $L^2(\Omega;\,C(\bar{Q}))$ and the weak convergence of η_ε to η implies that

$$E\int_{\bar{Q}}(v-u_\varepsilon)\,d\eta_\varepsilon\to E\int_{\bar{Q}}(v-u)\,d\eta.\tag{2.24}$$

But if $v\in C_+(\bar{Q})$, then

$$E\int_{\bar{Q}}(v-u_\varepsilon)u_\varepsilon^-\,dx\,dt\geq 0$$

and now (2.24) implies that (u, η) satisfies Definition 2.3(iii), i.e., the proof is complete. □

We saw in the proof that the solution u constructed above is $\mathcal{V}$-valued, so that

$$a(u(s), v) = (Au(s), v),$$

where $Au(s)$ is in H. From Definition 2.3(ii) it now follows that

$$\begin{aligned}\int_0 \int_0^t v(x)\eta(d\theta, dx) &= \int_0 v(x)\Big\{u(t, x) - u_0(x) - M_t(x) + \int_0^t [Au(s, x) - f(s, x)]\, ds \\ &\qquad - \int_0^t [B_i u(s, x) + g_i(s, x)]\, dW_s^i\Big\}\, dx\end{aligned}$$

so that in fact

$$\eta(dt, dx) = \eta'(dt, x)\, dx,$$

where $\eta'(dt, x)$ is in $L^2(\Omega; L^1(O; \mathcal{M}_+([0, T])))$ since

$$E\left\{\left(\int_0 \int_0^T \eta'(dt, x)\, dx\right)^2\right\} = E\{\eta(Q)^2\} < \infty.$$

Hence we have

Corollary 2.7. *Assume* (2.5), (2.6), *and* (2.7). *If* (u, η) *is a solution of* (2.8) *such that u is in* $L^2(\Omega; L^2((0, T); \mathcal{V}))$, *then there exists* η' *in* $L^2(\Omega; L^1(O; \mathcal{M}_+([0, T])))$ *such that*

$$\eta(dt, dx) = \eta'(dt, x)\, dx.$$

We can now address the uniqueness question.

Theorem 2.8. *Let* (u, η), (v, ν) *be two solutions of* (2.8) *such that* u, v *are elements of* $L^2(\Omega \times (0, T); \mathcal{V})$. *Then*

$$E \sup_t |u(t) - v(t)|^2 = 0, \qquad E\|\eta - \nu\|_{\bar{Q}} = 0.$$

Proof. Let $\bar{u} = u - v$, $\bar{\eta} = \eta - \nu$, $\bar{\eta}' = \eta' - \nu'$. Then for $\bar{v}$ in V

$$(\bar{u}(t), \bar{v}) + \int_0^t (A\bar{u}(s), \bar{v})\, ds = \int_0^t (B_i\bar{u}(s), \bar{v})\, dW_s^i + \int_0^t (\bar{v}, \bar{\eta}(ds)). \tag{2.25}$$

We shall now establish a pointwise version of (2.25). A similar result can be found in [5], but our proof is more direct. Using Fubini's theorem for the interchange of the dx and ds integrals, and well-known results on Hilbert-space-valued stochastic integrals for the interchange of the dx and dW_s integrals, we deduce for almost all t

$$(\bar{u}(t), \bar{v}) + \left(\int_0^t A\bar{u}(s)\, ds, \bar{v}\right) = \left(\int_0^t B_i\bar{u}(s)\, dW_s^i, \bar{v}\right) + (\bar{\eta}'(t), \bar{v}),$$

where $\bar{\eta}'(t)$ stands for $\bar{\eta}'([0, t], \cdot)$. In other words, we have the following inequality between continuous $L^2(O)$-valued processes:

$$\bar{u}(t)+\int_0^t A\bar{u}(s)\,ds=\int_0^t B_i\bar{u}(s)\,dW_s^i+\bar{\eta}'(t). \tag{2.26}$$

Moreover, $t\to\int_0^t B_i\bar{u}(s)\,dW_s^i$ is a.s. continuous with values in V, hence

$$(t, x)\to\int_0^t B_i\bar{u}(s, x)\,dW_s^i$$

is a.s. continuous. On the other hand, $\bar{u}\in C(\bar{Q})$ a.s. and

$$t\to\int_0^t A\bar{u}(s, x)\,ds$$

is continuous $dx\,dP$ a.e. Hence it follows from (2.26) that $t\to\bar{\eta}'(t, x)$ is continuous $dx\,dP$ a.e. and for (x, ω) not in some null set

$$\bar{u}(t, x)+\int_0^t A\bar{u}(s, x)\,ds=\int_0^t B_i\bar{u}(s, x)\,dW_s^i+\bar{\eta}'(t, x), \qquad \forall t. \tag{2.27}$$

We now apply Itô's formula to (2.27) for each x fixed. In one dimension we have no difficulty with the bounded variation term. Thus for almost all (ω, x)

$$\bar{u}(t, x)^2+2\int_0^t A\bar{u}(s, x)\bar{u}(s, x)\,ds$$
$$=2\int_0^t B_i\bar{u}(s, x)\bar{u}(s, x)\,dW_s^i+2\int_0^t \bar{u}(s, x)\bar{\eta}'(ds, x)+\sum_i\int_0^t [B_i\bar{u}(s, x)]^2\,ds.$$

Now integrating over x and using Fubini's theorem we find

$$|\bar{u}(t)|^2+2\int_0^t a(\bar{u}(s), \bar{u}(s))\,ds$$
$$=2\int_0^t (B_i\bar{u}(s), \bar{u}(s))\,dW_s^i+2\int_0^t (\bar{u}(s), \bar{\eta}'(ds))+\sum_i\int\int_0^t |B_i\bar{u}(s)|^2\,ds. \tag{2.28}$$

Note that the equality

$$\int_0^1\left(\int_0^t [B_i\bar{u}(s, x)]\bar{u}(s, x)\,dW_s^i\right)dx=\int_0^t (B_i\bar{u}(s), \bar{u}(s))\,dW_s^i$$

follows from the properties of Hilbert-space-valued stochastic integrals since

$$\bar{u}\in L^0(\Omega; C(\bar{Q})), \qquad B_i\bar{u}\in L^0(\Omega; L^2((0, T); H)),$$

so that

$$[B_i\bar{u}]\bar{u}\in L^0(\Omega; L^2((0, T); H))$$

and $1\in H$.

From Definition 2.3(iii) we have

$$\int_0^T (\bar{u}(s)-u(s), \eta'(ds))\geq 0 \qquad \text{a.e.} \tag{2.29}$$

for $\tilde{u}$ in $C_+(\bar{Q})$. Let $\{w_n\}$ be a sequence in $C_+([0, T])$ such that $w_n(s)$ increases to $1_{\{s\le t\}}$ pointwise and define

$$\tilde{u}_n(s) = u(s) + w_n(s)[\tilde{u}(s) - u(s)].$$

Replacing $\tilde{u}$ by $\tilde{u}_n$ in (2.29) and letting $n \to \infty$ yields

$$\int_0^t (\tilde{u}(s) - u(s), \eta'(ds)) \ge 0 \qquad \text{a.e.} \tag{2.30}$$

by the bounded convergence theorem. Now taking $\tilde{u}(s, x) = v(s, x, \omega)$ for ω fixed in (2.29), we find

$$\int_0^t (v(s) - u(s), \eta'(ds)) \ge 0 \qquad \text{a.e.}$$

and similarly

$$\int_0^t (u(s) - v(s), \nu'(ds)) \ge 0 \qquad \text{a.e.}$$

so that

$$\int_0^t (\bar{u}(s), \bar{\eta}'(ds))$$
$$= \int_0^t (u(s) - v(s), \eta'(ds)) + \int_0^t (v(s) - u(s), \nu'(ds)) \le 0 \qquad \text{a.e.}$$

But then (2.28) implies

$$|\bar{u}(t)|^2 + 2\int_0^t a(\bar{u}(s), \bar{u}(s))\, ds \le 2\int_0^t (B_i\bar{u}(s), \bar{u}(s))\, dW_s^i + \sum_i \int_0^t |B_i\bar{u}(s)|^2\, ds.$$

From Lemma 3.5 it follows that

$$E \sup_t |u(t) - v(t)|^2 = 0.$$

Now (2.27) implies that for almost all (ω, x) and all t

$$\int_0^t \bar{\eta}(ds, x) = 0$$

so that

$$\|\bar{\eta}'(\cdot, x)\|_{[0,T]} = 0 \qquad \text{a.e. } (\omega, x)$$

and hence

$$E\|\eta - \nu\|_{\bar{Q}}^2 = 0.$$

We are done. □

Remark 2.9. The above result only gives uniqueness in $L^2(\Omega \times (0, T); H^2(O))$, whereas the solutions of an S.V.I. are defined as elements of $L^2(\Omega; L^2((0, T); V))$, see Definition 2.3. The problem is that we do not have available an Itô formula for H-valued semimartingales where the bounded variation term has the form

$$\int v_1\, dt + \int dv_2$$

with v_1 assuming values in V' and v_2 not absolutely continuous with respect to dt. The result of Pardoux [11] requires $v_2 = 0$ and that of Métivier [10] requires v_1 to be H-valued with v_2 to have bounded variation with values in H. Nevertheless, we believe that the results of Pardoux could be extended to the case $v_2 \neq 0$ to provide uniqueness of solutions u in

$$L^2(\Omega; L^2((0, T); V) \cap C_+(\bar{Q})).$$

Remark 2.10. We have chosen the reflecting boundary at $u = 0$. Suppose we wish more generally that $u(t, x) \geq \psi(t, x)$ a.s. Then in Definition 2.3 we must replace $C_+(\bar{Q})$ by

$$C_\psi(\bar{Q}) = \{u \in C(\bar{Q}): u(t, x) \geq \psi(t, x)\}.$$

If ψ is random then so is $C_\psi(\bar{Q})$. Now assume

$$d\psi = \bar{f}\, dt + \bar{g}_i\, dW_t^i + d\bar{M}_t$$

and let $\tilde{u} = u - \psi$. If (u, η) satisfies (2.8) with $u \geq \psi$, then $(\tilde{u}, \eta)$ satisfies

$$d\tilde{u} + A\tilde{u}\, dt = (B_i\tilde{u} + \tilde{g}_i)\, dW_t^i + d\tilde{M}_t + \tilde{f}\, dt + d\eta \tag{2.31}$$

with $\tilde{u} \geq 0$, where

$$\tilde{g}_i = g_i - \bar{g}_i + B_i\psi, \qquad \tilde{f} = f - \bar{f} - A\psi, \qquad \tilde{M} = M - \bar{M},$$

and vice versa. Consequently if $\tilde{f}$, $\tilde{g}$, $\tilde{u}_0 = u_0 - \psi(0)$ and $\tilde{M}$ satisfy (2.6), (2.7) and a, b, c satisfy (2.5), then a unique solution $(\tilde{u}, \eta)$ of (2.31) exists, and hence $(\tilde{u} + \psi, \eta)$ is the unique solution of (2.8) with $u \geq \psi$. It is necessary that ψ be $V \cap H^2(O)$-valued but we can allow ψ to be piecewise continuous in t by breaking $(0, T)$ into subintervals, on each of which we solve (2.30). For example, if we require

$$u(t, x) \geq \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ 1 & \text{if } 1 \leq t \leq 2, \end{cases}$$

we solve (2.8) on $0 \leq t \leq 1$ to obtain (u^1, η^1). Next we solve (2.8) on $1 \leq t \leq 2$ with $u \geq 1$ and with initial condition $u(1, x) = \max\{1, u^1(1, x)\}$, to obtain (u^2, η^2). Then (u, η) solves the original problem where

$$u(t, x) = \begin{cases} u^1(t, x) & \text{if } 0 \leq t < 1, \\ u^2(t, x) & \text{if } 1 \leq t \leq 2, \end{cases}$$

$$d\eta(t, x) = \begin{cases} d\eta^1(t, x) & \text{if } 0 \leq t < 1, \\ d\eta^2(t, x) + [1 - u^1(t, x)]^+\, dx\, \delta_1(t)\, dt & \text{if } 1 \leq t \leq 2. \end{cases}$$

Observe that here u and $t \to \int_0^t \eta'(ds, x)$ are discontinuous at $t = 1$ so we must relax slightly Definition 2.3.

Remark 2.11. Suppose (u, η) is the solution of (2.8) with $B_i \equiv 0$, $g_i \equiv 0$, and $u \geq \psi$. Define $\tilde{z} = u - M$. Then pathwise $\tilde{z}$ satisfies

$$\left(\frac{dz}{dt}, v - z\right) + (A_z, v - z) \geq (f - AM, v - z), \qquad \forall v \in V, \quad v \geq \psi,$$

$$z \geq \psi - M.$$

This is a deterministic variational inequality as considered by Bensoussan and Lions [1] which has a unique minimal weak solution (in their terminology). This solution is obtained by the same penalization process as is used in our work, so we can conclude that this minimal solution is $\tilde{z}$. But we obtain a strong solution with an increasing process $\eta(t)$ which is in general not absolutely continuous in t.

Remark 2.12. If $M \equiv 0$ and $g_i \equiv 0$, $i = 1, \ldots, d$, then η is absolutely continuous in both t and x, i.e., for some η''

$$\eta(dt, dx) = \eta'(dt, x)\, dx = \eta''(t, x)\, dt\, dx.$$

Indeed, from the arguments leading to (2.27) we have that $dx\, dP$ a.e.

$$u(t, x) + \int_0^t Au(s, x)\, ds$$
$$= u_0(x) + \int_0^t B_i u(s, x)\, dW_s^i + \int_0^t f(s, x)\, ds + \eta'(t, x), \qquad t \geq 0, \tag{2.32}$$

where $\eta'(t, x) = \eta'([0, t], x)$. Since $u(t, x) \geq 0$, then $B_i u(s, x) = 0$ a.e. on $\{(s, x): u(s, x) = 0\}$. Therefore (2.32) may be rewritten as

$$u(t, x) = u_0(x) + \int_0^t [f(s, x) - Au(s, x)]\, ds$$
$$+ \int_0^t 1_{\{u(s,x)>0\}} B_i u(s, x)\, dW_s^i + \eta'(t, x), \qquad t \geq 0. \tag{2.33}$$

Moreover, it follows easily from (iii) in Definition 2.3 that $dx\, dP$ a.e.

$$\int_0^t [\varphi(s) - u(s, x)]\eta'(ds, x) \geq 0, \qquad \forall \varphi \in C_+(R_+), \quad \forall t \geq 0. \tag{2.34}$$

From (2.33) and (2.34), it follows that, for almost all x, $\{(u(t, x), \eta'(t, x)): t \geq 0\}$ is the unique solution of the Skorohod problem (see [8]) associated with the process

$$\left\{u_0(x) + \int_0^t [f(s, x) - Au(s, x)]\, ds + \int_0^t 1_{\{u(s, x)>0\}} B_i u(s, x)\, dW_s^i : t \geq 0\right\}.$$

It is then easily seen that $\eta'(dt, x) = \eta''(t, x)\, dt$, with

$$\eta''(t, x) = 1_{\{u(t,x)=0\}}[f(t, x) - Au(t, x)]^-.$$

Remark 2.13. The condition $\{\alpha(x)=0: x\in\Gamma_0\}$ is imposed to obtain

$$|((Bu, u))|\le c\|u\|^2$$

which is used in the proof of Lemma 3.4 in order to estimate the fourth-order moments which appear in (2.14). We could in fact proceed somewhat differently; we obtain two estimates:

$$E\sup_{t\le T}\|u^\varepsilon(t)\|^2+E\int_0^T\|u^\varepsilon(t)\|_{H^2}^2\,dt\le c \tag{2.14}'$$

which will require no hypothesis on α, and

$$E\sup_{t\le T}|u^\varepsilon(t)|^4+E\left\{\left(\int_0^T\|u^\varepsilon(t)\|^2\,dt\right)^2\right\}\le c \tag{2.14}''$$

which will require that

$$|(Bu, u)|\le c|u|^2$$

for which we need $\{\alpha(x)=0: x\in\Gamma_1\}$. If α does not satisfy either of these conditions, the result can still be deduced under an appropriate stronger coercivity hypothesis. We observe that in both the Neumann and the Dirichlet problem either Γ_0 or Γ_1 is empty so the condition on α is satisfied.

Remark 2.14. We can obtain our result even with $O=(0,1)$ replaced by R (now $\Gamma=\varnothing$). There are two arguments which need to be changed. In step 1 of the existence proof we cannot use the basis $\{e_i\}$ since $(I-\Delta)$ now has a continuous spectrum. However, the estimates (2.14) and (2.15) can be obtained using the coercivity condition on the energy inequality both for u_ε, see (2.9), and for ∇u_ε which satisfies the differentiated (with respect to x) equation of (2.9) (see [12]). Furthermore, (2.21) has been established with the aid of a lemma of Lions which requires O to be bounded. However, for any $\eta>0$ there exists a constant $c(\eta)$ such that for any $u\in H^1(R)$

$$\sup_x|u(x)|<\eta\|u\|+c(\eta)|u|.$$

This follows readily from

$$u(x)^2=\int_{-\infty}^x 2\nabla u(y)u(y)\,dy.$$

3. Technical Lemmas

We now prove some technical results used in Section 2.

Lemma 3.1. *Assume $u, v\in H$, and define u^n by*

$$u^n=\sum_{j=1}^n (u, e_j)e_j. \tag{3.1}$$

Then

$$\sum_{j=1}^{n} \lambda_j(v, e_j)(u, e_j) = (v, u^n - \Delta u^n).$$

Proof.

$$\sum_{j=1}^{n} \lambda_j(v, e_j)(u, e_j) = \sum_{j=1}^{n} \lambda_j(v, e_j)(u^n, e_j) = \sum_{j=1}^{n} (v, e_j)[(u^n, e_j) + (\nabla u^n, \nabla e_j)]$$

$$= \sum_{j=1}^{n} (v, e_j)(u^n - \Delta u^n, e_j) = (v, u^n - \Delta u^n),$$

where the third equality follows since in the integration by parts the boundary term is zero due to (2.11) and (3.1), and where the last equality follows because $u^n - \Delta u^n$ is in $\text{Span}\{e_1, \ldots, e_n\}$. □

Corollary 3.2. *Assume u is in H and v is in V. Then*

$$\sum_{j=1}^{n} \lambda_j(v, e_j)(u, e_j) = ((v, u^n)),$$

where u^n is defined by (3.1).

Proof. $(v, u^n - \Delta u^n) = (v, u^n) + (\nabla v, \nabla u^n) = ((v, u^n))$ because the boundary terms in the integration by parts are zero by (2.11), (3.1), and the fact that v is in V. The result follows from Lemma 3.1. □

Corollary 3.3. *Assume that u is in V. Then*

$$\sum_{j=1}^{n} \lambda_j(u, e_j)^2 = \|u^n\|^2 \le \|u\|^2,$$

where u^n is defined by (3.1).

Proof. Since $(\nabla e_i, \nabla e_j) = 0$ if $i \ne j$, then

$$((u, u^n)) = ((u^n, u^n)) = \|u^n\|^2 \le \|u\|^2$$

and the result follows from Corollary 3.2. □

Lemma 3.4. *Assume that $u \in L^2(\Omega; L^2((0, T); V) \cap C([0, T]; H))$ satisfies*

$$u(t) = \sum_{j=1}^{n} (u(t), e_j)e_j, \tag{3.2}$$

$$(u(t), e_j) + \int_0^t a(u(s), e_j)\, ds$$

$$= (u_0, e_j) + \int_0^t (B_i u(s) + g_i(s), e_j)\, dW_s^i + (M_t, e_j)$$

$$+ \int_0^t (f(s), e_j)\, ds + \frac{1}{\varepsilon} \int_0^t (u(s)^-, e_j)\, ds, \qquad j = 1, \ldots, n. \tag{3.3}$$

Then there exists a constant c depending only on $E\langle M\rangle_T^2$, $\|u_0\|_{L^4(\Omega;V)}$, $\|f\|_{L^4(\Omega\times(0,T);H)}$, $\|g_i\|_{L^4(\Omega\times(0,T);V)}$, $i=1,\dots,d$, *such that*

$$E\left\{\sup_{t\le T}\|u(t)\|^4+\left[\int_0^T \|u(t)\|_{H^2}^2\,dt\right]^2\right\}\le c,$$

$$E\left\{\left[\int_0^T \|u(t)^-\|^2\,dt\right]^2\right\}\le \varepsilon^2 c.$$

Proof. It follows from (3.3) and Itô's lemma that for fixed $j\le n$

$$\begin{aligned}(u(t),e_j)^2&+2\int_0^t a(u(s),e_j)(u(s),e_j)\,ds\\&=(u_0,e_j)^2+2\int_0^t (B_iu(s)+g_i(s),e_j)(u(s),e_j)\,dW_s^i\\&\quad+2\int_0^t (u(s),e_j)d(M_s,e_j)\\&\quad+2\int_0^t (f(s),e_j)(u(s),e_j)\,ds+\frac{2}{\varepsilon}\int_0^t (u(s)^-,e_j)(u(s),e_j)\,ds\\&\quad+\sum_{i=1}^d\int_0^t (B_iu(s)+g_i(s),e_j)^2\,ds+\langle M_j\rangle_t,\end{aligned}\tag{3.4}$$

where $\langle M_j\rangle_t$ is the quadratic variation of the martingale (M_t,e_j).

Let us multiply (3.4) by λ_j and sum over j. From Lemma 3.1 and its corollaries we find that

$$\begin{aligned}&\sum_{j=1}^n \lambda_j(u(t),e_j)^2=\|u(t)\|^2,\\&\sum_{j=1}^n \lambda_j(B_iu(s)+g_i(s),e_j)(u(s),e_j)=((B_iu(s)+g_i(s),u(s))),\\&\sum_{j=1}^n \lambda_j(f(s),e_j)(u(s),e_j)=(f(s),u(s)-\Delta u(s)),\\&\sum_{j=1}^n \lambda_j(B_iu(s)+g_i(s),e_j)^2\le\|B_iu(s)+g_i(s)\|^2,\\&\sum_{j=1}^n \lambda_j(u(s)^-,e_j)(u(s),e_j)=((u(s)^-,u(s)))=-\|u(s)^-\|^2.\end{aligned}\tag{3.5}$$

Moreover, by Lemma 3.1

$$\begin{aligned}\sum_j \lambda_j a(u(s),e_j)(u(s),e_j)&=\sum_j \lambda_j(-\nabla(a\nabla u(s))+b\nabla u(s)+cu(s),e_j)(u(s),e_j)\\&=(-\nabla(a\nabla u(s))+b\nabla u(s)+cu(s),u(s)-\Delta u(s))\\&=(a\nabla u(s),\nabla u(s))+(\nabla(a\nabla u(s)),\Delta u(s))\\&\quad+(b\nabla u(s)+cu(s),u(s)-\Delta u(s)).\end{aligned}\tag{3.6}$$

Next let

$$M^n = \sum_{j=1}^{n} (M, e_j)e_j.$$

Then M^n is a V-valued martingale—in fact it is $H^2(O)$-valued. If $\Pi = \{t_0, t_1, \dots, t_m\}$ is any partition of $[0, t]$ we set $\delta_i M = M_{t_i} - M_{t_{i-1}}$. Then by Corollary 3.3,

$$\sum_{j=1}^{n} \lambda_j \langle M_j \rangle_t = \sum_j \lambda_j \lim_{|\Pi|\to 0} \sum_{i=1}^{m} (\delta_i M, e_j)^2 \le \lim_{|\Pi|\to 0} \sum_i \|\delta_i M\|^2 = \langle M \rangle_t. \tag{3.7}$$

Finally, by first working with simple (in s) functions it can be established that

$$\sum_{j=1}^{n} \lambda_j (u(s), e_j) d(M_s, e_j) = ((u(s), dM_s)). \tag{3.8}$$

Note that $\int ((u(s), dM_s))$ is well defined because

$$\|u(s)\|^2 = \sum_{j=1}^{n} (u(s), e_j)^2 \lambda_j \le \max_{1\le j\le n} \lambda_j |u(s)|^2$$

so that

$$E \int_0^T \|u(s)\|^2 \, d\langle M \rangle_s \le \max_{1\le j\le n} \lambda_j E \int_0^T |u(s)|^2 \, d\langle M \rangle_s$$
$$\le \max_{1\le j\le n} \lambda_j \|u\|^2_{L^2(\Omega; C([0,T]; H))} E\langle M \rangle_T.$$

Combining (3.4)-(3.8) gives

$$\begin{aligned}
&\|u(t)\|^2 + 2 \int_0^t (a\nabla u, \nabla u) + (\nabla a \nabla u, \Delta u) + (a\Delta u, \Delta u) \, ds \\
&\quad + 2 \int_0^t (b\nabla u + cu, u - \Delta u) \, ds \\
&\le \|u_0\|^2 + 2 \int_0^t ((B_i u + g_i, u)) \, dW_s^i + 2 \int_0^t ((u(s), dM_s)) \\
&\quad + 2 \int_0^t (f, u - \Delta u) \, ds - \frac{2}{\varepsilon} \int_0^t \|u^-\|^2 \, ds + \langle M \rangle_t \\
&\quad + \sum_{i=1}^{d} \int_0^t [(\alpha_i^2 \Delta u, \Delta u) + (\alpha_i \Delta u, (\nabla \alpha_i + \beta_i)\nabla u + u\nabla \beta_i + \nabla g_i) \\
&\qquad + |(\nabla \alpha_i + \beta_i)\nabla u + u \nabla \beta_i + \nabla g_i|^2] \, ds,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\|u(t)\|^2 + \theta \int_0^t \|u(s)\|^2_{H^2} \, ds + \frac{2}{\varepsilon} \int_0^t \|u(s)^-\|^2 \, ds \\
&\le \|u_0\|^2 + \int_0^t ((B_i u + g_i, u)) \, dW_s^i + 2 \int_0^t ((u, dM_s)) + 2 \int_0^t |f(s)| \, \|u(s)\|_{H^2} \, ds \\
&\quad + \langle M \rangle_t + c_0 \int_0^t \left[\|u\|_{H^2} \left(\|u\| + \sum_i \|g_i\| \right) + \|u\|^2 + \sum_i \|g_i\|^2 \right] ds,
\end{aligned}$$

where $\|\cdot\|_{H^2}$ is the norm in $H^2(O)$. If

$$t_N = \min\left\{T, \inf\left\{t \geq 0: \|u(t)\|^2 + \int_0^t \|u(s)\|_{H^2}^2 \, ds \geq N\right\}\right\}$$

and if t is a stopping time, $t \leq t_N$, then it follows that

$$\begin{aligned} E \sup_{s\leq t} \|u(s)\|^4 &+ E\left\{\left(\int_0^t \|u(s)\|_{H^2}^2 \, ds\right)^2\right\} + \varepsilon^{-2} E\left\{\left(\int_0^t \|u(s)^-\|^2 \, ds\right)^2\right\} \\ &\leq c_1 \left\{ E\|u_0\|^4 + E \sum_i \int_0^t ((B_i u + g_i, u))^2 \, ds + E \int_0^t \|u(s)\|^2 \, d\langle M\rangle_s \right. \\ &\quad + E\left\{\left(\int_0^t |f(s)| \, \|u(s)\|_{H^2} \, ds\right)^2\right\} + E\langle M\rangle_t^2 \\ &\quad + E\left\{\left(\int_0^t \|u\|_{H^2}\left(\|u\| + \sum_i \|g_i\|\right) ds\right)^2\right\} \\ &\quad \left. + E \int_0^t \left(\|u\|^4 + \sum_i \|g_i\|^4\right) ds\right\}. \end{aligned} \tag{3.9}$$

We have used the fact that

$$\begin{aligned} E \int_0^t ((B_i u + g_i, u))^2 \, ds &\leq \max_{0\leq j\leq n} \lambda_j E \sup_{s\leq t_N} \|u(s)\|^2 \int_0^{t_N} (\kappa\|u\|_{H^2}^2 + \|g_i\|^2) \, ds \\ &\leq \kappa_0 N(N+1) < \infty. \end{aligned}$$

Now observe that

$$(\nabla(\alpha_i \nabla u), \nabla u) = (\nabla\alpha_i \nabla u, \nabla u) + (\alpha_i \Delta u, \nabla u) = (\nabla\alpha_i \nabla u, \nabla u) - (\nabla u, \nabla(\alpha_i \nabla u))$$

since $\alpha_i(x)\nabla u(x) = 0$ if $x \in \Gamma$, by (2.11) and (3.2). Hence for some c_2, $\tilde{c}_1$

$$(\nabla(\alpha_i \nabla u), \nabla u) = \tfrac{1}{2}(\nabla\alpha_i \nabla u, \nabla u),$$

$$((B_i u + g_i, u)) \leq c_2 \|u\|(\|u\| + \|g_i\|),$$

$$\sum_i \int_0^t ((B_i u + g_i, u))^2 \, ds \leq \tilde{c}_1 \int_0^t \|u(s)\|^4 + \sum_i \|g_i\|^4 \, ds.$$

We now require a version of Young's inequality (see p. 138 of [4]): for any real positive x, y, η, p, q with $p^{-1} + q^{-1} = 1$ there exists $c < \infty$ such that

$$xy \leq \eta x^p + c y^q. \tag{3.10}$$

Then we can conclude that for $\tilde{c}_1$ sufficiently large

$$c_1 \int_0^t \|u(s)\|^2 \, d\langle M\rangle_s \leq \tfrac{1}{2} \sup_{s\leq t} \|u(s)\|^4 + \tilde{c}_1 \langle M\rangle_t^2,$$

$$c_1\left(\int_0^t |f(s)| \, \|u(s)\|_{H^2} \, ds\right)^2 \leq \tfrac{1}{4}\left(\int_0^t \|u(s)\|_{H^2}^2 \, ds\right)^2 + \tilde{c}_1 \int_0^t |f(s)|^4 \, ds,$$

$$\begin{aligned} c_1&\left(\int_0^t \|u(s)\|_{H^2}\left(\|u\| + \sum_i \|g_i\|\right) ds\right)^2 \\ &\leq \tfrac{1}{4}\left(\int_0^t \|u(s)\|_{H^2}^2 \, ds\right)^2 + \tilde{c}_1 \int_0^t \left(\|u(s)\|^4 + \sum_i \|g_i(s)\|^4\right) ds, \end{aligned}$$

and hence (3.9) implies

$$E \sup_{s \le t} \|u(s)\|^4 + E\left\{\left(\int_0^t \|u(s)\|_{H^2}^2 \, ds\right)^2\right\} + \varepsilon^{-2} E\left\{\left(\int_0^t \|u(s)^-\|^2 \, ds\right)^2\right\}$$

$$\le c_2 E\|u_0\|^4 + E\int_0^t \|u(s)\|^4 \, ds + \sum_i \|g_i\|_4^4 + \|f\|_4^4 + E\langle M\rangle_T^2,$$

where

$$\|g_i\|_4 = \|g_i\|_{L^4(\Omega\times(0,T);V)}, \qquad \|f\|_4 = \|f\|_{L^4(\Omega\times(0,T);H)}.$$

From Gronwall's lemma we obtain

$$E\left\{\sup_{t \le t_N} \|u(t)\|^4 + \left(\int_0^{t_N} \|u(t)\|_{H^2}^2 \, dt\right)^2\right\} \le c,$$

$$E\left\{\left(\int_0^{t_N} \|u^-(t)\|^2 \, dt\right)^2\right\} \le \varepsilon^2 c.$$

Since it now follows that $t_N \to T$ a.s. as $N \to \infty$ then the result follows. □

Lemma 3.5. *If* $v \in L^2(\Omega; C([0, T]; H) \cap L^2((0, T); V))$, $\Phi \in L^1(\Omega \times (0, T); R)$ *with* $\Phi(s) \ge 0$ *a.e., a.s. and if* v *satisfies*

$$|v(t)|^2 + 2\int_0^t a(v(s), v(s)) \, ds$$

$$\le 2\int_0^t (B_i v(s), v(s)) \, dW_s^i + \sum_{i=1}^d \int_0^t |B_i v(s)|^2 \, ds + \int_0^t \Phi(s) \, ds, \tag{3.11}$$

then there exists $c < \infty$ *depending on* T *and the coefficient of* A, B_i, *but not on* Φ *such that*

$$E \sup_{t \le T} |v(t)|^2 \le cE\int_0^T \Phi(t) \, dt.$$

Proof. There exists a constant c_1 such that

$$2a(v, v) - \sum_i |B_i v|^2 \ge \theta|\nabla v|^2 + 2(b\nabla v, v) + 2(cv, v) - 2(\alpha_i \nabla v, \beta_i v) - \sum |\beta_i v|^2$$

$$\ge \frac{\theta}{2}|\nabla v|^2 - c_1|v|^2.$$

Moreover,

$$E\left\{\left[\int_0^T (B_i v(t), v(t))^2 \, dt\right]^{1/2}\right\} \le E\left\{\sup_{t \le T}|v(t)|\left[\int_0^T |B_i v(t)|^2 \, dt\right]^{1/2}\right\}$$

$$\le \tfrac{1}{2}E\left\{\sup_{t \le T} |v(t)|^2\right\} + \frac{c}{2} E\int_0^T \|v(t)\|^2 \, dt < \infty;$$

$$i = 2, \ldots, d.$$

It then follows that the stochastic integrals in (3.11) are martingales, and taking the expectation in (3.11) we obtain

$$E|v(t)|^2+\frac{\theta}{2}E\int_0^t \|v(s)\|^2\,ds\leq c_2E\int_0^t |v(s)|^2\,ds+E\int_0^t \Phi(s)\,ds.$$

From this and Gronwall's lemma it follows that

$$E\int_0^T \|v(t)\|^2\,dt\leq\frac{2}{\theta}(c_1Te^{c_1T}+1)E\int_0^T \Phi(t)\,dt.$$

Now again from (3.11)

$$\begin{aligned}E\Big\{\sup_{t\leq T}|v(t)|^2\Big\}&\leq c_1E\int_0^T |v(t)|^2\,dt+E\int_0^T \Phi(t)\,dt\\&\quad+c_2\sum_{i=1}^d\left[E\left\{\int_0^T (B_iv(t),v(t))^2\,dt\right\}^{1/2}\right]\\&\leq\tfrac{1}{2}E\Big\{\sup_{t\leq T}|v(t)|^2\Big\}+E\int_0^T \Phi(t)\,dt+c_3E\int_0^T \|v(t)\|^2\,dt.\end{aligned}$$

The result follows from the last two inequalities. □

Lemma 3.6.

$$\lim_{\varepsilon\to 0} E\sup_{t\leq T}|u_\varepsilon^-(t)|^2=0.$$

Proof. Choose ρ in $C_b^2(R)$ such that for some constant $c>0$

$$|\rho(y)|+|\rho'(y)|+|\rho''(y)|\leq c,$$

$$\rho(y)\geq 0,\qquad y\in R,$$

$$\rho(0)=\rho'(0)=0,$$

where ρ' and ρ'' stand for the first and second derivatives of ρ, respectively. Define

$$\Phi(u)=\left\{\int_0^1 \rho(u(x))\,dx\right\}^2,$$

$$\Phi'(u)=\left\{2\int_0^1 \rho(u(x))\,dx\right\}\rho'(u(\cdot)),$$

$$\Phi''(u)=\left\{2\int_0^1 \rho(u(x))\,dx\right\}\rho''(u(\cdot))+2\rho'(u(\cdot))\otimes\rho'(u(\cdot)).$$

If u is in H, then $\rho'(u(\cdot))$ and hence $\Phi'(u)$ are in H. Observe that $v\to(\Phi'(u),v)$ is the (first) Fréchet derivative of Φ at u. Similarly, the bilinear form

$$[v,w]\to\left(2\int_0^1 \rho(u(x))\,dx\right)(\rho''(u)v,w)+2(\rho'(u),v)(\rho'(u),w)$$

is the second Fréchet derivative of Φ at u. Note that

$$|\Phi(u)|\le c^2, \qquad |\Phi'(u)|\le 2c^2,$$
$$|\Phi''(u)|_{\mathscr{L}(H,H)}\le 4c^2, \qquad \|\Phi'(u)\|\le 2c^2(1+\|u\|).$$

Moreover, $\Phi(\cdot)$, $\Phi'(\cdot)$, and $\Phi''(\cdot)$ are continuous on H and Φ' is continuous on V. We can now apply Itô's formula to Φ and (2.9) (see p. 65 of [11], or Theorem 1.2 of [12] in the case $M=0$). If we write $\psi(u)=\sqrt{\Phi(u)}$ we have

$$\begin{aligned}
&\Phi(u_\varepsilon(t))+2\int_0^t \psi(u_\varepsilon(s))(Au_\varepsilon(s),\rho'(u_\varepsilon(s)))\,ds\\
&\quad=\Phi(u_0)+2\int_0^t \psi(u_\varepsilon(s))(B_iu_\varepsilon(s)+g_i(s),\rho'(u_\varepsilon(s)))\,dW_s^i\\
&\qquad+2\int_0^t \psi(u_\varepsilon(s))(\rho'(u_\varepsilon(s)),dM_s)+2\int_0^t \psi(u_\varepsilon(s))(f(s),\rho'(u_\varepsilon(s)))\,ds\\
&\qquad+\frac{2}{\varepsilon}\int_0^t \psi(u_\varepsilon(s))(u_\varepsilon^-(s),\rho'(u_\varepsilon(s)))\,ds\\
&\qquad+\int_0^t \psi(u_\varepsilon(s))(\rho''(u_\varepsilon(s))[B_iu_\varepsilon(s)+g_i(s)],B_iu_\varepsilon(s)+g_i(s))\,ds\\
&\qquad+\sum_i\int_0^t (B_iu_\varepsilon(s)+g_i(s),\rho'(u_\varepsilon(s)))^2\,ds\\
&\qquad+\int_0^t \psi(u_\varepsilon(s))\,\mathrm{tr}[\rho''(u_\varepsilon(s))q_s^M]\,d\langle M\rangle_s^H\\
&\qquad+\int_0^t (q_s^M\rho'(u_\varepsilon(s)),\rho'(u_\varepsilon(s)))\,d\langle M\rangle_s^H.
\end{aligned}\tag{3.12}$$

Now observe that

$$\begin{aligned}
2(Au_\varepsilon,\rho'(u_\varepsilon))-(\rho''(u_\varepsilon)B_iu_\varepsilon,B_iu_\varepsilon)&=\left(\rho''(u_\varepsilon)\left[2a-\sum_i\alpha_i^2\right]\nabla u_\varepsilon,\nabla u_\varepsilon\right)\\
&\quad+2(b\nabla u_\varepsilon+cu_\varepsilon,\rho'(u_\varepsilon))-\left(\sum_i\beta_i^2u_\varepsilon+2\alpha_i\beta_i\nabla u_\varepsilon,\rho''(u_\varepsilon)u_\varepsilon\right)
\end{aligned}\tag{3.13}$$

because $u_\varepsilon\nabla u_\varepsilon=0$ on Γ, see Lemma 2.6, and $\rho'(0)=0$. Moreover, for some c_1 (independent of ρ)

$$\begin{aligned}
&2\int_0^t \psi(u_\varepsilon(s))(\rho''(u_\varepsilon(s))B_iu_\varepsilon(s),g_i(s))\,ds\\
&\quad\le\frac{\theta}{2}\int_0^t \psi(u_\varepsilon(s))(|\rho''(u_\varepsilon(s))|\nabla u_\varepsilon(s),\nabla u_\varepsilon(s))\,ds\\
&\qquad+c_1\int_0^t \psi(u_\varepsilon(s))(|\rho''(u_\varepsilon(s))|u_\varepsilon(s),u_\varepsilon(s))\,ds\\
&\qquad+c_1\int_0^t \psi(u_\varepsilon(s))(|\rho''(u_\varepsilon(s))|g_i(s),g_i(s))\,ds
\end{aligned}\tag{3.14}$$

so that (3.12)-(3.14) yield

$$
\begin{aligned}
\Phi(u_\varepsilon(t)) &+ \frac{\theta}{2}\int_0^t \psi(u_\varepsilon(s))(|\rho''(u_\varepsilon(s))|\nabla u_\varepsilon(s), \nabla u_\varepsilon(s))\,ds \\
&+2\theta\int_0^t \psi(u_\varepsilon(s))(\rho''(u_\varepsilon(s))1_{\{\rho''(u_\varepsilon(s))<0\}}\nabla u_\varepsilon(s), \nabla u_\varepsilon(s))\,ds \\
&\le \Phi(u_0) - 2\int_0^t \psi(u_\varepsilon(s))(b\nabla u_\varepsilon(s)+cu_\varepsilon(s), \rho'(u_\varepsilon(s)))\,ds \\
&+2\int_0^t \psi(u_\varepsilon(s))(\alpha_i\beta_i u_\varepsilon(s)\rho''(u_\varepsilon(s), \nabla u_\varepsilon(s)))\,ds \\
&+c_2\int_0^t \psi(u_\varepsilon(s))(|\rho''(u_\varepsilon(s))|u_\varepsilon(s), u_\varepsilon(s))\,ds \\
&+c_2\sum_i\int_0^t \psi(u_\varepsilon(s))(|\rho''(u_\varepsilon(s))|g_i(s), g_i(s))\,ds \\
&+2\int_0^t \psi(u_\varepsilon(s))(f(s), \rho'(u_\varepsilon(s)))\,ds \\
&+\frac{2}{\varepsilon}\int_0^t \psi(u_\varepsilon(s))(u_\varepsilon^-(s), \rho'(u_\varepsilon(s)))\,ds \\
&+\sum_i\int_0^t (B_i u_\varepsilon(s)+g_i(s), \rho'(u_\varepsilon(s)))^2\,ds \\
&+\int_0^t \psi(u_\varepsilon(s))\,\mathrm{tr}[\rho''(u_\varepsilon(s))q_s^M]\,d\langle M\rangle_s^H \\
&+\int_0^t (q_s^M\rho'(u_\varepsilon(s)), \rho'(u_\varepsilon(s)))\,d\langle M\rangle_s^H \\
&+2\int_0^t \psi(u_\varepsilon(s))(B_i u_\varepsilon(s)+g_i(s), \rho'(u_\varepsilon(s)))\,dW_s^i \\
&+2\int_0^t \psi(u_\varepsilon(s))(\rho'(u_\varepsilon(s)), dM_s),
\end{aligned}
\tag{3.15}
$$

where c_2 is independent of ρ. We define

$$
\rho_n(y)=\begin{cases} y^2 & \text{if } \; y\le -n^{-1}, \\ -n^3y^5-3n^2y^4-3ny^3 & \text{if } \; -n^{-1}\le u\le 0, \\ 0 & \text{if } \; y\ge 0. \end{cases}
$$

Note that ρ_n is in $C^2(R)$,

$$
0\le \rho(y)\le 7(y^-)^2, \qquad -26y^-\le \rho_n'(y)\le 0, \qquad 0\le \rho_n''(y)\le 74.
$$

Then $\rho_n''(y)=2$ if $y<-n^{-1}$. Define $\tilde\rho_n''$ by

$$\tilde\rho_n''(y)=\begin{cases}\rho''(y) & \text{if } y\geq -n,\\ 2(y+n+1) & \text{if } -(n+1)\leq y\leq -n,\\ \rho_n''(y-2n-2) & \text{if } y\leq -(n+1),\end{cases}$$

and let $\tilde\rho_n$ be the function whose second derivative is $\tilde\rho_n''$ and which agrees with ρ_n on $[-n,\infty)$. Then $\tilde\rho_n$ is in $C_b^2(R)$ and

$$0\leq \tilde\rho_n(y)\leq 7(y^-)^2,\qquad -26y^-\leq \tilde\rho_n'(y)\leq 0,$$

$$|\tilde\rho_n''(y)|\leq 74,\qquad \tilde\rho_n''(y)\geq 0\qquad \text{if}\quad y\geq -n$$

and, for each y in R,

$$\tilde\rho_n(y)\to(y^-)^2,\qquad \tilde\rho_n'(y)\to -2y^-,\qquad \tilde\rho_n''(y)\to 2\ 1_{\{y<0\}},$$

$$y\tilde\rho_n''(y)\to -2y^-,\qquad y|\tilde\rho_n''(y)|\to -2y^-$$

as $n\to\infty$. We take $\rho=\tilde\rho_n$ in (3.15) and then we let $n\to\infty$. Observe that $u_0^-=0$.

$$\begin{aligned}
&|u_\varepsilon^-(t)|^4+\theta\int_0^t |u_\varepsilon^-(s)|^2(1_{\{u_\varepsilon<0\}}\nabla u_\varepsilon(s),\nabla u_\varepsilon(s))\,ds\\
&\quad\leq 4\int_0^t |u_\varepsilon^-(s)|^2(b\nabla u_\varepsilon(s)+cu_\varepsilon(s),u_\varepsilon^-(s))\,ds\\
&\qquad -4\int_0^t |u_\varepsilon^-(s)|^2(\alpha_i\beta_i u_\varepsilon^-(s),\nabla u_\varepsilon(s))\,ds\\
&\qquad -2c_2\int_0^t |u_\varepsilon^-(s)|^2(u_\varepsilon^-(s),u_\varepsilon(s))\,ds\\
&\qquad +2c_2\int_0^t |u_\varepsilon^-(s)|^2\sum_i|g_i(s)|^2\,ds-4\int_0^t |u_\varepsilon^-(s)|^2(f(s),u_\varepsilon^-(s))\,ds\\
&\qquad -\frac{4}{\varepsilon}\int_0^t |u_\varepsilon^-(s)|^4\,ds+4\sum_i\int_0^t (B_iu_\varepsilon(s)+g_i(s),u_\varepsilon^-(s))^2\,ds\\
&\qquad +2\int_0^t |u_\varepsilon^-(s)|^2\operatorname{tr}\{1_{\{u_\varepsilon<0\}}q_s^M\}\,d\langle M\rangle_s^H\\
&\qquad +4\int_0^t (q_s^M u_s^-(s),u_s^-(s))\,d\langle M\rangle_s^H\\
&\qquad -4\int_0^t |u_\varepsilon^-(s)|^2(B_iu_\varepsilon(s)+g_i(s),u_\varepsilon^-(s))\,dW_s^i\\
&\qquad -4\int_0^t |u_\varepsilon^-(s)|^2(u_\varepsilon^-(s),dM_s).
\end{aligned}\tag{3.16}$$

Now observe that

$$(1_{\{u_\varepsilon<0\}}\nabla u_\varepsilon,\nabla u_\varepsilon)=|\nabla u_\varepsilon^-|^2$$

and for some constants c_θ, c_3

$$4(b\nabla u_\varepsilon, u_\varepsilon^-) = -4(b\nabla u_\varepsilon^-, u_\varepsilon^-) \le \frac{\theta}{2}|\nabla u_\varepsilon^-|^2 + c_\theta |u_\varepsilon^-|^2,$$

$$-4(\alpha_i\beta_i u_\varepsilon^-, \nabla u_\varepsilon) = 4(\alpha_i\beta_i u_\varepsilon^-, \nabla u_\varepsilon^-) \le \frac{\theta}{2}|\nabla u_\varepsilon^-|^2 + c_\theta |u_\varepsilon^-|^2,$$

$$4\sum_i \int_0^t (B_i u_\varepsilon(s) + g_i(s), u_\varepsilon^-(s))^2\, ds$$

$$\le 8\sum_i \int_0^t (\alpha_i \nabla u_\varepsilon(s), u_\varepsilon^-(s))^2 + (b_i u_\varepsilon(s) + g_i(s), u_\varepsilon^-(s))^2\, ds$$

$$\le \tfrac{1}{4}\sup_{s\le t}|u_\varepsilon^-(s)|^4 + c_3\left(\int_0^t \|u_\varepsilon^-(s)\|^2\, ds\right)^2 + c_3\int_0^t |u_\varepsilon^-(s)|^4\, ds$$

$$+ c_3 \sum_i \int_0^t |g_i(s)|^2 |u_s^-(s)|^2\, ds,$$

$$\int_0^t \{2|u_\varepsilon^-(s)|^2 \operatorname{tr}[1_{\{u_\varepsilon<0\}} q_s^M] + 4(q_s^M u_\varepsilon^-(s), u_\varepsilon^-(s))\}\, d\langle M\rangle_s^H$$

$$\le 6\int_0^t |u_\varepsilon^-(s)|^2\, d\langle M\rangle_s^H.$$

But for some constant κ

$$(B_i u_\varepsilon(s), u_\varepsilon^-(s))^2 \le \kappa \|u_\varepsilon^-(s)\|^2 |u_\varepsilon^-(s)|^2$$

and by (3.10) for any $\eta > 0$ there exists $\kappa' < \infty$ such that for any ε

$$\left(\int_0^t |u_\varepsilon^-(s)|^6 \|u_\varepsilon^-(s)\|^2\, ds\right)^{1/2} \le \sup_{s\le t}|u_\varepsilon^-(s)|^3 \left(\int_0^t \|u_\varepsilon^-(s)\|^2\, ds\right)^{1/2}$$

$$\le \eta \sup_{s\le t}|u_\varepsilon^-(s)|^4 + \kappa'\left(\int_0^t \|u_\varepsilon^-(s)\|^2\, ds\right)^2,$$

so that for c_3 sufficiently large

$$4\left(\int_0^t |u_\varepsilon^-(s)|^4 (B_i u_\varepsilon(s) + g_i(s), u_\varepsilon^-(s))^2\, ds\right)^{1/2}$$

$$= \tfrac{1}{4}\sup_{s\le t}|u_\varepsilon^-(s)|^4 + c_3\left(\int_0^t \|u_\varepsilon^-(s)\|^2\, ds\right)^2 + c_3\sum_i \int_0^t |g_i(s)|^2 |u_\varepsilon^-(s)|^2\, ds.$$

Similarly,

$$4\left(\int_0^t |u_\varepsilon^-(s)|^6\, d\langle M\rangle_s^H\right)^{1/2} \le \tfrac{1}{4}\sup_{s\le t}|u_\varepsilon^-(s)|^4 + c_3\int_0^t |u_\varepsilon^-(s)|^2\, d\langle M\rangle_s^H.$$

These last two inequalities and (2.14) and (2.15) imply that the stochastic integrals in (3.16) are martingales and so (3.16) and the above imply that

$$E\left\{\tfrac{1}{4}\sup_{s\le t}|u_\varepsilon^-(s)|^4+\frac{4}{\varepsilon}\int_0^t|u_\varepsilon^-(s)|^4\,ds\right\}$$
$$\le c_4\left\{E\int_0^t|u_\varepsilon^-(s)|^4\,ds+E\int_0^t|u_\varepsilon^-(s)|^2\sum|g_i(s)|^2\,ds\right.$$
$$+E\int_0^t|u_\varepsilon^-(s)|^3|f(s)|\,ds+E\left[\left(\int_0^t\|u_\varepsilon^-(s)\|^2\,ds\right)^2\right]$$
$$\left.+E\int_0^t|u_\varepsilon^-(s)|^2\,d\langle M\rangle_s^H\right\}. \tag{3.17}$$

But (3.10) applied to $|u^-|^3|f|$, and (2.14) and (2.15) as well as the square integrability of $m(s)=d\langle M\rangle_s^H/ds$ imply that the right-hand side of (3.17) is bounded by a constant, hence

$$E\int_0^t|u_\varepsilon^-(s)|^4\,dt=O(\varepsilon). \tag{3.18}$$

From Hölder's inequality and (2.15), (3.17), and (3.18) it now follows that

$$E\sup_{s\le t}|u_\varepsilon^-(s)|^4\le c_5(\varepsilon+\varepsilon^{1/2}+\varepsilon^{3/4}+\varepsilon^2+\varepsilon^{1/2})$$

and we are done. □

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